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A Direct Optimization Method for Low Group Delay FIR Filter Design

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Abstract

This paper studies the design of FIR filter with low group delay, where the desired phase response is not being approximated. It is formulated as a constrained optimization problem, which is then solved globally. Numerical experiments show that our design method can produce a filter with smaller group delay than that obtained by the existing convex optimization method used in conjunction with a minimum phase spectral factorization method under the same design criteria. Furthermore, our formulation offers us the flexibility for the trade-off between the group delay and the magnitude response directly. It also allows the feasibility of imposing constraints on the group delay.

Keywords: Low group delay, duality, semi-definite programming

1. Introduction

For filters with the same filter lengths, it is well known that nonlinear phase FIR filters can have smaller group delay and achieve better frequency selectivity than linear phase FIR filters if they are designed properly. Thus, nonlinear phase FIR filters have a wide range of applications in communications [1, 2]. A representative example is in telecommunications where long delays are known to cause problems of echo and singing on voice lines [3].

There are many methods available in the literature for nonlinear phase FIR filter design [1-13]. For the design of FIR filter with desired phase response, it can be formulated either as a convex quadratic optimization problem with convex quadratic constraints [4] or as a nonconvex optimization problem [2], depending on whether or not the constraints on the group delay are imposed directly. For the case without constraints being imposed on its group delay, the formulated quadratic convex optimization problems in [4] are easy to solve by existing software packages. However, if the constraints on the group delay are imposed directly, the formulated optimization problem in [2] is highly complicated and only local optimal solutions can be ensured. To overcome this difficulty, the non-convex constraints on group delay are relaxed to convex constraints such that the relaxed optimization problem can be solved by the semi-definite programming (SDP) [5]. Although a good solution can be obtained in some cases, there is no guarantee that it will do

the same in general.

For the design of FIR filter without taking into consideration of its group delay, it can be formulated as a semi-definite quadratic optimization problem with linear constraints (see [6] and [7]). By solving this convex programming problem, the autocorrelation of the designed filter is obtained. Then, a spectral factorization procedure is applied to construct the original filter. If a low group delay is required, then a minimum phase extraction method is applied. Traditionally, the design of such a filter is referred to as the minimum phase FIR filter design. There are many methods, which are available for the minimum phase filter design [3-8]. The methods in [8] and [9] are mainly based on Herrmann's design procedure [10] in which the frequency response of odd-length equiripple linear-phase prototype filter is shifted up by one-half of its stopband's peak-to-peak ripple to acquire the second-order zeros on the unit circle of the z -plane. In [3], a design procedure is proposed to design minimum-phase filter based on root moments while keeping the same magnitude response. However, it requires to start from a linear-phase filter. In [11], a design procedure from a mixed-phase filter is proposed to fill the drawback in [3]. All the aforementioned methods for the minimum phase filter design are required to have a given FIR filter. On this basis, its minimum phase part are being extracted. The smallest group delay of the minimum phase filter is obtained from the family of filters with the same magnitude

response. For a general constrained filter design, the smallest group delay cannot be obtained through the design of a minimum phase filter.

In this paper, we study the design of a low group delay FIR filter without imposing constraints on its phase response. Unlike the conventional minimum phase FIR filter design [3-8], we formulate it as an optimization problem directly and solve it globally. Numerical experiments show that our design method can produce a filter with smaller group delay than those obtained by the existing convex optimization method used in conjunction with a minimum phase extraction method under the same design criteria. Our formulation offers us the flexibility for the trade-off between the group delay and the magnitude response. Furthermore, it allows the feasibility of imposing constraints on the group delay. To achieve this task, we first derive a new simple criterion for group delay. Based on this newly derived criterion, the design of a low group delay nonlinear phase FIR filter is formulated as a fourth order polynomial optimization problem with quadratic constraints. However, the solution of this problem is difficult to obtain, as solving the p -norm FIR filter design problem is a challenge. The main contribution of this paper is a novel and simple computational scheme to solve this difficult optimization problem based on the canonical duality developed in [15]. Since the objective function and the constraints for our formulated problem are either quadratic or the composite of two quadratic functions, the solution of the for-

mulated optimization problem may be non-unique, causing difficulty in its convex reformulation [16]. To tackle this issue, we introduce a small linear perturbation term to break its symmetry. Under the canonical duality theory [15], the global solution of this linearly perturbed optimization problem is obtained via solving a simple SDP problem. Since the perturbation term is very small, the global solution of the perturbed problem can be viewed as an approximate global solution of the original problem. To achieve better performance, a gradient-based optimization method can be applied to refine the current approximate global solution. Numerical experiments are presented to compare the results obtained by our method and those obtained by existing methods. From the comparison, it is clearly observed that our method offers flexibility on the trade-off between group delay and the magnitude response. This feature is not shared by the existing methods. Furthermore, our method gives rise to a smaller group delay than those obtained by existing methods under the same design criteria.

2. Problem Formulation

Consider an FIR digital filter of order K with transfer function $H(z)$ given by

$$H(z) = \sum_{k=0}^K h_k z^{-k}, \quad (1)$$

where h_k , $k = 0, 1, \dots, K$, are real filter coefficients. The frequency response $H(\omega)$ can be expressed as

$$H(\omega) = |H(\omega)|e^{j\phi(\omega)}, \quad (2)$$

where $|H(\omega)|$ denotes the magnitude of $H(\omega)$, while $\phi(\omega)$ is the phase response which is related to the group delay $\tau(\omega)$ as shown by

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}. \quad (3)$$

Let $|H_d(\omega)|$ be the desired magnitude response given by

$$|H_d(\omega)| = \begin{cases} 1, & \text{if } \omega \in B_p, \\ 0, & \text{if } \omega \in B_s, \end{cases} \quad (4)$$

where B_p is the passband and B_s is the stopband. Taking derivative on both sides of equation (2), we obtain

$$\frac{dH(\omega)}{d\omega} = -j\tau(\omega)H(\omega) + \frac{d|H(\omega)|}{d\omega}e^{j\phi(\omega)}. \quad (5)$$

In light of (1), we have

$$\sum_{k=0}^K kh_k e^{-jk\omega} = j\frac{dH(\omega)}{d\omega}. \quad (6)$$

By the Parseval equality, it gives

$$\begin{aligned} \sum_{k=0}^K kh_k^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(j\frac{dH(\omega)}{d\omega}\right) H^*(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \tau(\omega) |H(\omega)|^2 d\omega + j\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d|H(\omega)|}{d\omega} |H(\omega)| d\omega, \end{aligned} \quad (7)$$

where $H^*(\omega)$ is the conjugate of $H(\omega)$. Since $|H(\omega)|$ is an even function, it follows that

$$\int_{-\pi}^{\pi} \frac{d|H(\omega)|}{d\omega} |H(\omega)| d\omega = 0.$$

Thus,

$$\sum_{k=0}^K kh_k^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tau(\omega) |H(\omega)|^2 d\omega \approx \frac{1}{2\pi} \int_{B_p \cup B_t} \tau(\omega) |H(\omega)|^2 d\omega, \quad (8)$$

where B_t is the transition band. In (8), it is shown that $\sum kh_k^2$ is a good measurement of the group delay in the passband and the transition band. Our extensive numerical experiments show that minimization of $\sum kh_k^2$ can achieve a desired low group delay filter in the passband. This formula has been used in [17] to design narrow band lowpass filters with low group delay. Let $\mathbf{h} = [h_0, h_1, \dots, h_K]^T$ and $D = \text{diag}(0, 1, \dots, K)$. Now we pose the design of a low group delay filter with the prescribed magnitude responses given by (4) as the following optimization problem

$$\min_{\mathbf{h}} P(\mathbf{h}) = \alpha \mathbf{h}^T D \mathbf{h} + (1 - \alpha) \int_{B_p} (|H(\omega)|^2 - 1)^2 d\omega \quad (9)$$

$$\text{subject to } |H(\omega)|^2 \leq \varepsilon, \forall \omega \in B_s, \quad (10)$$

where α is a weighting factor and ε is a ripple specification in the stopband. Let this problem be referred to as Problem (P). In the absence of the first quadratic term in $P(\mathbf{h})$, Problem (P) can be reformulated as a new quadratic optimization problem with linear constraints expressed in terms of the auto-correlation of the designed filter $H(z)$ ([6], [7]). By solving this reformulated

quadratic optimization problem with linear constraints together with spectral factorization, the original filter coefficients are obtained. During the computation process, only linear matrix inequality (LMI) and spectral factorization are involved. Thus, the method proposed in [6] and [7] is effective. However, due to the presence of the first quadratic term in (9), this method is not applicable to Problem (P). In the following sections, we will develop an efficient method to solve Problem (P).

3. Solution Method

During the computational process, the one-dimensional integral that appeared in (9) is evaluated by using Gaussian quadrature with M nodes, i.e.,

$$\int_{B_p} (|H(\omega)|^2 - 1)^2 d\omega = \sum_{m=1}^M \nu_m (|H(\omega_m^p)|^2 - 1)^2 = \sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2, \quad (11)$$

where ν_m , $m = 1, \dots, M$, are weights, ω_m^p , $m = 1, \dots, M$, are the Gaussian quadrature points, $\Psi(\omega) = C(\omega)C^T(\omega) + S(\omega)S^T(\omega)$, $C(\omega) = [1, \cos(\omega), \dots, \cos(K\omega)]^T$ and $S(\omega) = [0, \sin(\omega), \dots, \sin(K\omega)]^T$. The functional inequality constraint (10) is discretized into finite number of constraints given below:

$$|H(\omega_n^s)|^2 = \mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} \leq \varepsilon, \quad n = 1, \dots, N, \quad (12)$$

where $\omega_n^s \in B_s$, $n = 1, \dots, N$, and N is the number of the discretization points. Now Problem (P) is approximated by the following optimization

problem.

$$\min_{\mathbf{h}} \tilde{P}(\mathbf{h}) = \alpha \mathbf{h}^T D \mathbf{h} + (1 - \alpha) \sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2 \quad (13)$$

$$\text{subject to } \mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} \leq \varepsilon, \quad n = 1, \dots, N. \quad (14)$$

Let this problem be referred to as Problem (P_I) . Since (13) and (14) are all symmetric with respect to \mathbf{h} , the solution of Problem (P_I) may not be unique which could lead to some difficulties in its convex reformulation. To break the symmetry, we introduce a linear small perturbation to (13) as it was suggested in [18]. Under this linear perturbation, (13) becomes

$$\tilde{P}_g(\mathbf{h}) = \alpha \mathbf{h}^T D \mathbf{h} + (1 - \alpha) \sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2 - \mathbf{g}^T \mathbf{h}, \quad (15)$$

where \mathbf{g} is a small perturbation vector. Replacing (13) by (15) in Problem (P_I) , we obtain a perturbed optimization problem which is referred to as Problem (P_{Ig}) . If \mathbf{g} is small enough, then the global solution of Problem (P_{Ig}) can be viewed as an approximate global solution of Problem (P_I) . Thus, it is of crucial importance if Problem (P_{Ig}) can be solved efficiently. In the following, we introduce the canonical duality developed in [15] to solve Problem (P_{Ig}) .

The Lagrangian function for Problem (P_{Ig}) is defined as

$$\begin{aligned} L(\mathbf{h}, \boldsymbol{\lambda}) = & \alpha \mathbf{h}^T D \mathbf{h} + (1 - \alpha) \sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2 - \mathbf{g}^T \mathbf{h} \\ & + \sum_{n=1}^N \lambda_n (\mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} - \varepsilon), \end{aligned} \quad (16)$$

where $\boldsymbol{\lambda} \geq 0$. \mathbf{h}^* is called a KKT point of Problem (P_{Ig}) if there exists a $\boldsymbol{\lambda}^*$ such that

$$\begin{cases} \nabla_{\mathbf{h}} L(\mathbf{h}^*, \boldsymbol{\lambda}^*) = 0, \\ \lambda_n^* ((\mathbf{h}^*)^T \Psi(\omega_n^s) \mathbf{h}^* - \varepsilon) = 0, \quad n = 1, \dots, N, \\ \boldsymbol{\lambda}^* = [\lambda_1^*, \dots, \lambda_N^*]^T \geq 0. \end{cases} \quad (17)$$

Let $\xi_m = \mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1$, $m = 1, \dots, M$, and $V(\boldsymbol{\xi}) = \boldsymbol{\xi}^T B \boldsymbol{\xi}$, where $\boldsymbol{\xi} = [\xi_1, \dots, \xi_M]^T$ and $B = (1 - \alpha) \text{diag} \{\nu_1, \dots, \nu_M\}$. Clearly, $V(\boldsymbol{\xi})$ is a strictly convex function. Thus, (16) can be rewritten as

$$\bar{L}(\mathbf{h}, \boldsymbol{\xi}, \boldsymbol{\lambda}) = \alpha \mathbf{h}^T D \mathbf{h} + V(\boldsymbol{\xi}) + \sum_{n=1}^N \lambda_n (\mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} - \varepsilon) - \mathbf{g}^T \mathbf{h}. \quad (18)$$

Let $\boldsymbol{\varsigma} = \nabla V(\boldsymbol{\xi}) = 2B\boldsymbol{\xi}$. The Fenchel conjugate of $V(\boldsymbol{\xi})$ is defined [19] as

$$V^*(\boldsymbol{\varsigma}) = \boldsymbol{\varsigma}^T \boldsymbol{\xi} - V(\boldsymbol{\xi}) = \frac{1}{4} \boldsymbol{\varsigma}^T B^{-1} \boldsymbol{\varsigma}$$

Replacing $V(\boldsymbol{\xi})$ by $\boldsymbol{\varsigma}^T \boldsymbol{\xi} - V^*(\boldsymbol{\varsigma})$ in (18), the following total complementary function [15] is obtained.

$$\Xi(\mathbf{h}, \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \sum_{m=1}^M \varsigma_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1) - \frac{1}{4} \boldsymbol{\varsigma}^T B^{-1} \boldsymbol{\varsigma} + \sum_{n=1}^N \lambda_n (\mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} - \varepsilon) + \alpha \mathbf{h}^T D \mathbf{h} - \mathbf{g}^T \mathbf{h}. \quad (19)$$

Taking $\nabla_{\mathbf{h}} \Xi(\mathbf{h}, \boldsymbol{\varsigma}, \boldsymbol{\lambda}) = 0$, we have

$$2G(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) \mathbf{h} = \mathbf{g},$$

where

$$G(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = \alpha D + \sum_{m=1}^M \varsigma_m \Psi(\omega_m^p) + \sum_{n=1}^N \lambda_n \Psi(\omega_n^s).$$

If $G(\boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is invertible, then $\mathbf{h} = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}, \boldsymbol{\lambda})\mathbf{g}$. Substituting this \mathbf{h} into (19), we obtain the canonical dual function

$$P_{I_g}^d(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) = -\frac{1}{4}\mathbf{g}^T G^{-1}(\boldsymbol{\varsigma}, \boldsymbol{\lambda})\mathbf{g} - \frac{1}{4}\boldsymbol{\varsigma}^T B^{-1}\boldsymbol{\varsigma} - \sum_{m=1}^M \varsigma_m - \varepsilon \sum_{n=1}^N \lambda_n. \quad (20)$$

The canonical dual problem ($P_{I_g}^d$) is defined as follows:

$$\max_{\boldsymbol{\varsigma}, \boldsymbol{\lambda}} \{P_{I_g}^d(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) | \boldsymbol{\lambda} \geq 0\}. \quad (21)$$

The Lagrangian function $\tilde{L}(\boldsymbol{\varsigma}, \boldsymbol{\lambda}, \boldsymbol{\sigma})$ of Problem ($P_{I_g}^d$) is defined as

$$\tilde{L}(\boldsymbol{\varsigma}, \boldsymbol{\lambda}, \boldsymbol{\sigma}) = P_{I_g}^d(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) + \boldsymbol{\sigma}^T \boldsymbol{\lambda}, \quad (22)$$

where $\boldsymbol{\sigma} \geq 0$. $(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ is called a KKT point of Problem ($P_{I_g}^d$) if there exists a $\boldsymbol{\sigma}^*$ such that

$$\begin{cases} \nabla_{\boldsymbol{\varsigma}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0, \nabla_{\boldsymbol{\lambda}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0, \\ (\boldsymbol{\sigma}^*)^T \boldsymbol{\lambda}^* = 0, \\ \boldsymbol{\sigma}^* = [\sigma_1^*, \dots, \sigma_N^*]^T \geq 0, \boldsymbol{\lambda}^* = [\lambda_1^*, \dots, \lambda_N^*]^T \geq 0. \end{cases} \quad (23)$$

For Problem (P_{I_g}) and Problem ($P_{I_g}^d$), we have the following theorem.

Theorem 1 *Suppose that $(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ is a KKT point of Problem ($P_{I_g}^d$). If $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \succ 0$, then $\mathbf{h}^* = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)\mathbf{g}$ is the global minimizer of Problem (P_{I_g}). Furthermore, $\tilde{P}_g(\mathbf{h}^*) = P_{I_g}^d(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$.*

Proof. 1). We first show that \mathbf{h}^* is a feasible solution of Problem (P_{I_g}). Furthermore, $\tilde{P}_g(\mathbf{h}^*) = P_{I_g}^d(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$.

Since $(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ is a KKT point of Problem (P_{Ig}^d) , there exists a $\boldsymbol{\sigma}^*$ such

that

$$\begin{cases} \nabla_{\boldsymbol{\varsigma}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0, \nabla_{\boldsymbol{\lambda}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0, \\ (\boldsymbol{\sigma}^*)^T \boldsymbol{\lambda}^* = 0, \\ \boldsymbol{\sigma}^* \geq 0, \boldsymbol{\lambda}^* \geq 0. \end{cases} \quad (24)$$

By direct computation, we have

$$\nabla_{\boldsymbol{\varsigma}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = -\frac{1}{2} B^{-1} \boldsymbol{\varsigma}^* - \mathbf{1}_M + \begin{bmatrix} \frac{1}{4} \mathbf{g}^T G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \Psi(\omega_1^p) G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g} \\ \dots \\ \frac{1}{4} \mathbf{g}^T G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \Psi(\omega_M^p) G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g} \end{bmatrix}, \quad (25)$$

and

$$\nabla_{\boldsymbol{\lambda}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = -\varepsilon \mathbf{1}_N + \boldsymbol{\sigma}^* + \begin{bmatrix} \frac{1}{4} \mathbf{g}^T G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \Psi(\omega_1^s) G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g} \\ \dots \\ \frac{1}{4} \mathbf{g}^T G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \Psi(\omega_N^s) G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g} \end{bmatrix}. \quad (26)$$

where $\mathbf{1}_M \in \mathbb{R}^M = [1, 1, \dots, 1]^T$. Substituting $\mathbf{h}^* = \frac{1}{2} G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g}$ into (26)

and noting that $\nabla_{\boldsymbol{\lambda}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0$ and $\boldsymbol{\sigma}^* \geq 0$, we obtain

$$(\mathbf{h}^*)^T \Psi(\omega_n^s) (\mathbf{h}^*) - \varepsilon = -\sigma_n^* \leq 0.$$

Thus, $\mathbf{h}^* = \frac{1}{2} G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{g}$ is a feasible point of Problem (P_{Ig}) . Furthermore,

$$\lambda_n^* ((\mathbf{h}^*)^T \Psi(\omega_n^s) (\mathbf{h}^*) - \varepsilon) = -\lambda_n^* \sigma_n^* = 0. \quad (27)$$

Let $\boldsymbol{\xi}^* = [(\mathbf{h}^*)^T \Psi(\omega_1^p) \mathbf{h}^* - 1, \dots, (\mathbf{h}^*)^T \Psi(\omega_M^p) \mathbf{h}^* - 1]^T$. Then, it follows from (25) and $\nabla_{\boldsymbol{\varsigma}} \tilde{L}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) = 0$ that

$$\boldsymbol{\varsigma}^* = 2B\boldsymbol{\xi}^* = \nabla V(\boldsymbol{\xi}^*).$$

Thus, $\boldsymbol{\varsigma}^*$ and $\boldsymbol{\xi}^*$ is a conjugate duality pair and the following equality holds

$$(\boldsymbol{\varsigma}^*)^T \boldsymbol{\xi}^* = V(\boldsymbol{\xi}^*) + V^*(\boldsymbol{\varsigma}^*). \quad (28)$$

Therefore,

$$\begin{aligned} \tilde{P}_g(\mathbf{h}^*) &= \alpha(\mathbf{h}^*)^T D\mathbf{h}^* + V(\boldsymbol{\xi}^*) - \mathbf{g}^T \mathbf{h}^* \\ &= \bar{L}(\mathbf{h}^*, \boldsymbol{\xi}^*, \boldsymbol{\lambda}^*) \quad (\text{by (27)}) \\ &= \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \quad (\text{by (28)}). \end{aligned}$$

It is easy to verify that $\Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) = P_{Ig}^d(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ since $\mathbf{h}^* = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)\mathbf{g}$.

Thus,

$$\tilde{P}_g(\mathbf{h}^*) = \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) = P_{Ig}^d(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*). \quad (29)$$

2). Now we show that $\mathbf{h}^* = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)\mathbf{g}$ is the global minimizer of Problem (P_{Ig}) . Since \mathbf{h}^* is a feasible solution, we only need to show that for any \mathbf{h} , we have

$$\tilde{P}_g(\mathbf{h}) \geq \tilde{P}_g(\mathbf{h}^*), \quad \forall \mathbf{h} \text{ satisfying (14)}.$$

In fact, for any \mathbf{h} satisfying (14), we have

$$\begin{aligned}
\tilde{P}_g(\mathbf{h}) &\geq \alpha \mathbf{h}^T D \mathbf{h} + (1 - \alpha) \sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2 \\
&\quad - \mathbf{g}^T \mathbf{h} + \sum_{n=1}^N \lambda_n^* (\mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} - \varepsilon) \quad (\mathbf{h} \text{ satisfying (14)}) \\
&\geq \sum_{m=1}^M \varsigma_m^* (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1) - \frac{1}{4} (\boldsymbol{\varsigma}^*)^T B^{-1} \boldsymbol{\varsigma}^* \\
&\quad + \sum_{n=1}^N \lambda_n^* (\mathbf{h}^T \Psi(\omega_n^s) \mathbf{h} - \varepsilon) + \alpha \mathbf{h}^T D \mathbf{h} - \mathbf{g}^T \mathbf{h} \\
&\quad (\text{by the inequality } V(\boldsymbol{\xi}) + V(\boldsymbol{\varsigma}^*) \geq (\boldsymbol{\varsigma}^*)^T \boldsymbol{\xi}) \\
&= \Xi(\mathbf{h}, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*). \tag{30}
\end{aligned}$$

Note that $\Xi(\mathbf{h}, \boldsymbol{\varsigma}, \boldsymbol{\lambda})$ is a quadratic function with respect to \mathbf{h} and that

$$\nabla_{\mathbf{h}} \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) = 2G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \mathbf{h}^* - \mathbf{g} = 0.$$

By these together with (29) and (30), it yields

$$\begin{aligned}
\tilde{P}_g(\mathbf{h}) - \tilde{P}_g(\mathbf{h}^*) &\geq \Xi(\mathbf{h}, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) - \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \\
&= (\nabla_{\mathbf{h}} \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*))^T (\mathbf{h} - \mathbf{h}^*) \\
&\quad + \frac{1}{2} (\mathbf{h} - \mathbf{h}^*)^T \nabla_{\mathbf{h}\mathbf{h}}^2 \Xi(\mathbf{h}^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) (\mathbf{h} - \mathbf{h}^*) \\
&= (\mathbf{h} - \mathbf{h}^*)^T G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) (\mathbf{h} - \mathbf{h}^*) \\
&\geq 0, \quad \forall \mathbf{h} \text{ satisfying (14)}.
\end{aligned}$$

We complete the proof. ■

Theorem 1 shows that the global solution of Problem (P_{Ig}) may be obtained through solving its canonical dual problem (P_{Ig}^d) . Since Problem (P_{Ig}) is a small perturbation of Problem (P_I) , we can use a gradient-based optimization technique to obtain an improved solution. After obtaining the solution of Problem (P_I) , we check whether this solution satisfies the functional inequality constraint of Problem (P). If it does, this solution is further refined by a gradient-based optimization method and the corresponding solution obtained is viewed as an approximate solution of Problem (P). Otherwise, we increase the discretization number and repeat the above process.

To proceed further, we need Schur complement lemma [20].

Lemma 2 *Let*

$$A = \begin{pmatrix} B_1 & B_2^T \\ B_2 & B_3 \end{pmatrix},$$

where B_1 and B_3 are square matrices and B_2 is a matrix with proper dimension. If $B_1 \succ 0$, then A is positive semi-definite if and only if the matrix $B_3 - B_2 B_1^{-1} B_2^T$ is positive semi-definite.

To solve Problem (P_{Ig}^d) , we note that the matrix inverse is involved in the objective function (20). By exploiting its structure, the main theorem of the paper is obtained by combining Theorem 1 and Lemma 2 as follows.

Theorem 3 *Let $(t_1^*, t_2^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ be the solution of the following semi-definite*

programming problem which is referred to as Problem (RSDP)

$$\min \quad \frac{1}{4}t_1 + \frac{1}{4}t_2 + \sum_{m=1}^M \varsigma_m + \varepsilon \sum_{n=1}^N \lambda_n \quad (31)$$

$$\text{subject to} \quad \begin{bmatrix} G(\boldsymbol{\varsigma}, \boldsymbol{\lambda}) & \mathbf{g} \\ \mathbf{g}^T & t_1 \end{bmatrix} \succeq 0, \quad (32)$$

$$\begin{bmatrix} B & \boldsymbol{\varsigma} \\ \boldsymbol{\varsigma}^T & t_2 \end{bmatrix} \succeq 0, \quad (33)$$

$$\boldsymbol{\lambda} \geq 0. \quad (34)$$

If $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \succ 0$, then $\mathbf{h}^* = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)\mathbf{g}$ is the global minimizer of Problem (P_{I_g}) .

Proof. The validity of the results follows readily from Lemma 2 and Theorem 1. We complete the proof. ■

Problem (RSDP) is a linear SDP problem which can be easily solved by existing SDP software packages, for example, SeDuMi, [21]. Theorem 3 shows that solving Problem (RSDP) produces a global minimizer of Problem $(P_{I_g}^d)$ if $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \succ 0$. However, if $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$ is singular, there is still no way to dig out a global solution of Problem (P_{I_g}) from the solution of Problem (RSDP). Fortunately, this case is rather rare for our problem. Now we are in a position to present an algorithm for solving Problem (P) as follows.

Algorithm 1

Step 1. Initialization K , M and N .

Step 2. Randomly produce the small perturbation vector \mathbf{g} .

Step 3. Solve Problem (RSDP) and output solution $(t_1^*, t_2^*, \boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)$. If $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \succ 0$, let $\mathbf{h}_{ig}^* = \frac{1}{2}G^{-1}(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*)\mathbf{g}$, go to Step 4. Otherwise, go to Step 2.

Step 4. Using \mathbf{h}_{ig}^* as an initial condition, solve Problem (P) with the functional inequality constraint (10) replaced by the approximate discretized constraints (14) by a gradient-based optimization method. Check whether the functional inequality constraint (10) is satisfied in a dense set of B_s . If it is, stop and output the solution \mathbf{h}^* . Otherwise, increase N and goto Step 2.

4. Numerical Experiments

Let the desired magnitude response be given by

$$|H_d(\omega)| = \begin{cases} 1, & \text{if } \omega \in [0, 0.12\pi], \\ 0, & \text{if } \omega \in [0.24\pi, \pi] \end{cases}$$

and α is chosen from $[10^{-8}, 10^{-4}]$.

During our simulation, we set $K = 32$, $M = 50$, $N = 200$ and $\varepsilon = -35\text{dB}$. Algorithm 1 is used to solve the optimization problem (13) and (14) with different values of α . The small perturbation vector is generated by $10^{-5}\text{randn}(K + 1, 1)$. The weights ν_m and the Gaussian quadrature points

ω_m^p , $m = 1, \dots, M$, can be found from [22]. Problem (RSDP) is solved by SeDuMi [21] together with YALMIP [23] and the nonlinear constrained optimization is solved by “fmincon” in the Optimization Toolbox within the Matlab environment.

For comparison, we use the existing convex optimization methods used in [7, 24] in conjunction with a minimum phase spectral factorization method [25] to design a low group delay filter under the same criteria. More specifically, taking $\alpha = 0$ in (13), replacing $|H(\omega)|^2$ by $R(\omega) = r_0 + \sum_{k=1}^K 2r_k \cos(k\omega) = \mathbf{r}^T \boldsymbol{\psi}(\omega)$ (where $\mathbf{r} = [r_0, \dots, r_K]^T$ and $\boldsymbol{\psi}(\omega) = [1, 2 \cos(\omega), \dots, 2 \cos((K\omega))]$) and imposing a linear matrix inequality [24] to ensure the invertibility from \mathbf{r} to \mathbf{h} , a quadratic optimization problem with a linear matrix inequality and linear constraints in terms of \mathbf{r} and an auxiliary matrix variable P_c is obtained as follows:

$$\begin{aligned} & \min_{\mathbf{r}, P_c} \sum_{m=1}^M \nu_m (\mathbf{r}^T \boldsymbol{\psi}(\omega_m^p) - 1)^2 \\ & \text{subject to } \mathbf{r}^T \boldsymbol{\psi}(\omega_n^s) \leq \varepsilon, \quad n = 1, \dots, N. \\ & \begin{bmatrix} P_c - A_c^T P_c A_c & C_c^T - A_c^T P_c B_c \\ C_c - B_c^T P_c A_c & D_c + D_c^T - B_c^T P_c B_c \end{bmatrix} \succeq 0, \end{aligned} \quad (35)$$

where A_c, B_c, C_c, D_c are given in (7) in [24]. After finding the solution of the optimization problem defined by (35), the minimum phase spectral factorization method introduced in [25] is used to extract \mathbf{h} from \mathbf{r} . Let the solution obtained be denoted as \mathbf{h}_c^* . The computational complexity to obtain

\mathbf{h}_α^* and \mathbf{h}_c^* are similar as both Problem (RSDP) and Problem (35) involve a linear matrix inequality. However, Problem (35) contains more decision variables. Furthermore, after the solution of Problem (35) is obtained, a minimum phase spectral factorization is required to be used to extract \mathbf{h}_c^* from the solution \mathbf{r}^* .

We first set $\alpha_1 = 10^{-6}$ and use Algorithm 1 to solve Problem (P_I). Let the solution obtained be denoted as $\mathbf{h}_{\alpha_1}^*$. During the solution process, the function “fmincon” just takes two iterations to stop. In fact, the performance of the solution obtained by solving Problem (RSDP) has no visible difference from that refined by “fmincon”. Thus, the gradient-based optimization refinement seems not necessary from our computational experience. Nonetheless, it is included in Algorithm 1, as it may be needed in other problems.

The results corresponding to $\mathbf{h}_{\alpha_1}^*$ and \mathbf{h}_c^* are depicted in Fig. 1. From Fig. 1 (b), we see that there is no visible difference between the magnitude response of $\mathbf{h}_{\alpha_1}^*$ and that of \mathbf{h}_c^* in the passband. Fig. 1 (c) shows that both $\mathbf{h}_{\alpha_1}^*$ and \mathbf{h}_c^* are with the same stopband ripples. However, it is observed in Fig. 1 (d) that $\mathbf{h}_{\alpha_1}^*$ achieves a much smaller group delay than \mathbf{h}_c^* . Fig. 1 (e) shows that all zeros of $\mathbf{h}_{\alpha_1}^*$ are within the unit circle. Thus, the filter $\mathbf{h}_{\alpha_1}^*$ designed by our method is automatically a minimum phase filter without having to performing a minimum phase extraction step. This is not shared by any existing method. From this example, it is clear that our method can

achieve a much smaller group delay while keeping the same design criteria.

To investigate the impact of the parameter α , we increase α from α_1 to $\alpha_2 = 10^{-5}$. Let the solution obtained by our method be denoted as $\mathbf{h}_{\alpha_2}^*$. The results corresponding to $\mathbf{h}_{\alpha_2}^*$ and \mathbf{h}_c^* are depicted in Fig. 2. Fig. 2 shows that the group delay of $\mathbf{h}_{\alpha_2}^*$ becomes much smaller at the cost of some increase in its magnitude response in the passband and stopband.

Now we decrease α from α_1 to $\alpha_3 = 10^{-7}$. Let the solution obtained by our method be denoted as $\mathbf{h}_{\alpha_3}^*$. The results corresponding to $\mathbf{h}_{\alpha_3}^*$ and \mathbf{h}_c^* are depicted in Fig. 3. Fig. 3 shows that both the magnitude response and the group delay of $\mathbf{h}_{\alpha_3}^*$ are close to \mathbf{h}_c^* . In theory, \mathbf{h}_c^* can be obtained from \mathbf{h}_{α}^* by setting $\alpha \rightarrow 0$. All the coefficients of $\mathbf{h}_{\alpha_1}^*$, $\mathbf{h}_{\alpha_2}^*$, $\mathbf{h}_{\alpha_3}^*$ and \mathbf{h}_c^* are presented in Table 1.

Clearly, the parameter α plays an important role during our design process. In fact, the role of α is two-fold. On one hand, it is used to control the trade-off between the group delay and the magnitude response. The larger the α is, the smaller the group delay is achieved. Since the term $\sum_{m=1}^M \nu_m (\mathbf{h}^T \Psi(\omega_m^p) \mathbf{h} - 1)^2$ is much smaller than the term $\mathbf{h}^T D \mathbf{h}$, we usually take the value of α starting from 10^{-4} . If the resulting magnitude response does not meet our desired specification, we decrease the value of α by $\alpha = \alpha/10$ or to a smaller one. On the other hand, α plays the role of regularization during the process of solving Problem (RSDP). If the value of

α is too small (for example, smaller than 10^{-8}), Problem (RSDP) may not admit a solution such that $G(\boldsymbol{\varsigma}^*, \boldsymbol{\lambda}^*) \succ 0$. At the first glance, the adjustment of α appears to increase the complexity of solving the problem. In fact, due to the presence of α , we have more freedom to control the trade-off between the magnitude response and the group delay. Furthermore, a proper choice of α can lead to a smaller group delay while keeping all the design criteria the same (see Fig. 1).

5. Conclusion

In this paper, the design of a low group delay FIR filter is studied. Unlike the conventional minimum phase method, we formulate it as a non-convex optimization problem directly. Our main contribution is that a novel numerical scheme is developed to solve this formulated non-convex optimization problem globally. Numerical experiments show that our design method can produce a filter with smaller group delay than those obtained by the existing convex optimization method used in conjunction with a minimum phase extraction method under the same design criteria.

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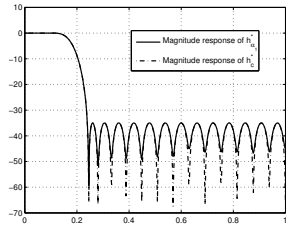
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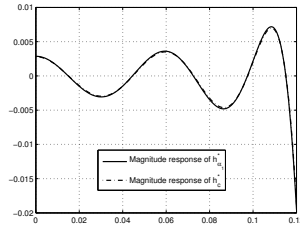
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Table 1: The coefficients of $h_{\alpha_1}^*$, $h_{\alpha_2}^*$, $h_{\alpha_3}^*$, and h_c^*

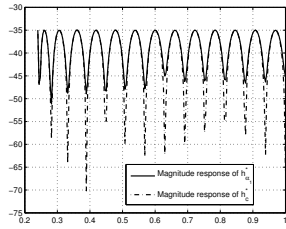
$h_{\alpha_1}^*$	$h_{\alpha_2}^*$	$h_{\alpha_3}^*$	h_c^*
0.0285116206006799	0.0387740383889875	0.0262107023409039	0.0252472337305588
0.0509866524174902	0.0614443952270321	0.0486333750106783	0.0473450207730468
0.0856088083718892	0.0978828806389367	0.0822097183380445	0.0810379492029237
0.123034390901257	0.134719509319526	0.119931194418926	0.118315645903861
0.156084821647626	0.164035324986959	0.153337081728234	0.152127711650623
0.17639025618915	0.178537316915211	0.175252349657358	0.174612661066523
0.178278138164555	0.173044184628575	0.178831620770272	0.179080069523033
0.158868245465607	0.146945417351412	0.16138455335902	0.162414973421189
0.120358748692553	0.104316502480046	0.124389155519067	0.12617170171523
0.0698444119645312	0.0532464367514738	0.0745058664743655	0.0766929272231663
0.0171576799182006	0.00358296855479145	0.0216618169750614	0.0236538095680745
-0.0273048888932891	-0.03523190002442	-0.0240601445359459	-0.0224972156907095
-0.055408731398201	-0.0569798418610146	-0.0540575198345802	-0.0532595049087454
-0.063955965089311	-0.0597728525689495	-0.0642092858615103	-0.0643882215963533
-0.0542237462602092	-0.0468743219339164	-0.0560442066213009	-0.0567683880468747
-0.0322817257959678	-0.024592722813039	-0.0348589697777236	-0.0358287251589483
-0.00627605177132923	-0.000912831822104257	-0.00842407441488813	-0.00965476219362919
0.0158289990856661	0.0177061601908936	0.014330704665103	0.0135807646076558
0.0286195549008114	0.0269920614387458	0.0280939492720026	0.0279162706674796
0.0302151504788049	0.0265448718017831	0.0309081516924011	0.0310432735978916
0.0227853602802973	0.0185592874179371	0.0239044204015682	0.0244040175111291
0.0102400250794694	0.00737664537854957	0.011449384382796	0.0120974536216775
-0.0022899340896227	-0.00297170649287577	-0.00127364695662734	-0.0008551248303386
-0.0109135754408663	-0.00952784396092289	-0.0105402761284925	-0.0103020083473471
-0.0139437136272584	-0.0114233791995216	-0.0141124838925119	-0.0141101882320104
-0.0116738095909395	-0.00928896815384196	-0.0121975868915368	-0.0124765774288458
-0.00633401566033477	-0.00495774606814332	-0.00708446025529165	-0.00735359258744947
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0.00366739459678194	0.00252396781666946	0.00329482070235785	0.00326212338320078
0.00540636197896201	0.00360183317036392	0.00574885041723123	0.00551582780234008
0.00510626880733615	0.00295381264416561	0.00534983833018261	0.00571431177419455
0.00392200136977837	0.00152181737685866	0.00482044693188607	0.00515350178800323
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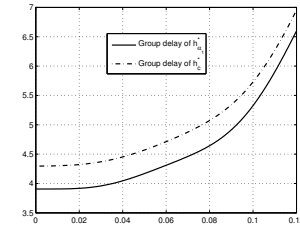
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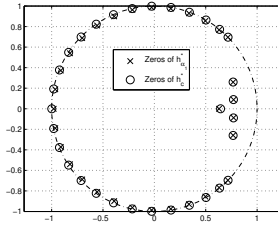
(b)



(c)

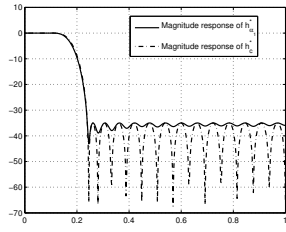


(d)

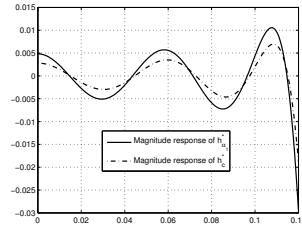


(e)

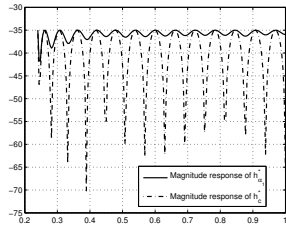
Figure 1: Magnitude responses and group delays of $\mathbf{h}_{\alpha_1}^*$ and \mathbf{h}_c^* : (a) The magnitude response (in dB), (b) Zoom-in of the magnitude response in the passband (in dB), (c) Zoom-in of the magnitude response in the stopband (in dB), (d) The group delay in the passband, (e) Zeros of FIR filters of $\mathbf{h}_{\alpha_1}^*$ and \mathbf{h}_c^*



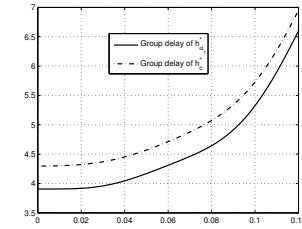
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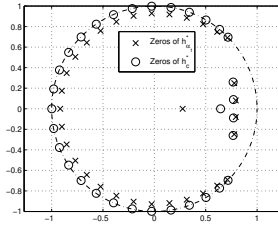
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(c)

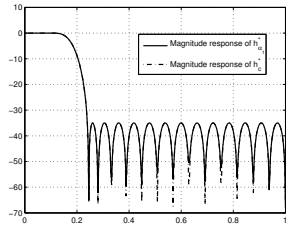


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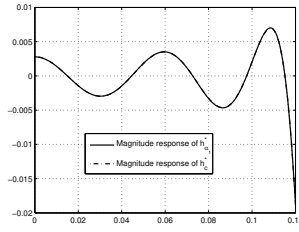


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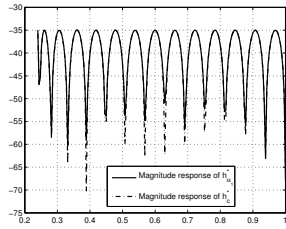
Figure 2: Magnitude responses and group delays of $\mathbf{h}_{\alpha_2}^*$ and \mathbf{h}_c^* : (a) The magnitude response (in dB), (b) Zoom-in of the magnitude response in the passband (in dB), (c) Zoom-in of the magnitude response in the stopband (in dB), (d) The group delay in the passband, (e) Zeros of FIR filters of $\mathbf{h}_{\alpha_2}^*$ and \mathbf{h}_c^*



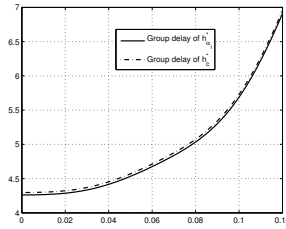
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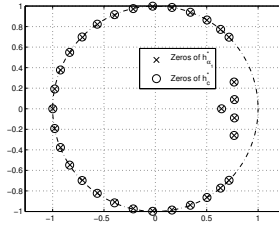
(b)



(c)



(d)



(e)

Figure 3: Magnitude responses and group delays of $\mathbf{h}_{\alpha_3}^*$ and \mathbf{h}_c^* : (a) The magnitude response (in dB), (b) Zoom-in of the magnitude response in the passband (in dB), (c) Zoom-in of the magnitude response in the stopband (in dB), (d) The group delay in the passband, (e) Zeros of FIR filters of $\mathbf{h}_{\alpha_3}^*$ and \mathbf{h}_c^*