

Conditioned Invariance and Unknown-Input Observation for Two-Dimensional Fornasini-Marchesini Models

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Abstract—The concept of conditioned invariance is extended for the class of 2-D systems described by Fornasini-Marchesini models in the most general form proposed by Kurek. Then, the use of this concept is investigated within the context of estimation in the presence of unknown inputs.

I. INTRODUCTION

Conditioned invariant subspaces were introduced by Basile and Marro in [1] as the dual of controlled invariant subspaces. Their role in relation to the problem of estimation in the presence of unknown input signals was investigated by the same authors in [2]. An alternative definition of conditioned invariance was proposed by Willems in terms of the existence of observers, [20], also see the recent textbooks [3, Chapter 4] and [19, Chapter 5].

The purpose of this paper is: (i) to extend the definition of conditioned invariance and input-containing subspaces given for 1-D systems in [1], to Fornasini-Marchesini models [7], [9] in the general form

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{i,j} \\ &\quad + B_1 u_{i+1,j} + B_2 u_{i,j+1} \\ y_{i,j} &= C x_{i,j} + D u_{i,j} \end{aligned} \quad (1)$$

of Kurek [14]; and (ii) to provide a characterisation of such subspaces in terms of the existence of certain observers. By general we intend that the Kurek model (1) encompasses both the so-called first and second forms of Fornasini-Marchesini models, FM-I [7] and FM-II [9], respectively. The first (FM-I) can be recovered from (1) by taking $B_1 = B_2 = 0$, while the second (FM-II) can be recovered by taking $A_0 = 0$ and $B_0 = 0$.

Over the last twenty years, several extensions of important geometric concepts, such as controlled invariance, have been proposed for 2-D latent variable models such as the Fornasini-Marchesini and Roesser state-space models, [5], [6], [12], [13]. While definitions of controlled invariance and output-nulling subspaces are not difficult to establish for FM-I, FM-II or Kurek models, a definition for conditioned invariant subspaces is less natural, since duality cannot be exploited as in the 1-D case. In fact, Fornasini-Marchesini models are such that their dual is not in the form described by

(1), or the class corresponding to the FM-I or FM-II models. To overcome this difficulty, definitions of conditioned invariance were proposed in [12] for two particular classes of models, motivated by the search for duality properties similar to the 1-D case. The first is yet another subclass of (1) with $A_0 = 0$, $B_1 = B_2 = 0$, see also [8]. The second is a variation of the FM-II with the output modelled as

$$y_{i,j} = C_1 x_{i+1,j} + C_2 x_{i,j+1} + D_1 u_{i+1,j} + D_2 u_{i,j+1},$$

thus displaying a symmetry between the state and the output equations (both models were introduced in [12] in the descriptor form). This model is non-standard, as it involves mixed dynamics in both the state and in the output equation, and its relevance in the context of state-space theory of 2-D systems is yet to be understood. For these two model classes, conditioned invariant subspaces were defined as the dual of controlled invariant subspaces, and their role in the state observation was investigated, see also [15].

In this paper, a new definition of conditioned invariance is provided for the more general class of Fornasini-Marchesini models in Kurek form. This is not, therefore, dual to the definitions of controlled invariance presented in the literature so far. The extension to singular models in Kurek form, introduced by Kaczorek in [11], can be carried out along the lines of [15].

Notation. Throughout this paper, we will denote by \mathbb{N} the positive integers including zero. The symbol 0_n will stand for the origin of the vector space \mathbb{R}^n . The image and the kernel of matrix $M \in \mathbb{R}^{n \times m}$ will be denoted by $\text{im}M$ and $\text{ker}M$, respectively. The $n \times m$ zero matrix is denoted by $0_{n \times m}$. Denote by M^\top and by M^\dagger the transpose and the Moore-Penrose pseudoinverse of M , respectively. For the sake of brevity we define $M_D := \text{diag}(M, M, M)$, and, accordingly, given a subspace \mathcal{J} of \mathbb{R}^n , the symbol \mathcal{J}_D will identify the subspace $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ of \mathbb{R}^{3n} , where the symbol \times has been used to denote the Cartesian product. Given the vector $\xi \in \mathbb{R}^n$, the symbol ξ/\mathcal{J} denotes the canonical projection of ξ on the quotient space \mathbb{R}^n/\mathcal{J} . Finally, given a triple of matrices $M_0, M_1, M_2 \in \mathbb{R}^{n \times m}$, we define $M_H := [M_0 \ M_1 \ M_2]$ and $M_V := [M_0^\top \ M_1^\top \ M_2^\top]^\top$.

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II. CONDITIONED INVARIANCE AND INPUT-CONTAINING SUBSPACES

Consider a linear 2-D system Σ described by the difference equations (1) where, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, $x_{i,j} \in \mathbb{R}^n$ is the local state, $u_{i,j} \in \mathbb{R}^m$ is the control input, $y_{i,j} \in \mathbb{R}^p$ is the output, $A_k \in \mathbb{R}^{n \times n}$ and $B_k \in \mathbb{R}^{n \times m}$ for $k \in \{0, 1, 2\}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$. For *boundary conditions* of (1) we intend assignments of the form $x_{i,j} = \bar{x}_{i,j} \in \mathbb{R}^n$ for all $(i, j) \in \mathbb{B}$, where

$$\mathbb{B} := (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\}).$$

The model described by (1) is usually referred to as a Kurek model, and it was first introduced in [14] as a generalisation of the classic models FM-I and FM-II, [7], [8], [9]. In order to develop a geometric control theory for 2-D systems, the notions of controlled invariant and output-nulling subspaces were adapted for FM-I in [5], [13], [17]. These subspaces play a key role in the so-called exact decoupling problems.

By straightforwardly extending such definitions to the Kurek model described by (1), we say that a *controlled invariant subspace* \mathcal{V} is a subspace of \mathbb{R}^n satisfying the inclusion¹

$$A_V \mathcal{V} \subseteq \mathcal{V}_D + \text{im} B_V. \quad (2)$$

In the 1-D case, a controlled invariant subspace is such that, for all initial states lying on it, an input function exists such that the local state trajectory lies completely on that subspace. For one-dimensional systems, the converse is true as well: if given an initial state an input function can be found such that the state lies on a subspace, such subspace is controlled invariant. It was shown in [5] that this last implication does not hold in the 2-D case.

An *output-nulling subspace* \mathcal{V} of Σ is a subspace of \mathbb{R}^n satisfying the inclusion

$$\begin{bmatrix} A_V \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}_D \times 0_p) + \text{im} \begin{bmatrix} B_V \\ D \end{bmatrix}. \quad (3)$$

From any point of an output-nulling subspaces, there exists a static feedback control function $u_{i,j} = F x_{i,j}$, $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $x_{i,j} \in \mathcal{V}$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$ and the output y is identically zero. Clearly, when $A_0 = 0$, $B_0 = 0$ and $D = 0$, such definition reduces to that given in [5].

While the concepts of controlled invariance and output-nulling subspaces are useful when solving exact decoupling problems, [5], [17], the concept of conditioned invariance and input-containing subspaces are useful within the context of estimation in the presence of unknown inputs. Below we provide a definition and a characterisation for conditioned invariant and input-containing subspaces. While existing definitions of conditioned invariance only hold for particular versions of Fornasini-Marchesini models with a *self-dual* structure, the definition proposed here holds for the general

class of Kurek models, whose duals are not in Kurek form. As such, this definition does not make use of duality.

Definition 1: A *conditioned invariant subspace* \mathcal{S} is a subspace of \mathbb{R}^n satisfying

$$A_H (\mathcal{S} \cap \ker [C \ 0_{p \times 2n}]) \subseteq \mathcal{S}. \quad (4)$$

Lemma 1: Given the s -dimensional subspace \mathcal{S} of \mathbb{R}^n , let $Q \in \mathbb{R}^{(n-s) \times n}$ be such that $\ker Q = \mathcal{S}$ with Q of full row-rank. The following statements are equivalent:

- 1) the subspace \mathcal{S} is conditioned invariant for Σ ;
- 2) two matrices $\Gamma \in \mathbb{R}^{(n-s) \times 3(n-s)}$ and $\Lambda \in \mathbb{R}^{(n-s) \times p}$ exist such that

$$Q A_H = \Gamma Q_D + \Lambda [C \ 0_{p \times 2n}]; \quad (5)$$

- 3) a matrix $G \in \mathbb{R}^{n \times p}$ exists such that

$$(A_H + G [C \ 0_{p \times 2n}]) \mathcal{S}_D \subseteq \mathcal{S}. \quad (6)$$

The proof of this lemma follows as a particular case of that of Lemma 2 in the sequel, and therefore it is omitted.

Now, it is shown that conditioned invariant subspaces are related to the existence of quotient observers for the autonomous model Σ_0 described by

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ y_{i,j} &= C x_{i,j}. \end{aligned} \quad (7)$$

Given a subspace \mathcal{S} of \mathbb{R}^n , we define an \mathcal{S} -quotient observer to be a finite-dimensional system of the form

$$\begin{aligned} \omega_{i+1,j+1} &= K_0 \omega_{i,j} + K_1 \omega_{i+1,j} + K_2 \omega_{i,j+1} + L y_{i,j} \\ \zeta_{i,j} &= \omega_{i,j}, \end{aligned} \quad (8)$$

such that if $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ for all $(i, j) \in \mathbb{B}$, then $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. In other words, an \mathcal{S} -quotient observer is such that the information modulo \mathcal{S} of the state of Σ_0 is maintained. In fact, if the boundary conditions of the observer and of the system Σ_0 are such that $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ for all $(i, j) \in \mathbb{B}$, then on the basis of the informations $y_{i,j}$, $i \leq k$ and $j \leq l$, the output of the observer at (k, l) takes the value $x_{k,l}/\mathcal{S}$. Obviously, given an arbitrary subspace \mathcal{S} of \mathbb{R}^n , a \mathcal{S} -quotient observer does not necessarily exist. But if this subspace is conditioned invariant, the existence of such an observer is ensured by the following theorem.

Theorem 1: If the subspace \mathcal{S} is conditioned invariant for Σ_0 , there exists a \mathcal{S} -quotient observer for Σ_0 .

Proof: Let Γ and Λ be such that (5) holds. Let the system (8) be defined by $K_H = \Gamma$ and $L = \Lambda$. Moreover, let $Q \in \mathbb{R}^{(n-s) \times n}$ be such that $\ker Q = \mathcal{S}$ with Q of full row-rank, where s is the dimension of \mathcal{S} . Define the new variable $e_{i,j} = Q x_{i,j} - \omega_{i,j}$, along with the vectors $\hat{x}(i, j) = [x_{i,j}^\top \ x_{i+1,j}^\top \ x_{i,j+1}^\top]^\top$ and $\hat{\omega}(i, j) = [\omega_{i,j}^\top \ \omega_{i+1,j}^\top \ \omega_{i,j+1}^\top]^\top$, $i, j \geq 0$. It follows that

$$\begin{aligned} e_{i+1,j+1} &= Q A_H \hat{x}(i, j) - \Gamma \hat{\omega}(i, j) - \Lambda C x_{i,j} \\ &= (Q A_H - \Lambda [C \ 0_{p \times 2n}]) \hat{x}(i, j) - \Gamma \hat{\omega}(i, j) \\ &= \Gamma Q \hat{x}(i, j) - \Gamma \hat{\omega}(i, j) \\ &= \Gamma_0 e_{i,j} + \Gamma_1 e_{i+1,j} + \Gamma_2 e_{i,j+1}, \end{aligned}$$

¹Recall that $A_V := [A_0^\top \ A_1^\top \ A_2^\top]^\top$ and $B_V := [B_0^\top \ B_1^\top \ B_2^\top]^\top$.

where (5) has been used and where $\Gamma = [\Gamma_0 \ \Gamma_1 \ \Gamma_2]$ has been partitioned conformably with A_H . Now, if for all $(i, j) \in \mathbb{B}$ there holds $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ - i.e. if $e_{i,j}$ is zero on \mathbb{B} - then $e_{i,j} = 0$ for all $i, j \geq 0$. As such, it follows that with this choice of K_i , $i = 0, 1, 2$, and L , system (8) is indeed a \mathcal{S} -quotient observer. ■

As a result of Theorem 1, when it is possible to find Γ such that the triple $(\Gamma_0, \Gamma_1, \Gamma_2)$ is stable, [9], [14], [10], the error $e_{i,j}$ goes to zero asymptotically as the index (i, j) moves away from \mathbb{B} . In that case, not only can the observer maintain information on $x_{i,j}$ modulo \mathcal{S} , if $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ on \mathbb{B} , but it can also recover such information with an error that decreases as (i, j) evolves away from the boundary \mathbb{N} when $\zeta_{i,j}$ are not equal to $x_{i,j}/\mathcal{S}$ on \mathbb{B} .

So far, the relation between quotient observers for the autonomous 2-D system Σ_0 and conditioned invariant subspaces has been analysed. When the 2-D system is not autonomous, i.e. when its structure is given by (1), we need the notion of *input-containing* subspaces in order to guarantee the existence of a quotient observer in the form given by (8), i.e., which only has the signal y as its input. Let $\bar{C} := [C \ 0_{p \times 2n}]$ and $\bar{D} := [D \ 0_{p \times 2m}]$. In the sequel we concisely identify Σ with the set $(A_H, B_H, \bar{C}, \bar{D})$.

Definition 2: We define an *input-containing subspace* \mathcal{S} as a subspace of \mathbb{R}^n satisfying

$$[A_H \ B_H] \left((\mathcal{S}_D \times \mathbb{R}^{3m}) \cap \ker [\bar{C} \ \bar{D}] \right) \subseteq \mathcal{S}. \quad (9)$$

The set of input-containing subspaces of Σ will be herein denoted by the symbol $\mathcal{S}(\Sigma)$. As for the 1-D case, it is easy to see that the intersection of two input-containing subspaces is input-containing. It follows that the set $\mathcal{S}(\Sigma)$ is closed under subspace intersection. The same is not true for subspace addition. This is due to the fact that the Grassman manifold of \mathbb{R}^n is a non-distributive lattice with respect to the operations of sum and intersection (and with respect to the partial ordering given by the standard subspace inclusion \subseteq), [3]. As a result of these considerations, it turns out that the set $\mathcal{S}(\Sigma)$ is a (modular) lower semilattice with respect to subspace intersection. Thus, the intersection of all the input-containing subspaces of Σ is the smallest input-containing subspace of Σ , and is usually denoted by \mathcal{S}^* . For input-containing subspaces, a generalised version of Lemma 1 holds.

Lemma 2: Given the s -dimensional subspace \mathcal{S} of \mathbb{R}^n , let $Q \in \mathbb{R}^{(n-s) \times n}$ be such that $\ker Q = \mathcal{S}$ with Q of full row-rank. The following statements are equivalent:

- 1) the subspace \mathcal{S} is input-containing for Σ .
- 2) two matrices $\Gamma \in \mathbb{R}^{(n-s) \times 3(n-s)}$ and $\Lambda \in \mathbb{R}^{(n-s) \times p}$ exist such that

$$Q [A_H \ B_H] = \Gamma [Q_D \ 0] + \Lambda [\bar{C} \ \bar{D}]. \quad (10)$$

- 3) a matrix $G \in \mathbb{R}^{n \times p}$ exists such that

$$\begin{aligned} (A_H + G\bar{C})\mathcal{S}_D &\subseteq \mathcal{S} \\ \text{im}(B_H + G\bar{D}) &\subseteq \mathcal{S} \end{aligned} \quad (11)$$

Proof: We prove that (9) and (10) are equivalent. Since $\ker Q = \mathcal{S}$, it follows that the subspace $\mathcal{S}_D \times \mathbb{R}^{3m}$ can be written as the null-space of $[Q_D \ 0]$, so that (9) can be written as

$$Q [A_H \ B_H] \left(\ker [Q_D \ 0] \cap \ker [\bar{C} \ \bar{D}] \right) = 0,$$

which in turn leads to the inclusion $\ker [Q_D \ 0] \cap \ker [\bar{C} \ \bar{D}] \subseteq \ker Q [A_H \ B_H]$. It follows that (10) holds for some matrices Γ and Λ .

We now prove that (9) and (11) are equivalent. To this end, let us first write (11) in the form

$$[A_H + G\bar{C} \ B_H + G\bar{D}] (\mathcal{S}_D \times \mathbb{R}^{3m}) \subseteq \mathcal{S}. \quad (12)$$

Let $\Xi_i = [\xi_0^{i\top} \ \xi_1^{i\top} \ \xi_2^{i\top}]^\top$ and $W_i = [w_0^{i\top} \ w_1^{i\top} \ w_2^{i\top}]^\top$ be such that the vectors $\begin{bmatrix} \Xi_1 \\ W_1 \end{bmatrix}, \begin{bmatrix} \Xi_2 \\ W_2 \end{bmatrix}, \dots, \begin{bmatrix} \Xi_s \\ W_s \end{bmatrix}$ are a basis of $\mathcal{S}_D \times \mathbb{R}^{3m}$ adapted to $(\mathcal{S}_D \times \mathbb{R}^{3m}) \cap \ker [\bar{C} \ \bar{D}]$, i.e., the vectors $\begin{bmatrix} \Xi_1 \\ W_1 \end{bmatrix}, \begin{bmatrix} \Xi_2 \\ W_2 \end{bmatrix}, \dots, \begin{bmatrix} \Xi_r \\ W_r \end{bmatrix}$, $r \leq s$, are a basis of $(\mathcal{S}_D \times \mathbb{R}^{3m}) \cap \ker [\bar{C} \ \bar{D}]$. Define $y^i = C \xi_0^i + D w_0^i$ for $i = 1, \dots, s$. It follows that $y^i = 0$ for $i = 1, \dots, r$ and the vectors y^{r+1}, \dots, y^s are linearly independent. Define G so that $G y^i = -A_H \Xi_i - B_H W_i$ for $i = r+1, \dots, s$ and let

$$\begin{aligned} n^i &:= [A_H + G\bar{C} \ B_H + G\bar{D}] \begin{bmatrix} \Xi_i \\ W_i \end{bmatrix} \\ &= A_H \Xi_i + B_H W_i + G y^i. \end{aligned}$$

It follows that $n^i = A_H \Xi_i + B_H W_i \in \mathcal{S}$ for all $i = 1, \dots, r$ and $n^i = 0$ for all $i = r+1, \dots, s$ by the definition of G . Hence, (12) readily follows. ■

Since (10) is linear, if the subspace \mathcal{S} is input-containing, the set of all matrices Γ and Λ satisfying (10) are parameterised by

$$[\Gamma \ \Lambda] = Q [A_H \ B_H] \begin{bmatrix} Q_D & 0 \\ \bar{C} & \bar{D} \end{bmatrix}^\dagger + H K$$

where H is a matrix of suitable dimensions such that $\text{im} H = \ker \begin{bmatrix} Q_D & \bar{C}^\top \\ 0 & \bar{D}^\top \end{bmatrix}$ and K is an arbitrary matrix. As such, the matrices Γ and Λ satisfying (10) are unique if and only if the map $\begin{bmatrix} Q_D & 0 \\ \bar{C} & \bar{D} \end{bmatrix}$ is epic. In the following theorem, we show that input-containing subspaces for system (1) are associated to the existence of \mathcal{S} -quotient observers that are still in the form governed by (8).

Theorem 2: If the subspace \mathcal{S} is input-containing for Σ , there exists a \mathcal{S} -quotient observer for Σ .

Proof: In view of Lemma 2, given an input-containing subspace \mathcal{S} , two matrices G and Γ' exist such that

$$Q [A_H + G\bar{C} \ B_H + G\bar{D}] = \Gamma' [Q_D \ 0], \quad (13)$$

where $Q \in \mathbb{R}^{(n-s) \times n}$ is such that $\ker Q = \mathcal{S}$ with Q of full row-rank, and s is the dimension of \mathcal{S} . Suppose that in (8) $K_H = \Gamma$ and $L = -QG$. Define the new variable $e_{i,j} = Qx_{i,j} - \omega_{i,j}$, along with the vectors $\hat{x}(i,j) = [x_{i,j}^\top \ x_{i+1,j}^\top \ x_{i,j+1}^\top]^\top$, $\hat{u}(i,j) = [u_{i,j}^\top \ u_{i+1,j}^\top \ u_{i,j+1}^\top]^\top$ and $\hat{\omega}(i,j) = [\omega_{i,j}^\top \ \omega_{i+1,j}^\top \ \omega_{i,j+1}^\top]^\top$, $i, j \geq 0$. It follows that

$$\begin{aligned} e_{i+1,j+1} &= QA_H \hat{x}(i,j) + QB_H \hat{u}(i,j) - \Gamma' \hat{\omega}(i,j) \\ &\quad - QGCx_{i,j} - QGDu_{i,j} \\ &= Q \begin{bmatrix} A_H + G\bar{C} & B_H + G\bar{D} \end{bmatrix} \begin{bmatrix} \hat{x}(i,j) \\ \hat{u}(i,j) \end{bmatrix} - \Gamma' \hat{\omega}(i,j) \\ &= \Gamma' Q \hat{x}(i,j) - \Gamma' \hat{\omega}(i,j) \\ &= \Gamma'_0 e_{i,j} + \Gamma'_1 e_{i+1,j} + \Gamma'_2 e_{i,j+1}, \end{aligned}$$

where (13) has been used and where $\Gamma' = [\Gamma'_0 \ \Gamma'_1 \ \Gamma'_2]$ has been partitioned conformably with A_H . Now, if for all $(i,j) \in \mathbb{B}$ the identity $\zeta_{i,j} = x_{i,j}/\mathcal{S}$ holds, then $e_{i,j} = 0$ for all $i, j \geq 0$. As such, it follows that with this choice of K_i , $i = 0, 1, 2$, and L , system (8) is indeed a \mathcal{S} -quotient observer. ■

In the case where Γ' can be found such that the triple $(\Gamma'_0, \Gamma'_1, \Gamma'_2)$ is stable, the error $e_{i,j}$ goes to zero asymptotically as the index (i,j) evolves away from \mathbb{B} .

The following theorem provides the algorithm for the computation of the smallest input-containing subspace \mathcal{S}^* .

Lemma 3: The sequence of subspaces $(\mathcal{S}_i)_{i \in \mathbb{N}}$ described by the recurrence

$$\begin{aligned} \mathcal{S}^0 &= 0_n \\ \mathcal{S}^i &= [A_H \ B_H] \left((\mathcal{S}_D^{i-1} \times \mathbb{R}^{3m}) \cap \ker [\bar{C} \ \bar{D}] \right), \quad i > 0, \end{aligned}$$

is monotonically non-increasing. An integer $k \leq n-1$ exists such that $\mathcal{S}^{k+1} = \mathcal{S}^k$. For such k , the identity $\mathcal{S}^* = \mathcal{S}^k$ holds.

The proof of Lemma 3 is similar to that in [3, p.203], and is therefore omitted.

III. UNKNOWN-INPUT OBSERVATION

Consider a linear shift-invariant (LSI) system Σ described by the following difference equations

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{i,j} \\ &\quad + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \\ y_{i,j} &= C_1 x_{i,j} + D_1 u_{i,j}, \\ z_{i,j} &= C_2 x_{i,j} + D_2 u_{i,j}, \end{aligned} \tag{14}$$

where, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, $x_{i,j} \in \mathbb{R}^n$ is the local state, $u_{i,j} \in \mathbb{R}^m$ is an input which is not accessible for measurement. The variable $y_{i,j} \in \mathbb{R}^{p_1}$ represents an output that can be measured and the variable $z_{i,j} \in \mathbb{R}^{p_2}$ is an output that we want to estimate on the basis of the information represented by the measurement of the output y . Matrices $A_k, B_k, k = 0, 1, 2$, and $C_l, D_l, l = 1, 2$, are of appropriate dimensions. Let $\bar{C}_k := [C_k \ 0_{p_k \times 2n}]$ and $\bar{D}_k := [D_k \ 0_{p_k \times 2m}]$ for $k = 1, 2$. In the sequel we concisely identify Σ with the set $(A_H, B_H, \bar{C}_1, \bar{D}_1, \bar{C}_2, \bar{D}_2)$.

Consider the block diagram depicted in Figure 1. Let the observer Σ_0 be described by the equations

$$\begin{aligned} \omega_{i+1,j+1} &= K_0 \omega_{i,j} + K_1 \omega_{i+1,j} + K_2 \omega_{i,j+1} + Ly_{i,j}, \\ \zeta_{i,j} &= M \omega_{i,j} + Ny_{i,j}, \end{aligned} \tag{15}$$

and let $\hat{\Sigma}$ denote the overall system from the input u to the output $e := z - \zeta$, as shown in Figure 1. Notice that with the choice of the structure of the observer Σ_0 , the overall system $\hat{\Sigma}$ is still in Kurek form, and is governed by

$$\begin{aligned} \begin{bmatrix} x_{i+1,j+1} \\ \omega_{i+1,j+1} \end{bmatrix} &= \begin{bmatrix} A_0 & 0 \\ LC_1 & K_0 \end{bmatrix} \begin{bmatrix} x_{i,j} \\ \omega_{i,j} \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & K_1 \end{bmatrix} \begin{bmatrix} x_{i+1,j} \\ \omega_{i+1,j} \end{bmatrix} \\ &\quad + \begin{bmatrix} A_2 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} x_{i,j+1} \\ \omega_{i,j+1} \end{bmatrix} + \begin{bmatrix} B_0 \\ LD_1 \end{bmatrix} u_{i,j} \\ &\quad + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_{i+1,j} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u_{i,j+1}, \\ e_{i,j} &= [C_2 - NC_1 \quad -M] \begin{bmatrix} x_{i,j} \\ \omega_{i,j} \end{bmatrix} + (D_2 - ND_1) u_{i,j}. \end{aligned}$$

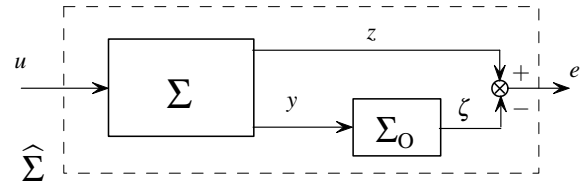


Fig. 1. Block diagram of the unknown input observation scheme.

Roughly speaking, the *unknown-input observation* problem considered in this paper consists of finding an observer Σ_0 ruled by (15) and connected as in Figure 1, such that $\zeta_{i,j} = z_{i,j}$ for $(i,j) \in \mathbb{B}$ implies $\zeta_{i,j} = z_{i,j}$ for all $i, j \geq 0$. This problem is equivalent to finding an observer Σ_0 such that the input u has no influence on the output e . Such problem has been addressed in [4] for FM-2 models using polynomial techniques.

Theorem 3: The unknown-input observation problem admits solutions if

$$\ker [C_2 \ D_2] \supseteq (\mathcal{S}^* \times \mathbb{R}^m) \cap \ker [C_1 \ D_1], \tag{16}$$

where \mathcal{S}^* is the smallest input-containing subspace of the system $(A_H, B_H, \bar{C}_1, \bar{D}_1)$.

The result given in Theorem 3 will be proved on the basis of the following considerations.

- 1) Given any input-containing \mathcal{S} subspace of \mathbb{R}^m such that

$$\ker [C_2 \ D_2] \supseteq (\mathcal{S} \times \mathbb{R}^m) \cap \ker [C_1 \ D_1], \tag{17}$$

two matrices Φ and Ψ exist such that $[C_2 \ D_2] = \Phi [Q \ 0] + \Psi [C_1 \ D_1]$, where Q is a full row-rank matrix such that $\ker Q = \mathcal{S}$. Notice that the former can be written as

$$[\bar{C}_2 \ \bar{D}_2] = \hat{\Phi} [Q_D \ 0] + \Psi [\bar{C}_1 \ \bar{D}_1], \tag{18}$$

where $\widehat{\Phi} := [\Phi \ 0 \ 0]$.

- 2) If condition (17) is satisfied, matrices Φ and Ψ satisfying $[C_2 \ D_2] = \Phi[Q \ 0] + \Psi[C_1 \ D_1]$ are parameterised by

$$[\Phi \ \Psi] = [C_2 \ D_2] \begin{bmatrix} Q & 0 \\ C_1 & D_1 \end{bmatrix}^\dagger + HK,$$

where H is such that $\text{im}H = \ker \begin{bmatrix} Q^\top & C_1^\top \\ 0 & D_1^\top \end{bmatrix}$ and K is an arbitrary matrix. As such, the matrices Φ and Ψ satisfying (18) are unique if and only if the map $\begin{bmatrix} Q & 0 \\ C_1 & D_1 \end{bmatrix}$ is epic. When $[C_1 \ D_1]$ is full row-rank, this condition is equivalent to $C_1 \mathcal{S} + \text{im}D_1 = \mathbb{R}^{p_1}$ or alternatively $\mathcal{S} + C_1^{-1} \text{im}D_1 = \mathbb{R}^n$.

Now, since it can be straightforwardly established that the kernel of $\begin{bmatrix} Q^\top & C_1^\top \\ 0 & D_1^\top \end{bmatrix}$ is zero if and only if such is the kernel of $\begin{bmatrix} Q^\top & \bar{C}_1^\top \\ 0 & \bar{D}_1^\top \end{bmatrix}$, it turns out that in the case where $C_1 \mathcal{S} + \text{im}D_1 = \mathbb{R}^{p_1}$, equations (10) (with $\bar{C} = \bar{C}_1$ and $\bar{D} = \bar{D}_1$) and (16) admit a unique solution, so that the four matrices Γ, Λ, Φ and Ψ can be uniquely determined.

Proof of Theorem 1: Let \mathcal{S} be any input-containing subspace of the system $(A_H, B_H, \bar{C}_1, \bar{D}_1)$ for which (16) holds with \mathcal{S} in place of \mathcal{S}^* , and let $\widehat{\Phi} := [\Phi \ 0 \ 0]$ and Ψ be such that (18) holds. Denote by $\Gamma := [\Gamma_0 \ \Gamma_1 \ \Gamma_2]$ and Λ two matrices such that (10) holds with $\bar{C} = \bar{C}_1$ and $\bar{D} = \bar{D}_1$. We show that the observer Σ_o ruled by (15) with $K_k = \Gamma_k$ ($k = 0, 1, 2$), $M = \Phi$, $L = -\Lambda$ and $N = \Psi$ solves the unknown-input observation problem. First, note that

$$\begin{aligned} e_{i,j} &= z_{i,j} - \zeta_{i,j} \\ &= \left([C_2 \ D_2] - \Psi [C_1 \ D_1] \right) \begin{bmatrix} x_{i,j} \\ u_{i,j} \end{bmatrix} - \Phi \omega_{i,j} \\ &= \Phi [Q \ 0] \begin{bmatrix} x_{i,j} \\ u_{i,j} \end{bmatrix} - \Phi \omega_{i,j} = \Phi(Qx_{i,j} - \omega_{i,j}). \end{aligned}$$

Define $\varepsilon_{i,j} := Qx_{i,j} - \omega_{i,j}$. Given the signal $s : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}^l$ for some l , let also $\hat{l}(i, j) = [l_{i,j}^\top \ l_{i+1,j}^\top \ l_{i,j+1}^\top]^\top$, $i, j \geq 0$. Then, by using (10) and (16), it is found

$$\begin{aligned} \varepsilon_{i+1,j+1} &= \left(Q [A_H \ B_H] - \Lambda [\bar{C}_1 \ \bar{D}_1] \right) \begin{bmatrix} \hat{x}(i, j) \\ \hat{u}(i, j) \end{bmatrix} - \Gamma \hat{\omega}(i, j) \\ &= \Gamma [Q \ 0] \begin{bmatrix} \hat{x}(i, j) \\ \hat{u}(i, j) \end{bmatrix} - \Gamma \hat{\omega}(i, j) \\ &= \Gamma_0 \varepsilon_{i,j} + \Gamma_1 \varepsilon_{i+1,j} + \Gamma_2 \varepsilon_{i,j+1}. \end{aligned}$$

As a result of this, the signal $\varepsilon_{i,j}$ is independent of $u_{i,j}$, and since $e_{i,j} = \Phi \varepsilon_{i,j}$, such is also the error dynamic $e_{i,j}$, so that the transfer function from the input u to the output e is zero. ■

The definition given above for unknown-input observation is weaker than the one usually adopted in the 1-D framework. In fact, while a quotient observer guarantees that the

information on z is maintained if we assume that its value on \mathbb{B} is known exactly, it is not possible in general to recover information on z in the case where the values of z and ζ on \mathbb{B} are not equal. In other words, finding a 2-D observer Σ_o such that the transfer function matrix from the input u to the output e is zero does not guarantee that the estimation ζ obtained is asymptotic. However, it is easily established that if a stable triple $(\Gamma_0, \Gamma_1, \Gamma_2)$ can be found such that (10) holds, the observer given in Theorem 3 is asymptotic, i.e., it recovers the latent variable $x_{i,j}$ with greater accuracy as the spatial index (i, j) evolves away from \mathbb{B} . Conditions would be desirable for the existence of such triple in terms of the problem data. Unfortunately, while in the 1-D framework stability is easily embedded in the geometric concept of controlled and conditioned invariance, in the 2-D context providing a definition and a characterisation to *internally and externally stabilisable* controlled or conditioned invariant subspaces (and hence to internally and externally stabilisable output-nulling or input-containing subspaces) is not an easy task, and to date remains an open problem.

Remark 1: The solution proposed here for the unknown-input observation problem can be utilised for the solution of the 2-D counterpart of the so-called *fixed-lag smoothing*, where the delay between the measurement and the generation of the estimate is here represented by a (finite) double shift. Consider system (14), where now the error is defined as $e_{i,j} = z_{i-N,j-M} - \zeta_{i,j}$. The (N, M) -shift accounts for the *delay* tolerated for the estimation of z , see Figure 2.

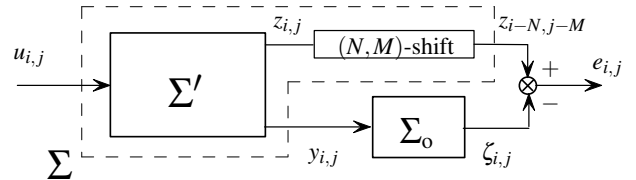


Fig. 2. Block diagram of the fixed-shift smoothing scheme.

It is easily seen that this problem can be turned into an unknown-input observation problem. Consider the following realisation of the (N, M) -shift with a Kurek model: if $M > N$ let

$$A_o^d = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B_o^d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$A_1^d = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \end{array} \right], \quad A_2^d = 0,$$

$$B_1^d = B_2^d = 0, \quad C^d = [0 \ 0 \ \dots \ 0 \ 1],$$

where the size the block submatrices in the top left-hand of A_0^d and A_1^d is $N \times N$, while the dimension of this realisation is M . When $N > M$, the structure of the matrices A_1^d and A_2^d swaps. The dynamic of the delay is governed by

$$d_{i+1,j+1} = A_0^d d_{i,j} + A_1^d d_{i+1,j} + A_2^d d_{i,j+1} + B_0^d z_{i,j},$$

$$z_{i-N,j-M} = C^d d_{i,j}.$$

As such, a latent variable realisation of the series connection Σ of the system Σ' and of the double shift (N, M) is given by the following matrices:

$$A_0^\Sigma = \begin{bmatrix} A_0 & 0 \\ B_0^d C_2 & A_0^d \end{bmatrix}, \quad A_1^\Sigma = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^d \end{bmatrix}, \quad A_2^\Sigma = \begin{bmatrix} A_2 & 0 \\ 0 & A_2^d \end{bmatrix},$$

$$B_0^\Sigma = \begin{bmatrix} B_0 \\ B_0^d D_2 \end{bmatrix}, \quad B_1^\Sigma = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_2^\Sigma = \begin{bmatrix} B_2 \\ 0 \end{bmatrix},$$

$$C_1^\Sigma = [C_1 \ 0], \quad D_1^\Sigma = D_1$$

$$C_2^\Sigma = [0 \ C^d], \quad D_2^\Sigma = 0.$$

The fixed-shift smoothing has therefore solutions if the following condition holds:

$$\ker [C_2^\Sigma \ D_2^\Sigma] \supseteq (\mathcal{S}^* \times \mathbb{R}^m) \cap \ker [C_1^\Sigma \ D_1^\Sigma], \quad (19)$$

where \mathcal{S}^* is the smallest input-containing subspace of the system $(A_H^\Sigma, B_H^\Sigma, \bar{C}_1^\Sigma, \bar{D}_1^\Sigma)$. Since D_2^Σ is zero, condition (19) can be alternatively written as

$$\ker C_2^\Sigma \supseteq \mathcal{S}^* \cap (C_1^\Sigma)^{-1} \text{im} D_1^\Sigma.$$

If such condition is satisfied, the design procedure for the smoother Σ_0 can be carried out as in the proof of Theorem 3 with the obvious substitutions.

A. Conclusions

A new definition has been proposed for conditioned invariant and input-containing subspaces for Fornasini-Marchesini models in Kurek form. Moreover, the problem of estimation in presence of unknown inputs has been investigated. The possibility of providing a characterisation of conditioned invariant and input-containing subspaces in terms of stability of the associated observers is under investigation, and will be dealt with in a forthcoming journal paper.

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