On the Performance of the Minimum VaR Portfolio[*]

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Abstract

Alexander and Baptista (2002) develop the concept of mean-VaR efficiency for portfolios and demonstrate its very close connection with mean-variance efficiency. In particular they identify the *minimum VaR portfolio* as a special type of mean-variance efficient portfolio. Our empirical analysis finds that, for commonly used VaR breach probabilities, minimum VaR portfolios yield *ex post* returns that conform well with the specified VaR breach probabilities and with return/risk expectations. These results provide a considerable extension of evidence supporting the empirical validity and tractability of the mean-VaR efficiency concept.

JEL Classification: G11

Key Words: Portfolio Optimization, Mean-Variance Efficiency, Value-at-Risk, Fama-French Portfolios, iShares.

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1 Introduction

Portfolios with a discrete investment horizon are routinely constructed and assessed with consideration for two important concepts. First, mean-variance efficiency: for an efficient portfolio, expected return is maximized for a given level of risk as indicated by return variance (Markowitz, 1952). Second, value-at-risk (VaR), being the level of portfolio loss that has specified (small) probability of being breached.

In the spirit of Baumol (1963), Alexander and Baptista (2002) develop the concept of mean-VaR efficiency for portfolios and demonstrate its very close connection with mean-variance efficiency.[1] In particular they formulate the minimum VaR portfolio as a special type of mean-variance efficient portfolio. Alexander and Baptista initially assume jointly normally distributed asset returns and show that the global minimum VaR portfolio, when it exists, is mean-variance efficient (Lemma 1, p.1166). Thus they are able to explicitly identify the minimum VaR portfolio using the efficient frontier formulation of Merton (1972) for the case when short sales are allowed (Proposition 1, p.1167). They then extend their results to the multivariate Student's t distribution (Section 3, p.1177), and in fact to any distribution for which the VaR can be written as a linear function of the expectation and standard deviation of the returns. A Student's t distribution assumption has notable relevance in that the heavy-tailedness of returns distributions is an empirically-observed fact: in a recent study, Platen and Sidorowicz (2008) show that a Student's t distribution with four degrees of freedom provides a good fit to the returns of a large sample of widely varying world stock indices.

The Markowitz mean-variance efficiency paradigm for portfolio selection has some very attractive features, including ease of application and analysis, and a long history of theoretical understanding and practical experience, which we want to preserve. Thus we specify our returns distribution as-

sumption only with respect to mean-variance efficient portfolios, and proceed by finding the portfolio on the efficient frontier that minimizes VaR, given (small) probability $q^\star$ as the specification for the acceptable likelihood of a loss that exceeds VaR.[2] We call this the *minimum VaR portfolio*, and term $q^\star$ the *VaR breach probability* or just *breach probability*.

Our empirical analysis finds that minimum VaR portfolios with commonly used breach probabilities (formed *ex ante*) yield *ex post* returns that (i) generally conform to the *ex ante* VaR breach specifications, and (ii) conform, on average, to *ex ante* return/risk expectations, as indicated by ranking amongst the *ex post* returns of various other efficient and non-efficient portfolios. Our results, obtained with two very distinct datasets, provide a considerable extension of evidence supporting the empirical validity and tractability of the mean-VaR efficient portfolio framework originated by Alexander and Baptista (2002). Furthermore, the conformity of *ex post* return performance to *ex ante* specification/expectation for minimum VaR portfolios supports Alexander et al.'s (2009) proposal that constraining VaR reduces portfolio estimation risk.

Our contribution to the literature is best considered in comparison with Alexander et al. (2009). Firstly, Alexander et al. use simulation and empirical analyses to examine the *ex post* performance of VaR-constrained efficient portfolios in general; whereas we use empirical analysis to examine the performance specifically of the minimum VaR portfolio in comparison with mean-variance efficient minimum variance and tangency portfolios, and inefficient equally weighted and index portfolios. Secondly, our empirical analysis examines the minimum VaR portfolio with short sales allowed and disallowed; whereas Alexander et al. examine VaR-constrained portfolios with short sales allowed only. Thirdly, for their empirics, Alexander et al. use the six Fama-French size and book-to-market partitioned portfolios; whereas we use the cross-sectionally more diverse 25 Fama-French size and book-to-market port-

folios. Finally, our empirics extend to a second dataset comprised of iShares utilized with higher frequency (weekly) portfolio rebalancing (in contrast to monthly rebalancing using the Fama-French data).

This paper is organized as follows. Section 2 sets out the relation between an efficient portfolio and its VaR, and specifically identifies the minimum VaR portfolio. Section 3 introduces two disparate datasets and reviews the characteristics of the minimum VaR portfolios constructed from them, for varying breach probabilities, and with and without short sales allowed. We apply a rolling window investment process to the two datasets to represent an investor who uses recent historical data to period-to-period identify the *ex ante* minimum VaR portfolios. Then, in Section 4, we examine the time series *ex post* performance of this strategy, and demonstrate that the minimum VaR portfolios generally conform well to their *ex ante* VaR breach specifications. Favorable return/risk performance is also demonstrated in comparison with tangency, minimum variance, equally weighted and index portfolios. Finally, Section 5 concludes the paper.

2   Value-at-risk along the efficient frontier

For a designated discrete investment time horizon, let $\sigma > 0$ and μ be, respectively, the standard deviation ("volatility") of return and the expected return of a portfolio constructed from a specified universe of $N \geq 2$ *risky* assets. An *efficient risky asset portfolio* is a combination of the assets that dominates other possible combinations by offering the maximum possible expected return given its volatility of return, or, equivalently, the minimum possible volatility of return given its expected return. The range of efficient combinations (σ_p, μ_p) defines the *efficient frontier* in (σ, μ) space (Markowitz, 1952).

Define random variable R_p to be the realized return of an efficient portfolio

4

over the discrete investment time horizon, and let $Z_p = (R_p - \mu_p)/\sigma_p$ be the standardized realized return. We assume each Z_p to be identically distributed for all possible efficient portfolios, and use VaR considerations to single out a specific portfolio on the efficient frontier.[3] To do this, let $-Q$ be the benchmark rate of return below which a return will fall with probability q, $0 < q < 1$ (for VaR considerations, q will be small, i.e. $0 < q < 0.5$). So we have the relations

$$\Pr\left(R_p < -Q\right) = \Pr\left(Z_p < \frac{-Q - \mu_p}{\sigma_p}\right) = \Phi\left(\frac{-Q - \mu_p}{\sigma_p}\right) = q,$$

and, inversely,

$$-Q = \mu_p + \sigma_p \Phi^{-1}(q), \tag{1}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf)[4] of Z_p, and $\Phi^{-1}(\cdot)$ is its inverse function (assumed uniquely defined). For a typical small value of q, we expect that $\Phi^{-1}(q)$ will be negative and Q will be positive.

The efficient frontier formulation of Merton (1972) specifies an explicit relation between μ_p and σ_p when short sales are allowed. Using this relation, we can eliminate μ_p from (1), reducing it to a functional equation of the form

$$Q = Q(\sigma_p, q). \tag{2}$$

Our suggested strategy can now be summarized as follows. Let q^* denote a specified value of the VaR breach probability, q. Substitute $q^\star$ for q in (2), then use (2) to select that value of σ_p which *minimizes* the VaR, Q. This yields a value, denoted Q_{min}, and a corresponding VaR-minimizing value of σ_p, denoted σ_{Qmin}. This procedure identifies the *minimum VaR strategy*, with Q_{min} the minimum value-at-risk corresponding to $q^\star$. Precise details of the procedure, with short sales allowed, are set out in Section 2.1 below.

As an aid to intuition, Figure 1 displays example probability density func-

tions for two efficient portfolios (designated A and B) and their comparative VaRs, Q_A and Q_B, for a given $q^\star$. To minimize VaR, we choose the portfolio that maximizes $-Q$: Figure 1 indicates that portfolio A is preferable to portfolio B in terms of VaR.

INSERT FIGURE 1 HERE

2.1 The minimum VaR strategy

In this section we obtain Alexander and Baptista's (2002) minimum VaR portfolio result, alluded to with equation (2). Substitute a given VaR breach probability $q^\star$ for q in (1), and use equation (17) of Merton's (1972) analytic representation of the efficient frontier to eliminate μ_p. This yields an explicit expression for the relation between Q and σ_p, namely,

$$-Q = \frac{\left[A + \sqrt{D(C\sigma_p^2 - 1)}\right]}{C} + \sigma_p \Phi^{-1}(q^\star), \tag{3}$$

where $A = \boldsymbol{i'\Sigma^{-1}\mu}$, $B = \boldsymbol{\mu'\Sigma^{-1}\mu}$, $C = \boldsymbol{i'\Sigma^{-1}i}$, and $D = BC - A^2 > 0$, as per Merton (1972, p.1853). This relation is valid for $\sigma_{mvp} < \sigma_p < \infty$, where σ_{mvp} is the volatility of the minimum variance portfolio, given by $\sigma_{mvp} = \sqrt{1/C}$.

Here $\boldsymbol{\mu}$ is the N-vector of expected returns and $\boldsymbol{\Sigma}$ is the $N \times N$ non-singular variance-covariance matrix of the returns for the N risky assets, $\boldsymbol{i}$ is an N-vector of ones, and the prime denotes a vector or matrix transpose. Merton's (1972) approach allows short-selling of individual risky assets but requires a "net positive" investment overall, which we can normalize to one unit of investment; hence $\boldsymbol{i'x}_p = 1$, where $\boldsymbol{x}_p$ is the N-vector of portfolio allocation weights corresponding to the selected portfolio.

Some calculus shows that the existence of a definable minimum VaR portfolio depends on the value of the "excess gradient criterion" (EGC), defined

6

by

$$EGC = \sqrt{D/C} - \left(-\Phi^{-1}(q^\star)\right).$$

For $EGC < 0$, VaR is minimized when $\mathrm{d}Q/\mathrm{d}\sigma_p = 0$. Calculating $\mathrm{d}Q/\mathrm{d}\sigma_p$ from (3) yields the coordinates of the minimizing portfolio as

$$\sigma_{Qmin} = \frac{|\Phi^{-1}(q^\star)|}{\sqrt{C\left[\Phi^{-1}(q^\star)\right]^2 - D}}, \quad \text{and} \quad \mu_{Qmin} = \frac{A}{C} + \frac{D}{C\sqrt{C\left[\Phi^{-1}(q^\star)\right]^2 - D}}.$$

The corresponding minimized value of Q satisfies

$$-Q_{min} = \frac{A}{C} + \frac{C|\Phi^{-1}(q^\star)|\Phi^{-1}(q^\star) + D}{C\sqrt{C\left[\Phi^{-1}(q^\star)\right]^2 - D}}, \tag{4}$$

and some further algebra gives the corresponding portfolio allocation as

$$\begin{aligned}
\boldsymbol{x}_{Qmin} &= \left(\frac{A}{CD} + \frac{1}{C\sqrt{C\left[\Phi^{-1}(q^\star)\right]^2 - D}}\right)\left(C\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - A\boldsymbol{\Sigma}^{-1}\boldsymbol{i}\right) \\
&\quad + \frac{1}{D}\left(B\boldsymbol{\Sigma}^{-1}\boldsymbol{i} - A\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right).
\end{aligned} \tag{5}$$

Equations (4) and (5) respectively correspond to equations (11) and (10) of Alexander and Baptista (2002) if $\Phi^{-1}(q^\star)$ (in our notation) is substituted for $-t^*$ (in theirs). Our requirement that $EGC < 0$ is equivalent to Alexander and Baptista's Proposition 1 requirement that $t > \Phi(\sqrt{D/C})$, where t (in their notation) equals $1 - q^\star$ (in ours).[5]

As explained by Alexander and Baptista (2002, Corollary 3), the minimum variance portfolio is notionally a special case of the minimum VaR strategy obtained in the limit as $q^\star \to 0$. If we accept that our distributional assumption for standardized efficient portfolio returns entails an infinite left tail, then $q^\star \to 0$ implies $\Phi^{-1}(q^\star) \to -\infty$ and $\sigma_{Qmin} \to \sigma_{mvp} = \sqrt{1/C}$. Since we wish

to work with discrete returns, as is appropriate for Markowitz-style portfolio optimization, the reasonableness of our assumption that standardized efficient portfolio returns have identical distributions will certainly break-down at the extreme left tail (i.e. as $q^\star \to 0$). However this is not necessarily of concern. For the purpose of VaR minimization, given a reasonable choice of $q^\star > 0$, we are only concerned that the tail area (i.e. $q^\star$) of our single assumed distribution matches the actual tail areas to the left of $-Q(\sigma_p, q^\star)$ of the true returns distributions of different efficient portfolios; this is generally borne out by the empirical *ex post* performance of the minimum VaR portfolios assessed in this paper.

For $EGC \geq 0$, the minimum VaR portfolio can only be approached in the limit as $\sigma_p \to \infty$.[6] For the case when $\Phi^{-1}(q^\star) = -\sqrt{D/C}$, we have $Q_{min} \to -A/C$, as $\sigma_p \to \infty$; and for the case when $\Phi^{-1}(q^\star) > -\sqrt{D/C}$, we have $Q_{min} \to -\infty$, as $\sigma_p \to \infty$. For our datasets and realistic choices of $q^\star$, there are no instances of $EGC \geq 0$. If such an instance occurs in a dataset, the technique of Maller and Turkington (2002) can be used to approach the minimum VaR portfolio as closely as desired.

If the portfolio allocation decision requires short sales to be restricted (or any other such imposition of lower or upper limits on individual risky asset positions), numerical techniques can be used to identify the minimum VaR portfolio. However, equation (5) will still be valid when short sales are disallowed if the calculated portfolio allocation weights do not involve short positions.

Alexander and Baptista (2004) revisit the mean-VaR efficient portfolio framework and additionally consider conditional value-at-risk (CVaR) as the integral risk measure for portfolio optimization. The CVaR metric, in contrast to VaR, is attractive for being a coherent risk measure as per Artzner et al. (1999).[7] Appendix A demonstrates that the Section 2 setup also extends to the identification of a minimum CVaR portfolio.

3   Empirical analysis

It has been observed many times in practice and in the literature that portfolios selected by the Markowitz procedure frequently fail to perform *ex post* as they were expected to *ex ante*. However, we find that efficient portfolios constructed using the minimum VaR strategy do yield performance close to that expected, as demonstrated by the empirical analysis that follows.

Given a universe of risky assets investable for a series of equal-length time periods, and a specified VaR breach probability $q^\star$, specific implications of the setup and definitions given in Section 2 are that a time series strategy of investing in the minimum VaR portfolio should result in:

(i) *a series of portfolio return realizations that breach the time-varying minimum value-at-risk according to a binomial distribution with "success" probability $q^\star$; furthermore,*

(ii) *a lesser number of portfolio return realizations breaching the time-varying minimum value-at-risk than for any other efficient portfolio strategy.*

Such expectations will only be borne out in reality if the world behaves in accordance with the assumptions made in deriving the relations in Section 2. In particular, the distribution assumed for the realized returns will, of course, be critical to the VaR calculations.

We test the minimum VaR strategy on two particular "real-world" asset return datasets, each observed over different periods of time. The potential disadvantage of this approach is a lack of generality, because the specifics of the data samples limit the extent to which inference can be made about wider populations, as compared with the alternative of a simulation approach, with which we could apply and evaluate procedures with full knowledge of the population parameters underlying the data.[8] The disadvantage of a simulation approach is that simulated data can never fully capture the variations inherent in real-world data: we are particularly interested in extreme downside as-

set return events, where the minimum VaR strategy should prove attractive. We attempt to curtail the lack of generality associated with our approach by selecting widely representative samples of assets for our datasets.

The performance of the minimum VaR strategy, along with some "standard" portfolios, is assessed out-of-sample using datasets of returns for two different universes of investable assets. The first set of assets is the 25 Fama and French size (market equity, ME) and book-to-market (book equity-to-market equity, BE/ME) partitioned and value-weighted stock portfolios (which we term FF5×5), and the second is an international collection comprised of iShares.

For investors limiting themselves to domestic US assets, the optimization problem is daunting due to the very large number of individual securities that might be considered. Furthermore, straightforward Markowitz optimization via a sample variance-covariance matrix is impossible when there are more assets than there are time series observations of returns on those assets (in which case the matrix would be singular). Reducing the problem to a choice between asset classes is a viable compromise. The question then arises as to which asset classes might appropriately be used. In order to reduce the dimensionality of the problem to a manageable level, we use the 25 value-weighted size and book-to-market partitioned US stock portfolios inspired by the approach of Fama and French (1993), thereby making a strategic choice of asset classes.[9]

Fama and French (1993) demonstrate the value-relevance of size and book-to-market factors in comparison to the market risk-premium factor of the standard CAPM. It thus seems appropriate to use size and book-to-market derived stock portfolios as asset classes underlying an investment allocation decision. Indeed these portfolios have previously been used to study portfolio optimization (e.g. by Jagannathan and Ma, 2003). An investor implementing a strategy based on allocations to the FF5×5 portfolios needs simply to

allocate funds to assets representative of these portfolios.[10]

Portfolio optimization naturally lends itself to application for international diversification. Giving regard to such purpose, we utilize exchange-traded funds in the form of iShares, together with Standard & Poor's depository receipts, to represent the opportunity set available to an investor who is not constrained to invest only in domestic securities.

While our portfolio investment setup is single-period, we incorporate a dynamic aspect in the empirical implementation via the use of a rolling investment allocation process. We take the position of an investor allocating wealth to a portfolio of assets with perfect hindsight about what has happened, but with very imperfect foresight about how the portfolio will perform in the future. Portfolio estimation and investment allocation is undertaken based on sample returns data for a "window" of time, and portfolio performance is calculated out-of-sample over the next unit of time. The window is then "rolled forward" by one unit of time, and the procedure repeated. In this way a time series of realized portfolio returns is generated, which can be used to evaluate the overall performance of the portfolio strategy. This procedure is described in detail in Appendix B.

3.1 FF5×5 and iShares datasets

Monthly returns for the 25 Fama and French size and book-to-market partitioned and value-weighted portfolios were downloaded from Ken French's website.[11] The 25 time series consist of 978 monthly returns from July 1926 to December 2007. Using a rolling estimation period of length 200 months, 778 vectors of mean monthly returns and matrices of monthly return variances and covariances were obtained as time series estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.[12]

This approach was repeated for the iShares dataset; however the shorter time series of iShares data requires more economy in terms of the estimation

and holding periods for application of portfolio optimization. From the CRSP database we obtained time series of Wednesday fortnightly discrete total returns for 23 international stock index iShares,[13] and Standard & Poor's depository receipts tracking the US S&P 500 stock index (SPY) and the US S&P MidCap 400 stock index (MDY). The iShares and the SPY and MDY exchange-traded funds offer investors a convenient method of creating an internationally diversified equity portfolio. For 17 of the iShares, and the SPY and MDY indices, the time series consist of 307 fortnightly returns from fortnight-ending 3 April 1996 to 26 December 2007. The returns time series for another six, newer, iShares have various later commencement dates, but the same terminal date.[14] Using a rolling estimation period of 100 fortnights, a time series of 207 sets of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ was obtained.[15]

For both the FF5×5 and iShares datasets, for each estimation period, three minimum VaR portfolios were identified using the methodology outlined in Section 2, corresponding to VaR breach probabilities ($q^\star$) equal to 0.01, 0.05 and 0.10. Furthermore three "standard" portfolios were identified: the minimum variance portfolio; the tangency portfolio that maximizes μ_p/σ_p (i.e. the efficient frontier portfolio that is tangent to a ray from the origin in (σ_p, μ_p) space); and the equally weighted portfolio.[16] Finally, an appropriate index asset was also identified as a fourth standard portfolio: specifically we use the S&P500 index and the MSCI World investable index, for which returns time series were obtained from CRSP. The realized return of each of these seven portfolios was calculated for the unit of time (month or fortnight) subsequent to each estimation period.

3.2   Minimum VaR portfolio characteristics

We considered two probability distributions for the efficient portfolio standardized realized returns: a Student's t distibution with four degrees of free-

dom (t_4), and a standard normal distribution. In the empirical evaluations reported in more detail in the next section, we found the Student's t_4 distribution to be appropriate for portfolios when short sales are allowed, whereas the imposition of a no-short-sales constraint resulted in realized returns distributions with very light tails at the lefthand end, for which the standard normal was more appropriate. Consequently, we only present results assuming a Student's t_4 distribution for portfolios with short sales allowed, and assuming a standard normal distribution for portfolios with short sales disallowed, and we restrict discussion to these cases.[17]

<center>INSERT FIGURE 2 HERE</center>

Figure 2 shows an example *ex ante* efficient frontier (top panel) and the associated VaR along the efficient frontier (bottom panel) as estimated from a subset of the FF5×5 dataset. The bottom panel of Figure 2 depicts the relation given by equation (3) for three choices of $q^\star = \{0.01, 0.05, 0.10\}$ and specifically identifies each minimum VaR portfolio, which correspond to locations on the efficient frontier shown in the top panel. Figure 2 illustrates that, for practical values of VaR breach probability ($q^\star \leq 0.10$), the minimum VaR portfolios are generally located towards the lefthand end of the efficient frontier, while remaining quite distinct from the minimum variance portfolio. The top panel in Figure 2 also reveals that both the S&P500 index and the equally-weighted combination of the FF5×5 portfolios are located below the efficient frontier.

<center>INSERT FIGURE 3 HERE</center>

Figure 3 shows that the expected return-risk ratios (μ_p/σ_p) of the three minimum VaR portfolios corresponding to $q^\star = \{0.01, 0.05, 0.10\}$ lie between those of the minimum variance and tangency portfolios for the entire time series of 778 portfolio sets obtained from the FF5×5 dataset, with or without

<center>13</center>

short sales allowed. With short sales allowed (see the top panel of Figure 3), the expected return-risk ratios of the minimum VaR portfolios are readily distinguishable from each other and from those of the minimum variance and tangency portfolios. With short sales disallowed (see the bottom panel of Figure 3), the minimum VaR portfolios are often located very close to each other and to the minimum variance portfolio.[18]

The tangency portfolio, by its nature, should produce the highest return/risk outcomes. However, as demonstrated by Alexander et al. (2009), overestimation error for expected return increases sharply as portfolio choice moves up along the efficient frontier. As illustrated by Figure 2, the location of the minimum VaR portfolio moves up along the efficient frontier as breach probability increases. Therefore, as breach probability increases, the *ex post* return/risk performance of the minimum VaR portfolio will benefit in terms of higher (estimated) expected performance, but will suffer due to higher overestimation error. Thus an "optimal" breach probability may be identified as the point when this trade-off becomes zero at the margin. Additional results not presented in this paper for both of our datasets with short sales allowed indicate that this optimal breach probability occurs at a level considerably higher than our maximum choice of $q^\star = 0.10$.

4 Ex post portfolio performance

In this section we summarize the results obtained from the rolling window investment process applied to our two datasets. The 200 month rolling estimation window (with monthly steps) applied to the FF5×5 dataset yielded a monthly time series of 778 sets of *ex ante* portfolios and their *ex post* realized returns. Similarly, the 100 fortnight rolling estimation window (with fortnightly steps) applied to the iShares/depository receipts dataset yielded a fortnightly time series of 207 sets of *ex ante* portfolios and their *ex post*

realized returns. Note that the overlapping estimation periods associated with the rolling window investment process imparts some serial correlation on the portfolio time series (see Appendix B).

4.1 Ex post portfolio performance: FF5×5 data

For an overall view of the data, refer to Figure 4, which shows the expected and realized returns of the minimum VaR portfolios constructed from the FF5×5 data, along with the *negatives* of their time varying VaRs, with and without short sales allowed. Using the technique detailed in Section 2, the portfolios are derived using a Student's t_4 distribution when short sales are allowed and a standard normal distribution when short sales are disallowed.

INSERT FIGURE 4 HERE

The different minimum VaR bounds indicated in Figure 4 all start off wide, relative to expected return, early in the time series (due to the presence of the 1929 stockmarket crash and subsequent volatility in the rolling estimation window), but tighten as the time series moves into more benign periods of stockmarket volatility. Breaches of minimum VaR (i.e. incidents of realized return less than the negative of minimum VaR, $R_p < -Q_{min}$) are clearly revealed in Figure 4 when they occur. For example, the October 1987 realized return is clearly identifiable as a breach for each minimum VaR portfolio. With short sales allowed and a VaR breach probability of $q^\star = 0.01$, the minimum VaR portfolio exhibits only two VaR breaches (see panel (a) of Figure 4), but when the VaR breach probability is loosened to $q^\star = 0.10$ (see panel (c)) the number of VaR breaches increases to 76. Later (see Figure 5 and Table 2) we assess whether the observed VaR breaches conform to the implications of our setup in Section 2.

Table 1 provides summary statistics for the minimum VaR and standard portfolios over the 778 rolling window observations for the FF5×5 data. The

first four columns summarize the characteristics of the portfolio weights. A favorable characteristic of the minimum VaR portfolios is that the portfolio weights for individual assets entail only modest extremes, even when short sales are allowed. By contrast, the portfolio weights for the tangency portfolio are highly variable and extreme in the short sales case. For example, the time series of portfolio weights (60 sets of 24 weights plus 718 sets of 25 weights) for the minimum VaR portfolio with $q^\star = 0.10$ has a global minimum of -1.62, a median minimum of -0.58, a global maximum of 1.25, and a median maximum of 0.84; whereas the respective values for the tangency portfolio are -142.5, -1.68, 137.3 and 1.42. That is, when short sales are allowed, the minimum VaR portfolio strategy is not overly prone to extreme long and short positions, whereas the tangency portfolio strategy is.

INSERT TABLE 1 HERE

Disallowing short sales limits extreme positions, as all portfolio weights must lie between zero and one, which also limits the scope for differentiation between efficient portfolios. Consequently the differences between the portfolio weight extremes of the minimum variance, minimum VaR and tangency portfolios are reduced (see columns 1–4, rows "without short sales allowed", of Table 1). The smaller differentiation is also observable in terms of expected return-risk ratio by comparing the top and bottom panels of Figure 3 (noting the difference in the scales of the vertical axes).

In columns 5–11 of Table 1, summary statistics for the expected return-risk ratios and realized returns of the minimum variance, minimum VaR and tangency portfolios are listed, and seen to be consistent with their relative efficient frontier positions. That is, the time series minima, maxima and means of the *expected* return-risk ratios (columns 5–7) all increase in the order of the efficient frontier positions of the portfolios: again, the incremental differences are much greater with short sales allowed than without. Interest-

16

ingly, although the tangency portfolio has the greatest extremes of portfolio weights, it has the lowest variability (standard deviation) of expected return-risk ratio (see column 8). The time series means and standard deviations of the *realized* returns (columns 9 and 10) also generally increase in the order of the efficient frontier positions of the portfolios: only the mean realized return of the tangency portfolio strategy with short sales allowed deviates from this order (with a very low mean realized return). Measured by the ratio of mean realized return to standard deviation of realized return (column 11), the tangency portfolio with short sales allowed produces, by far, the worst performance, followed by the inefficient S&P500 and equally weighted portfolios with the next worst performances.

From comparison of columns 9–11 with columns 5–8 of Table 1, the conformity of the rankings of *ex post* return performance with the rankings of *ex ante* return-risk expectation for the minimum VaR portfolios is supportive of Alexander et al.'s (2009) proposal that constraining VaR reduces portfolio estimation risk. Constraining short sales is also known to reduce portfolio estimation risk (e.g. see Jagannathan and Ma, 2003); but comparison of the *ex post* results (columns 9–11) with and without short sales allowed indicates that minimizing VaR reduces portfolio estimation risk without the loss of return performance associated with additionally constraining short sales.

For each portfolio strategy, columns 12–17 of Table 1 provide: summary statistics for realized returns standardized by their *ex ante* expected values and standard deviations (μ_p and σ_p), so that they represent observations on Z_p; and test statistics for the fits of the assumed distributions. With short sales allowed, we do not reject the Student's t_4 distribution for the standardized realized returns of the minimum variance and minimum VaR portfolios at the 5% level of significance using a Kolmogorov-Smirnov test (column 16), though we do reject it for the tangency portfolio. With short sales disallowed, we also reject the standard normal distribution as a fit for

17

all portfolios (column 17).[19] However, a caveat is in order here. For VaR considerations we need only be concerned with the left tail of the returns distributions. Figure 5 shows a magnified view of the extreme left tails of the P-P (probability) plots for minimum VaR portfolio standardized realized returns versus the Student's t_4 distribution when short sales are allowed, and versus the standard normal distribution with short sales disallowed.

INSERT FIGURE 5 HERE

With short sales allowed: up to the $q^\star = 0.05$ and 0.10 probability levels (see panels (a) and (b) of Figure 5) the Student's t_4 distribution conforms reasonably closely with the empirical distributions of the commensurate minimum VaR portfolio standardized realized returns. When short sales are disallowed: up to the 0.05 probability level (see panel (c)) the standard normal distribution provides a good fit with the commensurate empirical distribution; and up to the 0.10 probability level (see panel (d)) the standard normal distribution is marginally more heavy tailed than the commensurate empirical distribution.[20]

Our next set of results, shown in Table 2, allows us to assess whether the numbers of breaches of minimum VaR observed in the time series in Figure 4 conform to the distributional assumptions we have made for standardized realized returns.

INSERT TABLE 2 HERE

Table 2 has two diagonal series of boxed and bolded numbers: these tally the minimum VaR portfolios' breaches of minimum VaR (occurences of $R_p < -Q_{min}$ depicted in Figure 4), and can be compared within columns to the tallies of breaches by the other portfolios. Our expectations are that the boxed and bolded numbers (i) should conform to a Binomial($778,q^\star$) distribution, and (ii) should be the lowest within their own columns. Numbers

that are less than the boxed and bolded numbers within a column are under-lined, indicating portfolios that were more successful (i.e. had fewer breaches of minimum VaR) than the minimum VaR portfolio. The Binomial(778,$q^\star$) expectation and 90% (two-tailed) confidence interval for VaR breaches for each minimum VaR portfolio are indicated in brackets under the boxed and bolded observed VaR breaches.

The minimum VaR portfolios' VaR breach tallies shown in Table 2 gener-ally conform well to their binomial distribution expectations. For $q^\star = 0.01$, the deviation of breaches from expectation is attributable to a difference in the tail areas (up to the 0.01 probability level) of the empirical and assumed distributions for standardized realized returns.

We note that the minimum VaR portfolio for a given $q^\star$ does not always attain the minimum number of breaches of minimum VaR in comparison to other portfolios, as indicated by the underlined numbers within columns of Table 2. However this underperformance only occurs with respect to nearby efficient portfolios, i.e. the minimum variance portfolio or other minimum VaR portfolios with close $q^\star$ values, and the differences are slight. That other portfolios are found, *ex post*, to serendipitously produce fewer breaches of minimum VaR than the minimum VaR strategy, implemented *ex ante*, might not be particularly worrying to an investor relying on the minimum VaR to achieve desired, or required, outcomes.

For each portfolio, Table 3 presents the average size of the breaches of minimum VaR tallied in Table 2. The size of a breach is calculated as the negative of the portfolio's realized return (i.e. realized loss), minus minimum VaR (i.e. $-R_p - Q_{min}$). For example, with short sales allowed and $q^\star = 0.05$, the minimum VaR portfolio breaches minimum VaR (i.e. breaches its own VaR) 37 times by an average of 0.018 (see panel (b) of Figure 4 for a depiction of these breaches and see Table 2 for the tally); since each breach is for a monthly investment period, we multiply by 12 to present an annualized

average breach size of 22% (see Table 3). For comparison, again with short sales allowed and $q^\star = 0.05$, the tangency portfolio breaches minimum VaR 101 times (see Table 2) with an annualized average breach size of 198% (see Table 3). Note that, in describing average breach size, the % symbol represents a difference in percentages rather than a percentage difference.

INSERT TABLE 3 HERE

Like Table 2, Table 3 has two diagonal series of boxed and bolded values: these are the average size of breaches of minimum VaR specifically for the minimum VaR portfolios, and can be compared within columns to the average size of breaches of minimum VaR by the other portfolios. Our expectation is that the boxed and bolded values should be the lowest within their own columns. Values that are less than the boxed and bolded values within a column are underlined, indicating portfolios that were more successful (had a lower average size of breaches of minimum VaR) than the minimum VaR portfolio. Similar to the results depicted in Table 2, when the minimum VaR portfolio does not achieve the lowest average size of breaches of minimum VaR, as indicated by the underlined values within columns of Table 3, almost always this underperformance only occurs with respect to nearby portfolios, i.e. the minimum variance portfolio or other minimum VaR portfolios with close $q^\star$ values. Testing for differences of means within columns, there is no portfolio that has an average size of breaches of minimum VaR significantly lower than that of the minimum VaR portfolio.

The rolling window investment process utilized in this empirical analysis seeks to represent an investment procedure that commits an investor to an uncertain outcome each period. Implemented under our assumptions, the minimum VaR strategy generally performs as expected, and thereby generally distinguishes itself favorably in terms of breaches of minimum VaR – certainly in comparison with the tangency, equally weighted and index portfolios.

20

4.2 Ex post portfolio performance: iShares data

The analysis reported in Section 4.1 for the FF5×5 data considers a particular universe of domestic US assets. This section's analysis considers a different universe of international assets represented by (up to) 23 international stock index iShares and two US stock index Standard & Poor's depository receipts. For this dataset the time span of the returns data is shorter but more frequently observed than for the FF5×5 dataset. Thus, in order to achieve a reasonable time series length of portfolio estimates and realized returns, fortnightly portfolio rebalancing was applied instead of monthly rebalancing.

Figure 6 (which is the counterpart for the iShares data of Figure 4 for the FF5×5 data) shows the expected and realized returns of the minimum VaR portfolios constructed from the iShares data, along with the *negatives* of their time varying VaRs, with and without short sales allowed. Breaches of minimum VaR (i.e. incidents of realized return less than the negative of minimum VaR, $R_p < -Q_{min}$) are clearly revealed in Figure 6 when they occur.

INSERT FIGURE 6 HERE

Table 4 provides summary statistics for the minimum VaR and standard portfolios over the 207 rolling window observations for the iShares data. These results are generally in accordance with the FF5×5 results presented in Table 1. Beginning with columns 1–4 of Table 4, the minimum VaR portfolios' investment weights for individual assets entail only modest extremes of long and short positions in comparison to the tangency portfolio when short sales are allowed. Next, columns 5–11 show that the time series statistics for the expected return-risk ratios and realized returns of the minimum variance, minimum VaR and tangency portfolios are very much consistent with their relative efficient frontier positions. As was the case with the FF5×5 data, it is only the mean realized return of the tangency portfolio strategy

with short sales allowed that deviates from the expected order of performance with a very large negative mean realized return (column 9). Measured by the ratio of mean realized return to standard deviation of realized return (column 11), the tangency portfolio with short sales allowed produces, by far, the worst performance, followed by the mean-variance inefficient MSCI World investable index with the next worst performance. In contrast to the FF5×5 results, disallowing short sales almost always improves portfolio performance as measured by the ratio of mean realized return to standard deviation of realized return (column 11). Finally, the Kolmogorov-Smirnov test p-values given in columns 16–17 do not reject the Student's t_4 distribution as a fit for the standardized realized returns of the minimum VaR portfolios with short sales allowed; nor do they reject the standard normal distribution as a fit for the standardized realized returns of the minimum VaR portfolios without short sales allowed.[21]

<div align="center">INSERT TABLE 4 HERE</div>

Figure 7 shows a magnified view of the extreme left tails of the P-P (probability) plots for minimum VaR portfolio standardized realized returns versus the Student's t_4 distribution when short sales are allowed, and versus the standard normal distribution with short sales disallowed. With short sales allowed: up to the $q^\star = 0.05$ probability level (see panel (a) of Figure 7) the Student's t_4 distribution is slightly more heavy tailed than the empirical distribution of the commensurate minimum VaR portfolio standardized realized returns; and up to the 0.10 probability level (see panel (b)) the Student's t_4 distribution is less heavy tailed than the commensurate empirical distribution. When short sales are disallowed: up to the 0.05 probability level (see panel (c)) the standard normal distribution is less heavy tailed than the commensurate empirical distribution; and up to the 0.10 probability level (see panel (d)) the standard normal distribution has a tail weight similar to the

<div align="center">22</div>

commensurate empirical distribution.

Tables 5 and 6 are the counterparts for the iShares data of Tables 2 and 3 for the FF5×5 data. The conclusions are similar. The tallies of the minimum VaR portfolios' breaches of minimum VaR shown by the boxed and bolded numbers in Table 5 conform well with our expectations: except for the minimum VaR portfolio without short sales and with $q^\star = 0.01$, the numbers of breaches are within the 90% confidence interval for a Binomial(207,$q^\star$) random variable. As with the FF5×5 results, we find *ex post* that there are portfolios that breach minimum VaR on fewer occasions than the minimum VaR portfolio, as indicated by the underlined numbers within columns of Table 5, but they are nearby efficient portfolios and the differences are slight. Table 6 presents the average size of the breaches of minimum VaR tallied in Table 5. When the minimum VaR portfolio does not achieve the lowest average size of breaches of minimum VaR, as indicated by the underlined values within columns of Table 6, this underperformance usually only occurs with respect to nearby efficient portfolios (although notably the equally weighted portfolio outperforms in two of the six columns). Testing for differences of means within columns of Table 6, there is no portfolio that has an average size of breaches of minimum VaR significantly lower than that of the minimum VaR portfolio.

5 Conclusion

That investors can, and should, maximize the return to risk ratios of their portfolios is a well established principle. Investors might also minimize or

limit the size of the VaRs of their portfolios. These important portfolio features can be analyzed and managed jointly.

We provide empirical support for the applicability and validity of Alexander and Baptista's (2002) relation linking the concepts of portfolio efficiency and VaR minimization via analyses based on two datasets of asset returns. The first of these represents investment opportunities for investors restricting themselves to US securities only. The second is of shorter duration, observed at higher frequency, comprising returns on a set of assets useful to investors seeking an internationally diversified portfolio. Portfolios were formed *ex ante* and need not have behaved as expected *ex post*. Nevertheless we find that, for commonly used VaR breach probabilities, minimum VaR portfolios yield returns that conform well with the specified VaR breach probabilities and with return/risk expectations.

The analysis shows that investors can achieve a mean-variance efficient portfolio and simultaneously minimize VaR. It also highlights the fact that an accurate estimate of the distribution function of the returns is essential, but only for the lefthand tail near the VaR breach probability designated by the user.

Notes

[1]See also Alexander (2009), Alexander and Baptista (2004, 2006, 2008), and Alexander et al. (2009) for further development and analysis of this concept. Fabozzi et al. (2009) present a recent review of this concept and other related research.

[2]The VaR of a portfolio conventionally refers to the threshold *dollar* loss the portfolio is at risk of suffering over a discrete holding period with a specified small probability, $q^\star$, say (which might be, for example, the 1% level specified by the Basel Committee on Banking Supervision, 2006). We deviate slightly from this convention and quantify VaR relative to initial portfolio value; i.e. our VaR represents a relative loss rather than an absolute loss. With this interpretation, the VaR is the negative of the return at the $q^\star$-quantile on the lefthand end of the portfolio's returns distribution.

[3]Working within the Markowitz (1952) paradigm it is desirable to use discrete returns corresponding to the designated discrete portfolio holding period, as we do. This implies a potential lower bound of minus one for realized returns, with consequently varying lower bounds for the standardized realized returns of different portfolios. In principle this violates our assumption of identically distributed standardized realized returns of efficient portfolios, but, practically speaking, the lower bound is seldom if ever achieved and exerts no effect on the analysis.

[4]The symbol Φ is reminiscent of the standard normal cdf, but we do not restrict ourselves to this distribution.

[5]Similarly relevant is Lemma 1 of Alexander and Baptista (2006).

[6]Equation (3) shows that $-Q$ is the sum of two functions of σ_p: Merton's (1972) efficient frontier, added to a line through the origin with slope $\Phi^{-1}(q^\star)$. The slope of the efficient frontier decreases from plus infinity at $\sigma_p = \sigma_{mvp}$ to $\sqrt{D/C} > 0$ as $\sigma_p \to \infty$. Thus a definable maximum for $-Q$ (minimum for VaR) will obtain for choices of $q^\star$ sufficiently small that $\Phi^{-1}(q^\star) < -\sqrt{D/C}$, which is indicated by a negative value for the excess gradient criterion, EGC. Dependent on the choice of $q^\star$, as one moves further up along the efficient frontier, it is the case that $EGC \geq 0$ when the progressive efficient portfolio returns distributions persistently move to the right (i.e. offer increased expected return, contributing to higher $-Q$) at a rate faster than their $q^\star$-tails broaden away to the left of the mean (thereby lowering $-Q$).

[7]If asset returns are jointly normally distributed, then VaR is a coherent risk measure as explained by Artzner et al. (1999, Remark 3.6). The same argument applies for a Student's t distribution.

[8]Alexander et al. (2009) provides simulation results for the *ex post* performance of VaR-constrained optimal portfolios, including the minimum VaR portfolio.

[9]The importance of asset classes relative to individual securities is highlighted by Ibbotson and Kaplan (2000), who find, on average, that the returns of investment fund benchmarks explain around 90% of fund returns, and that funds do not add "value above their policy benchmarks because of a combination of timing, security selection, management fees, and expenses" (p.32).

[10]The analysis is unaffected by survival bias (although the strategy may incur transaction costs over and above those involved in rebalancing the portfolio to achieve the desired allocations). Stocks which disappear, or shift to a different size and/or book-to-market partition, are simply replaced by others.

[11]http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[12]The big-ME/high-BE/ME portfolio has missing data for July 1930 to June 1931 and was thus excluded from portfolio optimization for initial estimation periods that overlap these dates. That is, with roll-forward of the 200 month estimation period, the number of assets available for portfolio optimization was initially 24 (for the 60 months from March 1943 to February 1948), and finally 25 (for the 718 months from March 1948 to December 2007).

[13]iShares are units of exchange-traded funds managed to track various stock indices. We chose all non-US, non-overlapping (i.e. primarily country specific rather than regional) international equity iShares with returns histories longer than two years. The resulting set of 23 iShares track the MSCI international stock indices for: Australia, Canada, Sweden, Germany, Hong Kong, Italy, Japan, Belgium, Switzerland, Malaysia, Netherlands, Austria, Spain, France, Singapore, Taiwan, United Kingdom, Mexico, South Korea, Brazil and South Africa; the FTSE/Xinhua China 26 index; and the S&P Latin America 40 Index. The respective tickers are EWA, EWC, EWD, EWG, EWH, EWI, EWJ, EWK, EWL, EWM, EWN, EWO, EWP, EWQ, EWS, EWT, EWU, EWW, EWY, EWZ, EZA, FXI, ILF.

[14]The returns time series for EWT, EWY, EWZ, EZA, FXI and ILF respectively commence with fortnight-ending 19 July 2000, 24 May 2000, 19 July 2000, 12 February 2003, 20 October 2004 and 7 November 2001.

[15]For each estimation period, iShares without a full 100 fortnight returns history were excluded. Thus, with roll-forward of the estimation period, the number of assets increases from 19 to 24 (note that the returns history for FXI is too short for inclusion).

[16]The methods for identifying the standard portfolios are generally well-known and are thus not detailed here. However note that the tangency portfolio is occasionally non-existent. With short sales allowed, Merton (1972), Theorem II, p.1865 ff., gives the condition for existence of the tangency portfolio; in the case of non-existence, the technique of Maller and Turkington (2002) was used to approximate a tangency portfolio. With short sales disallowed, the efficient frontier terminates at the highest-risk single asset, which limits the possible existence of the tangency portfolio; in the case of non-existence, the tangency portfolio was taken to be the highest-risk single asset.

[17]Our assumptions for standardized realized returns are supported by analyses using the Kolmogorov-Smirnov test reported in Table 1 below. The Student's t_4 distribution and the standard normal are at extreme ends of a spectrum, in terms of heaviness of tails. As noted previously, a Student's t_4 distibution has been found to be appropriate for world stock indices (Platen and Sidorowicz, 2008), and Jagannathan and Ma (2003) found the

standard normal distribution to be appropriate for portfolios with short sales disallowed.

[18]We repeated the analysis in Figure 3 for the iShares data; as the corresponding diagrams convey the same impression as those for the FF5×5 dataset, we have not included them in this paper.

[19]Only "like-for-like" comparisons are reported in columns 16–17 of Table 1 for the minimum VaR portfolios, by which we mean that minimum VaR portfolios formed under a t_4 distribution assumption have their standardized realized returns compared with the t_4 distribution (for cases when short sales are allowed), and similarly for the normal distribution (when short sales are not allowed). We recognize that the rolling window portfolio estimation process introduces autocorrelation for the time-series of μ_p and σ_p, and thereby also for Z_p, which undermines the validity of our Kolmogorov-Smirnov test statistics. However we still provide these results as background to our main "numbers of breaches of minimum VaR" results, which we expect to be binomially distributed.

[20]An alternative to making distributional assumptions such as Student's t_4 or standard normal is to use the empirical cdf to locate quantiles for the calculation of VaRs, but it is usually desirable to do some form of smoothing so as not to be too dependent on the vagaries of a particular set of historical information. One could use the "peaks over threshold" method (see Embrechts et al., 1997, Section 6.5) to fit a generalized Pareto distribution to the extreme left tail of the returns distribution; but this introduces some subjectivity into deciding on a threshold, and would involve a more complex data analysis, different for each case considered. We feel that the t_4 and normal assumptions are a reasonable compromise for our datasets. Distributions that better fit the empirical left tail would, of course, only improve the performance of the VaR procedure.

[21]See note 19.

Appendix A: Extension to minimizing CVaR

Symbolize conditional value-at-risk (CVaR) as $\widetilde{Q}$ and define it to be the expected relative loss a portfolio will suffer conditional on a VaR breach. The CVaR of an efficient portfolio with VaR quantile Q is given by

$$-\widetilde{Q} = \frac{\int_{-\infty}^{-Q} r \ \mathrm{dPr}(R_p < r)}{\mathrm{Pr}(R_p < -Q)}.$$

In our setup, Q is related to μ_p, σ_p and q by equation (1). Set $q = q^\star$ and substitute for Q with equation (1), and change variable from r to $z = (r - \mu_p)/\sigma_p$ to get

$$-\widetilde{Q} = \mu_p + \sigma_p \frac{\int_{-\infty}^{\Phi^{-1}(q^\star)} z \ \mathrm{d}\mathrm{Pr}(Z_p < z)}{q^\star}.$$

This is of the same form as for $-Q$ in equation (1), but with the quantile level shifted via a more negative coefficient of σ_p (so that $-\widetilde{Q} < -Q$). Hence the minimization approach demonstrated for VaR similarly applies for CVaR.

Alexander and Baptista (2004) obtain this relation under the assumption of normally distributed returns, and demonstrate the potential for a CVaR constraint to dominate a VaR constraint in terms of risk management by providing an efficient portfolio opportunity set with both lower maximum risk and lower minimum risk.

Arguably the distribution assumption for efficient portfolio returns entailed in our setup will be less reasonable for minimizing CVaR (compared to minimizing VaR) because the "spread" of the assumed and real distribution tails will need to coincide (not just the areas of the assumed and real tails as is the case for minimizing VaR).

Appendix B: Rolling window investment methodology

Here we describe the methodology used for the empirical analysis. Rolling time series windows of width m were used for portfolio estimation, where $m = 200$ months for the FF5×5 data and $m = 100$ fortnights for the iShares data.

Rolling window investment allocation

Let $\mathbf{S}_t$ denote the N-vector of asset prices as observed from market closing prices at time t, $t = 0, 1, 2, \ldots, T$, where T denotes the length of the entire time series of asset prices. Let $\boldsymbol{\delta}_t$ denote the N-vector of asset dividends at time t, $t = 1, 2, \ldots, T$. $T = 978$ (months) for the FF5×5 data and $T = 307$ (fortnights) for the iShares data. For an asset to be included amongst the N assets, it requires a minimum return history of length m. Returns are calculated from market close to close, thus $(\mathbf{S}_t + \boldsymbol{\delta}_t - \mathbf{S}_{t-1})/\mathbf{S}_{t-1}$ (with the divisions taken component-wise) is the *discrete raw return vector*, $\mathbf{R}_t$, at time t, $t = 1, 2, \ldots, T$. Let $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ be the expected value and variance-covariance matrix of $\mathbf{R}_{t+1}$, considered as a random variable; in general, we allow for them to change with time.

At a given time t, for a window of width m, we estimate $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ by the sample mean and sample variance-covariance matrix of the returns based on the previous m trading periods' observations; thus

$$\widehat{\boldsymbol{\mu}}_t = \frac{1}{m} \sum_{s=t-m+1}^{t} \mathbf{R}_s,$$

and similarly for $\widehat{\boldsymbol{\Sigma}}_t$. Extension from this sample variance-covariance estimation approach to other estimation approaches is of interest but left for future research.

We begin the series of estimates at time m and conclude it at time T; e.g. the first FF5×5 window extends from July 1926 to February 1943, from which we calculate $\widehat{\boldsymbol{\mu}}_m$ and $\widehat{\boldsymbol{\Sigma}}_m$. The window is then rolled forward one time period to get $\widehat{\boldsymbol{\mu}}_{m+1}$ and $\widehat{\boldsymbol{\Sigma}}_{m+1}$, etc., and we continue this process until we finally obtain $\widehat{\boldsymbol{\mu}}_T$ and $\widehat{\boldsymbol{\Sigma}}_T$.

The minimum VaR portfolio allocation for time $t = m$, denoted $\widehat{\boldsymbol{x}}_{Qmin,m}$, corresponding to VaR breach probability $q^\star$, is calculated from equation (5)

by substituting $\widehat{\boldsymbol{\mu}}_m$ and $\widehat{\boldsymbol{\Sigma}}_m$ for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Rolling the window forward and repeating for values $t = m+1, m+2, \ldots, T$ gives the time series of minimum VaR portfolio allocations.

Rolling window investment evaluation

At each of times $t = m+1, m+2, \ldots, T$ we calculate the realized return on a minimum VaR portfolio with allocation vector $\widehat{\boldsymbol{x}}_{Qmin,t}$ from

$$R_{p,t} = \widehat{\boldsymbol{x}}'_{Qmin,t-1}\mathbf{R}_t.$$

This gives a time series of observations $R_{p,m+1}, \ldots, R_{p,T}$ as the returns on a minimum VaR portfolio.

Although it might be reasonable to assume the return vectors $\mathbf{R}_1, \ldots, \mathbf{R}_T$ are independent, the realized portfolio returns $R_{p,m+1}, \ldots, R_{p,T}$ are correlated by virtue of the overlaps occurring in the calculation of the $\widehat{\boldsymbol{\mu}}_t$ and $\widehat{\boldsymbol{\Sigma}}_t$, and hence in the corresponding portfolio allocations, as the window is rolled forward. Nevertheless the degree of dependence is negligible for portfolio returns separated by m or more units of time, as there is then no overlap in the observations used to calculate the $\widehat{\boldsymbol{\mu}}_t$ and $\widehat{\boldsymbol{\Sigma}}_t$. Where necessary, we can use the limit theory worked out for "m-dependent" observations in Herrndorf (1984), to justify significance tests.

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Figure 1: Comparative VaRs for two hypothetical efficient portfolios

Example probability densities for the realized returns of two efficient portfolios (A and B) and their comparative VaRs (corresponding to benchmark returns $-Q_A$ and $-Q_B$) for nominal VaR breach probability $q^\star$. $\phi_R(\cdot)$ is the probability density function for efficient portfolio realized return, R_p. (σ_A, μ_A) and (σ_B, μ_B) are the volatility and expected return combinations for efficient portfolios A and B. B is above and to the right of A on the efficient frontier (i.e. $\mu_B > \mu_A$ and $\sigma_B > \sigma_A$).

33
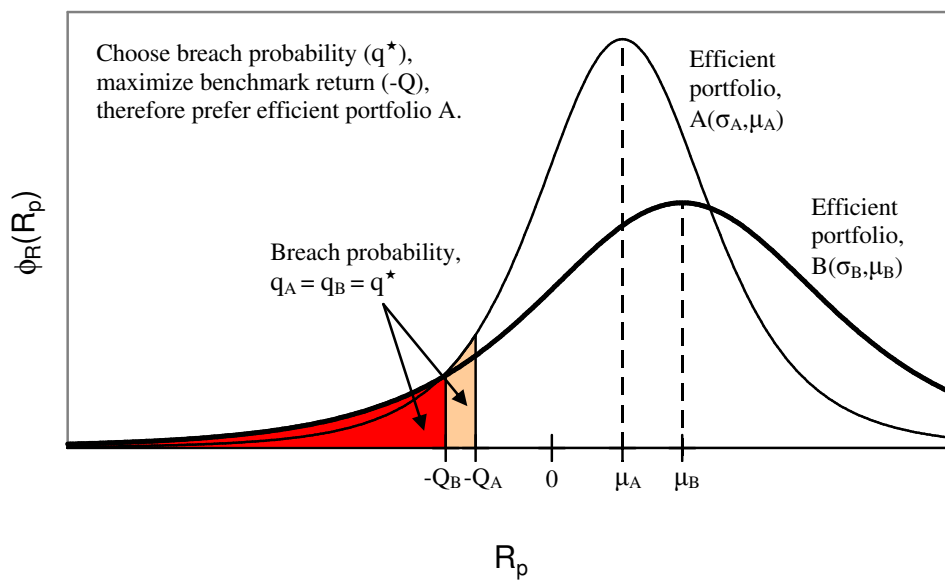
Figure 2: FF5 × 5 Efficient frontier and minimum VaR portfolios

Ex ante (σ_p, μ_p) frontier of efficient portfolios with locations of the minimum VaR and "standard" portfolios (top panel), and corresponding efficient frontier VaR (Q, given by equation (3)) (bottom panel), with short sales allowed and assuming a Student's t_4 distribution for efficient portfolio standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios, and estimated from the 200 month period from June 1983 to January 2000.
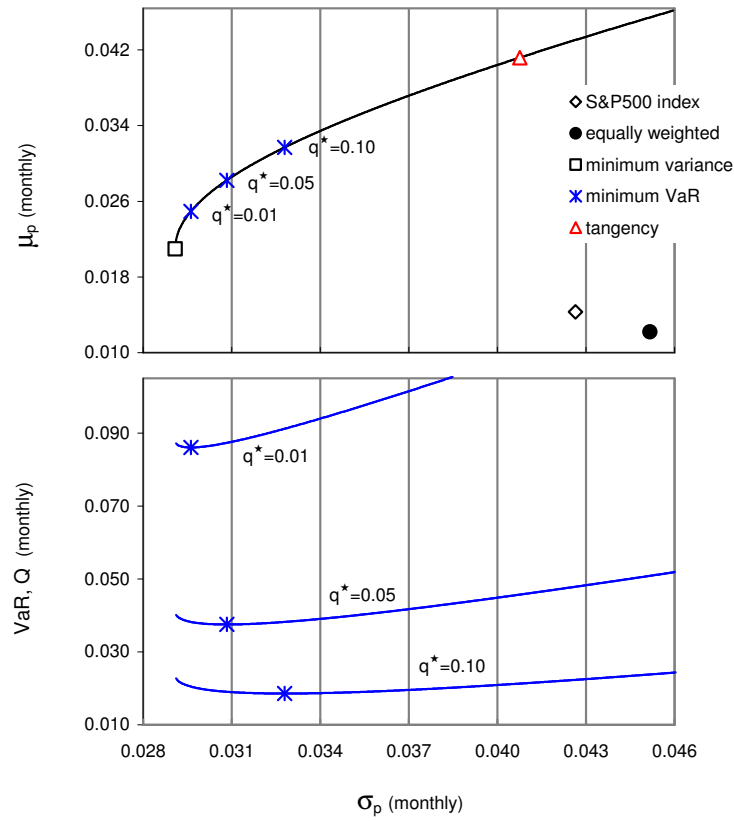
Figure 3: FF5 × 5 Expected return-risk ratio

Monthly expected return-risk ratio (μ_p/σ_p) for the tangency and minimum variance portfolios, and the minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.
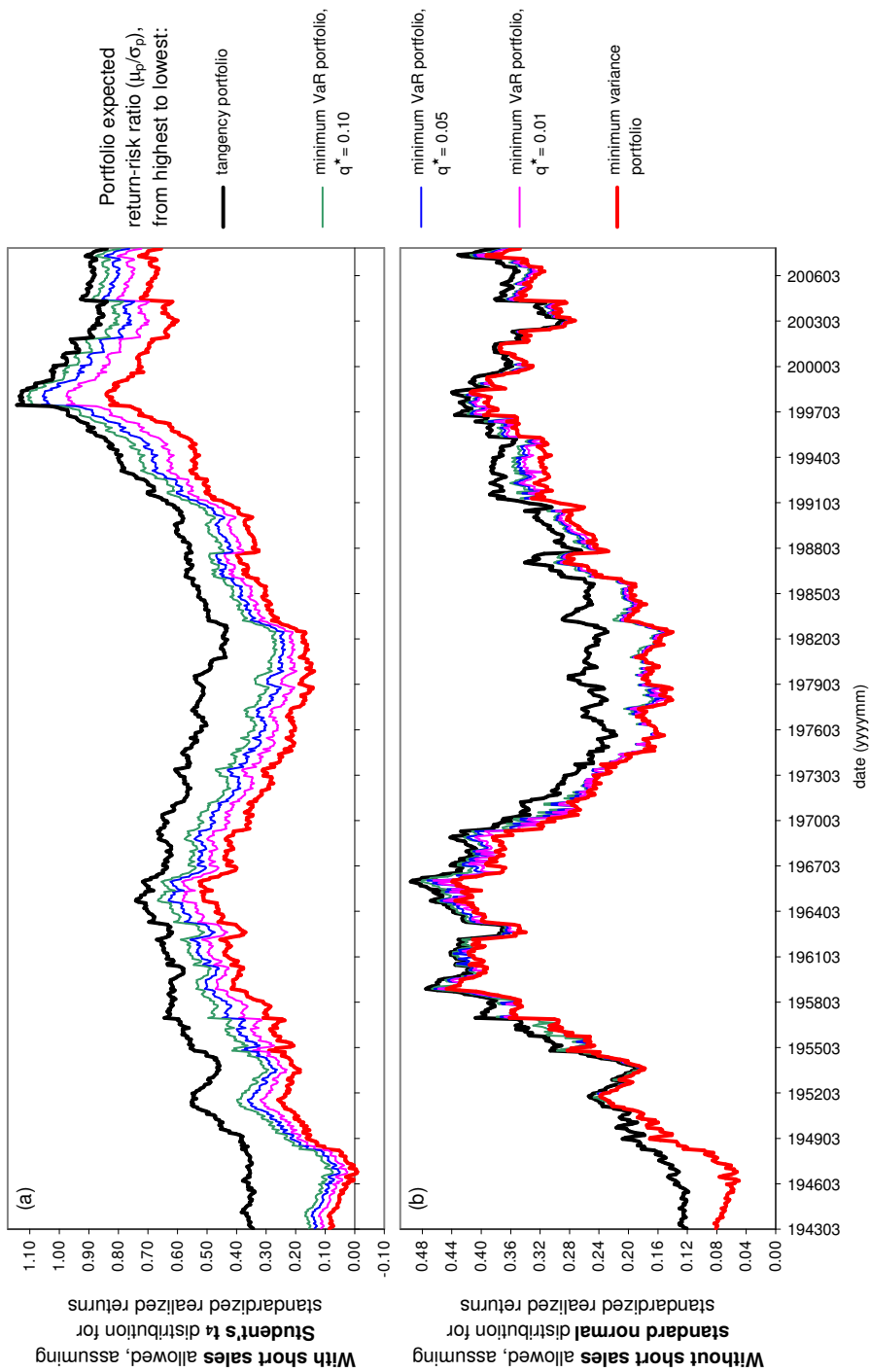
Figure 4: FF5 × 5 Expected and realized returns and negative VaRs

Monthly expected and realized returns and negative VaR ($-Q$) for the minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.
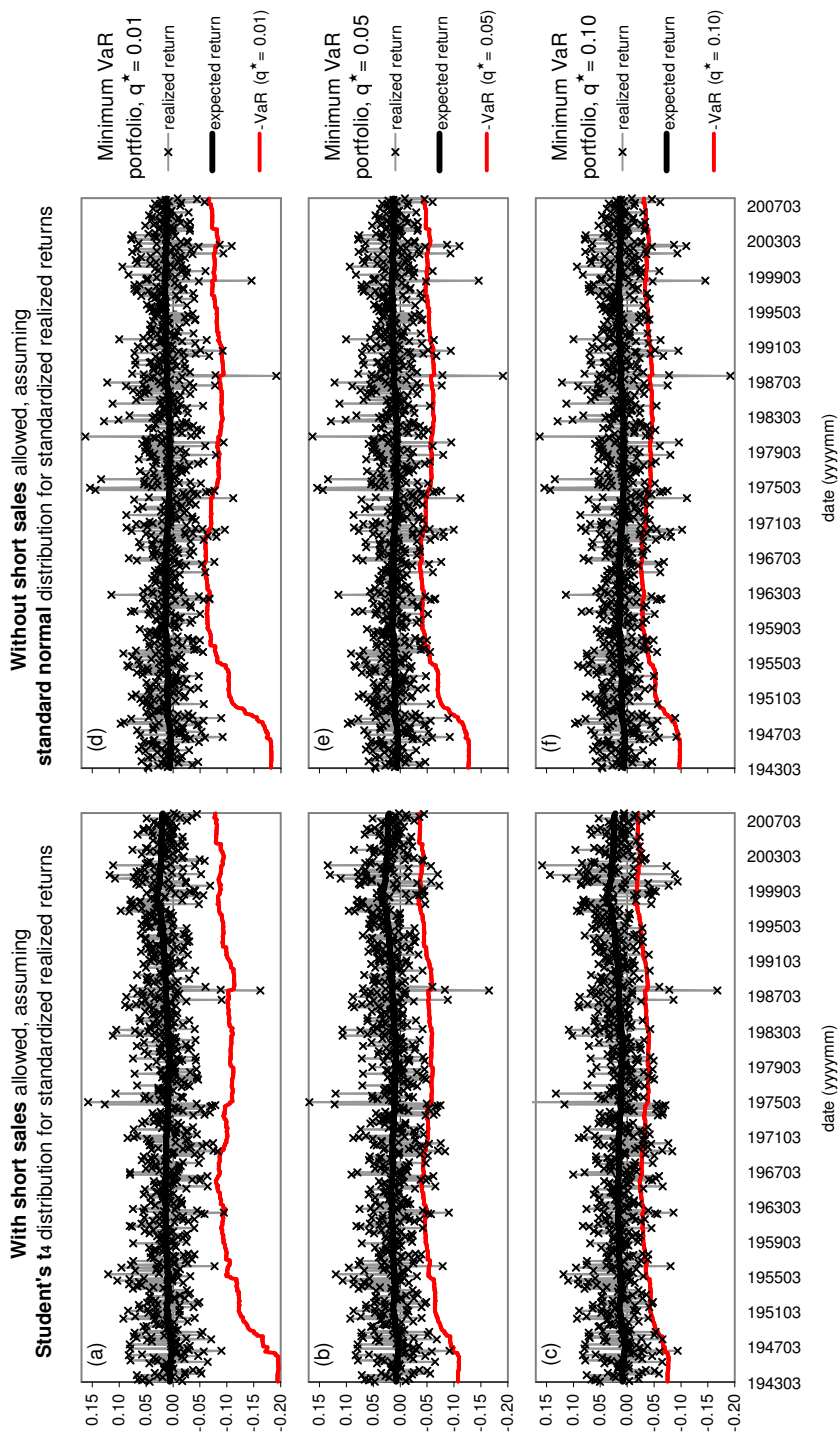
Figure 5: FF5×5 P-P plots

Left tail P-P plots of monthly standardized realized returns of minimum VaR portfolios with $q^\star = \{0.05, 0.10\}$, with and without short sales allowed respectively versus Student's t_4 and standard normal distributions; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.
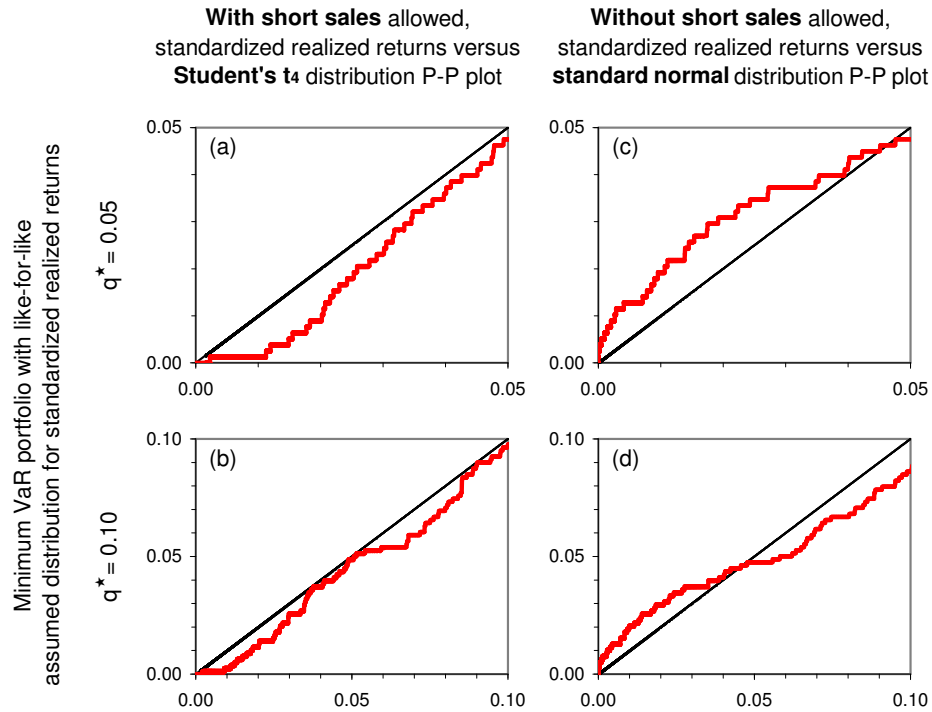
Figure 6: iShares expected and realized returns and negative VaRs

Fortnightly expected and realized returns and negative VaR ($-Q$) for the minimum VaR portfolios with $q^* = \{0.01, 0.05, 0.10\}$, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating 22 international stock index iShares and the SPY and MDY with 207 rolling 100 fortnight estimation periods.
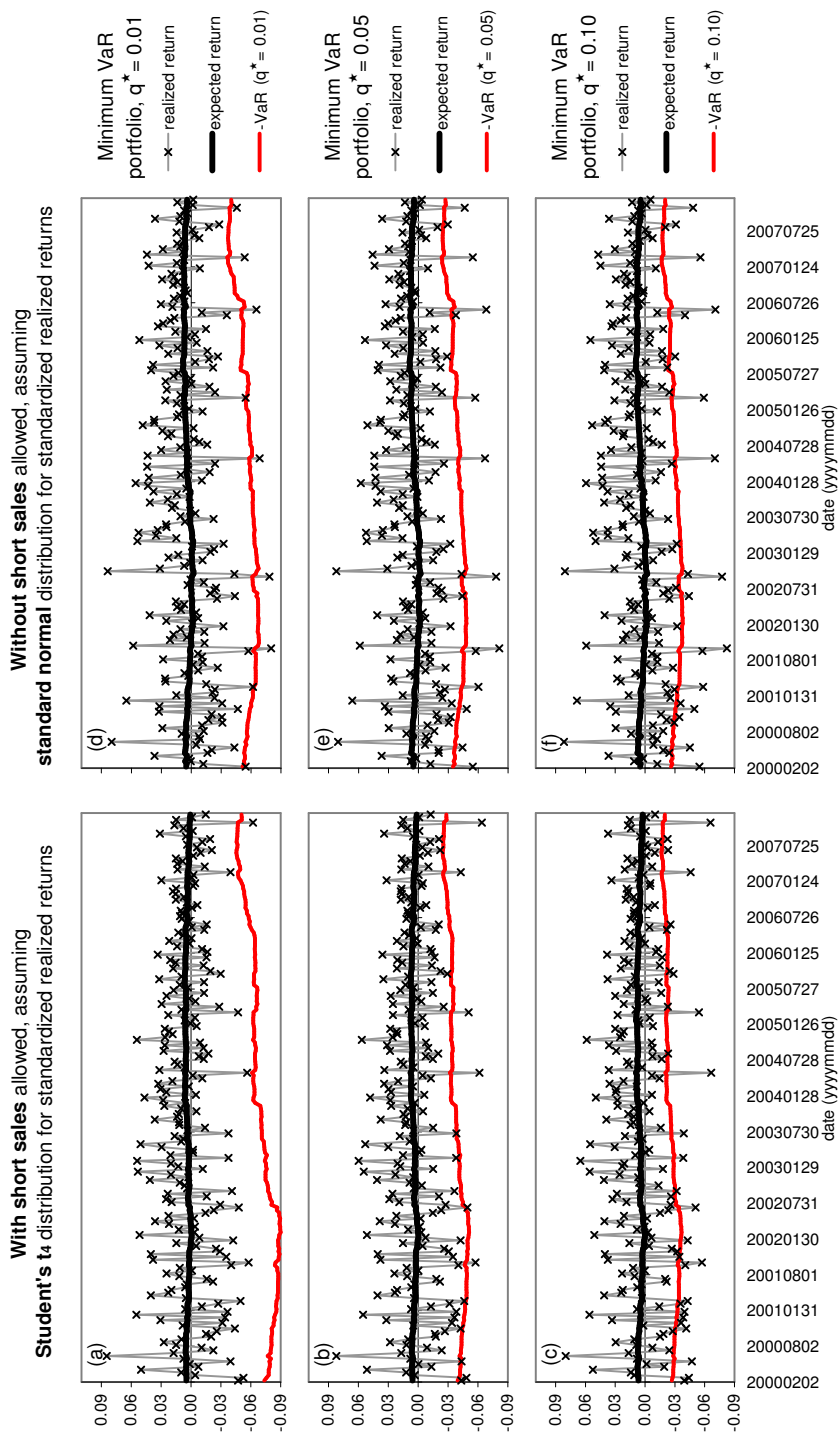
Figure 7: iShares P-P plots

Left tail P-P plots of fortnightly standardized realized returns of minimum VaR portfolios with $q^\star = \{0.05, 0.10\}$, with and without short sales allowed respectively versus Student's t_4 and standard normal distributions; incorporating 22 international stock index iShares and the SPY and MDY with 207 rolling 100 fortnight estimation periods.

Table 1: FF5 × 5 descriptive statistics

Descriptive statistics for monthly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.
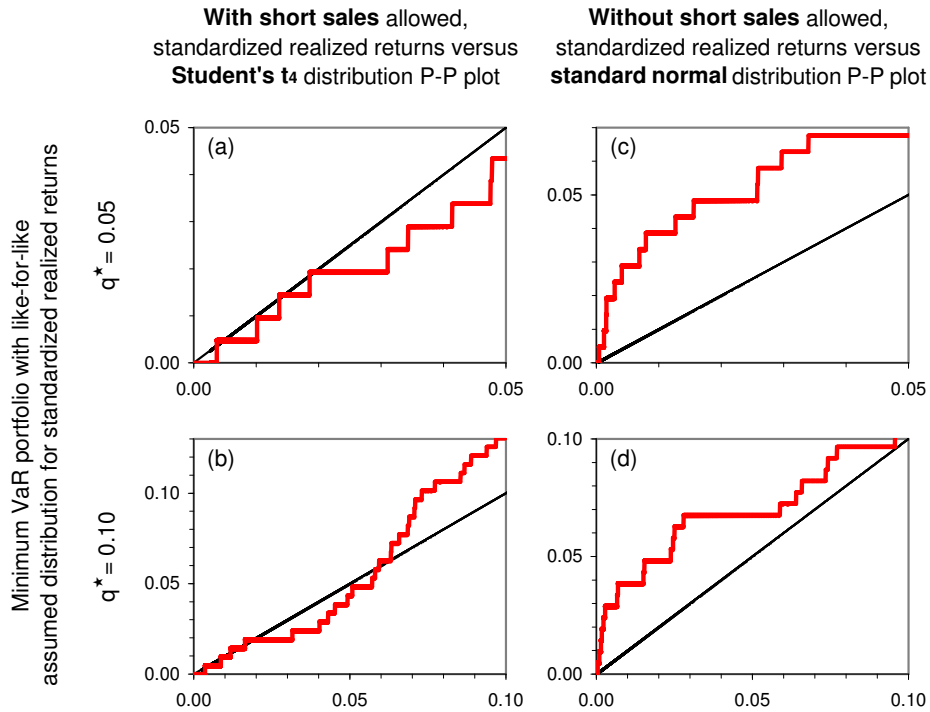
For each portfolio, descriptive statistics for 778-month time series of: sets of portfolio weights; expected return-risk ratios; realized returns; and standardized realized returns

| portfolio (rebalanced monthly) | monthly portfolio weights | | | | monthly expected return-risk ratio, μ_p/σ_p | | | | monthly realized return, R_p | | | monthly standardized realized return, Z_p | | | | Kolm.-Smir. p-value: | |
| | min. of monthly min's | median of monthly min's | max. of monthly max's | median of monthly max's | min. | max. | mean | st. dev. | mean ×12 (annualized) | st. dev. ×√12 (annualized) | mean/st. dev. (monthly) | mean | st. dev. | skewness | excess kurt. | Student's t_4 | standard normal |
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
S&P500 index	n.a.	n.a.	n.a.	n.a.	0.06	0.40	0.23	0.09	12.4%	14.1%	0.25	-0.01	0.94	-0.35	2.25	0.00	0.00
equally weighted	0.04	0.04	0.04	0.04	0.09	0.39	0.24	0.07	14.8%	16.5%	0.26	-0.01	0.94	-0.46	3.27	0.00	0.00
with short sales allowed — minimum VaR portfolio, assuming **Student's t_4** distribution for standardized realized returns — minimum variance	-0.79	-0.51	1.21	0.74	-0.01	0.84	0.37	0.20	15.6%	12.7%	0.35	0.05	1.19	-0.20	1.30	0.11	0.06
$q^\star$=0.01	-0.98	-0.52	1.23	0.77	0.03	0.98	0.44	0.22	16.7%	12.7%	0.38	0.01	1.19	-0.15	1.27	0.26	n.a.
$q^\star$=0.05	-1.29	-0.55	1.24	0.80	0.05	1.06	0.48	0.23	17.5%	13.0%	0.39	-0.02	1.18	-0.12	1.32	0.57	n.a.
$q^\star$=0.10	-1.62	-0.58	1.25	0.84	0.07	1.12	0.52	0.24	18.4%	13.4%	0.40	-0.05	1.18	-0.10	1.47	0.48	n.a.
tangency	-142.5	-1.68	137.3	1.42	0.34	1.14	0.63	0.18	12.0%	83.0%	0.04	-0.22	1.12	-0.04	2.58	0.00	0.00
without short sales allowed — minimum VaR portfolio, assuming **standard normal** distribution for standardized realized returns — minimum variance	0.00	0.00	0.94	0.54	0.05	0.45	0.27	0.10	13.2%	13.5%	0.28	0.00	0.99	-0.23	2.32	0.00	0.01
$q^\star$=0.01	0.00	0.00	0.93	0.55	0.05	0.46	0.27	0.10	13.2%	13.5%	0.28	-0.01	0.99	-0.23	2.32	n.a.	0.02
$q^\star$=0.05	0.00	0.00	0.93	0.55	0.05	0.47	0.28	0.11	13.3%	13.5%	0.28	-0.01	0.99	-0.23	2.33	n.a.	0.02
$q^\star$=0.10	0.00	0.00	0.95	0.58	0.05	0.48	0.28	0.11	13.4%	13.5%	0.29	-0.01	0.98	-0.24	2.37	n.a.	0.03
tangency	0.00	0.00	1.00	0.67	0.12	0.50	0.31	0.09	15.0%	15.2%	0.28	-0.06	0.94	-0.33	3.21	0.00	0.00

Table 2: FF5 × 5 VaR breaches

Numbers of breaches of minimum VaR, for monthly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.

	Breaches of minimum VaR for a time series of 778 realized returns					
	with short sales allowed, assuming **Student's t_4** distribution for standardized realized returns			**without short sales** allowed, assuming **standard normal** distribution for standardized realized returns		
portfolio	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10
S&P500 index	6	50	103	19	50	79
FF5x5 portfolios:						
equally weighted	13	63	117	24	62	90
minimum variance	2	34	73	16	37	74
minimum VaR portfolio $q^\star$=0.01	**2** (4, 7.8, 13)*	37	73	**17** (4, 7.8, 13)*	37	72
minimum VaR portfolio $q^\star$=0.05	1	**37** (29, 38.9, 49)*	75	15	**37** (29, 38.9, 49)*	70
minimum VaR portfolio $q^\star$=0.10	3	38	**76** (64, 77.8, 92)*	16	37	**67** (64, 77.8, 92)*
tangency	34	101	142	17	40	71

*(0.05 probable maximum breaches, expected breaches = 778$q^\star$, 0.95 probable maximum breaches), for a Binomial(778,$q^\star$) random variable.

Table 3: FF5 × 5 average size of VaR breaches

Average size of breaches of minimum VaR, for monthly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating the 25 Fama and French ME and BE/ME portfolios with 778 rolling 200 month estimation periods.

| | Average size of breaches of minimum VaR for a time series of 778 monthly realized returns,* **annualized** (x 12) | | | | | |
| | **with short sales** allowed, assuming **Student's t₄** distribution for standardized realized returns | | | **without short sales** allowed, assuming **standard normal** distribution for standardized realized returns | | |
portfolio	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10
S&P500 index	44%	32%	32%	28%	30%	32%
FF5x5 portfolios:						
equally weighted	51%	38%	37%	47%	36%	38%
minimum variance	38%	27%	27%	27%	32%	27%
minimum VaR portfolio, with like-for-like assumed distribution for standardized realized returns — $q^\star$=0.01	36%	22%	26%	26%	32%	28%
$q^\star$=0.05	75%	22%	26%	30%	33%	29%
$q^\star$=0.10	30%	24%	27%	28%	32%	30%
tangency	470%	198%	162%	41%	38%	35%

*Size of breach is the negative of realized return (i.e. realized loss) minus minimum VaR.
N.B. the % symbol represents a difference in percentages rather than a percentage difference.

Table 4: iShares descriptive statistics

Descriptive statistics for fortnightly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating 22 international stock index iShares and the SPY and MDY with 207 rolling 100 fortnight estimation periods.

For each portfolio, descriptive stat's for 207-fortnight time series of: sets of portfolio weights; expected return-risk ratios; realized returns; and standardized realized returns

| portfolio (rebalanced fortnightly) | fortnightly portfolio weights | | | | fortnightly expected return-risk ratio, μ_p/σ_p | | | | fortnightly realized return, R_p | | | fortnightly standardized realized return, Z_p | | | | Kolm.-Smir. p-value: | |
| | min. of fort-nightly min's | median of fort-nightly min's | max. of fort-nightly max's | median of fort-nightly max's | min. | max. | mean | st. dev. | mean x 26 (annualized) | st. dev. x √26 (annualized) | mean st. dev. (fort-nightly) | mean | st. dev. | skewness | excess kurt. | Student's t_4 | standard normal |
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
MSCI World investable index	n.a.	n.a.	n.a.	n.a.	-0.06	0.40	0.12	0.12	8.8%	31.7%	0.05	0.01	1.53	6.47	82.04	0.00	0.00
equally weighted	0.04	0.05	0.05	0.05	-0.09	0.43	0.13	0.13	11.6%	16.6%	0.14	0.00	0.96	-0.33	0.75	0.15	0.21
with short sales allowed — minimum VaR portfolio, assuming **Student's t_4** distribution for standardized realized returns — minimum variance	-0.78	-0.51	1.30	0.50	-0.05	0.28	0.14	0.08	7.7%	12.1%	0.12	0.02	1.21	-0.37	0.98	0.41	0.25
$q^\star$=0.01	-0.81	-0.53	1.21	0.46	-0.03	0.35	0.19	0.09	8.3%	12.4%	0.13	-0.02	1.24	-0.38	0.98	0.75	n.a.
$q^\star$=0.05	-0.83	-0.54	1.13	0.44	-0.01	0.41	0.24	0.10	8.9%	12.7%	0.14	-0.06	1.25	-0.39	0.96	0.55	n.a.
$q^\star$=0.10	-0.85	-0.55	1.06	0.48	0.00	0.47	0.27	0.11	9.4%	13.1%	0.14	-0.09	1.27	-0.40	0.90	0.43	n.a.
tangency	-146.9	-1.81	110.3	2.26	0.28	0.66	0.49	0.09	-74.8%	454.9%	-0.03	-0.32	1.23	-0.57	0.87	0.02	0.00
without short sales allowed — minimum VaR portfolio, assuming **standard normal** distribution for standardized realized returns — minimum variance	0.00	0.00	0.92	0.37	-0.10	0.38	0.13	0.12	9.9%	13.7%	0.14	0.03	1.05	-0.30	1.01	0.21	0.71
$q^\star$=0.01	0.00	0.00	0.92	0.35	-0.09	0.40	0.16	0.13	10.4%	14.1%	0.14	0.01	1.08	-0.34	0.78	n.a.	0.71
$q^\star$=0.05	0.00	0.00	0.92	0.35	-0.09	0.41	0.17	0.13	10.7%	14.2%	0.15	0.00	1.09	-0.36	0.71	n.a.	0.60
$q^\star$=0.10	0.00	0.00	0.92	0.35	-0.09	0.42	0.18	0.13	10.7%	14.5%	0.14	-0.01	1.10	-0.38	0.64	n.a.	0.69
tangency	0.00	0.00	1.00	0.60	0.02	0.49	0.26	0.12	13.6%	18.0%	0.15	-0.09	1.05	-0.45	0.46	0.27	0.93

Table 5: iShares VaR breaches

Numbers of breaches of minimum VaR, for fortnightly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating 22 international stock index iShares and the SPY and MDY with 207 rolling 100 fortnight estimation periods.

| | Breaches of minimum VaR for a time series of 207 realized returns | | | | | |
| | **with short sales** allowed, assuming **Student's t_4** distribution for standardized realized returns | | | **without short sales** allowed, assuming **standard normal** distribution for standardized realized returns | | |
portfolio	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10
MSCI World index	3	16	31	8	16	26
iShares portfolios:						
equally weighted	6	17	36	10	19	32
minimum variance	1	10	22	8	12	17
minimum VaR portfolio, with like-for-like assumed distribution for standardized realized returns $q^\star$=0.01	**1** (0, 2.1, 5)*	8	24	**7** (0, 2.1, 5)*	14	20
$q^\star$=0.05	2	**9** (5, 10.4, 16)*	24	8	**14** (5, 10.4, 16)*	21
$q^\star$=0.10	2	9	**28** (14, 20.7, 28)*	8	14	**21** (14, 20.7, 28)*
tangency	38	56	72	12	20	33

*(0.05 probable maximum breaches, expected breaches = 207$q^\star$, 0.95 probable maximum breaches), for a Binomial(207,$q^\star$) random variable.

Table 6: iShares average size of VaR breaches

Average size of breaches of minimum VaR, for fortnightly minimum VaR portfolios with $q^\star = \{0.01, 0.05, 0.10\}$ and "standard" portfolios, with and without short sales allowed respectively assuming Student's t_4 and standard normal distributions for standardized realized returns; incorporating 22 international stock index iShares and the SPY and MDY with 207 rolling 100 fortnight estimation periods.

| | Average size of breaches of minimum VaR for a time series of 207 fortnightly realized returns,* **annualized** (x 26) | | | | | |
| | **with short sales** allowed, assuming **Student's t₄** distribution for standardized realized returns | | | **without short sales** allowed, assuming **standard normal** distribution for standardized realized returns | | |
portfolio	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10	$q^\star$=0.01	$q^\star$=0.05	$q^\star$=0.10
MSCI World index	365%	119%	86%	175%	123%	96%
iShares portfolios:						
equally weighted	31%	61%	49%	44%	53%	50%
minimum variance	34%	33%	38%	27%	53%	59%
minimum VaR portfolio, with like-for-like assumed distribution for standardized realized returns — $q^\star$=0.01	40%	39%	35%	32%	50%	54%
$q^\star$=0.05	23%	36%	36%	30%	52%	54%
$q^\star$=0.10	32%	40%	33%	34%	55%	56%
tangency	1245%	916%	741%	58%	68%	60%

*Size of breach is the negative of realized return (i.e. realized loss) minus minimum VaR.
N.B. the % symbol represents a difference in percentages rather than a percentage difference.