Operation Properties and δ-Equalities of Complex Fuzzy Sets

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Abstract

A complex fuzzy set is a fuzzy set whose membership function takes values in the unit circle in the complex plane. This paper investigates various operation properties and proposes a distance measure for complex fuzzy sets. The distance of two complex fuzzy sets measures the difference between the grades of two complex fuzzy sets as well as that between the phases of the two complex fuzzy sets. This distance measure is then used to define δ-equalities of complex fuzzy sets which coincide with those of fuzzy sets already defined in the literature if complex fuzzy sets reduce to real-valued fuzzy sets. Two complex fuzzy sets are said to be δ-equal if the distance between them is less than 1-δ. This paper shows how various operations between complex fuzzy sets affect given δ-equalities of complex fuzzy sets. An example application of signal detection demonstrates the utility of the concept of δ-equalities of complex fuzzy sets in practice.

Keywords: Complex fuzzy set, Fuzzy set, Distance measure, δ-equality, Operation

1. Introduction

Since the seminal paper in 1965 by Zadeh proposed *Fuzzy Sets* [12], a huge amount of literature has appeared on different aspects of fuzzy sets and their applications. Ramot et al. [10] recently proposed an important extension of these ideas, the *Complex Fuzzy Sets,* where the membership function μ instead of being a real valued function with the range [0,1] is replaced by a complex-valued function of the form

$$r_S(x) \cdot e^{i\omega_S(x)} \ (i = \sqrt{-1})$$

where $r_s(x)$ and $\omega_s(x)$ are both real valued giving the range as the unit circle. However, this concept is different from fuzzy complex number introduced and discussed by Buckley [1 - 4] and Zhang [13 - 15]. Essentially as explained in [10] this still retains the characterization of the uncertainty through the amplitude of the grade of membership having a value in the range of [0,1] whilst adding the membership phase captured by fuzzy sets $\omega_S(x)$. As explained in Ramot et al [10], the key feature of complex fuzzy sets is the presence of phase and its membership. This gives those complex fuzzy sets wavelike properties which could result in constructive and destructive interference depending on the phase value. Thus property distinguishes these complex fuzzy sets from conventional fuzzy sets, fuzzy complex sets, and type 2 fuzzy sets [10, 12] (a brief comparison of them in Appendix). Several examples are given in [10] which demonstrate the utility of these complex fuzzy sets. They also define several important concepts such as the complement, union, intersection and fuzzy relations for such complex fuzzy sets.

On the other hand, with an attempt to show that 'precise membership values should normally be of no practical significance', Pappis [9] introduced firstly the notion of 'proximity measure'. Hong and Hwang [8] then presented an important generalization. Further, Cai [5, 6] introduced and discussed *δ-equalities of Fuzzy Sets* and their properties. Two fuzzy sets are said to be δ-equal if they are equal to an extent of δ. The concept of δ-equalities of fuzzy sets was then employed in synthesis of real-time fuzzy systems by Virant [11], for assessing the robustness of fuzzy reasoning by Cai [6], and generalized in theory to the so-called (*,δ)-equalities of fuzzy sets by Georgescu [7]. In this paper, we build on the results obtained in Cai's papers by introducing some operations on complex fuzzy sets and their properties and then investigate the important concept of *δ-equalities* which allows us to systematically develop measures of distance between, equality and similarity for complex fuzzy sets.

This paper is a continuing work of the papers of Ramot et.al [10] and Cai [5, 6]. We follow the philosophy of Ramot et.al [10] and will not argue for the rationale of introducing the concept of complex fuzzy sets in this paper. In Section 2, after reviewing the concept of complex fuzzy set, some operations of complex fuzzy sets are introduced, and their properties are discussed. Section 3 introduces δ-equalities of complex fuzzy sets and discusses δ-equalities for various implication operators. An example application is presented in Section 4 to demonstrate the utility of δ-equalities of complex fuzzy sets in practice. Conclusion is given in Section 5.

2. Operations of Complex Fuzzy Sets

Definition 2.1[10] A complex fuzzy set *C*, defined on a universe of discourse *U*, is characterized by a membership function $\mu_C(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in *C*.

By definition, the values $\mu_C(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_C(x) \cdot e^{iArg_C(x)}$, ($i = \sqrt{-1}$), $r_C(x)$ is a real-valued function such that $r_C(x) \in [0,1]$ and $e^{iArg_C(x)}$ is a periodic function whose periodic law and principal period are, respectively, 2π and $0 < arg_C(x) \le 2\pi$, *i.e.*, $Arg_C(x) = arg_C(x) + 2k\pi$, $k = 0, \pm 1, \pm 2, \cdots$, where $arg_C(x)$ is the principal argument. The principal argument $arg_C(x)$ will used on the following text.

Let *F**(*U*) be the set of all complex fuzzy sets on *U*. The complex fuzzy set *C* may be represented as the set of ordered pairs

$$C = \{(x, \mu_C(x)) \mid x \in U\}. \tag{2.1}$$

Definition 2.2 (1) A quasi-triangular norm T is a function $(0, 1]\times(0, 1] \to [0, 1]$ that satisfies the following conditions:

(i) $T(1, 1) = 1$;

(ii) $T(a, b) = T(b, a)$;

(iii) $T(a, b) \le T(c, d)$ whenever $a \le c, b \le d$;

(iv) $T(T(a, b)) = T(a, T(b, c))$.

(2) A triangular norm T is a function $[0, 1]\times[0, 1] \to [0, 1]$ the satisfies conditions (i) – (iv) and the following condition:

(v) $T(0, 0) = 0$.

We said T is an s-norm, if a triangular norm T satisfies

(vi) $T(a, 0) = a$;

We said T is a t-norm, if a triangular norm T satisfies

(vii) $T(a, 1) = a$.

(3) We said a binary function $\tilde{T}$:

$$\tilde{T} : F^*(U) \times F^*(U) \to F^*(U)$$

$$\tilde{T}(A, B) \mapsto \sup_{x\in U} T_1(r_A(x), r_B(x)) e^{i2\pi \sup_{x\in U} T_2\left(\frac{arg_A(x)}{2\pi}, \frac{arg_B(x)}{2\pi}\right)}.$$

is a triangular norm if T_1 is a triangular norm and T_2 is a quasi-triangular norm; we said $\tilde{T}$ is an s-norm if T_1 an s-norm; we said $\tilde{T}$ is a t-norm if T_1 a t-norm.

Definition 2.3 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy union of A and B, denoted by $A \cup B$, is specified by a function

$$\mu_{A\cup B}(x) = r_{A\cup B}(x) \cdot e^{iarg_{A\cup B}(x)} = \max(r_A(x),\ r_B(x)) \cdot e^{i\max(arg_A(x),\ arg_B(x))}. \qquad (2.2)$$

Example 2.1 Let $A = \dfrac{0.6e^{i1.2\pi}}{-1} + \dfrac{1.0e^{i2\pi}}{0} + \dfrac{0.8e^{i1.6\pi}}{1} + \dfrac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \cup B = \dfrac{0.6e^{i1.2\pi}}{-1} + \dfrac{1e^{i2\pi}}{0} + \dfrac{1e^{i2\pi}}{1} + \dfrac{0.8e^{i1.6\pi}}{2}$.

Proposition 2.1 The complex fuzzy union on $F^*(U)$ is an s-norm.

Proof. Properties (i), (ii), (v) and (vi) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B and C be three complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$, and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We suppose $|\mu_A(x)| \le |\mu_B(x)|$, $arg_A(x) \le arg_B(x), \forall x \in U$. Thus

$$|\mu_{A\cup C}(x)| = \max(r_A(x),\ r_C(x)) \le \max(r_B(x),\ r_C(x)) = |\mu_{B\cup C}(x)|, \quad \forall x \in U.$$

$$arg_{A\cup C}(x) = \max(arg_A(x),\ arg_C(x)) \le \max(arg_B(x),\ arg_C(x)) = arg_{B\cup C}(x), \quad \forall x \in U.$$

(iv) Suppose A, B and C are three complex fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x)\cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x)\cdot e^{iarg_C(x)}$ their membership functions, respectively. We have

$$
\begin{aligned}
\mu_{A\cup(B\cup C)}(x) &= \max\left(r_A(x), r_{B\cup C}(x)\right)\cdot e^{i\max\left(arg_A(x),\, arg_{B\cup C}(x)\right)} \\
&= \max\left(r_A(x), \max(r_B(x), r_C(x))\right)\cdot e^{i\max\left(arg_A(x),\, \max(arg_B(x),\, arg_C(x))\right)} \\
&= \max\left(\max(r_A(x), r_B(x)), r_C(x)\right)\cdot e^{i\max\left(\max(arg_A(x),\, arg_B(x)),\, arg_C(x)\right)} \\
&= \mu_{(A\cup B)\cup C}(x)
\end{aligned}
$$

Corollary 2.1 Let $C_\alpha \in F^*(U)$, $\alpha \in I$, and $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where I is an arbitrary index sets. Then $\cup_{\alpha\in I} C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{\underset{\alpha\in I}{\cup} C_\alpha}(x) = \sup_{\alpha\in I} r_{C_\alpha}(x)\cdot e^{i\sup_{\alpha\in I} arg_{C_\alpha}(x)}.$$

Proof. This is straightforward from Definition 2.2 and Proposition 2.1.

Definition 2.4 Let C be a complex fuzzy set on U, and $\mu_C(x) = r_C(x)\cdot e^{iarg_C(x)}$ its membership function. The complex fuzzy complement of C, denoted $\overline{C}$ is specified by a function

$$\mu_{\overline{C}}(x) = r_{\overline{C}}(x)\cdot e^{iarg_{\overline{C}}(x)} = \left(1 - r_C(x)\right)\cdot e^{i(2\pi - arg_C(x))}. \tag{2.3}$$

Example 2.2 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$, then,

$$\overline{A} = \frac{0.4e^{i0.8\pi}}{-1} + \frac{0}{0} + \frac{0.2e^{i0.4\pi}}{1} + \frac{0.5e^{i\pi}}{2}.$$

Proposition 2.2 Let C be a complex fuzzy set on U. Then $\overline{\overline{C}} = C$.

Proof. By Definition 2.4, we have

$$
\begin{aligned}
\mu_{\overline{\overline{C}}}(x) &= r_{\overline{\overline{C}}}(x)\cdot e^{iarg_{\overline{\overline{C}}}(x)} = \left(1 - r_{\overline{C}}(x)\right)\cdot e^{i(2\pi - arg_{\overline{C}}(x))} \\
&= \left(1 - \left(1 - r_C(x)\right)\right)\cdot e^{i(2\pi - (2\pi - arg_C(x)))} \\
&= r_C(x)\cdot e^{iarg_C(x)} \\
&= \mu_C(x).
\end{aligned}
$$

Thus $\overline{\overline{C}} = C$.

Definition 2.5 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x)\cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x)\cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy intersection of A and B, denoted $A\cap B$, is specified by a function

$$\mu_{A\cap B}(x) = r_{A\cap B}(x)\cdot e^{iarg_{A\cap B}(x)} = \min\left(r_A(x),\ r_B(x)\right)\cdot e^{i\min\left(arg_A(x),\, arg_B(x)\right)}. \tag{2.4}$$

Example 2.3 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \cap B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i1\pi}}{2}.$

Proposition 2.3 Let A and B be two complex fuzzy sets on U. Then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof. By use of Definition 2.3-2.5, we have

$$\begin{aligned}
\mu_{\overline{A \cap B}}(x) &= r_{\overline{A \cap B}}(x) \cdot e^{iarg_{\overline{A \cap B}}(x)} = \left(1 - r_{A \cap B}(x)\right) \cdot e^{i(2\pi - arg_{A \cap B}(x))} \\
&= \left(1 - \min\left(r_A(x),\ r_B(x)\right)\right) \cdot e^{i(2\pi - \min(arg_A(x),\ arg_B(x)))} \\
&= \max\left(1 - r_A(x), 1 - r_B(x)\right) \cdot e^{i\max(2\pi - arg_A(x),\ 2\pi - arg_B(x))} \\
&= \max\left(r_{\overline{A}}(x), r_{\overline{B}}(x)\right) \cdot e^{i\max(arg_{\overline{A}}(x),\ arg_{\overline{B}}(x))} \\
&= \mu_{\overline{A} \cup \overline{B}}(x).
\end{aligned}$$

Proposition 2.4 The complex fuzzy intersection on $F^*(U)$ is a t-norm.

Proof. Properties (i) - (ii), (v) and (vii) can be easily verified from Definition 2.5. Here we only prove (iii) and (iv).

(iii) Let A, B and C be three complex fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We suppose $|\mu_A(x)| \le |\mu_B(x)|$, $arg_A(x) \le arg_B(x), \forall x \in U$. Thus

$$|\mu_{A \cap C}(x)| = \min\left(r_A(x),\ r_C(x)\right) \le \min\left(r_B(x),\ r_C(x)\right) = |\mu_{B \cap C}(x)|, \quad \forall x \in U.$$

$$arg_{A \cap C}(x) = \min\left(arg_A(x),\ arg_C(x)\right) \le \min\left(arg_B(x),\ arg_C(x)\right) = arg_{B \cap C}(x), \quad \forall x \in U.$$

(iv) Suppose A, B and C are three complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We have

$$\begin{aligned}
\mu_{A \cap (B \cap C)}(x) &= \min\left(r_A(x), r_{B \cap C}(x)\right) \cdot e^{i\min(arg_A(x),\ arg_{B \cap C}(x))} \\
&= \min\left(r_A(x), \min(r_B(x), r_C(x))\right) \cdot e^{i\min(arg_A(x),\ \min(arg_B(x),\ arg_C(x)))} \\
&= \min\left(\min(r_A(x), r_B(x)), r_C(x)\right) \cdot e^{i\min(\min(arg_A(x),\ arg_B(x)),\ arg_C(x))} \\
&= \mu_{(A \cap B) \cap C}(x).
\end{aligned}$$

Corollary 2.2 Let $C_\alpha \in F^*(U)$, $\alpha \in I$ and $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where I is an arbitrary index sets. Then $\bigcap_{\alpha \in I} C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{\bigcap_{\alpha \in I} C_\alpha}(x) = \inf_{\alpha \in I} r_{C_\alpha}(x) \cdot e^{i \inf_{\alpha \in I} arg_{C_\alpha}(x)}.$$

Proof. This is obvious from Definition 2.2 and Proposition 2.4.

Corollary 2.3 Let $C_{\alpha\beta} \in F^*(U),\ \alpha \in I_1, \beta \in I_2$ and $\mu_{C_{\alpha\beta}}(x) = r_{C_{\alpha\beta}}(x) \cdot e^{iarg_{C_{\alpha\beta}}(x)}$ their membership functions, respectively, where I_1 and I_2 are two arbitrary index sets. Then $\cup_{\alpha\in I_1} \cap_{\beta\in I_2} C_{\alpha\beta}, \cap_{\alpha\in I_1} \cup_{\beta\in I_2} C_{\alpha\beta} \in F^*(U)$ and their membership functions are

$$\mu_{\underset{\alpha\in I_1}{\cup}\underset{\beta\in I_2}{\cap} C_{\alpha\beta}}(x) = \sup_{\alpha\in I_1}\inf_{\beta\in I_2} r_{C_{\alpha\beta}}(x) \cdot e^{i\sup_{\alpha\in I_1}\inf_{\beta\in I_2} arg_{C_{\alpha\beta}}(x)},$$

$$\mu_{\underset{\alpha\in I_1}{\cap}\underset{\beta\in I_2}{\cup} C_{\alpha\beta}}(x) = \inf_{\alpha\in I_1}\sup_{\beta\in I_2} r_{C_{\alpha\beta}}(x) \cdot e^{i\inf_{\alpha\in I_1}\sup_{\beta\in I_2} arg_{C_{\alpha\beta}}(x)}.$$

Proof. Trivial.

Corollary 2.4 Let $C_k \in F^*(U),\ k = 1, 2, \cdots$ and $\mu_{C_k}(x) = r_{C_k}(x) \cdot e^{iarg_{C_k}(x)}$ their membership functions, respectively. Then $\overline{\lim}_{n\to\infty} C_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} C_k$, $\underline{\lim}_{n\to\infty} C_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty}$, $C_k \in F^*(U)$, and their membership functions are

$$\mu_{\overline{\lim}_{n\to\infty} C_k}(x) = \inf_{n\ge1}\sup_{k\ge n} r_{C_\alpha}(x) \cdot e^{i\inf_{n\ge1}\sup_{k\ge n} arg_{C_k}(x)}, \quad \mu_{\underline{\lim}_{n\to\infty} C_k}(x) = \sup_{n\ge1}\inf_{k\ge n} r_{C_\alpha}(x) \cdot e^{i\sup_{n\ge1}\inf_{k\ge n} arg_{C_k}(x)}.$$

Proof. Trivial.

Proposition 2.5 Let *A*, *B* and *C* be three complex fuzzy sets on *U*, Then

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Proof. Here we only prove first conclusion. For *A*, *B*, $C \in F^*(U)$, we have

$$\begin{aligned}
\mu_{(A\cup B)\cap C}(x) &= r_{(A\cup B)\cap C}(x) \cdot e^{iarg_{(A\cup B)\cap C}(x)} \\
&= \min\left(r_{A\cup B}(x), r_C(x)\right) \cdot e^{i\min\left(arg_{A\cup B}(x),\ arg_C(x)\right)} \\
&= \min\left(\max\left(r_A(x), r_B(x)\right), r_C(x)\right) \cdot e^{i\min\left(\max\left(arg_A(x),\ arg_B(x)\right),\ arg_C(x)\right)} \\
&= \max\left(\min\left(r_A(x), r_C(x)\right), \min\left(r_B(x), r_C(x)\right)\right) \\
&\quad \cdot e^{i\max\left(\min\left(arg_A(x),\ arg_C(x)\right),\ \min\left(arg_B(x),\ arg_C(x)\right)\right)} \\
&= \max\left(r_{A\cap C}(x), r_{B\cap C}(x)\right) \cdot e^{i\max\left(arg_{A\cap C}(x),\ arg_{B\cap C}(x)\right)} \\
&= r_{(A\cap C)\cup(B\cap C)}(x) \cdot e^{iarg_{(A\cap C)\cup(B\cap C)}(x)}.
\end{aligned}$$

Proposition 2.6 Let *A* and *B* be two complex fuzzy sets on *U*, Then

$$(A \cup B) \cap A = A, \quad (A \cap B) \cup A = A.$$

Proof. Here we only prove first conclusion. For *A*, $B \in F^*(U)$, we have

$$\begin{aligned}
\mu_{(A\cup B)\cap A}(x) &= r_{(A\cup B)\cap A}(x) \cdot e^{iarg_{(A\cup B)\cap A}(x)} \\
&= \min\left(r_{A\cup B}(x), r_A(x)\right) \cdot e^{i\min\left(arg_{A\cup B}(x),\ arg_A(x)\right)} \\
&= \min\left(\max\left(r_A(x), r_B(x)\right), r_A(x)\right) \cdot e^{i\min\left(\max\left(arg_A(x),\ arg_B(x)\right),\ arg_A(x)\right)} \\
&= r_A(x) \cdot e^{iarg_A(x)}.
\end{aligned}$$

Definition 2.6 Let *A* and *B* be two complex fuzzy sets on *U*, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy product of *A* and *B*, denoted $A \circ B$, is specified by a function

$$\mu_{A\circ B}(x) = r_{A\circ B}(x) \cdot e^{iarg_{A\circ B}(x)} = \left(r_A(x) \cdot r_B(x)\right) \cdot e^{i2\pi\left(\frac{arg_A(x)}{2\pi} \cdot \frac{arg_B(x)}{2\pi}\right)}. \qquad (2.5)$$

Example 2.4 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \circ B = \frac{0.36e^{i0.72\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.4e^{i0.8\pi}}{2}$.

Proposition 2.7 The complex fuzzy product on *F**(*U*) is a *t*-norm.

Proof. Properties (i) - (ii), (v) and (vii) can be easily verified from Definition 2.6. Here we only prove (iii) and (iv). (iii) Let *A*, *B* and *C* be three complex fuzzy sets on *U*, $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We suppose $|\mu_A(x)| \le |\mu_B(x)|$, $arg_A(x) \le arg_B(x), \forall x \in U.$ Thus

$$|\mu_{A\circ C}(x)| = |r_A(x)| \cdot |r_C(x)| \le |r_B(x)| \cdot |r_C(x)| = |(r_B(x),\ r_C(x))| = |\mu_{B\circ C}(x)|, \quad \forall x \in U.$$

$$arg_{A\circ C}(x) = 2\pi\left(\frac{arg_A(x)}{2\pi} \cdot \frac{arg_C(x)}{2\pi}\right) \le 2\pi\left(\frac{arg_B(x)}{2\pi} \cdot \frac{arg_C(x)}{2\pi}\right) = arg_{B\circ C}(x), \quad \forall x \in U.$$

(iv) Suppose A, *B* and *C* are three complex fuzzy sets on *U*, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We have

$$\begin{aligned}
\mu_{A\circ(B\circ C)}(x) &= \left(r_A(x) \cdot r_{B\circ C}(x)\right) \cdot e^{i2\pi\left(\frac{arg_A(x)}{2\pi} \cdot \frac{arg_{B\circ C}(x)}{2\pi}\right)} \\
&= \left(r_A(x) \cdot (r_B(x) \cdot r_C(x))\right) \cdot e^{i2\pi\left(\frac{arg_A(x)}{2\pi} \cdot \frac{2\pi\left(\frac{arg_B(x)}{2\pi} \cdot \frac{arg_C(x)}{2\pi}\right)}{2\pi}\right)} \\
&= \left((r_A(x) \cdot r_B(x)) \cdot r_C(x)\right) \cdot e^{i2\pi\left(\frac{2\pi\left(\frac{arg_A(x)}{2\pi} \cdot \frac{arg_B(x)}{2\pi}\right)}{2\pi} \cdot \frac{arg_C(x)}{2\pi}\right)} \\
&= \mu_{(A\circ B)\circ C}(x).
\end{aligned}$$

Corollary 2.5 Let $C_\alpha \in F^*(U),\ \alpha \in I$ and $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where *I* is an arbitrary index sets. Then $\prod_{\alpha \in I} C_\alpha = C_1 \circ C_2 \circ \cdots \circ C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{\prod_{\alpha \in I} C_\alpha}(x) = r_{C_1}(x) \cdot r_{C_2}(x) \cdots r_{C_\alpha}(x) \cdot e^{i2\pi\left(\frac{arg_{C_1}(x)}{2\pi} \cdot \frac{arg_{C_2}(x)}{2\pi} \cdots \frac{arg_{C_\alpha}(x)}{2\pi}\right)}.$$

Proof. Trivial.

Definition 2.7 Let A_n, n = 1, 2, …, N be N complex fuzzy sets on U, and $\mu_{A_n}(x) = r_{A_n}(x) \cdot e^{iarg_{A_n}(x)}$, n = 1, 2, …, N their membership functions, respectively. The complex fuzzy Cartesian product of A_n, n = 1, 2, …, N, denoted $A_1 \times A_2 \times \cdots \times A_N$, is specified by a function

$$\begin{aligned}\mu_{A_1 \times A_2 \times \cdots \times A_N}(x) &= r_{A_1 \times A_2 \times \cdots \times A_N}(x) \cdot e^{iarg_{A_1 \times A_2 \times \cdots \times A_N}(x)} \\ &= \min\left(r_{A_1}(x_1), r_{A_2}(x_2), \cdots, r_{A_N}(x_N)\right) \cdot e^{i\min\left(arg_{A_1}(x_1), arg_{A_2}(x_2), \cdots, arg_{A_N}(x_N)\right)},\end{aligned} \quad (2.6)$$

where $x = (x_1, x_2, \cdots, x_N) \in \underbrace{U \times U \times \cdots \times U}_{N}$.

Example 2.5 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \times B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$.

Definition 2.8 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy probabilistic sum of A and B, denoted $A \hat{+} B$, is specified by a function

$$\mu_{A\hat{+}B}(x) = r_{A\hat{+}B}(x) \cdot e^{iarg_{A\hat{+}B}(x)} = \left(r_A(x) + r_B(x) - r_A(x) \cdot r_B(x)\right) \cdot e^{i2\pi\left(\frac{arg_A(x)}{2\pi} + \frac{arg_B(x)}{2\pi} - \frac{arg_A(x)}{2\pi} \cdot \frac{arg_B(x)}{2\pi}\right)}. \quad (2.7)$$

Example 2.6 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \hat{+} B = \frac{0.84e^{i1.68\pi}}{-1} + \frac{1e^{i2\pi}}{0} + \frac{1e^{i2\pi}}{1} + \frac{0.9e^{i1.8\pi}}{2}$.

Proposition 2.8 The complex fuzzy probabilistic sum on $F^*(U)$ is an s-norm.

Proof. Properties (i), (ii), (v) and (vi) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B and C be three complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$, $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ and $\mu_C(x) = r_C(x) \cdot e^{iarg_C(x)}$ their membership functions, respectively. We suppose $|\mu_A(x)| \le |\mu_B(x)|$, $arg_A(x) \le arg_B(x)$, $\forall x \in U$. Thus

$$\begin{aligned}|\mu_{A\hat{+}C}(x)| &= |r_{A\hat{+}C}(x) \cdot e^{i\,arg_{A\hat{+}C}(x)}| = |r_A(x) + r_C(x) - r_A(x) \cdot r_C(x)| \\ &\le |r_B(x) + r_C(x) \cdot (1 - r_B(x))| = |\mu_{B\hat{+}C}(x)| \qquad \forall x \in U.\end{aligned}$$

$$arg_{A\hat{+}C}(x) = 2\pi\left(\frac{arg_A(x)}{2\pi} + \frac{arg_C(x)}{2\pi} - \frac{\arg_A(x)\cdot\arg_C(x)}{2\pi}\right)$$

$$\leq 2\pi\left(\frac{arg_B(x)}{2\pi} + \frac{arg_C(x)}{2\pi} - \frac{\arg_B(x)\cdot\arg_C(x)}{2\pi}\right) = arg_{B\hat{+}C}(x), \quad \forall x \in U.$$

(iv) Suppose A, B and C are three complex fuzzy sets on U, $\mu_A(x)=r_A(x)\cdot e^{iarg_A(x)}$, $\mu_B(x)=r_B(x)\cdot e^{iarg_B(x)}$ and $\mu_C(x)=r_C(x)\cdot e^{iarg_C(x)}$ their membership functions, respectively. We have

$$\mu_{A\hat{+}(B\hat{+}C)}(x) = \left(r_A(x)+r_{B\hat{+}C}(x)-r_A(x)r_{B\hat{+}C}(x)\right)\cdot e^{i2\pi\left(\frac{arg_A(x)+arg_{B\hat{+}C}(x)-\arg_A(x)\arg_{B\hat{+}C}(x)}{2\pi}\right)}$$

$$= \left(r_A(x)+\left(r_B(x)+r_C(x)-r_B(x)r_C(x)\right)-r_A(x)(r_B(x)+r_C(x)-r_B(x)r_C(x)\right)\cdot e^{i2\pi\left(\frac{arg_A(x)+arg_{B\hat{+}C}(x)-\mathrm{ar}}{2\pi}\right.}$$

$$= \left(\left(\left(r_A(x)+r_B(x)-r_A(x)r_B(x)\right)+r_C(x)-\left(\left(r_A(x)+r_B(x)-r_A(x)r_B(x)\right)r_C(x)\right)\right)\cdot e^{i2\pi\left(\frac{arg_A(x)+arg_{B\hat{+}C}(x)-\arg_A(x)\arg_{B\hat{+}C}(x}{2\pi}\right.}$$

$$= \left(r_{A\hat{+}B}(x)+r_C(x)-r_{A\hat{+}B}(x)r_C(x)\right)\cdot e^{i2\pi\left(\frac{arg_A(x)+arg_{B\hat{+}C}(x)-\arg_A(x)\arg_{B\hat{+}C}(x)}{2\pi}\right)}$$

$$= \ldots.$$

$$= \mu_{(A\hat{+}B)\hat{+}C}(x)$$

Corollary 2.6 Let $C_\alpha \in F^*(U)$, $\alpha \in I$ and $\mu_{C_\alpha}(x)=r_{C_\alpha}(x)\cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where I is an arbitrary index sets. Then $C_1 \hat{+} C_2 \hat{+} \cdots \hat{+}\ C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{C_1\hat{+}C_2\hat{+}\cdots\hat{+}C_\alpha}(x) = [(r_{C_1}(x)+r_{C_2}(x)+\cdots+r_{C_\alpha}(x))-\cdots+(-1)^{\alpha-1}(r_{C_1}(x)\cdot r_{C_2}(x)\cdots r_{C_\alpha}(x))]\cdot$$

$$e^{i2\pi[(\frac{arg_{C_1}(x)}{2\pi}+\frac{arg_{C_2}(x)}{2\pi}+\cdots+\frac{arg_{C_\alpha}(x)}{2\pi})-\cdots+\frac{(-1)^{\alpha-1}}{(2\pi)^\alpha}(arg_{C_1}(x)\cdot arg_{C_2}(x)\cdots arg_{C_\alpha}(x))]}.$$

Proof. Trivial.

Definition 2.9 Let A and B be two complex fuzzy sets on U, and $\mu_A(x)=r_A(x)\cdot e^{iarg_A(x)}$ and $\mu_B(x)=r_B(x)\cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy bold sum of A and B, denoted $A\dot{\cup}B$, is specified by a function

$$\mu_{A\dot{\cup}B}(x)=r_{A\dot{\cup}B}(x)\cdot e^{iarg_{A\dot{\cup}B}(x)}=\min\left(1, r_A(x)+r_B(x)\right)\cdot e^{i\min(2\pi,\, arg_A(x)+arg_B(x))}. \qquad (2.8)$$

Example 2.7 Let $A=\dfrac{0.6e^{i1.2\pi}}{-1}+\dfrac{1.0e^{i2\pi}}{0}+\dfrac{0.8e^{i1.6\pi}}{1}+\dfrac{0.5e^{i\pi}}{2}$

$$B=\frac{0.6e^{i1.2\pi}}{-1}+\frac{0.8e^{i1.6\pi}}{0}+\frac{1.0e^{i2\pi}}{1}+\frac{0.8e^{i1.6\pi}}{2},$$

then $A\dot{\cup}B=\dfrac{1e^{i2\pi}}{-1}+\dfrac{1e^{i2\pi}}{0}+\dfrac{1e^{i2\pi}}{1}+\dfrac{1e^{i2\pi}}{2}$.

Proposition 2.9 The complex fuzzy bold sum on $F^*(U)$ is an s-norm.

Proof. This proof is similar to the proof of Proposition 2.1.

Corollary 2.7 Let $C_\alpha \in F^*(U)$, $\alpha \in I$ and $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where I is an arbitrary index sets. Then $C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup}\ C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_\alpha}(x) = \min(1, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x)) \cdot e^{i \min(2\pi,\, arg_{C_1}(x) + arg_{C_2}(x) + \cdots + arg_{C_\alpha}(x))}.$$

Proof. Trivial.

Definition 2.10 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy bold intersection of A and B, denoted $A \dot{\cap} B$, is specified by a function

$$\mu_{A \dot{\cap} B}(x) = r_{A \dot{\cap} B}(x) \cdot e^{iarg_{A \dot{\cap} B}(x)} = \max\bigl(0, r_A(x) + r_B(x) - 1\bigr) \cdot e^{i \max\left(0,\, arg_A(x) + arg_B(x) - 2\pi\right)}. \tag{2.9}$$

Example 2.8 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A \dot{\cap} B = \frac{0.2e^{i0.4\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.3e^{i0.6\pi}}{2}$.

Proposition 2.10 The complex fuzzy bold intersection on $F^*(U)$ is a t-norm.

Proof. This proof is similar to the proof of Proposition 2.4.

Corollary 2.8 Let $C_\alpha \in F^*(U)$, $\alpha \in I$ and $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{iarg_{C_\alpha}(x)}$ its membership function, where I is an arbitrary index sets. Then $C_1 \dot{\cap} C_2 \dot{\cap} \cdots \dot{\cap}\ C_\alpha \in F^*(U)$, and its membership function is

$$\mu_{C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_\alpha}(x) = \max(0, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x) - \alpha + 1) \cdot e^{i \max(0,\, arg_{C_1}(x) + arg_{C_2}(x) + \cdots + arg_{C_\alpha}(x) - 2(\alpha - 1)\pi)}.$$

Definition 2.11 Let A and B be two complex fuzzy sets on U, and and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ fuzzy bounded difference of A and B, denoted $A| - |B$, is specified by a function

$$\mu_{A| - |B}(x) = r_{A| - |B}(x) \cdot e^{iarg_{A| - |B}(x)} = \max\bigl(0, r_A(x) - r_B(x)\bigr) \cdot e^{i \max\left(0,\, arg_A(x) - arg_B(x)\right)}. \tag{2.10}$$

Example 2.9 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A| - |B = \frac{0}{-1} + \frac{0.2e^{i0.4\pi}}{0} + \frac{0}{1} + \frac{0}{2}$.

Definition 2.12 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{iarg_B(x)}$ their membership functions, respectively. The complex fuzzy symmetrical difference of A and B, denoted $A\nabla B$, is specified by a function

$$\mu_{A\nabla B}(x) = r_{A\nabla B}(x) \cdot e^{i arg_{A\nabla B}(x)} = |r_A(x) - r_B(x)| \cdot e^{i|arg_A(x) - arg_B(x)|}. \tag{2.11}$$

Example 2.10 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A\nabla B = \frac{0}{-1} + \frac{0.2e^{i0.4\pi}}{0} + \frac{0.2e^{i0.4\pi}}{1} + \frac{0.3e^{i0.6\pi}}{2}$.

Definition 2.13 Let A and B be two complex fuzzy sets on U, and $\mu_A(x) = r_A(x) \cdot e^{i arg_A(x)}$ and $\mu_B(x) = r_B(x) \cdot e^{i arg_B(x)}$ their membership functions, respectively. The complex fuzzy convex linear sum of min and max of A and B, denoted $A\|_\lambda B\,(0 \le \lambda \le 1)$, is specified by a function

$$\mu_{A\|_\lambda B}(x) = r_{A\|_\lambda B}(x) \cdot e^{i arg_{A\|_\lambda B}(x)} = [\lambda \min(r_A(x),\ r_B(x)) + (1-\lambda)\max(r_A(x), r_B(x))] \cdot e^{i[\lambda \min(arg_A(x),\ arg_B(x)) + (1-\lambda)\max(arg_A(x), arg_B(x))]}. \tag{2.12}$$

Example 2.11 Let $A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$

$$B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.8e^{i1.6\pi}}{0} + \frac{1.0e^{i2\pi}}{1} + \frac{0.8e^{i1.6\pi}}{2},$$

then $A\|_\lambda B = \frac{0.6e^{i1.2\pi}}{-1} + \frac{0.9e^{i1.8\pi}}{0} + \frac{0.9e^{i1.8\pi}}{1} + \frac{0.65e^{i0.8\pi}}{2}$ when $\lambda = 0.5$.

3. δ-Equalities of Complex Fuzzy Sets

Definition 3.1 A distance of complex fuzzy sets is a function ρ: $(F^*(U), F^*(U)) \to [0, 1]$ with the properties: for any $A, B, C \in F^*(U)$

(1) $\rho(A, B) \ge 0,\ \ \rho(A, B) = 0$ if and only if $A = B$,
(2) $\rho(A, B) = \rho(B, A)$,
(3) $\rho(A, B) \le \rho(A, C) + \rho(C, B)$.

In the following, we introduce a function d, which plays a key role in the remainder of this paper. We define

$$d(A,\ B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)|\right) \tag{3.1}$$

Obviously, this function $d(.,\ .)$ is closure for any operations defined in Section 2, for example, complex fuzzy product, complex fuzzy probabilistic sum, complex fuzzy bold sum and complex fuzzy intersection, etc.

Theorem 3.1 $d(A, B)$ defined by the equality (3.1) is a distance function of complex fuzzy sets on U.

Proof. Trivial.

Example 3.1 Let

$$A = \frac{0.6e^{i1.2\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.8e^{i1.6\pi}}{1} + \frac{0.5e^{i\pi}}{2}$$
$$A' = \frac{0.7e^{i1.4\pi}}{-1} + \frac{1.0e^{i2\pi}}{0} + \frac{0.6e^{i1.2\pi}}{1} + \frac{0.4e^{i0.8\pi}}{2}$$

We see $\sup_{x\in U}|r_A(x) - r_{A'}(x)| = 0.2$ and $\frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_{A'}(x)| = 0.2.$ Therefore
$$d(A, A') = 0.2\,.$$

It is easy to see that, if *S* and *T* are two real fuzzy sets on *U*, then
$$d(A,\ B) = \sup_{x\in U}|\mu_A(x) - \mu_B(x)|.$$

Definition 3.2[5] Let *U* be a universe of discourse. Let *A* and *B* be two real fuzzy sets on *U*, and $\mu_A(x)$ and $\mu_B(x)$ their membership functions, respectively. Then *A* and *B* are said to be *δ*-equal, denoted by *A* = (*δ*) *B*, if and only if
$$\sup_{x\in U}|\mu_A(x) - \mu_B(x)| \le 1-\delta, \qquad 0 \le \delta \le 1. \tag{3.2}$$
In this way, we say *A* and *B* construct a *δ*-equality.

Lemma 3.1 Let
$$\delta_1 * \delta_2 = \max(0,\ \delta_1 + \delta_2 - 1); \qquad 0 \le \delta_1, \delta_2 \le 1. \tag{3.3}$$
Then

(1) $0 * \delta_1 = 0; \quad \forall \delta_1 \in [0,1],$
(2) $1 * \delta_1 = \delta_1; \quad \forall \delta_1 \in [0,1],$
(3) $0 \le \delta_1 * \delta_2 \le 1; \quad \forall \delta_1, \delta_2 \in [0,1],$
(4) $\delta_1 \le \delta_1' \Rightarrow \delta_1 * \delta_2 \le \delta_1' * \delta_2; \forall \delta_1, \delta_1', \delta_2 \in [0,1],$
(5) $\delta_1 * \delta_2 = \delta_2 * \delta_1; \quad \forall \delta_1, \delta_2 \in [0,1],$
(6) $(\delta_1 * \delta_2) * \delta_3 = \delta_1 * (\delta_2 * \delta_3); \quad \forall \delta_1, \delta_2, \delta_3 \in [0,1].$

Proof. Trivial.

Lemma 3.2 Let *f*, *g* be bounded, real valued functions on a set *U*. Then
$$\left|\sup_{x\in U} f(x) - \sup_{x\in U} g(x)\right| \le \sup_{x\in U}|f(x) - g(x)|, \quad \left|\inf_{x\in U} f(x) - \inf_{x\in U} g(x)\right| \le \sup_{x\in U}|f(x) - g(x)|. \tag{3.4}$$

Proof. See Ref. [Hong and Hwang 1994]

Definition 3.3 Let *A* and *B* be two complex fuzzy sets on *U*, and $\mu_A(x) = r_A(x)\cdot e^{iarg_A(x)}$ and $\mu_B(x) = r_B(x)\cdot e^{iarg_B(x)}$ their membership functions, respectively. Then *A* and *B* are said to be *δ*-equal, denoted by *A* = (*δ*)*B*, if and only if
$$d(A,\ B) \le 1-\delta; \quad 0 \le \delta \le 1. \tag{3.5}$$

Proposition 3.1 Let *A* and *B* be two complex fuzzy sets on *U*. Then

(1) *A* = (*0*)*B* ,
(2) *A* = (1)*B* ⇔ *A* = *B*,

(3) $A = (\delta)B \Leftrightarrow B = (\delta) A$,
(4) $A = (\delta_1)B$ and $\delta_2 \le \delta_1 \Rightarrow A = (\delta_2)B$,
(5) If $\forall \alpha \in I,\ A = (\delta_\alpha)B,$ where I is an index set, then $A = (\sup_{\alpha\in I} \delta_\alpha)B,$
(6) $\forall A,\ B,$ there exists a unique δ such that $A = (\delta)B$ and if $A = (\delta')B$ then $\delta' \le \delta$.

Proof. Properties (1) – (4) can be easily proved. Here we only prove properties (5) and (6). (5) Since $\forall \alpha \in I,\ A = (\delta_\alpha)B,$ we have

$$d(A,\ B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)| \right) \le 1 - \delta_\alpha, \quad \forall \alpha \in I.$$

Therefore

$$\sup_{x\in U}|r_A(x) - r_B(x)| \le 1 - \sup_{\alpha\in I}\delta_\alpha$$

and

$$\frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)| \le 1 - \sup_{\alpha\in I}\delta_\alpha.$$

So

$$d(A,\ B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)| \right) \le 1 - \sup_{\alpha\in I}\delta_\alpha.$$

Thus $A = (\sup_{\alpha\in I}\delta_\alpha)B.$

(6) Let $\delta = 1 - d(A, B)$. Then $A = (\delta)B$. If $A = (\delta')B$, we have $1 - \delta = d(A, B) \le 1 - \delta'$. Consequently $\delta' \le \delta$. Now suppose there exist two constants δ_1 and δ_2 which simultaneously satisfy the required properties, then $\delta_1 \le \delta_2$ and $\delta_2 \le \delta_1$. This implies $\delta_1 = \delta_2$. So the desired δ is unique.

Proposition 3.2 If $A = (\delta_1)B$ and $B = (\delta_2)C$, then $A = (\delta)C$, where $\delta = \delta_1 * \delta_2$.

Proof. Since $A = (\delta_1)B$ and $B = (\delta_2)C$, we have

$$d(A,\ B) = \max\left(\sup_{x\in U}|r_A(x) - r_B(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)| \right) \le 1 - \delta_1,$$

and

$$d(B,\ C) = \max\left(\sup_{x\in U}|r_B(x) - r_C(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_B(x) - arg_C(x)| \right) \le 1 - \delta_2.$$

Therefore

$$\sup_{x\in U}|r_A(x) - r_B(x)| \le 1 - \delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_B(x)| \le 1 - \delta_1,$$

and

$$\sup_{x\in U}|r_B(x) - r_C(x)| \le 1 - \delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|\arg_B(x) - \arg_C(x)| \le 1 - \delta_2.$$

Consequently, we have

$$d(A,\ C) = \max\left(\sup_{x\in U}|r_A(x) - r_C(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_C(x)| \right)$$
$$\le \max\left(\sup_{x\in U}|r_A(x) - r_B(x)| + \sup_{x\in U}|r_B(x) - r_C(x)|, \right.$$

$$\left.\frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_B(x)\right|+\frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_C(x)\right|\right)$$
$$\le \max\left(\left(1-\delta_1\right)+\left(1-\delta_2\right),\ \left(1-\delta_1\right)+\left(1-\delta_2\right)\right)$$
$$=\left(1-\delta_1\right)+\left(1-\delta_2\right)=1-\left(\delta_1+\delta_2-1\right),$$

and $d(A,C)\le 1$ from Definition 3.1. Therefore

$$d(A,C)\le 1-\delta_1*\delta_2=1-\delta.$$

That is $A = (\delta)C$.

Theorem 3.2 If $A = (\delta_1)A'$ and $B = (\delta_2)B'$, then $A\cup B=\left(\min(\delta_1,\ \delta_2)\right)A'\cup B'$.

Proof. Since $A = (\delta_1)A'$ and $B = (\delta_2)B'$, we have

$$d(A,\ A')=\max\left(\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|Arg_A(x)-Arg_{A'}(x)\right|\right)\le 1-\delta_1,$$

and

$$d(B,\ B')=\max\left(\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|Arg_B(x)-Arg_{B'}(x)\right|\right)\le 1-\delta_2.$$

Therefore

$$\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|\le 1-\delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|Arg_A(x)-Arg_{A'}(x)\right|\le 1-\delta_1,$$

and

$$\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|\le 1-\delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|Arg_B(x)-Arg_{B'}(x)\right|\le 1-\delta_2.$$

Consequently, we have

$$\sup_{x\in U}\left|r_{A\cup B}(x)-r_{A'\cup B'}(x)\right|=\sup_{x\in U}\left|\max\left(r_A(x),\ r_B(x)\right)-\max\left(r_{A'}(x),\ r_{B'}(x)\right)\right|$$
$$=\begin{cases}\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right| & \text{if } r_A(x)\ge r_B(x) \text{ and } r_{A'}(x)\ge r_{B'}(x)\\ \sup_{x\in U}\left|r_A(x)-r_{B'}(x)\right| & \text{if } r_A(x)\ge r_B(x) \text{ and } r_{B'}(x)> r_{A'}(x)\\ \sup_{x\in U}\left|r_B(x)-r_{A'}(x)\right| & \text{if } r_B(x)> r_A(x) \text{ and } r_{A'}(x)\ge r_{B'}(x)\\ \sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right| & \text{if } r_B(x)> r_A(x) \text{ and } r_{B'}(x)> r_{A'}(x)\end{cases}$$

$$\le\begin{cases}1-\delta_1 & \text{if } r_A(x)\ge r_B(x) \text{ and } r_{A'}(x)\ge r_{B'}(x)\\ \sup_{x\in U}\left|r_A(x)-r_{B'}(x)\right| & \text{if } r_A(x)\ge r_B(x) \text{ and } r_{B'}(x)> r_{A'}(x)\\ \sup_{x\in U}\left|r_B(x)-r_{A'}(x)\right| & \text{if } r_B(x)> r_A(x) \text{ and } r_{A'}(x)\ge r_{B'}(x)\\ 1-\delta_2 & \text{if } r_B(x)> r_A(x) \text{ and } r_{B'}(x)> r_{A'}(x)\end{cases}.$$

1. Consider the case $r_A(x)\ge r_B(x)$ and $r_{B'}(x)>r_{A'}(x)$.

(1) If $r_A(x)-r_{B'}(x)\ge 0$, then $r_A(x)-r_{A'}(x)\ge r_A(x)-r_{B'}(x)\ge 0$ from $r_{B'}(x)>r_{A'}(x)$. Therefore

$$\sup_{x\in U}\left|r_A(x)-r_{B'}(x)\right|=\sup_{x\in U}\left(r_A(x)-r_{B'}(x)\right)\le\sup_{x\in U}\left(r_A(x)-r_{A'}(x)\right)$$
$$\le\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|\le 1-\delta_1.$$

(2) If $r_A(x)-r_{B'}(x)\le 0$, then $r_{B'}(x)-r_B(x)\ge r_{B'}(x)-r_A(x)\ge 0$ from $r_B(x)\le r_A(x)$. Therefore

$$\sup_{x\in U}\left|r_A(x)-r_{B'}(x)\right| = \sup_{x\in U}\left(r_{B'}(x)-r_A(x)\right) \le \sup_{x\in U}\left(r_{B'}(x)-r_B(x)\right)$$
$$\le \sup_{x\in U}\left|r_{B'}(x)-r_B(x)\right| \le 1-\delta_2.$$

Thus, if $r_A(x) \ge r_B(x)$ and $r_{B'}(x) > r_{A'}(x)$, we have

$$\sup_{x\in U}\left|r_A(x)-r_{B'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right).$$

2. Similarly, We can prove

$$\sup_{x\in U}\left|r_B(x)-r_{A'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right),$$

if $r_B(x) > r_A(x)$ and $r_{A'}(x) \ge r_{B'}(x)$. So, we have

$$\sup_{x\in U}\left|r_{A\cup B}(x)-r_{A'\cup B'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right).$$

The other hand, we have also

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\cup B}(x)-arg_{A'\cup B'}(x)\right|$$
$$= \frac{1}{2\pi}\sup_{x\in U}\left|\max\left(arg_A(x),\ arg_B(x)\right)-\max\left(arg_{A'}(x),\ arg_{B'}(x)\right)\right|$$
$$= \begin{cases} \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{A'}(x)\right| & \text{if } arg_A(x) \ge arg_B(x) \text{ and } arg_{A'}(x) \ge arg_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{B'}(x)\right| & \text{if } arg_A(x) \ge arg_B(x) \text{ and } arg_{B'}(x) > arg_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_{A'}(x)\right| & \text{if } arg_B(x) > arg_A(x) \text{ and } arg_{A'}(x) \ge arg_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_{B'}(x)\right| & \text{if } arg_B(x) > arg_A(x) \text{ and } arg_{B'}(x) > arg_{A'}(x) \end{cases}$$
$$\le \begin{cases} 1-\delta_1 & \text{if } arg_A(x) \ge arg_B(x) \text{ and } arg_{A'}(x) \ge arg_{B'}(x) \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{B'}(x)\right| & \text{if } arg_A(x) \ge arg_B(x) \text{ and } arg_{B'}(x) > arg_{A'}(x) \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_{A'}(x)\right| & \text{if } arg_B(x) > arg_A(x) \text{ and } arg_{A'}(x) \ge arg_{B'}(x) \\ 1-\delta_2 & \text{if } arg_B(x) > arg_A(x) \text{ and } arg_{B'}(x) > arg_{A'}(x) \end{cases}.$$

1. Consider the case $arg_A(x) \ge arg_B(x)$ and $arg_{B'}(x) > arg_{A'}(x)$.

(1) If $arg_A(x)-arg_{B'}(x) \ge 0$, then $arg_A(x)-arg_{A'}(x) \ge arg_A(x)-arg_{B'}(x) \ge 0$ from $arg_{B'}(x) > arg_{A'}(x)$. Therefore

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in U}\left(arg_A(x)-arg_{B'}(x)\right)$$
$$\le \frac{1}{2\pi}\sup_{x\in U}\left(arg_A(x)-arg_{A'}(x)\right)$$
$$\le \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{A'}(x)\right|$$
$$\le 1-\delta_1.$$

(2) If $arg_A(x)-arg_{B'}(x) \le 0$, then $arg_{B'}(x)-arg_B(x) \ge arg_{B'}(x)-arg_A(x) \ge 0$ from $arg_B(x) \le Arg_A(x)$. Therefore

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{B'}(x)\right| = \frac{1}{2\pi}\sup_{x\in U}\left(arg_{B'}(x)-arg_A(x)\right)$$

$$\le \frac{1}{2\pi}\sup_{x\in U}\left(arg_{B'}(x) - arg_B(x)\right)$$
$$\le \frac{1}{2\pi}\sup_{x\in U}\left|arg_{B'}(x) - arg_B(x)\right|$$
$$\le 1-\delta_2.$$

Thus, if $arg_A(x) \ge arg_B(x)$ and $arg_{B'}(x) > arg_{A'}(x)$, we have

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{B'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right).$$

2. Similarly, We can prove

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{A'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right),$$

if $arg_B(x) > arg_A(x)$ and $arg_{A'}(x) \ge arg_{B'}(x)$. So, we have

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\cup B}(x) - arg_{A'\cup B'}(x)\right| \le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right).$$

Hence

$$d(A\cup B, A'\cup B') = \max\left(\sup_{x\in U}\left|r_{A\cup B}(x) - r_{A'\cup B'}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\cup B}(x) - arg_{A'\cup B'}(x)\right|\right)$$
$$\le \max\left(1-\delta_1, 1-\delta_2\right) = 1-\min\left(\delta_1, \delta_2\right).$$

That is $A\cup B = \left(\min(\delta_1, \delta_2)\right)A'\cup B'$.

Corollary 3.1 If $A_\alpha = (\delta_\alpha)B_\alpha,\ \alpha\in I$, where I is an index set, then $\cup_{\alpha\in I} A_\alpha = \left(\inf_{\alpha\in I}\delta_\alpha\right)\cup_{\alpha\in I} B_\alpha$.

Proof. This is because

$$d(\underset{\alpha\in I}{\cup} A_\alpha, \underset{\alpha\in I}{\cup} B_\alpha) = \max\left(\sup_{x\in U}\left|r_{\underset{\alpha\in I}{\cup} A_\alpha}(x) - r_{\underset{\alpha\in I}{\cup} B_\alpha}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|arg_{\underset{\alpha\in I}{\cup} A_\alpha}(x) - arg_{\underset{\alpha\in I}{\cup} B_\alpha}(x)\right|\right)$$
$$= \max\left(\sup_{x\in U}\left|\sup_{\alpha\in I} r_{A_\alpha}(x) - \sup_{\alpha\in I} r_{B_\alpha}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|\sup_{\alpha\in I} arg_{A_\alpha}(x) - \sup_{\alpha\in I} arg_{B_\alpha}(x)\right|\right)$$
$$\le \max\left(\sup_{x\in U}\sup_{\alpha\in I}\left|r_{A_\alpha}(x) - r_{B_\alpha}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\sup_{\alpha\in I}\left|arg_{A_\alpha}(x) - arg_{B_\alpha}(x)\right|\right)$$
$$= \max\left(\sup_{\alpha\in I}\sup_{x\in U}\left|r_{A_\alpha}(x) - r_{B_\alpha}(x)\right|, \sup_{\alpha\in I}\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A_\alpha}(x) - arg_{B_\alpha}(x)\right|\right)$$
$$= \max\left(\sup_{\alpha\in I}\left(1-\delta_\alpha\right), \sup_{\alpha\in I}\left(1-\delta_\alpha\right)\right) = \sup_{\alpha\in I}\left(1-\delta_\alpha\right) = 1-\inf_{\alpha\in I}\delta_\alpha,$$

from Lemma 3.2.

Theorem 3.3 If $A = (\delta)B$, then $\overline{A} = (\delta)\overline{B}$.

Proof. This is because of

$$d(\overline{A}, \overline{B}) = \max\left(\sup_{x\in U}\left|r_{\overline{A}}(x) - r_{\overline{B}}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_{\overline{A}}(x) - arg_{\overline{B}}(x)\right|\right)$$

$$= \max\left(\sup_{x\in U}\left|(1-r_A(x))-(1-r_B(x))\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|(2\pi - arg_A(x))-(2\pi - arg_B(x))\right| \right)$$
$$= \max\left(\sup_{x\in U}\left|r_A(x)-r_B(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_B(x)\right| \right)$$
$$= d(A, B) \le 1-\delta.$$

Theorem 3.4 If $A = (\delta_1)A'$ and $B = (\delta_2)B'$, then $A\cap B = (\min(\delta_1,\ \delta_2))A'\cap B'$.

Proof. By use of Theorem 3.2 and 3.3, we have $\overline{A} = (\delta_1)\overline{A'}$, $\overline{B} = (\delta_2)\overline{B'}$ and $\overline{A}\cup\overline{B} =$ $(\min(\delta_1,\delta_2))\overline{A'}\cup\overline{B'}$. Thus

$$A\cap B = \overline{\overline{A}\cup\overline{B}} = (\min(\delta_1,\ \delta_2))\overline{\overline{A'}\cup\overline{B'}} = (\min(\delta_1,\ \delta_2))A'\cap B',$$

from Proposition 2.2 and 2.3.

Corollary 3.2 If $A_\alpha = (\delta_\alpha)B_\alpha,\ \alpha\in I$, where I is an index set, then $\cap_{\alpha\in I} A_\alpha = (\inf_{\alpha\in I}\delta_\alpha)\cap_{\alpha\in I} B_\alpha$.

Proof. By using Corollary 3.1 and Theorem 3.3, we have $\overline{A}_\alpha = (\delta_\alpha)\overline{B}_\alpha,\ \forall\alpha\in I$ and $\cup_{\alpha\in I}\overline{A}_\alpha = (\inf_{\alpha\in I}\delta_\alpha)\cup_{\alpha\in I}\overline{B}_\alpha$ Thus

$$\bigcap_{\alpha\in I} A_\alpha = \overline{\bigcup_{\alpha\in I}\overline{A}_\alpha} = (\inf_{\alpha\in I}\delta_\alpha)\overline{\bigcup_{\alpha\in I}\overline{B}_\alpha} = (\inf_{\alpha\in I}\delta_\alpha)\bigcap_{\alpha\in I} B_\alpha.$$

Corollary 3.3 If $A_{\alpha\beta} = (\delta_{\alpha\beta})B_{\alpha\beta},\ \alpha\in I_1,\ \beta\in I_2$ where I_1 and I_2 are index sets, then

$$\bigcup_{\alpha\in I_1}\bigcap_{\beta\in I_2} A_{\alpha\beta} = \left(\inf_{\alpha\in I_1}\inf_{\beta\in I_2}\delta_{\alpha\beta}\right)\bigcup_{\alpha\in I_1}\bigcap_{\beta\in I_2} B_{\alpha\beta},$$

and

$$\bigcap_{\alpha\in I_1}\bigcup_{\beta\in I_2} A_{\alpha\beta} = \left(\inf_{\alpha\in I_1}\inf_{\beta\in I_2}\delta_{\alpha\beta}\right)\bigcap_{\alpha\in I_1}\bigcup_{\beta\in I_2} B_{\alpha\beta}.$$

Proof. This is due to Corollary 3.1 and 3.2.

Corollary 3.4 Let

$$\lim_{n\to\infty}\sup A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k,\quad \lim_{n\to\infty}\sup B_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}B_k,$$
$$\lim_{n\to\infty}\inf A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k,\quad \lim_{n\to\infty}\inf B_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}B_k.$$

If $A_k = (\delta_k)B_k,\ k = 1, 2, \cdots,$ then

$$\lim_{n\to\infty}\sup A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\sup B_n,$$
$$\lim_{n\to\infty}\inf A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\inf B_n.$$

Proof. From Corollary 3.3, we have

$$\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \left(\inf_{n\ge 1}\inf_{k\ge n_2}\delta_{\alpha\beta}\right)\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} B_k,\quad \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \left(\inf_{n\ge 1}\inf_{k\ge n_2}\delta_{\alpha\beta}\right)\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k.$$

This implies that

$$\lim_{n\to\infty}\sup A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\sup B_n, \quad \lim_{n\to\infty}\inf A_n = \left(\inf_{n\ge 1}\delta_n\right)\lim_{n\to\infty}\inf B_n.$$

Theorem 3.5 If $A = (\delta_1)A'$ and $B = (\delta_2)B'$, then $A\circ B = (\delta_1 * \delta_2)A'\circ B'$.

Proof. Since $A = (\delta_1)A'$ and $B = (\delta_2)B'$, we have

$$d(A,\ A') = \max\left(\sup_{x\in U}|r_A(x) - r_{A'}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_{A'}(x)|\right) \le 1-\delta_1,$$

and

$$d(B,\ B') = \max\left(\sup_{x\in U}|r_B(x) - r_{B'}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_B(x) - arg_{B'}(x)|\right) \le 1-\delta_2.$$

Therefore

$$\sup_{x\in U}|r_A(x) - r_{A'}(x)| \le 1-\delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|arg_A(x) - arg_{A'}(x)| \le 1-\delta_1,$$

and

$$\sup_{x\in U}|r_B(x) - r_{B'}(x)| \le 1-\delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|arg_B(x) - arg_{B'}(x)| \le 1-\delta_2.$$

Consequently, we have

$$d(A\circ B,\ A'\circ B') = \max\left(\sup_{x\in U}|r_{A\circ B}(x) - r_{A'\circ B'}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_{A\circ B}(x) - arg_{A'\circ B'}(x)|\right)$$

$$= \max\left(\sup_{x\in U}|r_A(x)r_B(x) - r_{A'}(x)r_{B'}(x)|,\right.$$

$$\left.\frac{1}{2\pi}\sup_{x\in U}\left|2\pi\left(\frac{arg_A(x)}{2\pi}\cdot\frac{arg_B(x)}{2\pi}\right) - 2\pi\left(\frac{arg_{A'}(x)}{2\pi}\cdot\frac{arg_{B'}(x)}{2\pi}\right)\right|\right)$$

$$= \max\left(\sup_{x\in U}|r_A(x)r_B(x) - r_A(x)r_{B'}(x) + r_A(x)r_{B'}(x) - r_{A'}(x)r_{B'}(x)|,\right.$$

$$\frac{1}{2\pi}\sup_{x\in U}\left|\frac{arg_A(x)arg_B(x)}{2\pi} - \frac{arg_A(x)arg_{B'}(x)}{2\pi} + \frac{arg_A(x)arg_{B'}(x)}{2\pi} - \frac{arg_{A'}(x)arg_{B'}(x)}{2\pi}\right|$$

$$= \max\left(\sup_{x\in U}|r_A(x)(r_B(x) - r_{B'}(x)) + (r_A(x) - r_{A'}(x))r_{B'}(x)|,\right.$$

$$\frac{1}{2\pi}\sup_{x\in U}\left|\frac{arg_A}{2\pi}(x)(arg_B(x) - arg_{B'}(x)) + (arg_A(x) - arg_{A'}(x))\frac{arg_{B'}(x)}{2\pi}\right|$$

$$\le \max\left(\sup_{x\in U}|r_B(x) - r_{B'}(x)| + \sup_{x\in U}|r_A(x) - r_{A'}(x)|,\right.$$

$$\left.\frac{1}{2\pi}\left(\sup_{x\in U}|arg_B(x) - arg_{B'}(x)| + \sup_{x\in U}|arg_A(x) - arg_{A'}(x)|\right)\right)$$

$$\le \max\left((1-\delta_2) + (1-\delta_1),\ (1-\delta_2) + (1-\delta_1)\right) = 1 - (\delta_1 + \delta_2 - 1).$$

Further we note that $d(A\circ B,\ A'\circ B') \le 1$. So

$$d(A\circ B,\ A'\circ B') \le 1 - \delta_1 * \delta_2.$$

Corollary 3.5 If $A_\alpha = (\delta_\alpha)B_\alpha,\ \alpha\in I$, where I is an index set, then $A_1\circ A_2\circ\cdots\circ A_\alpha = (\delta_1 * \delta_2 * \cdots * \delta_\alpha)B_1\circ B_2\circ\cdots\circ B_\alpha$.

Proof. Trivial from Theorem 3.5.

Theorem 3.6 If $A_n = (\delta_n)A_n^{'}, n = 1, 2, \cdots, N,$ then $A_1 \times A_2 \times \cdots \times A_N = \left(\inf_{1\le n\le N} \delta_n\right)A_1^{'} \times \times A_2^{'} \times \cdots \times A_N^{'}$.

Proof. Since $A_n = (\delta_n)A_n^{'}, n = 1, 2, \cdots, N,$ we have

$$d(A_n,\ A_n^{'}) = \max\left(\sup_{x\in U}\left|r_{A_n}(x_n) - r_{A_n^{'}}(x_n)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A_n}(x_n) - arg_{A_n^{'}}(x_n)\right|\right) \le 1-\delta_n,$$

for any n = 1, 2, …, N. Therefore

$$\sup_{x\in U}\left|r_{A_n}(x_n) - r_{A_n^{'}}(x_n)\right| \le 1-\delta_n \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A_n}(x_n) - arg_{A_n^{'}}(x_n)\right| \le 1-\delta_n.$$

Consequently, from Lemma 3.2, we have

$$d(A_1 \times \cdots \times A_N,\ A_1^{'} \times \cdots \times A_N^{'})$$
$$= \max\left(\sup_{x\in U\times\cdots XU}\left|r_{A_1\times\cdots\times A_N}(x) - r_{A_1^{'}\times\cdots\times A_N^{'}}(x)\right|, \frac{1}{2\pi}\sup_{x\in U\times\cdots XU}\left|arg_{A_1\times\cdots\times A_N}(x) - arg_{A_1^{'}\times\cdots\times A_N^{'}}(x)\right|\right)$$
$$= \max\left(\sup_{x\in U\times\cdots XU}\left|\min_{1\le n\le N} r_{A_n}(x_n) - \min_{1\le n\le N} r_{A_n^{'}}(x_n)\right|, \frac{1}{2\pi}\sup_{x\in U\times\cdots XU}\left|\min_{1\le n\le N} arg_{A_n}(x_n) - \min_{1\le n\le N} arg_{A_n^{'}}(x_n)\right|\right)$$
$$\le \max\left(\sup_{1\le n\le N}\sup_{x_n\in U_n}\left|r_{A_n}(x_n) - r_{A_n^{'}}(x_n)\right|, \frac{1}{2\pi}\sup_{1\le n\le N}\sup_{x_n\in U_n}\left|arg_{A_n}(x_n) - arg_{A_n^{'}}(x_n)\right|\right)$$
$$\le \max\left(\sup_{1\le n\le N}(1-\delta_n), \sup_{1\le n\le N}(1-\delta_n)\right) = 1 - \inf_{1\le n\le N}\delta_n.$$

Theorem 3.7 If A= (δ_1)A' and B = (δ_2)B', then $A \hat{+} B = \left(\delta_1 * \delta_2\right)A'\hat{+}B'$.

Proof. Since A = (δ_1)A' and B = (δ_2)B', we have

$$d(A,\ A') = \max\left(\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right|\right) \le 1-\delta_1,$$

and

$$d(B,\ B') = \max\left(\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right|\right) \le 1-\delta_2.$$

Therefore

$$\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right| \le 1-\delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right| \le 1-\delta_1,$$

and

$$\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right| \le 1-\delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right| \le 1-\delta_2.$$

Consequently, we have

$$d(A\hat{+}B,\ A'\hat{+}B') = \max\left(\sup_{x\in U}\left|r_{A\hat{+}B}(x) - r_{A'\hat{+}B'}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\hat{+}B}(x) - arg_{A'\hat{+}B'}(x)\right|\right)$$
$$= \max\left(\sup_{x\in U}\left|(r_A(x) + r_B(x) - r_A(x)\cdot r_B(x)) - (r_{A'}(x) + r_{B'}(x) - r_{A'}(x) r_{B'}(x))\right|,\right.$$
$$\frac{1}{2\pi}\sup_{x\in U}\left|2\pi(\frac{arg_A(x)}{2\pi} + \frac{arg_B(x)}{2\pi} - \frac{arg_A(x)}{2\pi}\frac{arg_B(x)}{2\pi})\right.$$

$$-2\pi\left(\frac{arg_{A'}(x)}{2\pi}+\frac{arg_{B'}(x)}{2\pi}-\frac{arg_{A'}(x)}{2\pi}\frac{arg_{B'}(x)}{2\pi}\right)\Bigg|\Bigg)$$

$$=\max\Bigg(\sup_{x\in U}\left|(1-r_B(x))(r_A(x)-r_{A'}(x))+(1-r_{A'}(x))(r_B(x)-r_{B'}(x))\right|,$$

$$\sup_{x\in U}\left|(1-\frac{arg_B(x)}{2\pi})(\frac{arg_A(x)}{2\pi}-\frac{arg_{A'}(x)}{2\pi})+(1-\frac{arg_{A'}(x)}{2\pi})(\frac{arg_B(x)}{2\pi}-\frac{arg_{B'}(x)}{2\pi})\right|\Bigg)$$

$$\le\max\Bigg(\sup_{x\in U}|1-r_B(x)||r_A(x)-r_{A'}(x)|+\sup_{x\in U}|1-r_{A'}(x)||r_B(x)-r_{B'}(x)|,$$

$$\sup_{x\in U}\left|1-\frac{arg_B(x)}{2\pi}\right|\left|\frac{arg_A(x)}{2\pi}-\frac{arg_{A'}(x)}{2\pi}\right|+\sup_{x\in U}\left|1-\frac{arg_{A'}(x)}{2\pi}\right|\left|\frac{arg_B(x)}{2\pi}-\frac{arg_{B'}(x)}{2\pi}\right|\Bigg)$$

$$\le\max\Bigg(\sup_{x\in U}|r_A(x)-r_{A'}(x)|+\sup_{x\in U}|r_B(x)-r_{B'}(x)|,$$

$$\frac{1}{2\pi}\left(\sup_{x\in U}|arg_A(x)-arg_{A'}(x)|+\sup_{x\in U}|arg_B(x)-arg_{B'}(x)|\right)\Bigg)$$

$$\le\max\left((1-\delta_2)+(1-\delta_1),\,(1-\delta_2)+(1-\delta_1)\right)=1-(\delta_1+\delta_2)-1.$$

Further we note that $d(A\hat{+}B,\ A'\hat{+}B')\le 1$. So

$$d(A\hat{+}B,\ A'\hat{+}B')\le 1-\delta_1*\delta_2.$$

Corollary 3.6 **If** $A_\alpha=(\delta_\alpha)B_\alpha,\ \alpha\in I$, where I is an index set, then $A_1\hat{+}A_2\hat{+}\cdots\hat{+}A_\alpha=(\delta_1*\delta_2*\cdots*\delta_\alpha)B_1\hat{+}B_2\hat{+}\cdots\hat{+}B_\alpha$.

Proof. Trivial from Theorem 3.7.

Theorem 3.8 If $A=(\delta_1)A'$ and $B=(\delta_2)B'$, then $A\dot{\cup}B=(\delta_1*\delta_2)A'\dot{\cup}B'$.

Proof. Since $A=(\delta_1)A'$ and $B=(\delta_2)B'$, we have

$$d(A,\ A')=\max\left(\sup_{x\in U}|r_A(x)-r_{A'}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_A(x)-arg_{A'}(x)|\right)\le 1-\delta_1,$$

and

$$d(B,\ B')=\max\left(\sup_{x\in U}|r_B(x)-r_{B'}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_B(x)-arg_{B'}(x)|\right)\le 1-\delta_2.$$

Therefore

$$\sup_{x\in U}|r_A(x)-r_{A'}(x)|\le 1-\delta_1\quad\text{and}\quad\frac{1}{2\pi}\sup_{x\in U}|arg_A(x)-arg_{A'}(x)|\le 1-\delta_1,$$

and

$$\sup_{x\in U}|r_B(x)-r_{B'}(x)|\le 1-\delta_2\quad\text{and}\quad\frac{1}{2\pi}\sup_{x\in U}|arg_B(x)-arg_{B'}(x)|\le 1-\delta_2.$$

Consequently, we have

$$\sup_{x\in U}|r_{A\dot{\cup}B}(x)-r_{A'\dot{\cup}B'}(x)|=\sup_{x\in U}\left|\min(1,\ r_A(x)+r_B(x))-\min(1,\ r_{A'}(x)+r_{B'}(x))\right|$$

$$
= \begin{cases} 0 & \text{if } r_A(x)+r_B(x)\ge 1 \text{ and } r_{A'}(x)+r_{B'}(x)\ge 1 \\ \sup_{x\in U}\left(1-r_{A'}(x)-r_{B'}(x)\right) & \text{if } r_A(x)+r_B(x)\ge 1 \text{ and } r_{A'}(x)+r_{B'}(x)< 1 \\ \sup_{x\in U}\left(1-r_{A}(x)-r_{B}(x)\right) & \text{if } r_A(x)+r_B(x)< 1 \text{ and } r_{A'}(x)+r_{B'}(x)\ge 1 \\ \sup_{x\in U}\left|r_A(x)+r_B(x)-r_{A'}(x)-r_{B'}(x)\right| & \text{if } r_A(x)+r_B(x)< 1 \text{ and } r_{A'}(x)+r_{B'}(x)< 1 \end{cases}
$$

$$
\le \begin{cases} 0 & \text{if } r_A(x)+r_B(x)\ge 1 \text{ and } r_{A'}(x)+r_{B'}(x)\ge 1 \\ \sup_{x\in U}\left(r_A(x)-r_{A'}(x)+r_B(x)-r_{B'}(x)\right) & \text{if } r_A(x)+r_B(x)\ge 1 \text{ and } r_{A'}(x)+r_{B'}(x)< 1 \\ \sup_{x\in U}\left(r_{A'}(x)-r_{A}(x)+r_{B'}(x)-r_{B}(x)\right) & \text{if } r_A(x)+r_B(x)< 1 \text{ and } r_{A'}(x)+r_{B'}(x)\ge 1 \\ \sup_{x\in U}\left|r_A(x)+r_B(x)-r_{A'}(x)-r_{B'}(x)\right| & \text{if } r_A(x)+r_B(x)< 1 \text{ and } r_{A'}(x)+r_{B'}(x)< 1 \end{cases}
$$

$$
\le \sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right| + \sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right| \le (1-\delta_1)+(1-\delta_1) = 1-(\delta_1+\delta_2-1).
$$

The other hand, we have also

$$
\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\dot\cup B}(x)-arg_{A'\dot\cup B'}(x)\right|
$$

$$
= \frac{1}{2\pi}\sup_{x\in U}\left|\min\left(2\pi, arg_A(x)+arg_B(x)\right)-\min\left(2\pi, arg_{A'}(x)+arg_{B'}(x)\right)\right|
$$

$$
= \begin{cases} 0 & \text{if } arg_A(x)+arg_B(x)\ge 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\ge 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(2\pi-arg_{A'}(x)-arg_{B'}(x)\right) & \text{if } arg_A(x)+arg_B(x)\ge 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)< 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(2\pi-arg_{A}(x)-arg_{B}(x)\right) & \text{if } arg_A(x)+arg_B(x)< 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\ge 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right| & \text{if } arg_A(x)+arg_A(x)< 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)< 2\pi \end{cases}
$$

$$
\le \begin{cases} 0 & \text{if } arg_A(x)+arg_B(x)\ge 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\ge 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right) & \text{if } arg_A(x)+arg_B(x)\ge 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)< 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_{A'}(x)+arg_{B'}(x)-arg_{A}(x)-arg_{B}(x)\right) & \text{if } arg_A(x)+arg_B(x)< 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\ge 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right| & \text{if } arg_A(x)+arg_B(x)< 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)< 2\pi \end{cases}
$$

$$
\le \frac{1}{2\pi}\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right| + \frac{1}{2\pi}\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right| \le (1-\delta_1)+(1-\delta_1) = 1-(\delta_1+\delta_2-1).
$$

Therefore

$$
d(A\dot\cup B, A'\dot\cup B') = \max\left(\sup_{x\in U}\left|r_{A\dot\cup B}(x)-r_{A'\dot\cup B'}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\dot\cup B}(x)-arg_{A'\dot\cup B'}(x)\right|\right)
$$

$$
\le 1-(\delta_1+\delta_2-1).
$$

Further we note that $d(A\dot\cup B, A'\dot\cup B')\le 1$. So

$$
d(A\dot\cup B, A'\dot\cup B')\le 1-\delta_1 * \delta_2 .
$$

Theorem 3.9 If $A = (\delta_1)A'$ and $B = (\delta_2)B'$, then $A \dot{\cap} B = (\delta_1 * \delta_2)A' \dot{\cap} B'$.

Proof. Since $A = (\delta_1)A'$ and $B = (\delta_2)B'$, we have

$$d(A,\ A') = \max\left(\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right| \right) \le 1 - \delta_1,$$

and

$$d(B,\ B') = \max\left(\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right| \right) \le 1 - \delta_2.$$

Therefore

$$\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right| \le 1 - \delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right| \le 1 - \delta_1,$$

and

$$\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right| \le 1 - \delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right| \le 1 - \delta_2.$$

Consequently, we have

$$\sup_{x\in U}\left|r_{A\dot{\cap}B}(x) - r_{A'\dot{\cap}B'}(x)\right| = \sup_{x\in U}\left|\max(0,\ r_A(x) + r_B(x) - 1) - \max(0,\ r_{A'}(x) + r_{B'}(x) - 1)\right|$$

$$= \begin{cases} 0 & \text{if } r_A(x) + r_B(x) \le 1 \text{ and } r_{A'}(x) + r_{B'}(x) \le 1 \\ \sup_{x\in U}\left(r_{A'}(x) + r_{B'}(x) - 1\right) & \text{if } r_A(x) + r_B(x) \le 1 \text{ and } r_{A'}(x) + r_{B'}(x) > 1 \\ \sup_{x\in U}\left(r_A(x) + r_B(x) - 1\right) & \text{if } r_A(x) + r_B(x) > 1 \text{ and } r_{A'}(x) + r_{B'}(x) \le 1 \\ \sup_{x\in U}\left|r_A(x) + r_B(x) - r_{A'}(x) - r_{B'}(x)\right| & \text{if } r_A(x) + r_B(x) < 1 \text{ and } r_{A'}(x) + r_{B'}(x) < 1 \end{cases}$$

$$\le \begin{cases} 0 & \text{if } r_A(x) + r_B(x) \le 1 \text{ and } r_{A'}(x) + r_{B'}(x) \le 1 \\ \sup_{x\in U}\left(r_{A'}(x) + r_{B'}(x) - r_A(x) - r_B(x)\right) & \text{if } r_A(x) + r_B(x) \le 1 \text{ and } r_{A'}(x) + r_{B'}(x) > 1 \\ \sup_{x\in U}\left(r_A(x) + r_B(x) - r_{A'}(x) - r_{B'}(x)\right) & \text{if } r_A(x) + r_B(x) > 1 \text{ and } r_{A'}(x) + r_{B'}(x) \le 1 \\ \sup_{x\in U}\left|r_A(x) + r_B(x) - r_{A'}(x) - r_{B'}(x)\right| & \text{if } r_A(x) + r_B(x) > 1 \text{ and } r_{A'}(x) + r_{B'}(x) > 1 \end{cases}$$

$$\le \sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right| + \sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right| \le (1 - \delta_1) + (1 - \delta_1) = 1 - (\delta_1 + \delta_2 - 1).$$

The other hand, we have also

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\dot{\cap}B}(x) - arg_{A'\dot{\cap}B'}(x)\right|$$

$$= \frac{1}{2\pi}\sup_{x\in U}\left|\max(0,\ arg_A(x) + arg_B(x) - 2\pi) - \max(0,\ arg_{A'}(x) + arg_{B'}(x) - 2\pi)\right|$$

$$= \begin{cases} 0 & \text{if } arg_A(x) + arg_B(x) \le 2\pi \\ & \quad arg_{A'}(x) + arg_{B'}(x) \le 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_{A'}(x) + arg_{B'}(x) - 2\pi\right) & \text{if } arg_A(x) + arg_B(x) \le 2\pi \\ & \quad arg_{A'}(x) + arg_{B'}(x) > 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_A(x) + arg_B(x) - 2\pi\right) & \text{if } arg_A(x) + arg_B(x) > 2\pi \\ & \quad arg_{A'}(x) + arg_{B'}(x) \le 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) + arg_B(x) - arg_{A'}(x) - arg_{B'}(x)\right| & \text{if } arg_A(x) + arg_B(x) < 2\pi \\ & \quad arg_{A'}(x) + arg_{B'}(x) > 2\pi \end{cases}$$

$$\le \begin{cases} 0 & \text{if } arg_A(x)+arg_B(x)\le 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\le 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_{A'}(x)+arg_{B'}(x)-arg_A(x)-arg_B(x)\right) & \text{if } arg_A(x)+arg_B(x)\le 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)> 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left(arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right) & \text{if } arg_A(x)+arg_B(x)> 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)\le 2\pi \\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right| & \text{if } arg_A(x)+arg_B(x)> 2\pi \\ & \quad arg_{A'}(x)+arg_{B'}(x)> 2\pi \end{cases}$$

$$\le \frac{1}{2\pi}\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|+\frac{1}{2\pi}\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|\le(1-\delta_1)+(1-\delta_1)=1-(\delta_1+\delta_2-1).$$

Therefore

$$d(A\dot\cap B,\ A'\dot\cap B')=\max\left(\sup_{x\in U}\left|r_{A\dot\cap B}(x)-r_{A'\dot\cap B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|Arg_{A\dot\cap B}(x)-Arg_{A'\dot\cap B'}(x)\right|\right)$$
$$\le 1-(\delta_1+\delta_2-1).$$

Further we note that $d(A\dot\cap B,\ A'\dot\cap B')\le 1.$ So

$$d(A\dot\cap B,\ A'\dot\cap B')\le 1-(\delta_1 * \delta_2).$$

Theorem 3.10 If $A=(\delta_1)A'$ and $B=(\delta_2)B'$, then $A|-|B=(\delta_1*\delta_2)A'|-|B'$.

Proof. Since $A=(\delta_1)A'$ and $B=(\delta_2)B'$, we have

$$d(A,\ A')=\max\left(\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{A'}(x)\right|\right)\le 1-\delta_1,$$

and

$$d(B,\ B')=\max\left(\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_{B'}(x)\right|\right)\le 1-\delta_2.$$

Therefore

$$\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|\le 1-\delta_1 \quad\text{and}\quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{A'}(x)\right|\le 1-\delta_1,$$

and

$$\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|\le 1-\delta_2 \quad\text{and}\quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x)-arg_{B'}(x)\right|\le 1-\delta_2.$$

Consequently, we have

$$\sup_{x\in U}\left|r_{A|-|B}(x)-r_{A'|-|B'}(x)\right|=\sup_{x\in U}\left|\max(0,\ r_A(x)-r_B(x))-\max(0,\ r_{A'}(x)-r_{B'}(x))\right|$$

$$=\begin{cases} 0 & \text{if } r_A(x)-r_B(x)\le 0 \text{ and } r_{A'}(x)-r_{B'}(x)\le 0 \\ \sup_{x\in U}\left(r_{A'}(x)-r_{B'}(x)\right) & \text{if } r_A(x)-r_B(x)\le 0 \text{ and } r_{A'}(x)-r_{B'}(x)> 0 \\ \sup_{x\in U}\left(r_A(x)-r_B(x)\right) & \text{if } r_A(x)-r_B(x)> 0 \text{ and } r_{A'}(x)-r_{B'}(x)\le 0 \\ \sup_{x\in U}\left|r_A(x)-r_B(x)-r_{A'}(x)+r_{B'}(x)\right| & \text{if } r_A(x)-r_B(x)> 0 \text{ and } r_{A'}(x)+r_{B'}(x)> 0 \end{cases}$$

$$\le \begin{cases} 0 & \text{if } r_A(x)-r_B(x)\le 0 \text{ and } r_{A'}(x)-r_{B'}(x)\le 0 \\ \sup\limits_{x\in U}\left(r_{A'}(x)-r_{B'}(x)-r_A(x)+r_B(x)\right) & \text{if } r_A(x)-r_B(x)\le 0 \text{ and } r_{A'}(x)-r_{B'}(x)> 0 \\ \sup\limits_{x\in U}\left(r_A(x)-r_B(x)-r_{A'}(x)+r_{B'}(x)\right) & \text{if } r_A(x)-r_B(x)> 0 \text{ and } r_{A'}(x)-r_{B'}(x)\le 0 \\ \sup\limits_{x\in U}\left|r_A(x)-r_B(x)-r_{A'}(x)+r_{B'}(x)\right| & \text{if } r_A(x)-r_B(x)> 0 \text{ and } r_{A'}(x)-r_{B'}(x)> 0 \end{cases}$$

$$\le \sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|+\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|\le (1-\delta_1)+(1-\delta_1)=1-(\delta_1+\delta_2-1).$$

The other hand, we have also

$$\frac{1}{2\pi}\sup_{x\in U}\left|arg_{A|-|B}(x)-arg_{A'|-|B'}(x)\right|$$

$$=\frac{1}{2\pi}\sup_{x\in U}\left|\max(0,\, arg_A(x)-arg_B(x))-\max(0,\, arg_{A'}(x)-arg_B(x))\right|$$

$$=\begin{cases} 0 & \text{if } arg_A(x)-arg_B(x)\le 0,\ arg_{A'}(x)-arg_{B'}(x)\le 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left(arg_{A'}(x)-arg_{B'}(x)\right) & \text{if } arg_A(x)-arg_B(x)\le 0,\ arg_{A'}(x)-arg_{B'}(x)> 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left(arg_A(x)-arg_B(x)\right) & \text{if } arg_A(x)-arg_B(x)> 0,\ arg_{A'}(x)-arg_{B'}(x)\le 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left|arg_A(x)-arg_B(x)-arg_{A'}(x)+arg_{B'}(x)\right| & \text{if } arg_A(x)-arg_B(x)> 0,\ arg_{A'}(x)-arg_{B'}(x)> 0 \end{cases}$$

$$\le\begin{cases} 0 & \text{if } arg_A(x)-arg_B(x)\le 0,\ arg_{A'}(x)-arg_{B'}(x)\le 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left(arg_{A'}(x)-arg_{B'}(x)-arg_A(x)+arg_B(x)\right) & \text{if } arg_A(x)-arg_B(x)\le 0,\ arg_{A'}(x)-arg_{B'}(x)> 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left(arg_A(x)-arg_B(x)-arg_{A'}(x)+arg_{B'}(x)\right) & \text{if } arg_B(x)-arg_A(x)> 0,\ arg_{A'}(x)-arg_{B'}(x)\le 0 \\ \frac{1}{2\pi}\sup\limits_{x\in U}\left|arg_A(x)+arg_B(x)-arg_{A'}(x)-arg_{B'}(x)\right| & \text{if } arg_A(x)-arg_B(x)> 0,\ arg_{A'}(x)-arg_{B'}(x)> 0 \end{cases}$$

$$\le \frac{1}{2\pi}\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|+\frac{1}{2\pi}\sup_{x\in U}\left|r_B(x)-r_{B'}(x)\right|\le (1-\delta_1)+(1-\delta_1)=1-(\delta_1+\delta_2-1).$$

Therefore

$$d(A|-|B,\ A'|-|B')=\max\left(\sup_{x\in U}\left|r_{A|-|B}(x)-r_{A'|-|B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A|-|B}(x)-arg_{A'|-|B'}(x)\right|\right)$$
$$\le 1-(\delta_1+\delta_2-1).$$

Further we note that $d(A|-|B,\ A'|-|B')\le 1.$ So

$$d(A|-|B,\ A'|-|B')\le 1-\delta_1 * \delta_2.$$

Theorem 3.11 If $A=(\delta_1)A'$ and $B=(\delta_2)B'$, then $A\nabla B=(\delta_1 * \delta_2)A'\nabla B'$.

Proof. Since $A=(\delta_1)A'$ and $B=(\delta_2)B'$, we have

$$d(A,\ A')=\max\left(\sup_{x\in U}\left|r_A(x)-r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x)-arg_{A'}(x)\right|\right)\le 1-\delta_1,$$

and

$$d(B,\ B') = \max\left(\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right|\right) \le 1 - \delta_2.$$

Therefore

$$\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right| \le 1 - \delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right| \le 1 - \delta_1,$$

and

$$\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right| \le 1 - \delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right| \le 1 - \delta_2.$$

Consequently, we have

$$d(A\nabla B,\ A'\nabla B') = \max\left(\sup_{x\in U}\left|r_{A\nabla B}(x) - r_{A'\nabla B'}(x)\right|, \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\nabla B}(x) - arg_{A'\nabla B'}(x)\right|\right)$$

$$= \max\left(\sup_{x\in U}\Big|\left|r_A(x) - r_B(x)\right| - \left|r_{A'}(x) - r_{B'}(x)\right|\Big|,\right.$$

$$\left.\frac{1}{2\pi}\sup_{x\in U}\Big|\left|arg_A(x) - arg_B(x)\right| - \left|arg_{A'}(x) - arg_{B'}(x)\right|\Big|\right)$$

$$= \max\left(\sup_{x\in U}\left|\max(r_A(x) - r_B(x), r_B(x) - r_A(x)) - \max(r_{A'}(x) - r_{B'}(x), r_{B'}(x) - r_{A'}(x))\right|,\right.$$

$$\frac{1}{2\pi}\sup_{x\in U}\Big|\max(arg_A(x) - arg_B(x), arg_B(x) - arg_A(x))$$

$$\left.- \max(arg_{A'}(x) - arg_{B'}(x), arg_{B'}(x) - arg_{A'}(x))\Big|\right)$$

$$\le \max\left(\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right| + \sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right|,\right.$$

$$\left.\frac{1}{2\pi}\left(\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right| + \sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right|\right)\right)$$

$$\le \max\big((1-\delta_2) + (1-\delta_1), (1-\delta_2) + (1-\delta_1)\big) = 1 - (\delta_1 + \delta_2 - 1).$$

Further we note that $d(A\nabla B,\ A'\nabla B') \le 1$. So

$$d(A\nabla B,\ A'\nabla B') \le 1 - \delta_1 * \delta_2.$$

Theorem 3.12 Let $A\Delta B = \left(A \cap \overline{B}\right) \cup \left(B \cap \overline{A}\right)$. If A= (δ_1)A' and B = (δ_2)B', then $A\Delta B = \left(\min(\delta_1, \delta_2)\right)A'\Delta B'$.

Proof. We can prove by using $A\Delta B = \left(A \cap \overline{B}\right) \cup (\overline{A} \cap B)$ and Theorem 3.2 – 3.4.

Theorem 3.13 If A= (δ_1)A' and B = (δ_2)B', then $A \|_\lambda\ B = \left(\min(\delta_1, \delta_2)\right)A' \|_\lambda\ B'$.

Proof. Since A = (δ_1)A' and B = (δ_2)B', we have

$$d(A,\ A') = \max\left(\sup_{x\in U}\left|r_A(x) - r_{A'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_A(x) - arg_{A'}(x)\right|\right) \le 1 - \delta_1,$$

and

$$d(B,\ B') = \max\left(\sup_{x\in U}\left|r_B(x) - r_{B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_B(x) - arg_{B'}(x)\right|\right) \le 1 - \delta_2.$$

Therefore

$$\sup_{x\in U}|r_A(x)-r_{A'}(x)|\le 1-\delta_1 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|arg_A(x)-arg_{A'}(x)|\le 1-\delta_1,$$

and

$$\sup_{x\in U}|r_B(x)-r_{B'}(x)|\le 1-\delta_2 \quad \text{and} \quad \frac{1}{2\pi}\sup_{x\in U}|arg_B(x)-arg_{B'}(x)|\le 1-\delta_2.$$

Consequently, we have

$$d(A\|_\lambda\ B,\ A'\|_\lambda\ B')=\max\left(\sup_{x\in U}\left|r_{A\|_\lambda B}(x)-r_{A'\|_\lambda B'}(x)\right|,\ \frac{1}{2\pi}\sup_{x\in U}\left|arg_{A\|_\lambda B}(x)-arg_{A'\|_\lambda B'}(x)\right|\right)$$

$$\begin{aligned}
&=\max\Big(\sup_{x\in U}\big|\lambda\min\big(r_A(x),\ r_B(x)\big)+(1-\lambda)\max\big(r_A(x), r_B(x)\big)\\
&\qquad -\lambda\min\big(r_{A'}(x),\ r_{B'}(x)\big)-(1-\lambda)\max\big(r_{A'}(x), r_{B'}(x)\big)\big|,\\
&\qquad \frac{1}{2\pi}\sup_{x\in U}\big|\lambda\min\big(arg_A(x),\ arg_B(x)\big)+(1-\lambda)\max\big(arg_A(x), arg_B(x)\big)\\
&\qquad -\lambda\min\big(arg_{A'}(x),\ arg_{B'}(x)\big)-(1-\lambda)\max\big(arg_{A'}(x), arg_{B'}(x)\big)\big|\Big)\\
&\le\max\Big(\sup_{x\in U}\Big(\lambda\big|\min(r_A(x), r_B(x))-\min(r_{A'}(x), r_{B'}(x))\big|\\
&\qquad +(1-\lambda)\big|\max(r_A(x), r_B(x))-\max(r_{A'}(x), r_{B'}(x))\big|\Big),\\
&\qquad \frac{1}{2\pi}\sup_{x\in U}\lambda\Big(\big|\min(arg_A(x), arg_B(x))-\min(arg_{A'}(x), arg_{B'}(x))\big|\\
&\qquad +(1-\lambda)\big|\max(arg_A(x), arg_B(x))-\max(arg_{A'}(x), arg_{B'}(x))\big|\Big)\Big)\\
&\le\max\Big(\sup_{x\in U}\Big(\lambda\max\big(|r_A(x)-r_{A'}(x)|, |r_B(x)-r_{B'}(x)|\big)\\
&\qquad +(1-\lambda)\max\big(|r_A(x)-r_{A'}(x)|, |r_B(x)-r_{B'}(x)|\big)\Big),\\
&\qquad \frac{1}{2\pi}\sup_{x\in U}\Big(\lambda\max\big(|arg_A(x)-arg_{A'}(x)|, |arg_B(x)-arg_{B'}(x)|\big)\\
&\qquad +(1-\lambda)\max\big(|arg_A(x)-arg_{A'}(x)|, |arg_B(x)-arg_{B'}(x)|\big)\Big)\Big)\\
&\le\max\big(\max(1-\delta_2, 1-\delta_1), \max(1-\delta_2, 1-\delta_1)\big)=1-\min(\delta_1, \delta_2).
\end{aligned}$$

Theorem 3.14 Let A_1, B_1, C_1, A_2, B_2 and C_2 be complex fuzzy sets on U, If

$$r_{A_1}(x)\le r_{B_1}(x)\le r_{C_1}(x),\quad r_{A_2}(x)\le r_{B_2}(x)\le r_{C_2}(x),$$

$$arg_{A_1}(x)\le arg_{B_1}(x)\le arg_{C_1}(x),\quad arg_{A_2}(x)\le arg_{B_2}(x)\le arg_{C_2}(x),\quad \forall x\in U$$

and

$$A_1=(\delta_a)A_2,\quad C_1=(\delta_c)C_2,\quad A_1=(\delta_{ac})C_1,$$

Then $B_1=(\delta_{ac}*\min(\delta_a, \delta_c))B_2$.

Proof. Since $A_1=(\delta_a)A_2$ and $C_1=(\delta_c)C_2$, we have

$$d(A_1,\ A_2)=\max\left(\sup_{x\in U}|r_{A_1}(x)-r_{A_2}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_{A_1}(x)-arg_{A_2}(x)|\right)\le 1-\delta_a,$$

$$d(A_1,\ C_1)=\max\left(\sup_{x\in U}|r_{A_1}(x)-r_{C_1}(x)|,\ \frac{1}{2\pi}\sup_{x\in U}|arg_{A_1}(x)-arg_{C_1}(x)|\right)\le 1-\delta_{ac},$$

and

$$d(C_1,\ C_2) = \max\left(\sup_{x\in U} \left| r_{C_1}(x) - r_{C_2}(x) \right|,\ \frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{C_2}(x) \right| \right) \le 1 - \delta_c.$$

Therefore

$$\sup_{x\in U} \left| r_{A_1}(x) - r_{A_2}(x) \right| \le 1 - \delta_a \quad \text{and} \quad \frac{1}{2\pi} \sup_{x\in U} \left| arg_{A1}(x) - arg_{A_2}(x) \right| \le 1 - \delta_a,$$

$$\sup_{x\in U} \left| r_{A_1}(x) - r_{C_1}(x) \right| \le 1 - \delta_{ac} \quad \text{and} \quad \frac{1}{2\pi} \sup_{x\in U} \left| arg_{A1}(x) - arg_{C_1}(x) \right| \le 1 - \delta_{ac},$$

and

$$\sup_{x\in U} \left| r_{C_1}(x) - r_{C_2}(x) \right| \le 1 - \delta_c \quad \text{and} \quad \frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{C_2}(x) \right| \le 1 - \delta_c.$$

From

$$r_{A_1}(x) - r_{C_2}(x) \le r_{B_1}(x) - r_{B_2}(x) \le r_{C_1}(x) - _{A_2}(x),$$

$$arg_{A_1}(x) - arg_{C_2}(x) \le arg_{B_1}(x) - arg_{B_2}(x) \le arg_{C_1}(x) - arg_{A_2}(x), \quad \forall x \in U$$

we have

$$\left| r_{B_1}(x) - r_{B_2}(x) \right| \le \max\left(\left| r_{A_1}(x) - r_{C_2}(x) \right|, \left| r_{C_1}(x) - _{A_2}(x) \right| \right),$$

$$\left| arg_{B_1}(x) - arg_{B_2}(x) \right| \le \max\left(\left| arg_{A_1}(x) - arg_{C_2}(x) \right|, \left| arg_{C_1}(x) - arg_{A_2}(x) \right| \right). \quad \forall x \in U$$

However

$$\left| r_{A_1}(x) - r_{C_2}(x) \right| \le \left| r_{A_1}(x) - r_{C_1}(x) \right| + \left| r_{C_1}(x) - r_{C_2}(x) \right| \le 1 - \delta_{ac} + 1 - \delta_c,$$

$$\left| r_{C_1}(x) - r_{A_2}(x) \right| \le \left| r_{C_1}(x) - r_{A_1}(x) \right| + \left| r_{A_1}(x) - r_{A_2}(x) \right| \le 1 - \delta_{ac} + 1 - \delta_a,$$

$$\begin{aligned} &\frac{1}{2\pi} \sup_{x\in U} \left| arg_{A_1}(x) - arg_{C_2}(x) \right| \\ &\le \frac{1}{2\pi} \sup_{x\in U} \left| arg_{A_1}(x) - arg_{C_1}(x) \right| + \frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{C_2}(x) \right| \\ &\le 1 - \delta_{ac} + 1 - \delta_c, \end{aligned}$$

$$\begin{aligned} &\frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{A_2}(x) \right| \\ &\le \frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{A_1}(x) \right| + \frac{1}{2\pi} \sup_{x\in U} \left| arg_{A_1}(x) - arg_{A_2}(x) \right| \\ &\le 1 - \delta_{ac} + 1 - \delta_a. \end{aligned}$$

Thus

$$\left| r_{B_1}(x) - r_{B_2}(x) \right| \le \max(2 - \delta_{ac} - \delta_c,\ 2 - \delta_{ac} - \delta_a) \le 2 - \delta_{ac} - \min(\delta_a,\ \delta_c),$$

$$\frac{1}{2\pi} \sup_{x\in U} \left| arg_{C_1}(x) - arg_{A_2}(x) \right| \le \max(2 - \delta_{ac} - \delta_c,\ 2 - \delta_{ac} - \delta_a) \le 2 - \delta_{ac} - \min(\delta_a,\ \delta_c).$$

Further we note that $d(B_1, B_2) \le 1$. So

$$d(B_1, B_2) \le 1 - \max(0, \delta_{ac} + \min(\delta_a, \delta_c) - 1).$$

4. Example Application

We consider a signal processing example below which involves the application of δ-equalities of complex fuzzy sets. In this section we are *not* intended to show the potential advantages of using complex fuzzy sets in comparison with existing alternative approaches. The reader should be referred to Ramot et al., [10] for the rationale of using complex fuzzy sets. Rather, we want to show how the theoretical results presented in this paper can be *applied* in reality. The example demonstrates the use of δ-equality theory of complex fuzzy sets in application that determines if any of the L different signals $S_l(k)$, $(1 \le l \le L)$ received by a digital receiver can be identified as similar to a given reference signal R. Each of the signals S_L and R are sampled N times i.e. $(1 \le k \le N)$. The discrete Fourier Transforms of both $S_l(k)$ and $R(k)$ can obtained.

The problem statement of the example is taken from Ref [10]. Let $S_l(k)$ denote each k-th sample $(1 \le k \le N)$ of the l-th signal $(1 \le l \le L)$.

Let $C_{l,n}$ $(1 \le n \le N)$ be the complex Fourier coefficients of S_l. Then $S_l(k)$ can be expressed as

$$S_l(k) = \frac{1}{N}\sum_{n=1}^{N} C_{l,n} \cdot e^{i\frac{2\pi(n-1)(k-1)}{N}}$$

Let the Fourier coefficients of R be $C_{R,n}$ where $(1 \le n \le N)$. Then

$$R(k) = \frac{1}{N}\sum_{n=1}^{N} C_{R,n} \cdot e^{i\frac{2\pi(n-1)(k-1)}{N}}.$$

The aforementioned sum may be rewritten in the form

$$S_l(k) = \frac{1}{N}\sum_{n=1}^{N} A_{l,n} \cdot e^{i\frac{2\pi(n-1)(k-1)+N\alpha_{l,n}}{N}} \tag{4.2}$$

where $C_{l,n} = A_{l,n} \cdot e^{i\alpha_{l,n}}$, with $A_{l,n}, \alpha_{l,n}$ real-valued and $A_{l,n} \ge 0$ for all n $(1 \le n \le N)$.

The purpose of the application is to determine which, if any, of the L signals received can be identified as the reference signal, R. The reference signal R has been similarly sampled N times, and its discrete Fourier transform is known. Let the Fourier coefficients of R be $C_{R,n}$, where $1 \le n \le N$, thus

$$R(k) = \frac{1}{N}\sum_{n=1}^{N} A_{R,n} \cdot e^{i\frac{2\pi(n-1)(k-1)+N\alpha_{R,n}}{N}} \tag{4.3}$$

where $C_{R,n} = A_{R,n} \cdot e^{i\alpha_{R,n}}$, with $A_{R,n}, \alpha_{R,n}$ real valued and $A_{l,n} \ge 0$ for all n.

Calculating a measure of the similarity between two signals is possible by comparing their Fourier transforms. Now we apply the following method supported by δ-Equality theory to compare the different signals.

Step 1) Normalize the amplitudes of all Fourier coefficients. Consider S_l, $(1 \le l \le L)$. Denote as $\boldsymbol{A_l}$ the (N-dimensional) vector of amplitudes of S_l's Fourier coefficients: $(A_{l,1}, A_{l,2},..., A_{l,N})$, and let $\boldsymbol{A_R}$ denote the vector of R's Fourier coefficients: $(A_{R,1}, A_{R,2},..., A_{R,N})$, let $\boldsymbol{B_l}$ denote the normalized vector 1/(norm($\boldsymbol{A_l}$))$\boldsymbol{A_l}$, where norm($\boldsymbol{A_l}$) = $\sqrt{\sum_{n=1}^{N}(A_{l,n})^2}$, and let $\boldsymbol{B_R}$ denote the normalized vector 1/(norm($\boldsymbol{A_R}$))$\boldsymbol{A_R}$. Thus, $\boldsymbol{B_l}$ = $(B_{l,1}, B_{l,2},..., B_{l,N})$ is the vector of normalized amplitudes of S_l's Fourier coefficients. Similarly, $\boldsymbol{B_R}$ = $(B_{R,1}, B_{R,2},..., B_{R,N})$ is the vector of normalized amplitudes of R's Fourier coefficients.

Step 2) Define complex fuzzy sets $\boldsymbol{S_{l,n}}$ $(1 \le n \le N, 1 \le l \le L)$ and $\boldsymbol{R_n}$ $(1 \le n \le N)$ such that their Cartesian product $S_l = S_{l,1} \times \cdots \times S_{l,N}$ and $R = R_1 \times \cdots \times R_N$ corresponding to S_l $(1 \le l \le L)$ and R, respectively, as

$$\mu_{S_{l,n}}(k) = B_{l,n} \cdot e^{i\frac{2\pi(n-1)(k-1)+N\alpha_{l,n}}{N}}, \quad 1 \le k, n \le N, 1 \le l \le L \tag{4.4}$$

and

$$\mu_{R,n}(k) = B_{R,n} \cdot e^{i\frac{2\pi(n-1)(k-1)+N\alpha_{R,n}}{N}}, \quad 1 \le k, n \le N. \tag{4.5}$$

Therefore

$$\mu_{S_l}(k_1, k_2, \cdots, k_N) = \min_{1\le n\le N} B_{l,n} \cdot e^{i \min_{1\le n\le N} \alpha_{l,n}}, \quad 1 \le l \le L \tag{4.6}$$

and

$$\mu_R(k_1, k_2, \cdots, k_N) = \min_{1\le n\le N} B_{R,n} \cdot e^{i \min_{1\le n\le N} \alpha_{l,n}}. \tag{4.7}$$

Step 3) Calculate the distances of two complex fuzzy sets $\boldsymbol{S_l}$ ($1 \le l \le L$) and $\boldsymbol{R}$,

$$d(S_l, R) = \max\left(\sup_{k_1,k_2,\cdots,k_N} \left| \min_{1\le n\le N} B_{l,n} - \min_{1\le n\le N} B_{R,n} \right|, \ \frac{1}{2\pi} \sup_{k_1,k_2,\cdots,k_N} \left| \min_{1\le n\le N} \alpha_{l,n} - \min_{1\le n\le N} \alpha_{l,n} \right| \right)$$
$$\le \max\left(\sup_{n\in\{1,2,\cdots,N\}} \left| B_{l,n} - B_{R,n} \right|, \ \frac{1}{2\pi} \sup_{n\in\{1,2,\cdots,N\}} \left| \alpha_{l,n} - \alpha_{R,n} \right| \right). \ 1 \le l \le L \tag{4.8}$$

Step 4) In order to conclude if S_l may be identified as R, compare: $1 - d(S_l, R)$ ($1 \le l \le L$) to a threshold δ. If $1 - d(S_l, R)$ exceeds the threshold, identify S_l as R.

By this method, a device for measuring the similarity between two signals is provided. The method can be of use for any signal analysis application in which the relative phase between the Fourier components of the signals under consideration is important.

Note that Step (1) above is the same as that in Ref [10]. However Steps (2) to (4) are different and utilize the results derived in this paper.

5. Conclusion

Up to this point we have investigated the properties of various operations on complex fuzzy sets and introduced a distance measure for complex fuzzy sets. This distance measure was then used to define δ-equalities of complex fuzzy sets which subsume δ-equalities of real-valued fuzzy sets defined in references [5, 6]. Two complex fuzzy sets are said to be δ-equal if the distance between them is less than 1-δ. Suppose $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$, and f is an operation on two complex fuzzy sets. In the preceding sections we have shown how δ varies with different form of f such that $f(A_1, A_2) = (\delta) f(B_1, B_2)$. An example application demonstrates that the concept of δ-equalities of complex fuzzy sets can be exploited to pick up the underlying reference signal from a set of noisy signals.

The importance of the work presented in this paper can be justified in theory as well as in practice. On the one hand, this paper shows that the δ-equalities of (complex) fuzzy sets can be defined and investigated in a general framework by introducing a distance measure for complex fuzzy sets. Such a distance measures the difference between the grades of two complex fuzzy sets as well as that between the phases of the two complex fuzzy sets. In this way δ-equalities may be further investigated from the perspective of a metric space of (complex) fuzzy sets. On the other hand, as shown in the example application of signal detection in Section 4, the concept of δ-equalities of complex fuzzy sets may be useful in various applications where errors of membership functions of (complex) fuzzy sets are of concern.

A lot of research work can be conducted for the δ-equalities of (complex) fuzzy sets in the future. For example, given $A_1 = (\delta_1)B_1$, $A_2 = (\delta_2)B_2$ and f, how to determine a maximal δ such that $f(A_1, A_2) = (\delta) f(B_1, B_2)$. How to apply to the concept of δ-equalities of complex fuzzy sets to synthesis of real-time fuzzy systems is another problem of interest.

Appendix

Comprison of the complex fuzzy sets, fuzzy sets (type 1 fuzzy sets), fuzzy complex sets, and type 2 fuzzy set is listed below.

	domain	Co-domain
Complex fuzzy sets	A given universe of discourse	Complex unit circle
Fuzzy sets	A given universe of discourse	Real unit interval
Fuzzy numbers	A given universe of discourse	Real unit interval
Fuzzy complex numbers	Complex numbers universe	Real unit interval
Type 2 fuzzy sets	A given universe of discourse	Fuzzy numbers

From above table, it can be seen that the four concepts have close relationships and also remarkable difference although the operations on them may be similar.

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