Dissipative Analysis and Control for Discrete-time State-space Symmetric Systems*

WANG Shuping[1], ZHANG Guoshan[1], LIU Wanquan[2]

1. School of Electrical Engineering and Automation, Tianjin University, Tianjin 300072, P.R.China
E-mail: wangshuping@tju.edu.cn, zhanggs@tju.edu.cn

2. The Department of Computing, Curtin University of Technology, WA, 6102, Australia
E-mail: W.Liu@curtin.edu.au

Abstract: In this paper, the problem of dissipativity analysis and output feedback control synthesis for discrete linear time-invariant systems with state-space symmetry is investigated. Firstly, an explicit expression of H_∞ norm for discrete-time symmetric system is given under the mixed H_∞ and positive real performance criterion, and this is a particular case of dissipative systems. Subsequently, we consider the control synthesis problems for such systems and obtain an explicit parameterized expression of the static output feedback controllers. Finally, two numerical examples are employed to show the effectiveness and reliableness of the proposed approach.
Key Words: Dissipativity Analysis and Control, Static Output Feedback Controller, H_∞ NORM, Positive Real, Discrete-Time State-Space Symmetric Systems

1 INTRODUCTION

Dissipative systems description based on energy-related consideration are of particular interest in engineering and physics. Typical examples of such systems are electrical network, visco-elastic systems, thermodynamic systems [1]. An abstract study of dissipative dynamical systems had been given in [2], which includes applications of dissipativeness in the stability analysis of linear systems with certain nonlinear feedback. In recent years, there are a wide range of topics involving in the dissipative theory, such as the passivity theorem [3], Kalman-Yakubovich lemma [4, 5], the bounded real lemma [6, 7], the circle criterion [8] and the references therein. We know only consider the bounded real (gain margin) or the positive real (phase margin) that is not a sufficient indicator for the robustness of such systems, especially while the gain and phase margin are changed at the same time [9]. So there are many study on the two performance criterions considered together. The researchers in [10] considers the quadratic dissipative control for linear time invariant discrete-time systems. In [11], the authors investigate the robust quadratic dissipative analysis and control for linear systems. The feedback quadratic dissipativity properties for general nonlinear discrete systems is considered in [12].

The state-space symmetric systems, as described in [13, 14], have also received considerable attention because of the special structure and better control properties. Such systems include large-space structures, electrical and power network systems, chemical reactions and so on. The researchers in [15] consider the model reduction problem for the state-space symmetric systems. In [16], the quadratic dissipative analysis and control problems for continuous-time systems with state-space symmetric are investigated. The obtained results show that the property of symmetry often offers some advantages in the analysis and synthesis for these systems.

In this paper, we focus on the problems of quadratic dissipative control for discrete linear time-invariant state-space

symmetric systems. First, based on the necessary and sufficient conditions for quadratic dissipativity of discrete linear time-invariant (LTI) state-space symmetric systems, we give an explicit expression for the H_∞ performance index γ only using the system parameters and the weighting parameter. Then we study the mixed H_∞ and positive real performance analysis and control problem for such systems. Our aim is to achieve an explicit solutions for those state-space symmetric systems. For this purpose, an explicit expressions of the optimally achieved closed-loop H_∞ performance and a parametric expressions of the optimal control gains are derived for the static output feedback control synthesis problem. Finally, we give two numerical examples to illustrate the results.

For the sake of the self-contained completeness of this article, we will introduce the following notations. Let M be a matrix of complex numbers with proper dimension, M^T and M^* stand for the transpose and the Hermitian of M, respectively; $M > 0$ (or $M < 0$) means that M is positive (or negative) definite; while, by $M \geqslant 0$ ($M \leqslant 0$), M is positive (or negative) semi-definite. $M^\perp$ is the orthogonal complementation of M, which can be computed from singular value decomposition. The Moore$-$Penrose generalized inverse of a matrix M will be denoted by M^+ and $\lambda_{\max}(M)$ denotes the maximum eigenvalue of square matrix M. The identity matrix with dimension $r \times r$ is denoted by I_r, and we simply use I to indicate any identity matrix with proper dimension.

2 PRELIMINARIES

In order to introduce some definitions for quadratic dissipativity, we first consider the following general dynamical system:

$$
\begin{aligned}
x(k+1) &= f((x(k), w(k)), \quad x(0) = x_0, \\
z(k) &= g((x(k), w(k)),
\end{aligned}
\tag{1}
$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^m$ is the input, $z \in \mathbb{R}^p$ is the output, $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are known vector functions.

Let us introduce the following quadratic energy supply function E with $T \geqslant 0$.

$$E(w,z,T) = \prec z, Qz \succ_T + 2 \prec z, Sw \succ_T + \prec w, Rw \succ_T, \tag{2}$$

where Q, S and R are real matrices of appropriate dimensions with matrices Q and R symmetric, and $\prec u,v \succ_T \triangleq \sum_{k=0}^{T} u'(k)v(k)$.

Definition 1 [10] Given matrices Q, S and R where Q and R are symmetric, the system (1) with energy supply function E is called $(Q,S,R)-$dissipative if for some real function $\beta(\cdot)$ with $\beta(0) = 0$,

$$E(w,z,T) + \beta(x_0) \geqslant 0, \tag{3}$$

for all $T \geqslant 0$ and all $w \in l_2[0,T]$. Further, if for any $T > 0$ and some sufficiently small scalar $\alpha > 0$,

$$E(w,z,T) + \beta(x_0) \geqslant \alpha \prec w, w \succ_T, \tag{4}$$

Then system (1) is called strictly (Q,S,R)-dissipative.

Remark 1 The notion of strict $(Q,S,R)-$dissipativity includes H_∞ and passivity as special cases:

(a) When $Q = -I, S = 0$ and $R = \gamma^2 I$, strict $(Q,S,R)-$dissipativity reduces to a H_∞ norm constraint;

(b) When $Q = 0$, $S = I$, $R = 0$, (4) reduces to the strict positive realness;

(c) When $Q = -\theta I$, $S = (1-\theta)I$, $R = \theta\gamma^2 I$, $\theta \in (0,1)$ or $Q = -\gamma^{-1}\theta I$, $S = (1-\theta)I$, $R = \gamma\theta I$, $\theta \in (0,1)$, (4) represents a mixed H_∞ and positive real performance;

(d) When $Q = -I$, $S = \frac{1}{2}(K_1 + K_2)^T$, $R = -\frac{1}{2}(K_1^T K_2 + K_2^T K_1)$, where K_1 and K_2 are constant matrices of appropriate dimensions, (4) corresponds to a sector bounded constraint.

Remark 2 Note that the extremum of θ are interpreted as $\theta \to 0$ which corresponds to positive realness of the system, and $\theta \to 1$ which corresponds to the H_∞ performance.

In the following of this section, we focus on the linear discrete time invariant systems described by:

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \ x(0) = x_0, \\ z(k) &= Cx(k) + Dw(k), \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^m$ is the input, $z \in \mathbb{R}^p$ is the output, and $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ are known real constant matrices.

We say that the state-space representation (5) is state-space symmetric if the following conditions hold:

$$A = A^T, \ B = C^T, \ D = D^T. \tag{6}$$

The transfer function of system (5) is given by

$$T(z) = C(zI - A)^{-1}B + D. \tag{7}$$

Then for given matrices Q, S and R, we need to derive the conditions for the system (5) such that it is $(Q,S,R)-$dissipative. To this end, we make the following assumptions concerning the system (5) and the matrices Q, S, R.

Assumption 1
(a) $M \triangleq R + D^T S + S^T D + D^T Q D > 0$;
(b) $Q_- \triangleq -Q \geqslant 0$.

Remark 3 In the case of H_∞ performance problem, (a) of assumption 1 simplifies to $M = \gamma^2 I - D^T D > 0$; In the case of strict positive realness problem, the part (a) of assumption 1 simplifies to $M = D + D^T > 0$, which is a necessary condition for solving those problems. Furthermore, it should be observed that part (b) of assumption 1 holds for all the special cases in Remark 1.

Lemma 1 [Finsler's Lemma] [17]
Consider matrices M and Q such that M has full column rank and $Q = Q^T$. Then the following statements are equivalent:

(i) There exists a scalar μ such that

$$\mu MM^T - Q > 0. \tag{8}$$

(ii) The following condition holds:

$$M^\perp Q M^{\perp T} < 0. \tag{9}$$

If the above statements hold, then all scalars μ satisfying (8) are given by

$$\mu > \mu_{\min} := \lambda_{\max}\{M^+(Q - QM^{\perp T}(M^\perp Q M^{\perp T})^{-1} M^\perp Q)M^{+T}\}. \tag{10}$$

where M^+ is Moore-Penrose generalized inverse of real matrix M, $M^\perp$ with maximum row rank that satisfies $M^\perp M = 0$ and $M^\perp M^{\perp T} > 0$ is the orthogonal complement of real matrix M.

Lemma 2 [18] Consider matrices $U \in \mathbb{R}^{n \times m}, V \in \mathbb{R}^{k \times n}, W = W^T \in \mathbb{R}^{n \times n}, \text{rank}U = m < n, \text{rank}V = k < n$. Then there exists a matrix G satisfying the following matrix inequality:

$$UGV + (UGV)^T + W < 0, \tag{11}$$

if and only if the following two conditions hold:

$$U^\perp W U^{\perp T} < 0, \ V^{T\perp}W V^{T\perp T} < 0. \tag{12}$$

If these conditions are satisfied then one such solution G is given by

$$G = -R^{-1}U^T \phi V^T (V\phi V^T)^{-1}, \tag{13}$$

where R is an arbitrary positive definite matrix such that

$$\phi = (UR^{-1}U^T - W)^{-1} > 0. \tag{14}$$

Lemma 3 [10] Let the matrices Q, S and R be given with Q and R symmetric. Consider the system (5) satisfying Assumption 1 and the matrix A is a Hurwitz matrix. Then, the following statements are equivalent:

(a) System (5) is strictly $(Q,S,R)-$dissipative with $\beta(0) = 0$.

(b) There exists a matrix $P = P^T > 0$ such that

$$\begin{bmatrix} -P^{-1} & -A & -B & 0 \\ -A^T & -P & -C^T S & C^T Q_-^{1/2} \\ -B^T & -S^T C & -D^T S - S^T D - R & D^T Q_-^{1/2} \\ 0 & Q_-^{1/2}C & Q_-^{1/2}D & -I \end{bmatrix} < 0. \tag{15}$$

Lemma 4 [17] Consider matrices Γ and Λ such that Γ has full column rank, and Λ is symmetric positive definite. Then $\Lambda \geqslant \Gamma\Gamma^T$ if and only if $\lambda_{\max}(\Gamma^T\Lambda^{-1}\Gamma) \leqslant 1$.

3 MAIN RESULTS

3.1 Dissipative Analysis of Discrete-time Symmetric Systems

In this section, we will provide the dissipativity analysis for discrete-time state-space symmetric systems. We first consider the case with $D = 0$. Then the results will be extended to the general case with $D \neq 0$.

Theorem 1 Consider the symmetric system (5) with the symmetry conditions (6) and $D = 0$. Q, S, R satisfy the Assumption 1 and (c) of Remark 1. Then the H_∞ norm of the system (5) can be explicitly obtained from the following form:

$$||T(z)||_\infty = \max\left[\lambda_{\max}(\frac{\alpha + \theta - 1}{\theta}\Omega_1), \lambda_{\max}(\frac{\alpha - \theta + 1}{\theta}\Omega_2)\right] \tag{16}$$

where $\alpha, \Omega_1, \Omega_2$ are given by

$$\alpha = \sqrt{\theta^2 + (1 - \theta)^2}, \Omega_1 = B^{\mathrm{T}}(I - A)^{-1}B, \\ \Omega_2 = B^{\mathrm{T}}(I + A)^{-1}B. \tag{17}$$

Proof From the Lemma 3, (c) of Remark 1 and $P = \alpha I$, we can derive following expression:

$$\begin{bmatrix} -\alpha^{-1}I & 0 & -A & -B \\ 0 & -I & \sqrt{\frac{\theta}{\gamma}}B^{\mathrm{T}} & 0 \\ -A^{\mathrm{T}} & B\sqrt{\frac{\theta}{\gamma}} & -\alpha I & (\theta - 1)B \\ -B^{\mathrm{T}} & 0 & (\theta - 1)B^{\mathrm{T}} & -\gamma\theta I \end{bmatrix} < 0. \tag{18}$$

Pre- and Post-multiply (18) by $\mathrm{diag}\{I, \sqrt{\gamma\theta}I, I, I\}$, then

$$\begin{bmatrix} \alpha^{-1}I & 0 & A & B \\ 0 & \gamma\theta I & -\theta B^{\mathrm{T}} & 0 \\ A^{\mathrm{T}} & -\theta B & \alpha I & (1 - \theta)B \\ B^{\mathrm{T}} & 0 & (1 - \theta)B^{\mathrm{T}} & \gamma\theta I \end{bmatrix} > 0. \tag{19}$$

This can be rewritten into the form as $\gamma MM^{\mathrm{T}} - Z > 0$, where M and Z are defined as follows:

$$M = \begin{bmatrix} 0 & \sqrt{\theta}I & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\theta}I \end{bmatrix}^{\mathrm{T}}, \\ Z = \begin{bmatrix} -\alpha^{-1}I & 0 & -A & -B \\ 0 & 0 & \theta B^{\mathrm{T}} & 0 \\ -A^{\mathrm{T}} & \theta B & -\alpha I & (\theta - 1)B \\ -B^{\mathrm{T}} & 0 & (\theta - 1)B^{\mathrm{T}} & 0 \end{bmatrix} \tag{20}$$

It is noteworthy that

$$M^\perp = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, M^+ = \begin{bmatrix} 0 & \frac{I}{\sqrt{\theta}} & 0 & 0 \\ 0 & 0 & 0 & \frac{I}{\sqrt{\theta}} \end{bmatrix} \tag{21}$$

Using (9) of the Finsler's Lemma, it results in

$$\begin{bmatrix} \alpha^{-1}I & A \\ A^{\mathrm{T}} & \alpha I \end{bmatrix} > 0.$$

Using the Schur Complement Lemma, we can obtain that $-I < A < I$. Obviously, A is a Hurwitz matrix.

The formula in (10) provides the expression for the H_∞ norm of the system (5) as follows:

$$\gamma > \lambda_{\max} \begin{bmatrix} \frac{\theta}{\alpha}B^{\mathrm{T}}(I - A^2)^{-1}B \\ B^{\mathrm{T}}(I - A^2)^{-1}AB + \frac{\theta - 1}{\alpha}B^{\mathrm{T}}(I - A^2)^{-1}B \end{bmatrix}$$

$$\begin{matrix} B^{\mathrm{T}}A(I - A^2)^{-1}B + \frac{\theta - 1}{\alpha}B^{\mathrm{T}}(I - A^2)^{-1}B \\ (\frac{\alpha}{\theta}B^{\mathrm{T}}(I - A^2)^{-1}B + \frac{\theta - 1}{\theta}B^{\mathrm{T}}A(I - A^2)^{-1}B \\ + \frac{\theta - 1}{\theta}B^{\mathrm{T}}(I - A^2)^{-1}AB + \frac{(1 - \theta)^2}{\theta\alpha}B^{\mathrm{T}}(I - A^2)^{-1}B) \end{matrix} .$$

Our goal is to get an explicit expression. So we can simplify the above of inequality into the following form:

$$\gamma > \lambda_{\max} \begin{bmatrix} I & 0 \\ \frac{\alpha + \theta - 1}{\theta}I & I \end{bmatrix} \\ \begin{bmatrix} \frac{\theta}{2\alpha}(\Omega_1 + \Omega_2) & \frac{\alpha + \theta - 1}{2\alpha}\Omega_1 + \frac{\theta - 1 - \alpha}{2\alpha}\Omega_2 \\ -\Omega_2 & \frac{\alpha - \theta + 1}{\theta}\Omega_2. \end{bmatrix} \tag{22}$$

From the algebra tools, we know that $\lambda_i(AB) = \lambda_i(BA)$. Therefore, we derive (22) into the following style.

$$\gamma > \lambda_{\max} \begin{bmatrix} \frac{\alpha + \theta - 1}{\theta}\Omega_1 & \frac{\alpha + \theta - 1}{2\alpha}\Omega_1 + \frac{\theta - 1 - \alpha}{2\alpha}\Omega_2 \\ 0 & \frac{\alpha - \theta + 1}{\theta}\Omega_2 \end{bmatrix} .$$

Since the above matrix is an upper triangular matrix, then the H_∞ norm of system (5) will be given by (16).

So the proof of this theorem is complete.

Remark 4 From Remark 2, we know that $\theta \to 1$ corresponding to the H_∞ performance. So when $\theta \to 1$, (16) can be computed in the following form:

$$||T(z)||_\infty = \max[\lambda_{\max}(\Omega_1), \lambda_{\max}(\Omega_2)],$$

which is consistent with the result of paper [13].

Theorem 2 Consider the state-space symmetric system (5) with the symmetry conditions (6). Q, S, R satisfy the Assumption 1 conditions and (c) of Remark 1. Then the H_∞ norm of the system (5) can be explicitly obtained from the following form:

$$||T(z)||_\infty = \max\left[\lambda_{\max}(\frac{\alpha + \theta - 1}{\theta}[D + \Omega_1]), \\ \lambda_{\max}(\frac{\alpha - \theta + 1}{\theta}[-D + \Omega_2])\right]. \tag{23}$$

where $\alpha, \Omega_1, \Omega_2$ are given as (17).

Proof As the proof of theorem 1, from the Lemma 3, (c) of Remark 1 and $P = \alpha I$, we can obtain that

$$\begin{bmatrix} \alpha^{-1}I & 0 & A & B \\ 0 & I & -\sqrt{\frac{\theta}{\gamma}}B^{\mathrm{T}} & -\sqrt{\frac{\theta}{\gamma}}D \\ A & -B\sqrt{\frac{\theta}{\gamma}} & \alpha I & (1 - \theta)B \\ B^{\mathrm{T}} & -\sqrt{\frac{\theta}{\gamma}}D & (1 - \theta)B^{\mathrm{T}} & 2(1 - \theta)D + \gamma\theta I \end{bmatrix} > 0. \tag{24}$$

Pre- and post-multiply (24) by $\mathrm{diag}\{I, \sqrt{\gamma\theta}I, I, I\}$, then (24) can be rewritten into the form as $\gamma MM^\mathrm{T} - Z' > 0$, where Z' are defined as follows:

$$Z' = \begin{bmatrix} -\alpha^{-1}I & 0 & -A & -B \\ 0 & 0 & \theta B^\mathrm{T} & \theta D \\ -A & \theta B & -\alpha I & (\theta-1)B \\ -B^\mathrm{T} & \theta D & (\theta-1)B^\mathrm{T} & 2(\theta-1)D \end{bmatrix} \quad (25)$$

Using (9) of the Finsler's Lemma and (21), it results in

$$\begin{bmatrix} -\alpha^{-1}I & -A \\ -A & -\alpha I \end{bmatrix} < 0.$$

Here, we can also obtain that matrix A is a Hurwitz matrix by using Schur Complement Lemma, and the formula in (10) provides an expression for the H_∞ norm of the system (5) as follows:

$$\gamma > \lambda_{\max} \begin{bmatrix} \dfrac{\theta}{2\alpha}(\Omega_1 + \Omega_2) \\ D + \dfrac{\alpha+\theta-1}{2\alpha}\Omega_1 + \dfrac{\theta-1-\alpha}{2\alpha}\Omega_2 \\ D + \dfrac{\alpha+\theta-1}{2\alpha}\Omega_1 + \dfrac{\theta-1-\alpha}{2\alpha}\Omega_2 \\ \dfrac{(\alpha+\theta-1)^2}{2\alpha\theta}\Omega_1 + \dfrac{(\alpha-\theta+1)^2}{2\alpha\theta}\Omega_2 + \dfrac{2(\theta-1)}{\theta}D \end{bmatrix}$$

Recall that our purpose is to obtain an explicit expression. So we must simplify above inequality.

From the matrix algebra tools, we know that similar matrices have the same characteristic polynomial and the eigenvalues. Then let $T = \begin{bmatrix} I & 0 \\ -\dfrac{\alpha+\theta-1}{\theta}I & I \end{bmatrix}$, we derive that the following expression is equal to the above inequality.

$$\gamma > \lambda_{\max} \begin{bmatrix} \dfrac{\alpha+\theta-1}{\theta}(D+\Omega_1) & * \\ 0 & \dfrac{\alpha-\theta+1}{\theta}(-D+\Omega_2) \end{bmatrix} \quad (26)$$

Since (26) is an upper triangular, then the H_∞ norm of system (5) will be given by (23).

So the proof of this theorem is complete.

Remark 5 If the weighting parameter $\theta \to 1$, from theorem 2, we can derive that

$$||T(z)||_\infty = \max[\lambda_{\max}(D+\Omega_1), \lambda_{\max}(-D+\Omega_2)].$$

which is consistent with the conclusion of [13].

3.2 Output Feedback Dissipative Control Synthesis Problem

In this section, we will provide an explicit solution to the strictly $(Q, S, R)-$dissipative output feedback control problem in terms of LMIs.

Now consider the following state-space system representation:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1\omega(k) + B_2u(k) \\ z(k) &= C_1x(k) + D_{11}\omega(k) \\ y(k) &= C_2x(k) \end{aligned} \quad (27)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control inputs, $\omega(k) \in \mathbb{R}^q$ is the exogenous input which belongs to $l_2[0, \infty)$, $z(k) \in \mathbb{R}^p$ is the controlled output, $y(k) \in \mathbb{R}^r$ is the measurement output, and $A, B_1, B_2, C_1, C_2, D_{11}$ are real constant matrices of appropriate dimensions and satisfy the following symmetry conditions

$$A = A^\mathrm{T}, B_1 = C_1^\mathrm{T}, B_2 = C_2^\mathrm{T}, D_{11} = D_{11}^\mathrm{T}. \quad (28)$$

The static output feedback mixed H_∞ and positive real control synthesis problem consists of designing a static feedback gain G such that the control law

$$u(k) = Gy(k), \ G = G^\mathrm{T}. \quad (29)$$

which make the closed-loop system stable and guarantees a mixed H_∞ and positive real performance.

Then we can obtain that the closed-loop system of the state-space symmetric systems (27) and the controller (29) as follows.

$$\begin{aligned} x(k+1) &= (A + B_2GC_2)x(k) + B_1\omega(k) \\ z(k) &= C_1x(k) + D_{11}\omega(k). \end{aligned} \quad (30)$$

Obviously, the closed-loop system (30) is still symmetric. The following results gives explicit expressions for the H_∞ norm of the system and the corresponding optimal controller gain for the mixed H_∞ norm and positive real performance synthesis problem. Without loss of generality, we consider the case of $D_{11} = 0$.

Lemma 5 Consider the state-space symmetric system (30), where $D_{11} = 0$. There exists a symmetric output feedback control law (29) such that the closed-loop system is stable if and only if

$$B_2^\perp(A^2 - I)B_2^{\perp\mathrm{T}} < 0.$$

If this condition is satisfied, all stabilizing symmetric output feedback gains G are given by

$$\begin{cases} B_2^+[(A+I)\Theta_1(A+I) - (A+I)]B_2^{+\mathrm{T}} < G \\ G < B_2^+[(A-I)\Theta_2(A-I) - (A-I)]B_2^{+\mathrm{T}}, \end{cases}$$

where $\Theta_1 = B_2^{\perp\mathrm{T}}(B_2^\perp(A+I)B_2^{\perp\mathrm{T}})^{-1}B_2^{\perp\mathrm{T}}$, $\Theta_2 = B_2^{\perp\mathrm{T}}(B_2^\perp(A-I)B_2^{\perp\mathrm{T}})^{-1}B_2^{\perp\mathrm{T}}$.

Proof We can use the Lyapunov inequality and the Generalized Finslers Lemma or theorem 8 of [13] to prove above lemma. Here the detail is omitted.

Theorem 3 Consider the state-space symmetric system (27) with the control law (29). Then the optimal H_∞ norm γ^* can be computed by

$$\gamma^* \geqslant \frac{1}{\alpha\theta}\lambda_{\max}\left\{ \begin{bmatrix} \alpha B_1^\mathrm{T} & (1-\theta)B_1^\mathrm{T} \\ 0 & -\theta B_1^\mathrm{T} \end{bmatrix} \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp\mathrm{T}} \right.$$
$$\left(\begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} I & A \\ A & I \end{bmatrix} \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp\mathrm{T}} \right)^{-1} \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp}$$
$$\left. \begin{bmatrix} \alpha B_1 & 0 \\ (1-\theta)B_1 & -\theta B_1 \end{bmatrix} \right\}$$

$$(31)$$

where $\theta \in (0, 1)$ represents the weighting between H_∞ performance and positive real performance.

For any $\gamma > \gamma^*$, a static output feedback H_∞ control gain which make the closed-loop system stable with H_∞ norm less than γ can be selected as

$$G = \frac{\overline{G} + \overline{G}^\mathrm{T}}{2}, \ \overline{G} = R^{-1}U^\mathrm{T}\Phi V^\mathrm{T}(V\Phi V^\mathrm{T})^{-1}, \quad (32)$$

where R is an arbitrary positive-definite matrix such that $\Phi = (UR^{-1}U^{\mathrm{T}} + W)^{-1} > 0$, and

$$U = \begin{bmatrix} B_2^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}, V = \begin{bmatrix} 0 & B_2^{\mathrm{T}} \end{bmatrix},$$

$$W = \begin{bmatrix} I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} & A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} \\ A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} & I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} \end{bmatrix}. \qquad (33)$$

Proof Based on the (c) of Lemma 4, we can obtain that the closed-loop system of controller $u = \overline{G}y$ satisfy the following conditions:

$$\begin{bmatrix} \alpha^{-1}I & K & B_1 & 0 \\ K^{\mathrm{T}} & \alpha I & (1-\theta)B_1 & -\sqrt{\dfrac{\theta}{\gamma}}B_1 \\ B_1^{\mathrm{T}} & (1-\theta)B_1^{\mathrm{T}} & \gamma\theta I & 0 \\ 0 & -\sqrt{\dfrac{\theta}{\gamma}}B_1^{\mathrm{T}} & 0 & I \end{bmatrix} > 0, \qquad (34)$$

where $K = A + B_2\overline{G}B_2^{\mathrm{T}}$.

Pre- and post-multiply (34) by $\mathrm{diag}\{\alpha I, I, I, \sqrt{\theta\gamma}I\}$ and its transpose, respectively, we get

$$\begin{bmatrix} \alpha I & \alpha K & \alpha B_1 & 0 \\ \alpha K^{\mathrm{T}} & \alpha I & (1-\theta)B_1 & -\theta B_1 \\ \alpha B_1^{\mathrm{T}} & (1-\theta)B_1^{\mathrm{T}} & \gamma\theta I & 0 \\ 0 & -\theta B_1^{\mathrm{T}} & 0 & \gamma\theta I \end{bmatrix} > 0. \qquad (35)$$

Applying the Schur Complement Lemma as well as further simplifying the (35), then we can derive that

$$\begin{bmatrix} 0 & B_2\overline{G}B_2^{\mathrm{T}} \\ (B_2\overline{G}B_2^{\mathrm{T}})^{\mathrm{T}} & 0 \end{bmatrix} + \\ \begin{bmatrix} I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} & A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} \\ A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} & I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} \end{bmatrix} > 0, \qquad (36)$$

which can be rewritten as follows:

$$\begin{bmatrix} B_2 \\ 0 \end{bmatrix} \overline{G} \begin{bmatrix} 0 & B_2^{\mathrm{T}} \end{bmatrix} + \left(\begin{bmatrix} B_2 \\ 0 \end{bmatrix} \overline{G} \begin{bmatrix} 0 & B_2^{\mathrm{T}} \end{bmatrix} \right)^{\mathrm{T}} + \\ \begin{bmatrix} I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} & A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} \\ A - \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} & I - \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} \end{bmatrix} > 0. \qquad (37)$$

Obviously, $\overline{G}^{\mathrm{T}}$ is a solution of (37) if $\overline{G}$ is a solution. Then we can observe that $G = (\overline{G} + \overline{G}^{\mathrm{T}})/2$ is also a solution of (37). So we always can obtain a symmetric solution of (37) from a nonsymmetric one. Using Lemma 2 on (37) will result in (32), and the solvability condition is obtained as follows:

$$U^{\perp}WU^{\perp T} > 0, \qquad (38)$$

where U, W defined as (33). Then we derive that

$$\begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} I & A \\ A & I \end{bmatrix} \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp T} \\ > \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp} \begin{bmatrix} \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} & \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} \\ \dfrac{1-\theta}{\theta\gamma}B_1B_1^{\mathrm{T}} & \dfrac{\alpha}{\theta\gamma}B_1B_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} B_2 \\ 0 \end{bmatrix}^{\perp T}. \qquad (39)$$

From previous results, we know that the right of (39) is semi-positive definite, so the left of the (39) is positive. Using Lemma 5, we can obtain (31).

Remark 6 The same as the theorem 13 of [9], if the state-space symmetric dynamic control law

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c y(k) \\ u(k) &= C_c x(k) + D_c y(k) \end{aligned} \qquad (40)$$

can make the state-space symmetric system satisfy the dissipativity with H_∞ norm and positive real performance, then so does the static output feedback control law (29).

4 NUMERICAL EXAMPLES

In this section, we will give two examples to demonstrate that the expression solutions of dissipative control is consistent with the direct solving linear matrix inequalities.

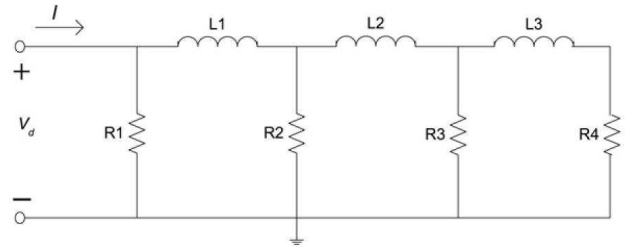**Example 1** Consider the RL circuit network systems shown in Fig. 1 as [16].

Fig. 1 RL Circuit Network

We choose the currents of the inductors L_1, L_2 and L_3 as state variable x_i, $i = 1, 2, 3$, the disturbance voltage V_d as the disturbance input w, and the current I as the output z. Assume that $L_1 = L_2 = L_3 = 1$, and $R_1 = 1$, $R_2 = 2$, $R_3 = 3$, $R_4 = 4$. Using the Kirchhoff's circuit laws, we can derive a symmetric state-space system with the following data:

$$\hat{A} = \hat{A}^{\mathrm{T}} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 3 & -7 \end{bmatrix}, \hat{B} = \hat{C}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{D} = 1.$$

Using the bilinear transformation of the RL circuit network system $\{\hat{A}; \hat{B}; \hat{C}; \hat{D}\}$ for a sampling interval $h = 0.02$ results in a discrete-time space representation $\{A; B; C; D\}$ with the following data:

$$A = A^{\mathrm{T}} = \begin{bmatrix} 0.9615 & 0.0374 & 0.0010 \\ 0.0374 & 0.9070 & 0.0535 \\ 0.0010 & 0.0535 & 0.8707 \end{bmatrix}, D = 1.0098,$$
$$C = B^{\mathrm{T}} = \begin{bmatrix} 0.1387 & 0.0026 & 0.0001 \end{bmatrix}.$$

Using the Matlab tools, we can derive that the RL circuit network is a stable system easily.

Let the weighting parameter θ vary in the interval $(0, 1)$, we can plot the relationships between θ and H_∞ norm of the open-loop system, which based on the theorem 2 and LMI Tools of Matlab, respectively. It is noteworthy that they are in perfect match. This comparison is illustrated in folllowing Fig.2.
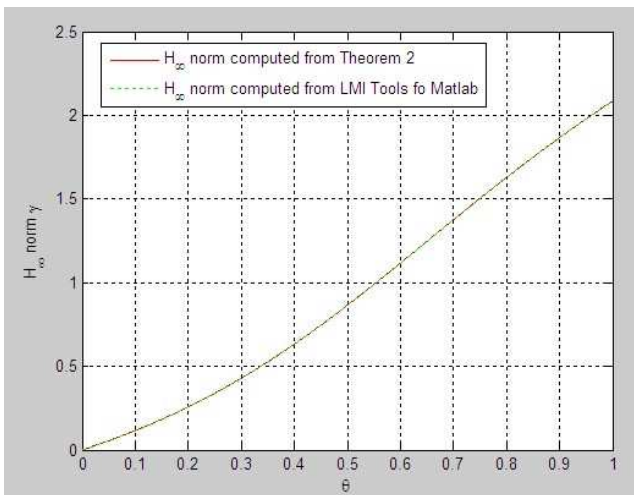
Fig. 2 The relationship between θ and γ

Example 2 Consider the discrete-time state-space symmetric system (27) with following data:

$$A = A^{\mathrm{T}} = \left[\begin{array}{cc} 1 & 0.5 \\ 0.5 & 1 \end{array} \right], B_1 = B_2 = C_1^{\mathrm{T}} = C_2^{\mathrm{T}} = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$$
$$D_{11} = 0.$$

Using the Matlab tools and based on the theorem 3, let the weighing parameter θ vary in the interval (0,1), we can obtain the H_∞ norm γ of the closed-loop system, respectively. There relationship is expressed as follows Fig. 3.
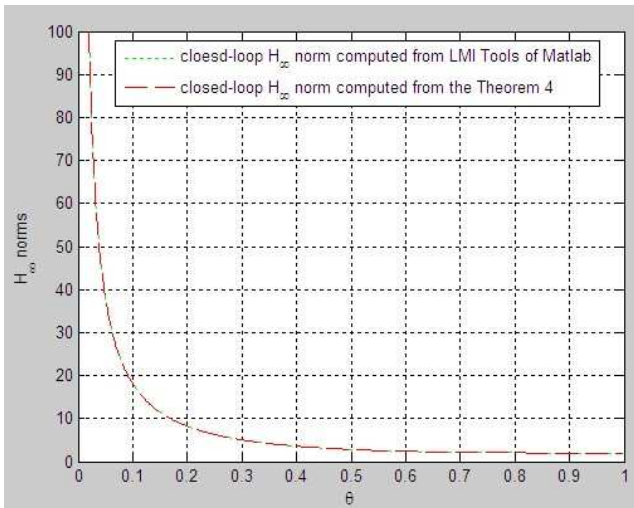

Fig. 3 The relationship between θ and γ

These two examples not only show that the calculated results by using the explicit expression of Theorem 2 and Theorem 3 are the same as the iterative solution by using the Matlab tools, but also have very clear advantages of the calculation.

5 CONCLUSION

In this paper we have addressed the dissipativity analysis and static output feedback control synthesis for discrete linear time-invariant systems with state-space symmetry. In the particular case of dissipative systems, an explicit expression of H_∞ norm for discrete-time symmetric system have been given. We also have derived the explicit parametrization of the output feedback controllers for such closed-loop systems. The results are expressed in terms of the state space form without need for solving the extremely cumbersome solution due to the matrix coupling. The results have more computational advantages than the existing ones.

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