A computational algorithm for a class of non-smooth optimal control problems arising in aquaculture operations

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Abstract

This paper introduces a computational approach for solving non-linear optimal control problems in which the objective function is a discontinuous function of the state. We illustrate this approach using a dynamic model of shrimp farming in which shrimp are harvested at several intermediate times during the production cycle. The problem is to choose the optimal harvesting times and corresponding optimal harvesting fractions (the percentage of shrimp stock extracted) to maximize total revenue. The main difficulty with this problem is that the selling price of shrimp is modelled as a piecewise constant function of the average shrimp weight and thus the revenue function is discontinuous. By performing a time-scaling transformation and introducing a set of auxiliary binary variables, we convert the shrimp harvesting problem into an equivalent optimization problem that has a smooth objective function. We then use an exact penalty method to solve this equivalent problem. We conclude the paper with a numerical example.

Keywords: Optimal control, Non-smooth optimization, Shrimp farming, Exact penalty function

1. Introduction

In [17], Yu and Leung proposed a mathematical model for shrimp farming over a single batch production cycle. This model was subsequently used to demonstrate that harvesting shrimps at intermediate times before the final harvest (called partial harvesting) is an optimal strategy for maximizing the overall profit. The dynamic equations in [17] describe shrimp growth by a uniform density-dependent growth rate and shrimp mortality by way of exponential decay. Research into aquaculture population dynamics shows that this mathematical model is appropriate in the shrimp farming context [10, 14].

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In this paper, we again consider the model proposed by Yu and Leung in [17], which describes the mortality and growth processes of the shrimp population. The dynamics in this model involve jumps in the state variables because there is a sudden change in the number of shrimps when an intermediate harvest takes place. This model has one major limitation: it assumes that the shrimp price is constant, when in fact the price of shrimp usually varies significantly with the average shrimp weight. In this paper, we improve the model by introducing a more realistic price function that is suitable for a commercial environment. More specifically, we have incorporated a piecewise constant price function, which depends on the average shrimp weight, into the revenue function.

The problem of choosing the harvest times and the harvest fractions (i.e. the percentage of shrimp stock extracted) to maximize the total revenue is an optimal control problem in which the objective function is discontinuous and the dynamic system experiences state jumps at variable time points. Such optimal control problems are called impulsive optimal control problems in the literature, and they have been an active area of research over the past decade [3, 7, 11, 13, 15, 16]. To handle the variable jump points, we apply the time-scaling technique [7, 11], which involves mapping the variable jump times to fixed integers, thus yielding an equivalent problem in a new time horizon. This transformation is necessary as most standard optimal control algorithms for impulsive systems can only handle jump times that are fixed [7, 9, 15]. Although the time-scaling transformation eliminates the variability in the jump times, it does not eliminate the discontinuity in the objective function. Hence, inspired by the relaxation approaches in [8, 12, 20], we introduce new binary variables into the objective function, together with linear and quadratic constraints, to transform the objective function into a smooth function. The resulting optimization problem can be solved using an exact penalty method. This involves adding continuous penalty terms to the cost function to transform the original non-smooth optimal control problem into an approximate unconstrained problem. The penalty terms are zero at feasible points and positive at infeasible points. Prior research [8, 19, 20] indicates that any local minimizer of the unconstrained problem will be a local minimizer of the original problem when the penalty parameter is sufficiently large.

In this paper, we illustrate the effectiveness of the proposed approach by applying it to the shrimp farming model. The time-scaling transformation and exact penalty approach result in a problem that can be readily solved using MISER 3.3 [2], which is an optimal control software based on the control parameterization technique [1, 4, 6]. The approach described in this paper can also be readily extended to more general optimal control problems involving discontinuous objective functions. To the best of our knowledge, this paper is the first attempt at generating numerical solutions such non-smooth optimal control problems.

2. Problem Formulation

We start with the dynamic model proposed by Yu and Leung [17] in which the biological mortality and growth processes of the shrimp stock are described by the following differential equations:

$$\dot{n}(t) = -mn(t), \qquad n(0) = n_0, \tag{1}$$

$$\dot{w}(t) = g[f(t), w(t), n(t), t], \qquad w(0) = w_0, \tag{2}$$

where

- t is the time in weeks;
- $w(t)$ is the average weight of an individual shrimp in grams at time t;
- $n(t)$ is the number of remaining shrimp at time t;
- m is a given constant representing the natural mortality rate of the shrimp;
- $f(t)$ is the feeding rate at time t;
- g is a given function that is differentiable with respect to each of its arguments;
- n_0 and w_0 are given initial conditions at $t = 0$.

Let $[0, T]$ denote the time horizon over which a single production cycle takes place (the final harvest occurs at time $t = T$).

Suppose that N harvests ($N - 1$ intermediate harvests and 1 final harvest) take place during the production cycle. Let $\tau_j \in [0, T]$ denote the time of the j^{th} harvest, with τ_N referring to the final harvest time. Then we have the following constraint:

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_N = T. \tag{3}$$

Let ν_j denote the fraction of shrimp stock harvested at time τ_j. Then clearly,

$$0 \leq \nu_j \leq 1, \qquad j = 1, \ldots, N. \tag{4}$$

The state variables n and w are subject to the following jump conditions at each harvest time $t = \tau_j$:

$$n(\tau_j^+) - n(\tau_j^-) = -\nu_j n(\tau_j^-), \tag{5}$$

$$w(\tau_j^+) - w(\tau_j^-) = 0, \tag{6}$$

where, for a general function $h(t)$, we adopt the notation $h(\tau^{\pm}) = \lim_{t \to \tau^{\pm}} h(t)$.

Equation (5) asserts that the difference in the number of shrimps before and after the j^{th} harvest is equal to the number of shrimps harvested at time $t = \tau_j$. Equation (6) simply states that the average shrimp weight is unchanged by the j^{th} harvest. This assumes, of course, that when harvesting the shrimp, a uniform cross-section of the shrimp stock is extracted.

In the general dynamics (1) and (2) proposed by Yu and Leung in [17], the feeding rate $f(t)$ is also a decision variable to be chosen optimally, in addition to the harvesting fractions ν_j and harvesting times τ_j. Note though, that no specific example of this general form was actually proposed by Yu and Leung and their numerical results were based on a simpler form of the dynamics not involving the feeding rate. Following their lead, we also ignore the feeding rate $f(t)$ and consider only the harvesting fractions and the corresponding harvesting times as decision variables. However, note that the computational approach in this paper can be easily extended to also optimize the feeding rate $f(t)$ using the control parameterization technique described in [1, 4, 6].

Yu and Leung proposed the following general model for the revenue obtained at the j^{th} harvest time [17]:

$$\text{Revenue} = R_j\{p[w(\tau_j)], w(\tau_j), n(\tau_j^-), \nu_j, c_j, h\}, \tag{7}$$

where

- c_j is the variable cost of the j^{th} harvest in dollars per kilogram;
- h is the fixed cost associated with each harvest;
- $p[w(\tau_j)]$ is the sale price of shrimp in dollars per kilogram (*as a function of the average weight of shrimp at the j^{th} harvest*);
- R_j is a given continuously differentiable function.

We adopt the following specific model suggested by Yu and Leung for the total revenue over the production cycle $[0, T]$:

$$J = \sum_{j=1}^{N} \left[10^{-3}\{p[w(\tau_j)] - c_j\} w(\tau_j) n(\tau_j^-) \nu_j - h \right]. \tag{8}$$

The numerical examples in [7, 17] consider the revenue function (8) for the simple case when the price function $p[w(\tau_j)]$ is a constant. More realistically, the sale price of shrimp is heavily dependent on the average weight of the shrimp. Thus, in this paper, we consider a more appropriate price function in which different prices are assigned to different weight ranges. This piecewise constant price function is defined as follows:

$$p[w(\tau_j)] = \alpha_i, \qquad \beta_{i-1} \leq w(\tau_j) < \beta_i, \qquad i = 1, \ldots, L, \tag{9a}$$

where

- L is the number of different price levels;
- β_0 is the left end point of the lowest weight range;
- β_i is the right end point of the i^{th} weight range;

- α_i is the sale price of shrimp stock in dollars per kilogram when the average weight lies in the interval $[\beta_{i-1}, \beta_i)$.

We assume without loss of generality that

$$\beta_0 < \beta_1 < \beta_2 < \cdots < \beta_L \tag{9b}$$

and

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_L. \tag{9c}$$

From a practical point of view, the price function (9a) is far more realistic than those used in [7, 17]. It is worth mentioning that [18] also considers the prize-size relationship of shrimp in a partial harvesting situation. [18] focuses on a network-flow approach for determining an efficient harvesting policy, whereas in our paper we focus on an optimal control approach for an impulsive system model .

Following the lead of Yu and Leung [17], we consider the following specific dynamics for the state variables n and w:

$$\dot{n}(t) = -mn(t), \qquad n(0) = n_0, \tag{10}$$

$$\dot{w}(t) = a - bw(t)n(t), \qquad w(0) = w_0, \tag{11}$$

where m, a and b are given constants.

We now formulate an optimal control problem as follows: Choose the harvesting fractions ν_j and the corresponding harvesting times τ_j to maximize the revenue function defined by (8) and (9a) subject to the dynamics described by equations (10) and (11), the constraints given by (3) and (4) and the jump conditions given by (5) and (6). We refer to this problem as Problem A.

When $L = 1$ (i.e. there is only one price level), Problem A reduces to the shrimp harvesting problem considered in [7, 17]. In this reduced problem, the objective is smooth, and therefore the problem can be solved effectively using the impulsive control techniques discussed in [7, 9]. In this paper, we are interested in the more difficult case when $L > 1$; that is, when there are distinct price levels. In this case, Problem A presents two major challenges for existing numerical solution methods:

- The jump conditions (5) and (6) occur at variable time points;
- The objective function is discontinuous and hence non-differentiable.

For these reasons, Problem A with $L > 1$ cannot be solved using standard optimal control software such as MISER 3.3. In this paper we use a time-scaling transformation to handle the variable jump points and a smoothing transformation to handle the discontinuous objective function. Both of these transformations are described in the next section.

3. Problem Transformation

3.1. Application of the Time-Scaling Technique

Problem A is an optimal control problem in which the harvesting times are decision variables to be chosen optimally. The state variable n undergoes an instantaneous jump at each harvesting time. From a computational point of view, it is well known that variable jump times cause major difficulties for standard optimal control algorithms [7, 9]. Such algorithms require that the jump times be fixed, whereas in Problem A, the jump times are decision variables to be optimized. Thus, we adopt the time-scaling transformation described in [5, 7], which enables us to map the harvesting times to fixed points in a new time horizon.

We first introduce a new time variable $s \in [0, N]$ and relate s to t through the following differential equation:

$$\dot{t}(s) = u(s), \qquad t(0) = 0, \tag{12a}$$

and

$$t(N) = T, \tag{12b}$$

where $u : [0, N] \to \mathbb{R}$ is a piecewise constant function satisfying the bounds

$$0 \leq u(s) \leq T, \qquad s \in [0, N]. \tag{13}$$

Let

$$\theta_j = \tau_j - \tau_{j-1}, \qquad j = 1, \dots, N. \tag{14}$$

That is, θ_j represents the duration between the $(j-1)^{th}$ and j^{th} harvest times. Thus,

$$\theta_j \geq 0, \qquad j = 1, \dots, N.$$

We express $u(s)$ mathematically as follows:

$$u(s) = \sum_{j=1}^{N} \theta_j \chi_{[j-1,j)}(s), \tag{15}$$

where $\chi_{[j-1,j)}(s) : \mathbb{R} \to \mathbb{R}$ is the indicator function defined by

$$\chi_{[j-1,\ j)}(s) = \begin{cases} 1, & \text{if } s \in [j-1, j), \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the control function $u(s)$ has fixed switching times at $s = 1, \dots, N$, and its height represents the duration between consecutive harvest times in the original time horizon. For $s \in [k-1,\ k]$, it follows from equations (12) and (15) that

$$t(s) = \int_0^s u(\eta) d\eta = \sum_{j=1}^{k-1} \theta_j + \theta_k (s - k + 1) \tag{16a}$$

and thus

$$t(k) = \sum_{j=1}^{k} \theta_j = \sum_{j=1}^{k} (\tau_j - \tau_{j-1}). \tag{16b}$$

In particular, $t(N) = T$, as required by equation (12b).

Applying the transformation defined by (12), the dynamics in Problem A become

$$\dot{\widetilde{n}}(s) = -m\widetilde{n}(s)u(s), \qquad \widetilde{n}(0) = n_0, \tag{17}$$

$$\dot{\widetilde{w}}(s) = [a - b\widetilde{w}(s)\widetilde{n}(s)]u(s), \qquad \widetilde{w}(0) = w_0, \tag{18}$$

where $\widetilde{n}(s) = n(t(s))$, $\widetilde{w}(s) = w(t(s))$, and $t(s)$ is the solution of the differential equation (12a). In the new time horizon, the state jumps occur at the fixed times $s = 1, \ldots, N-1$. Thus, the jump conditions (5) and (6) become

$$\widetilde{n}(j^+) - \widetilde{n}(j^-) = -\nu_j \widetilde{n}(j^-), \tag{19}$$

$$\widetilde{w}(j^+) - \widetilde{w}(j^-) = 0. \tag{20}$$

Problem A is thus transformed into the following problem: Choose the control $u(s)$ and the parameters ν_j, $j = 1, \ldots, N$ to maximize the transformed revenue function

$$\widetilde{J} = \sum_{j=1}^{N} \left[10^{-3}\{p[\widetilde{w}(j)] - c_j\}\widetilde{w}(j)\widetilde{n}(j^-)\nu_j - h \right] \tag{21}$$

subject to the dynamics (12), (17)-(18), the jump conditions (19)-(20) and the following bounds:

$$0 \le \nu_j \le 1, \quad j = 1, \ldots, N, \tag{22a}$$

$$0 \le u(s) \le T, \quad s \in [0, N]. \tag{22b}$$

We refer to this problem as Problem B. The control $u(s)$ in Problem B governs the harvesting times in the original time horizon.

Problems A and B are mathematically equivalent. The variable jump times in Problem A have been replaced by fixed times in Problem B. Although this makes the problem more amenable to solution via standard optimal control software packages such as MISER 3.3, the objective function is still a discontinuous function of the state. Thus, Problem B cannot be solved directly by MISER 3.3, which requires the objective to be smooth. In the next subsection, we overcome this difficulty by introducing new binary variables (adopting the transformation strategy described in [8, 19]) in addition to new linear and quadratic constraints.

3.2. Smoothing Transformation

Let z_{ij}, $i = 1, \ldots, L$, $j = 1, \ldots, N$, be new binary variables defined as follows:

$$z_{ij} = \begin{cases} 1, & \text{if } \beta_{i-1} \le \widetilde{w}(j) < \beta_i, \\ 0, & \text{otherwise.} \end{cases}$$

With this definition, we can write $p[\widetilde{w}(j)]$ as:

$$p[\widetilde{w}(j)] = \sum_{i=1}^{L} z_{ij}\alpha_i.$$

Thus, the revenue function may be written as:

$$\widehat{J}(\mathbf{z}, \boldsymbol{\nu}, u) = \sum_{j=1}^{N} \left[10^{-3} \left\{ \sum_{i=1}^{L} z_{ij}\alpha_i - c_j \right\} \widetilde{w}(j)\widetilde{n}(j^-)\nu_j - h \right], \tag{23}$$

where

$$\boldsymbol{\nu} = [\nu_1, \ldots, \nu_N]^\top \in \mathbb{R}^N$$
$$\mathbf{z}_j = [z_{1j}, z_{2j}, \ldots, z_{Lj}]^\top \in \mathbb{R}^L$$
$$\mathbf{z} = [(\mathbf{z}_1)^\top, (\mathbf{z}_2)^\top, \ldots, (\mathbf{z}_N)^\top]^\top \in \mathbb{R}^{LN}$$

Although we have defined each z_{ij} to be a binary variable, MISER 3.3 only allows us to define variables in a continuous domain. Hence, to ensure that each z_{ij} is a binary variable, we impose the following constraints:

$$H_j(\mathbf{z}) = \sum_{i=1}^{L} z_{ij} - 1 = 0, \qquad j = 1, \ldots, N, \tag{24a}$$

$$g_{ij}(\mathbf{z}) = z_{ij}(1 - z_{ij}) \leq 0, \qquad i = 1, \ldots, L, \qquad j = 1, \ldots, N, \tag{24b}$$

$$0 \leq z_{ij} \leq 1, \qquad i = 1, \ldots, L, \qquad j = 1, \ldots, N. \tag{24c}$$

It is clear that (24) ensures $z_{ij} \in \{0, 1\}$. However, (24) alone is not sufficient to ensure that z_{ij} is consistent with the definition given at the beginning of this section. Therefore, we impose the additional constraints given below:

$$G_{ij}(\mathbf{z}) = z_{ij}[\beta_{i-1} - \widetilde{w}(j)][\beta_i - \widetilde{w}(j)] \leq 0, \quad i = 1, \ldots, L, \quad j = 1, \ldots, N. \tag{25}$$

We now prove two important results.

Lemma 1. *Suppose that z_{ij}, $i = 1, \ldots, L$, $j = 1, \ldots, N$, satisfy constraints (24) and (25). For any $i \in \{1, \ldots, L\}$ and any $j \in \{1, \ldots, N\}$, if $z_{ij} = 1$, then $\beta_{i-1} \leq \widetilde{w}(j) \leq \beta_i$.*

Proof. Suppose $z_{ij} = 1$. Then inequality (25) reduces to

$$[\beta_{i-1} - \widetilde{w}(j)][\beta_i - \widetilde{w}(j)] \leq 0.$$

Since $\beta_{i-1} < \beta_i$ (recall (9b)), this inequality is only satisfied when we have

$$\beta_{i-1} \leq \widetilde{w}(j) \leq \beta_i.$$

□

Lemma 2. *Suppose that z_{ij}, $i = 1, \dots, L$, $j = 1, \dots, N$, satisfy constraints (24) and (25). For any $i \in \{1, \dots, L\}$ and any $j \in \{1, \dots, N\}$, if $\beta_{i-1} < \widetilde{w}(j) < \beta_i$, then $z_{ij} = 1$.*

Proof. Suppose $\beta_{i-1} < \widetilde{w}(j) < \beta_i$. Recall that inequalities (24b) and (24c) ensure that $z_{ij} \in \{0, 1\}$. Suppose that $z_{ij} \neq 1$. Then we must have $z_{ij} = 0$. It thus follows from (24a) that there exists a $k \in \{1, \dots, L\} \setminus \{i\}$ such that $z_{kj} = 1$. Then, by Lemma 1, we have $\beta_{k-1} \leq \widetilde{w}(j) \leq \beta_k$. Since $k \neq i$, and the weight intervals are disjoint (recall (9b)), this contradicts $\beta_{i-1} < \widetilde{w}(j) < \beta_i$.

Having arrived at a contradiction, it follows that $z_{ij} = 1$. □

Remark 1. *Lemma 2 implies that if $\widetilde{w}(j)$ lies in the interior of the i^{th} weight range, then we must have $z_{ij} = 1$. However, the converse of this result is not true in general* (i.e. Lemma 2 is not the direct converse of Lemma 1)*. According to Lemma 1 and* (9b)*, if $\widetilde{w}(j) = \beta_i$, then the only two possibilities are $z_{(i+1),j} = 1$ and $z_{ij} = 1$. Since the shrimp price increases with the average weight of shrimp, if $\widetilde{w}(j) = \beta_i$ at an optimal solution, then $z_{(i+1),j} = 1$ and $z_{ij} = 0$. Thus, the optimization process will push $\widetilde{w}(j)$ into a higher weight range as the revenue is maximized. It follows that, at an optimal solution, z_{ij} is an indicator variable equal to* 1 *if $\widetilde{w}(j)$ is in the i^{th} weight range and equal to zero otherwise.*

Our transformed problem can hence be described as follows: Choose the system parameters z_{ij}, the harvesting fractions ν_j and the control $u(s)$ to maximize (23) subject to:

- the dynamics (12) and (17)-(18);
- the jump conditions (19)-(20);
- the bounds (22);
- the constraints (24)-(25).

We refer to this problem as Problem C.

Although Problem C has a smooth objective function, the additional quadratic constraints imposed on the system (see inequalities (24) and (25)) give rise to a disjoint feasible region. Hence, standard optimization algorithms will struggle with these constraints. Indeed, when using MISER 3.3 to solve Problem C directly, we encountered a large number of numerical issues. This is expected, as MISER 3.3 assumes that the optimization problem has a continuous feasible region, an assumption that is violated in Problem C.

In the next section, we apply the exact penalty method proposed in [8] to transform Problem C into an unconstrained problem, which can then be solved readily by MISER 3.3.

4. An Exact Penalty Method

The exact penalty approach involves creating a pseudo-objective function by adding terms based on the constraints to the objective. Problem C, a constrained optimization problem, is subsequently transformed into an approximate unconstrained problem that can be readily solved using MISER 3.3.

The *constraint violation* is defined by:

$$\Delta(\mathbf{z},u) = \sum_{j=1}^{N}\left[H_j(\mathbf{z})\right]^2 + \sum_{i=1}^{L}\sum_{j=1}^{N}\left[\max\{0, G_{ij}(\mathbf{z})\}\right]^2 + \sum_{i=1}^{L}\sum_{j=1}^{N}\left[\max\{0, g_{ij}(\mathbf{z})\}\right]^2 + [t(N)-T]^2.$$

Note that $\Delta(\mathbf{z},u)=0$ if and only if the constraints in Problem C are satisfied.

Using the strategy introduced in [8], an exact penalty function $\widehat{J}_\sigma(\mathbf{z},\boldsymbol{\nu},u,\epsilon)$ is constructed as follows:

$$\widehat{J}_\sigma(\mathbf{z},\boldsymbol{\nu},u,\epsilon) = \begin{cases} -\widehat{J}(\mathbf{z},\boldsymbol{\nu},u), & \text{if } \epsilon = 0,\ \Delta(\mathbf{z},u)=0, \\ -\widehat{J}(\mathbf{z},\boldsymbol{\nu},u) + \epsilon^{-\lambda}\Delta(\mathbf{z},u) + \sigma\epsilon^{\gamma}, & \text{if } \epsilon > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where

- ϵ is a new decision variable;
- $\sigma > 0$ is the penalty parameter;
- λ and γ are positive constants satisfying $1 \le \gamma \le \lambda$.

The new decision variable ϵ is subject to the following bounds:

$$0 \le \epsilon \le \widetilde{\epsilon}, \tag{26}$$

where $\widetilde{\epsilon} > 0$ is a small positive number.

Our unconstrained penalty problem, to be referred to as Problem D, is defined as follows: Choose the system parameters z_{ij}, the harvesting fractions ν_j, the new decision variable ϵ and the control $u(s)$ to minimize $\widehat{J}_\sigma(\mathbf{z},\boldsymbol{\nu},u,\epsilon)$ subject to:

- the dynamics (12) and (17)-(18);
- the jump conditions (19)-(20);
- the bounds (22) and (26).

Note that when the penalty parameter σ is large, the term $\sigma\epsilon^\gamma$ in $\widehat{J}_\sigma$ forces ϵ to be small, thus causing the second term $\epsilon^{-\lambda}\Delta(\mathbf{z}, u)$ to severely penalize any constraint violations. It can be shown that $\widehat{J}_\sigma$ is an exact penalty function in the sense that when the penalty parameter σ is sufficiently large, any local solution of the approximate unconstrained problem (i.e. Problem D) is also a local solution of Problem C [8].

Problem D can be solved as a non-linear programming problem using the MISER 3.3 software. MISER automatically calculates the objective function gradients by integrating a costate system backwards in time; for more details see [1, 2, 6, 11].

In summary, the original non-smooth optimal control problem undergoes a series of transformations to overcome the challenges that would exist in realistic problems of this nature. This approach is summarized below.

- Our original problem included a complex revenue function that is a discontinuous function of the state variables. The decision variables need to be chosen optimally to maximize this revenue function.
- We use the time-scaling transformation (described in Section 3.1) to map the variable jump times to fixed times in a new time horizon, as standard optimal control algorithms can only deal with fixed jump times.
- We then use a smoothing transformation involving binary variables and quadratic constraints (described in Section 3.2) to overcome the challenge posed by the discontinuous objective function.
- Since standard optimal control software such as MISER 3.3 struggle with these quadratic constraints, we apply the exact penalty approach to transform the problem into an unconstrained problem. We thus arrive at a smooth impulsive optimal control problem with fixed jump times and only bound constraints. Such problems can be solved effectively using MISER 3.3, which solves optimal control problems using non-linear programming techniques.

In the next section, we demonstrate the efficiency of the proposed approach with a numerical example.

5. Numerical Results

We consider the shrimp farming model described in Section 2 with the following parameters:

- $N = 4$ (3 partial harvests and 1 final harvest);
- $L = 5$ (price function is based on 5 different weight ranges for the shrimp);
- $T = 13.2$;
- $c_j = 0$ for $j = 1, 2, 3, 4$ (no variable harvesting costs);

i	β_{i-1}	β_i	α_i
1	0	5	\$2
2	5	10	\$4
3	10	15	\$6
4	15	20	\$8
5	20	25	\$12

Table 1: Price Function Parameters

σ	ϵ^*	Penalty Function Value	Constraint Violation
10^4	4.0661×10^{-1}	5.601975×10^3	0.0000
10^5	2.6733×10^{-1}	4.180581×10^3	0.0000
10^6	2.8292×10^{-2}	3.107395×10^3	0.0000
10^7	5.0437×10^{-4}	3.110452×10^3	0.0000
10^8	7.5006×10^{-4}	3.110451×10^3	0.0000

Table 2: Numerical convergence using $\lambda = 4.01$ and $\gamma = 3.55$

- $h = 50$;
- $n_0 = 40,000$ and $w_0 = 1$;
- $a = 3.5$ and $b = 10^{-5}$;
- α_i, $i = 1, \ldots, 5$ and β_i, $i = 0, \ldots, 5$ are given in Table 1.

Recall from Section 4 that the parameters λ and γ in the exact penalty function must satisfy the condition $1 \leq \gamma \leq \lambda$. This is to ensure that the convergence results in [8] hold. Numerical testing reveals that the choice of λ and γ can significantly affect the results. Our best results were obtained using either $\lambda = 4.01$ and $\gamma = 3.55$ or $\lambda = 5.01$ and $\gamma = 3.55$. These results are presented below.

When running MISER 3.3, the initial values for ϵ and σ were 5.0×10^{-1} and 1.0×10^4 respectively. The penalty parameter σ was increased by a multiple of 10 for each subsequent MISER run. As expected this caused ϵ to decrease in value. Tables 2 and 3 show the penalty function value and the optimal value of ϵ (denoted by ϵ^*) for each run.

Note that the results in Tables 2 and 3 show a clear convergence of the objective function as σ is increased.

The optimal solution corresponding to the last line in Table 2 is:

$$\nu_1 = 6.8488 \times 10^{-1}, \qquad \nu_2 = 6.7209 \times 10^{-1}, \qquad \nu_3 = 0.0000, \qquad \nu_4 = 1.0000,$$

$$\tau_1 = 8.47906, \qquad \tau_2 = 13.2, \qquad \tau_3 = 13.2, \qquad \tau_4 = 13.2,$$

$$w(\tau_1) = 10.0, \qquad w(\tau_2) = 20.0, \qquad w(\tau_3) = 20.0, \qquad w(\tau_4) = 20.0.$$

σ	ϵ^*	Penalty Function Value	Constraint Violation
10^4	4.3980×10^{-1}	5.330159×10^3	0.0000
10^5	2.9917×10^{-1}	4.330777×10^3	0.0000
10^6	1.6207×10^{-2}	3.106807×10^3	0.0000
10^7	8.2666×10^{-3}	3.107617×10^3	0.0000
10^8	8.2661×10^{-3}	3.103999×10^3	0.0000

Table 3: Numerical convergence using $\lambda = 5.01$ and $\gamma = 3.55$

The optimal solution corresponding to the last line in Table 3 is:

$$\nu_1 = 6.4803 \times 10^{-1}, \qquad \nu_2 = 5.4889 \times 10^{-2}, \qquad \nu_3 = 8.6884 \times 10^{-2}, \qquad \nu_4 = 1.0000,$$

$$\tau_1 = 8.47906, \qquad \tau_2 = 8.89985, \qquad \tau_3 = 10.71427, \qquad \tau_4 = 13.2,$$

$$w(\tau_1) = 10.0, \qquad w(\tau_2) = 10.9935, \qquad w(\tau_3) = 15.0, \qquad w(\tau_4) = 20.0.$$

The optimal state variables corresponding to the solutions in Tables 2 and 3 are shown in Figures 1 and 2.

The results obtained in this section cannot be directly compared to the numerical results obtained in [7, 17]. This is because the price function used here is a weight dependent piecewise constant function (recall (9a)) and is not a fixed constant as in the numerical examples of [7, 17]. The method presented in this paper is more realistic from a commercial point of view, as the shrimp price is expected to be heavily dependent on the weight of the shrimp.

6. Concluding Remarks

We have developed an efficient computational algorithm for solving a class of optimization problems containing a discontinuous (and therefore non-differentiable) objective function subject to a dynamic system involving jump conditions at variable time points. The technique was successfully tested on a realistic shrimp farming problem. We note that that this algorithm could be adapted to other classes of problems to maximize or minimize non-smooth objective functions subject to various forms of constraints and continuous time dynamics.

Our results illustrate that we can obtain clear convergence of the objective function while optimally determining the partial harvesting fractions as well the corresponding partial harvest times, as shown by the numerical results. Future studies on the shrimp problem will consider variable initial conditions, variable harvesting costs and optimization of the feeding rate. Models for multiple continuous-production cycles of shrimp involving suitable cost functions for variable harvesting costs may also be examined in future research.

Moreover, research should be undertaken to study the relationship between the constants λ and γ in the exact penalty approach. Extensive numerical testing has shown that the optimal solution obtained is sometimes sensitive to

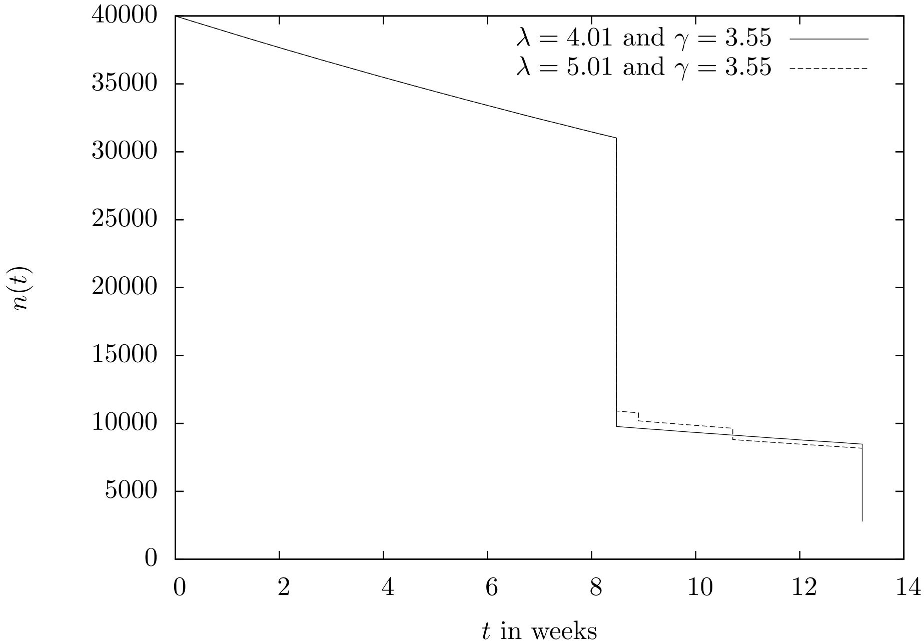

Figure 1: Number of Shrimps

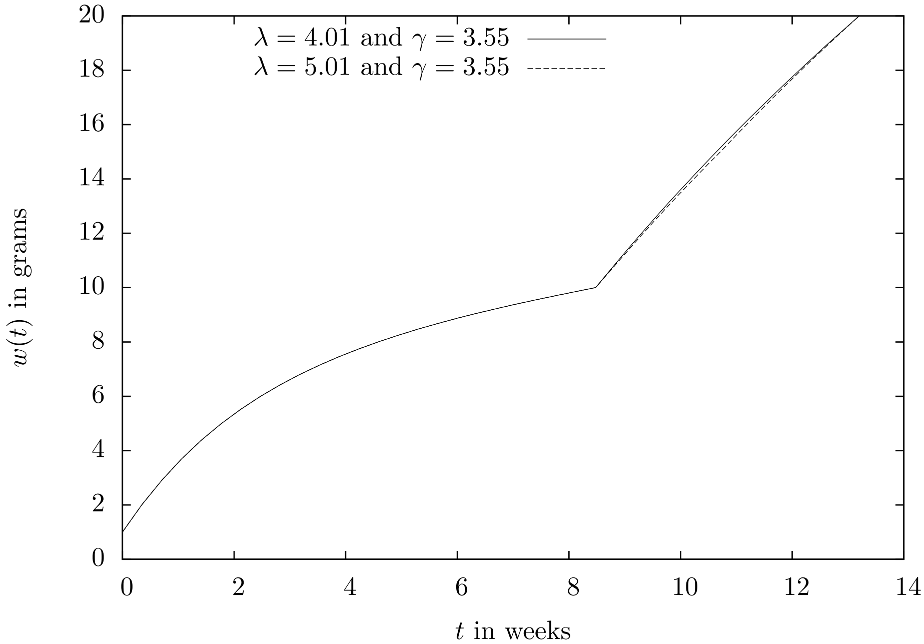

Figure 2: Average Weight of Shrimp

the values chosen for λ and γ. It would be of great benefit to develop a general guide as to the choice of values of these parameters, especially considering the vast array of applications of the exact penalty approach [1, 19].

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