

Optimal Discrete-Valued Control Computation

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Abstract In this paper, we consider an optimal control problem in which the control takes values from a discrete set and the state and control are subject to continuous inequality constraints. By introducing auxiliary controls and applying a time-scaling transformation, we transform this optimal control problem into an equivalent problem subject to additional linear and quadratic constraints. The feasible region defined by these additional constraints is disconnected, and thus standard optimization methods struggle to handle these constraints. We introduce a novel exact penalty function to penalize constraint violations, and then append this penalty function to the objective. This leads to an approximate optimal control problem that can be solved using standard software packages such as MISER. Convergence results show that when the penalty parameter is sufficiently large, any local solution of the approximate problem is also a local solution of the original problem. We conclude the paper with some numerical results for two difficult train control problems.

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1 Introduction

In many practical optimal control problems arising in engineering applications, the control is only allowed to assume a finite number of values. Such problems are called *optimal discrete-valued control problems*. Optimal discrete-valued control problems arise in many applications, including train control [2], switched amplifier design [11], submarine operation [10], sensor scheduling [16] and hybrid power system design [15, 13]. To solve an optimal discrete-valued control problem, we need to determine the order in which the different control values are operated, as well as the times at which the control switches from one value to another. Since the ordering of control values is a discrete variable, classical optimal control methods are not applicable to this type of problem.

In [2], the driving strategy for a diesel train traveling on a level track is considered. The train only has three modes of operation — accelerate, coast and brake — and thus the problem of controlling the train so that fuel consumption is minimized is an optimal discrete-valued control problem. An optimality condition is derived in [2] for solving this problem. However, this condition is only applicable to the train problem, and is not applicable to general optimal discrete-valued control problems.

In [6], a time-scaling transformation technique is developed for solving optimal discrete-valued control problems. Under this transformation, the original problem with variable control switching points is transformed into an ordinary optimal control problem with known and fixed switching points. Thus, the transformed problem can be easily solved by many existing optimal control methods. However, the time-scaling transformation introduces many additional switches, and therefore the transformed problem is not equivalent to the original problem.

In [14], a new approach is proposed for solving nonlinear mixed discrete programming problems. The idea is to introduce a set of new continuous variables and transform the discrete programming problem into an ordinary optimization problem. In principle, this new problem can be solved using many existing nonlinear programming techniques. However, the transformation introduces additional equality and inequality constraints, and these constraints are extremely difficult to satisfy in practice.

In [17], an exact penalty method is proposed for solving semi-infinite programming problems. This method is adapted in [5] to develop an effective algorithm for solving optimal control problems with continuous inequality constraints. It is shown that, under some mild assumptions, making the penalty parameter sufficiently large forces the continuous inequality constraints to be satisfied. In this case, a local optimal solution of the penalty problem is also a local optimal solution of the original problem.

In this paper, we consider a class of optimal discrete-valued control problems, where there is an upper bound on the maximum number of control switches. We first apply the transformation reported in [14], under which the discrete-valued control is expressed as a linear combination of piecewise constant controls subject to a linear equality constraint and a set of quadratic

inequality constraints. The original problem can then be written equivalently as an optimal control problem with piecewise constant controls subject to the original inequality constraints and the new constraints. Then, the time-scaling transformation [6] is applied to the transformed problem, yielding an optimal control problem with piecewise constant controls and fixed switching times. To solve this new problem, we introduce an exact penalty function, which is motivated by the work in [5,17–19], and construct a corresponding penalty problem. Convergence results show that when the penalty parameter is sufficiently large, the penalty problem is equivalent to the original problem. This penalty problem can be solved easily by many existing optimization software packages. Numerical results show that our approach can effectively solve two difficult train control problems.

2 Problem Formulation

2.1 A Discrete-Valued Control Problem

Consider the following dynamic system on the time horizon $[0, T]$:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

with the initial and terminal conditions

$$\mathbf{x}(0) = \mathbf{x}^0, \quad \mathbf{x}(T) = \mathbf{x}^f, \quad (2)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, T is a given terminal time, and \mathbf{x}^0 and \mathbf{x}^f are given vectors. We assume that the function $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$ is continuously differentiable with respect to each of its arguments.

Let

$$\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\},$$

where each $\mathbf{u}_j \in \mathbb{R}^r$ is a given vector. We assume that the control \mathbf{u} is a *discrete-valued control* taking values in \mathbf{U} . Thus, \mathbf{u} is completely determined by specifying:

- The order in which it assumes the different values in \mathbf{U} (the so-called *switching sequence*); and
- The times at which it switches from one value in \mathbf{U} to another (the so-called *switching times*).

In this paper, we assume that there is an upper bound N on the maximum number of control switches. A function $\mathbf{u} : [0, T] \rightarrow \mathbf{U}$ with at most N switches/discontinuities is called an *admissible control*. Let \mathcal{U} denote the class of all such admissible controls.

Our optimal discrete-valued control problem is stated as follows: Given the dynamic system (1)-(2), find an admissible control $\mathbf{u} \in \mathcal{U}$ such that the cost function

$$J(\mathbf{u}) = \int_0^T \mathcal{L}_0(\mathbf{x}(t), \mathbf{u}(t)) dt \quad (3)$$

is minimized subject to the constraints

$$g_i(\mathbf{x}(t), \mathbf{u}(t)) \leq 0, \quad t \in [0, T], \quad i = 1, 2, \dots, p. \quad (4)$$

Let this problem be referred to as Problem (P). Here, we assume that the functions \mathcal{L}_0 and g_i , $i = 1, \dots, p$, are continuously differentiable with respect to each of their arguments.

Most numerical techniques for solving nonlinear optimal control problems — for example, control parametrization (see [12]) and state discretization (see [1, 4]) — are only applicable when the control range is a continuous set. Thus, such methods are not applicable to Problem (P), in which the control range consists of a finite number of discrete points.

The time-scaling transform introduced in [6], which is also called the Control Parametrization Enhancing Technique (CPET), is an effective method for solving optimal discrete-valued control problems. This transformation involves expanding the number of control switches to allow for every possible switching sequence, and then mapping the switching times to fixed points in a new time horizon. This yields a new optimization problem that can be solved using standard optimization techniques, such as sequential quadratic programming with active set strategy (see [9]). However, the problem with this transformation is that it introduces many “artificial” switches, and thus the optimal control obtained may have more than the maximum allowable number of switches. Consequently, the transformed optimal control problem obtained by using the time-scaling transformation in [6] is not equivalent to the original problem. We will introduce an equivalent transformation in the next section.

2.2 Problem Transformation

Let \mathcal{V} denote the class of all piecewise constant functions mapping $[0, T]$ into \mathbb{R}^m and having no more than N switches/discontinuities. Let $\mathbf{v} \in \mathcal{V}$, where $\mathbf{v}(t) = (v_1(t), v_2(t), \dots, v_m(t))$, be an auxiliary control function.

We impose the following constraints:

$$\sum_{j=1}^m v_j(t) = 1, \quad t \in [0, T], \quad (5a)$$

$$v_j(t)(1 - v_j(t)) \leq 0, \quad t \in [0, T], \quad j = 1, 2, \dots, m, \quad (5b)$$

$$0 \leq v_j(t) \leq 1, \quad t \in [0, T], \quad j = 1, 2, \dots, m. \quad (5c)$$

The constraints (5) ensure that at each time $t \in [0, T]$, there exists exactly one $j \in \{1, \dots, m\}$ such that $v_j(t) = 1$ and $v_k(t) = 0$ for all $k \neq j$.

To continue, we let

$$\bar{\mathbf{u}}(t) = \sum_{j=1}^m v_j(t) \mathbf{u}_j. \quad (6)$$

Since $\mathbf{v} \in \mathcal{V}$ and constraints (5) hold, $\bar{\mathbf{u}}(t) \in \mathbf{U}$ for all $t \in [0, T]$. Moreover, since \mathbf{v} contains at most N switches, so does $\bar{\mathbf{u}}$. It follows that $\bar{\mathbf{u}}$ is an admissible control for Problem (P). In fact, it is easy to see that *any* admissible control for Problem (P) can be written in the form of (6). Thus, by substituting $\mathbf{u}(t) = \bar{\mathbf{u}}(t)$ into the dynamical system (1), we obtain

$$\dot{\mathbf{x}}(t) = \sum_{j=1}^m v_j(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}_j). \quad (7)$$

Similarly, the constraints (4) become

$$\sum_{j=1}^m v_j(t) g_i(\mathbf{x}(t), \mathbf{u}_j) \leq 0, \quad t \in [0, T], \quad i = 1, 2, \dots, p. \quad (8)$$

Our new optimal control problem is stated as follows: Given the dynamic system (7) with the initial and terminal conditions (2), find a control $\mathbf{v} \in \mathcal{V}$ such that the cost function

$$\bar{J}(\mathbf{v}) = J(\bar{\mathbf{u}}) = \sum_{j=1}^m \int_0^T v_j(t) \mathcal{L}_0(\mathbf{x}(t), \mathbf{u}_j) dt$$

is minimized subject to constraints (5) and (8). Let this problem be referred to as Problem (\bar{P}).

It is clear that Problems (\bar{P}) and (P) are equivalent. Thus, we have the following result.

Theorem 1 *Let $\mathbf{v}^* = (v_1^*, v_2^*, \dots, v_m^*) \in \mathcal{V}$ and*

$$\bar{\mathbf{u}}^*(t) = \sum_{j=1}^m v_j^*(t) \mathbf{u}_j.$$

Then \mathbf{v}^ is an optimal control for Problem (\bar{P}) if and only if $\bar{\mathbf{u}}^*$ is an optimal control for Problem (P).*

Problem (\bar{P}) is a standard optimal control problem subject to the continuous inequality constraints (8) and the new constraints (5). In principle, many optimal control software packages — for example, MISER [3] — can be used to solve this problem. However, in reality, there are two major difficulties that prevent us from solving Problem (\bar{P}) directly:

- The switching times for the new controls v_j are decision variables.
- The feasible region defined by the constraints (5) is a disconnected set.

We can overcome the first difficulty by applying the time-scaling transformation (see [6]), in which the variable switching times are mapped into fixed switching times. For the second difficulty, we will introduce an exact penalty function method that is inspired by our previous work in [5]. The details are provided in the next section.

3 Solution Procedure

3.1 Time-Scaling Transformation

Recall that the control $\mathbf{v} \in \mathcal{V}$ in Problem (\bar{P}) has at most N switches. Let τ_k denote the k th switching time. Then

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{N+1} = T.$$

We map these switching times to fixed time points as follows. Let $s \in [0, N+1]$ be a new time variable, and relate t to s by the following differential equation:

$$\begin{aligned} \dot{t}(s) &= \mu(s), \\ t(0) &= 0, \end{aligned} \tag{9}$$

where $\mu(s) = \theta_k = \tau_k - \tau_{k-1}$ for $s \in [k-1, k)$, $k = 1, \dots, N+1$. We can express the piecewise constant function μ as follows:

$$\mu(s) = \sum_{k=1}^{N+1} \theta_k \chi_{[k-1, k)}(s),$$

where χ_I is the indicator function of I defined by

$$\chi_I(s) = \begin{cases} 1, & \text{if } s \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{\theta} = [\theta_1, \dots, \theta_{N+1}]^T \in \mathbb{R}^{N+1}$, and note that $\theta_k = \tau_k - \tau_{k-1}$ is the duration of the k th control value. For each $k = 1, \dots, N+1$, we have

$$\begin{aligned} t(k) &= \int_0^k \mu(s) ds \\ &= \int_0^k [\theta_1 \chi_{[0,1)}(s) + \cdots + \theta_{N+1} \chi_{[N, N+1)}(s)] ds \\ &= \theta_1 + \cdots + \theta_k = \tau_k. \end{aligned}$$

This shows that the transformation (9) maps each integer k to the k th switching time. Furthermore,

$$t(N+1) = \int_0^{N+1} \mu(s) ds = \sum_{l=1}^{N+1} \theta_l = T. \tag{10}$$

Clearly,

$$0 \leq \theta_k = \tau_k - \tau_{k-1} \leq T, \quad k = 1, \dots, N+1. \tag{11}$$

Thus,

$$0 \leq \mu(s) \leq T, \quad s \in [0, N+1].$$

Under the time-scaling transform, the control v_j in Problem (\bar{P}) becomes

$$\tilde{v}_j(s) = v_j(t(s)) = \sum_{k=1}^{N+1} \xi_{jk} \chi_{[k-1,k)}(s),$$

where ξ_{jk} is the value of v_j on $[\tau_{k-1}, \tau_k)$. Constraints (5) become:

$$\sum_{j=1}^m \xi_{jk} = 1, \quad k = 1, \dots, N+1, \quad (12a)$$

$$\xi_{jk}(1 - \xi_{jk}) \leq 0, \quad j = 1, \dots, m, \quad k = 1, \dots, N+1, \quad (12b)$$

$$0 \leq \xi_{jk} \leq 1, \quad j = 1, \dots, m, \quad k = 1, \dots, N+1. \quad (12c)$$

Define

$$\boldsymbol{\xi}_j = [\xi_{j1}, \dots, \xi_{j(N+1)}]^T \in \mathbb{R}^{N+1}$$

and

$$\boldsymbol{\xi} = [\boldsymbol{\xi}_1^T, \dots, \boldsymbol{\xi}_m^T]^T \in \mathbb{R}^{m \times (N+1)}.$$

Now, by applying the time-scaling transform to Problem (\bar{P}), the dynamical system (7) becomes

$$\frac{d\tilde{\mathbf{x}}(s)}{ds} = \mu(s) \sum_{j=1}^m \tilde{v}_j(s) \mathbf{f}(\tilde{\mathbf{x}}(s), \mathbf{u}_j) = \sum_{k=1}^{N+1} \sum_{j=1}^m \theta_k \xi_{jk} \mathbf{f}(\tilde{\mathbf{x}}(s), \mathbf{u}_j) \chi_{[k-1,k)}(s), \quad (13)$$

where

$$\tilde{\mathbf{x}}(s) = \mathbf{x}(t(s)).$$

The initial and terminal conditions (2) become

$$\tilde{\mathbf{x}}(0) = \mathbf{x}^0, \quad \tilde{\mathbf{x}}(N+1) = \mathbf{x}^f. \quad (14)$$

Problem (\bar{P}) may now be written equivalently as the following problem, which we call Problem (\tilde{P}): Given the dynamic system (13)-(14), find $\boldsymbol{\theta} \in \mathbb{R}^{N+1}$ and $\boldsymbol{\xi} \in \mathbb{R}^{m \times (N+1)}$ such that the cost function

$$\tilde{J}(\boldsymbol{\theta}, \boldsymbol{\xi}) = \int_0^{N+1} \tilde{\mathcal{L}}_0(s, \tilde{\mathbf{x}}(s), \boldsymbol{\theta}, \boldsymbol{\xi}) ds,$$

where

$$\tilde{\mathcal{L}}_0(s, \tilde{\mathbf{x}}(s), \boldsymbol{\theta}, \boldsymbol{\xi}) = \sum_{k=1}^{N+1} \sum_{j=1}^m \theta_k \xi_{jk} \mathcal{L}_0(\tilde{\mathbf{x}}(s), \mathbf{u}_j) \chi_{[k-1,k)}(s),$$

is minimized subject to the constraints

$$\tilde{g}_i(s, \tilde{\mathbf{x}}(s), \boldsymbol{\xi}) = \sum_{k=1}^{N+1} \sum_{j=1}^m \xi_{jk} g_i(\tilde{\mathbf{x}}(s), \mathbf{u}_j) \chi_{[k-1,k)}(s) \leq 0, \quad (15)$$

$$s \in [0, N+1], \quad i = 1, \dots, p,$$

and constraints (10), (11) as well as (12).

In the next section, we will introduce an exact penalty function to develop an effective computational method for solving Problem (\tilde{P}).

3.2 An Exact Penalty Function

Problem (\tilde{P}) is an optimal control problem subject to the linear constraints (10), (12a) and (12c), the quadratic constraints (12b), and the nonlinear continuous inequality constraints (15). The continuous inequality constraints (15) are continuously differentiable with respect to each of their arguments. By adopting the idea introduced in [5], we construct the following exact penalty function:

$$F_\kappa(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) = \begin{cases} \tilde{J}(\boldsymbol{\theta}, \boldsymbol{\xi}), & \text{if } \epsilon = 0, \text{ and } (\boldsymbol{\theta}, \boldsymbol{\xi}) \text{ is feasible} \\ & \text{for Problem } (\tilde{P}), \\ \tilde{J}(\boldsymbol{\theta}, \boldsymbol{\xi}) + \epsilon^{-\alpha} \Delta(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) + \kappa \epsilon^\beta, & \text{if } \epsilon > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$ is a new decision variable, and the constraint violation $\Delta(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon)$ is defined by

$$\begin{aligned} \Delta(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) = & \sum_{k=1}^{N+1} \sum_{j=1}^m \max\{0, \xi_{jk}(1 - \xi_{jk}) - \epsilon^\gamma\}^2 + \sum_{k=1}^{N+1} \sum_{j=1}^m \max\{0, \xi_{jk} - 1 - \epsilon^\gamma\}^2 \\ & + \sum_{k=1}^{N+1} \sum_{j=1}^m \max\{0, -\xi_{jk} - \epsilon^\gamma\}^2 + \sum_{k=1}^{N+1} \left\{ \sum_{j=1}^m \xi_{jk} - 1 - \epsilon^\gamma \right\}^2 \\ & + \sum_{i=1}^p \int_0^{N+1} \max\{0, \tilde{g}_i(s, \tilde{\mathbf{x}}(s), \boldsymbol{\xi}) - \epsilon^\gamma\}^2 ds + (t(N+1) - T - \epsilon^\gamma)^2 \\ & + \sum_{k=1}^{N+1} \max\{0, -\theta_k - \epsilon^\gamma\}^2. \end{aligned}$$

Here, α , β and γ are positive real numbers, and κ is a penalty parameter. Next, we define

$$\begin{aligned} S_\epsilon = & \left\{ (\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) \in \mathbb{R}^{N+1} \times \mathbb{R}^{m \times (N+1)} \times [0, \infty) : \right. \\ & t(N+1) - T = \epsilon^\gamma \\ & -\theta_k \leq \epsilon^\gamma, \quad k = 1, \dots, N+1, \\ & \sum_{j=1}^m \xi_{jk} - 1 = \epsilon^\gamma, \quad k = 1, \dots, N+1, \\ & \xi_{jk}(1 - \xi_{jk}) \leq \epsilon^\gamma, \quad j = 1, \dots, m, \quad k = 1, \dots, N+1, \\ & \xi_{jk} - 1 \leq \epsilon^\gamma, \quad j = 1, \dots, m, \quad k = 1, \dots, N+1, \\ & -\xi_{jk} \leq \epsilon^\gamma, \quad j = 1, \dots, m, \quad k = 1, \dots, N+1, \\ & \left. \sum_{k=1}^{N+1} \sum_{j=1}^m \xi_{jk} g_i(\tilde{\mathbf{x}}(s), \mathbf{u}_j) \chi_{[k-1, k)}(s) \leq \epsilon^\gamma, \quad i = 1, 2, \dots, p \right\}. \end{aligned} \tag{16}$$

Now, consider the following problem: Given the dynamical system (13)-(14), find a triple $(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) \in \mathbb{R}^{N+1} \times \mathbb{R}^{m \times (N+1)} \times [0, \infty)$ such that the penalty function $F_\kappa(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon)$ is minimized. This problem is referred to as Problem (\tilde{P}_κ) .

In the next section, we will see that, under some mild assumptions, when the penalty parameter κ is sufficiently large, the satisfaction of the constraints (10), (11), (12) and (15) will be achieved, i.e. $\Delta(\boldsymbol{\theta}, \boldsymbol{\xi}, \epsilon) = 0$ for $\epsilon = 0$. Furthermore, an optimal solution of Problem (\tilde{P}_κ) is an optimal solution of Problem (\tilde{P}) .

3.3 Convergence Results

To obtain our main result, we need the following definition.

Definition 1 Suppose that the following implication holds:

$$\sum_{i=1}^p \int_0^{N+1} \varphi_i(s) \frac{\partial \tilde{g}_i(s, \tilde{\mathbf{x}}(s), \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} ds = 0 \implies \varphi_i(s) = 0$$

for all $s \in [0, N+1]$, $i = 1, \dots, p$. Then, we say that the *constraint qualification* is satisfied for the continuous inequality constraints (15).

Let $\{\kappa_l\}_{l=1}^\infty$ be an increasing sequence of penalty parameters such that $\kappa_l \rightarrow \infty$. Furthermore, let $(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}, \epsilon^{(l),*})$ denote a local optimal solution of Problem (\tilde{P}_{κ_l}) . We assume that the following hypotheses are satisfied.

(H_1) The constraint qualification defined in Definition 1 is satisfied at $(\boldsymbol{\theta}, \boldsymbol{\xi}) = (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$, where $(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$ is a local optimal solution of Problem (\tilde{P}) .

(H_2) There exists real numbers $\delta^1 > 0$ and $\delta^2 > 0$ such that

$$\lim_{l \rightarrow \infty} \frac{\max\{0, G_\iota(s, \tilde{\mathbf{x}}(s), \boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*})\}}{(\epsilon^{(l),*})^{\delta^1}} = 0$$

and

$$\lim_{l \rightarrow \infty} \frac{H_\eta(s, \tilde{\mathbf{x}}(s), \boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*})}{(\epsilon^{(l),*})^{\delta^2}} = 0,$$

where G_ι , $\iota = 1, \dots, p + (3m + 1)(N + 1)$ and H_η , $\eta = 1, \dots, N + 2$, are, respectively, the inequality constraints and the equality constraints of Problem (\tilde{P}) .

Theorem 2 Suppose that $(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}, \epsilon^{(l),*}) \rightarrow (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*, \epsilon^*)$ as $l \rightarrow +\infty$, and that the hypotheses (H_1) - (H_2) are satisfied. Then, $\epsilon^* = 0$ and $(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*) \in S_0$, where S_0 is defined by (16) with $\epsilon = 0$.

Theorem 3 Suppose that $\gamma > \alpha$, $\delta = \min(\delta^1, \delta^2) > \alpha$, $2\delta > \alpha + 1$, $2\gamma > \alpha + 1$. Then

$$F_{\kappa_l}(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}, \epsilon^{(l),*}) \xrightarrow[\substack{\epsilon^{(l),*} \rightarrow \epsilon^* = 0 \\ (\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}) \rightarrow (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*) \in S_0}]{\epsilon^{(l),*} \rightarrow \epsilon^* = 0} F_{\kappa_l}(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*, 0) = \tilde{J}(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$$

$$\nabla F_{\kappa_l}(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}, \epsilon^{(l),*}) \xrightarrow[\substack{\epsilon^{(l),*} \rightarrow \epsilon^* = 0 \\ (\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}) \rightarrow (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*) \in S_0}]{\epsilon^{(l),*} \rightarrow \epsilon^* = 0} \nabla F_{\kappa_l}(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*, 0) = (\nabla \tilde{J}(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*), 0).$$

Theorem 4 Let $\epsilon^{(l),*} \rightarrow \epsilon^* = 0$, $(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}) \rightarrow (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*) \in S_0$ be such that $F_{\kappa_l}(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*, \epsilon^*)$ is finite, where S_0 is as defined by (16) with $\epsilon = 0$. Then, $(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$ is a local optimal solution of Problem (\tilde{P}) .

Theorem 5 Suppose that $(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}, \epsilon^{(l),*}) \rightarrow (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*, \epsilon^*)$ as $l \rightarrow +\infty$, and that the parameters α and γ satisfy the same conditions as in Theorem 3. Then, there exists a $l_0 > 0$ such that $\epsilon^{(l),*} = 0$ and $(\boldsymbol{\theta}^{(l),*}, \boldsymbol{\xi}^{(l),*}) = (\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$, for all $l \geq l_0$. Furthermore $(\boldsymbol{\theta}^*, \boldsymbol{\xi}^*)$ is a local optimal solution of Problem (P) .

The proofs of these theorems are quite similar to the proofs of Theorems 5.1-5.4 in [5] and Theorem 4 in [8]. Thus, they are omitted. From the results above, we can conclude that under some mild assumptions, for a sufficiently large κ , a local optimal solution of Problem (\tilde{P}_κ) is a local optimal solution of Problem (\tilde{P}) . Such a solution can then be used to construct a corresponding local solution of Problem (P) .

Problem (\tilde{P}_κ) is a standard optimal control problem with fixed switching points and can be readily solved by various existing optimal control techniques. Here, the optimal control software package MISER 3.3 [3] is used. In the next section, two practical problems concerning optimal driving strategies for trains are solved by the proposed method.

4 Numerical Results

4.1 Optimal Train Control on a Level Track

The following model for the motion of a train is given in references [2, 6]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varphi(x_2)u_1 + \zeta_2 u_2 + \rho(x_2), \end{aligned}$$

where x_1 is the train's distance along the track, x_2 is the train's speed, u_1 is the fuel setting and u_2 models the deceleration applied to the train by the brakes. The function φ , which models the tractive effort, is defined by

$$\varphi(x_2) = \begin{cases} \zeta_1/x_2, & \text{if } x_2 \geq \zeta_3 + \zeta_4, \\ \zeta_1/\zeta_3 + \eta_1(x_2 - \zeta_3 + \zeta_4)^2 & \text{if } \zeta_3 - \zeta_4 \leq x_2 < \zeta_3 + \zeta_4, \\ \quad + \eta_2(x_2 - \zeta_3 + \zeta_4)^3, & \\ \zeta_1/\zeta_3, & \text{if } x_2 < \zeta_3 - \zeta_4, \end{cases}$$

ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6	ζ_7
1.5	1	1.4	0.1	-0.015	-0.00003	-0.000006

Table 1 Values of ζ_i , $i = 1, \dots, 7$.

where $\zeta_1, \zeta_2, \zeta_3$ and ζ_4 are constants, and

$$\eta_1 = \zeta_1 \left[\left(\frac{1}{\zeta_3 + \zeta_4} - \frac{1}{\zeta_3} \right) \frac{3}{4\zeta_4^2} + \frac{1}{2\zeta_4(\zeta_3 + \zeta_4)^2} \right],$$

and

$$\eta_2 = \zeta_1 \left[- \left(\frac{1}{\zeta_3 + \zeta_4} - \frac{1}{\zeta_3} \right) \frac{3}{4\zeta_4^3} - \frac{1}{4\zeta_4^2(\zeta_3 + \zeta_4)^2} \right].$$

The function ρ , which models the resistive deceleration due to friction, is given by

$$\rho(x_2) = \zeta_5 + \zeta_6 x_2 + \zeta_7 x_2^2.$$

The constants ζ_i , $i = 1, \dots, 7$, are defined in Table 1. The initial and terminal states are

$$\mathbf{x}(0) = [0, 0]^T, \quad \mathbf{x}(1500) = [18000, 0]^T.$$

This means that the train starts from the origin at rest and comes to rest again 18,000 meters away at $t = 1500$. Since the train is not allowed to go backwards, a non-negativity constraint is imposed on the speed,

$$x_2(t) \geq 0, \quad t \in [0, 1500].$$

The train driver can choose from three operation modes for the train: accelerate (powered by the engine), coast (no power), and brake (decelerate by the brakes). These three modes correspond to the following values for $\mathbf{u} = [u_1, u_2]^T$: $[1, 0]^T$, $[0, 0]^T$, and $[0, -1]^T$. Thus, we have

$$\mathbf{U} = \{[1, 0]^T, [0, 0]^T, [0, -1]^T\}.$$

The objective is to minimize the fuel consumption, i.e.,

$$\min J(\mathbf{u}) = \int_0^{1500} u_1(t) dt.$$

Here, we assume that the maximum number of switches is $N = 2$. We apply our method in conjunction with MISER 3.3 to solve the problem.

Figure 1 and Figure 2 show the optimal trajectory of x_1 and x_2 , respectively. From the figures, we see that the train accelerates for the first quarter of the journey, then coasts almost until the end. The brakes are applied briefly at the end before the train stops.

Figure 4 and Figure 5 show the optimal controls u_1 and u_2 , respectively. We see that the control u_2 stays zero for almost the entire time horizon, and assumes the value -1 less than two seconds before the end.

The minimum fuel consumption is 205.06. This is slightly higher than the result of 202.67 reported in [6], which was obtained using the control

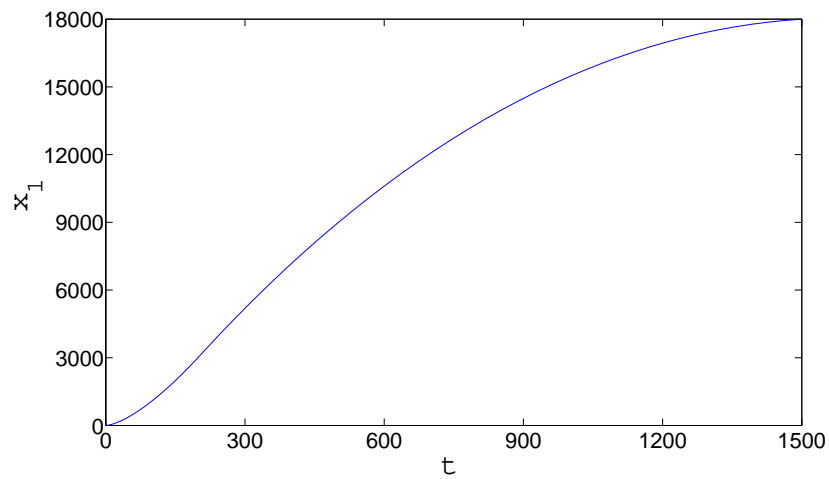


Fig. 1 The trajectory $x_1(t)$ against t

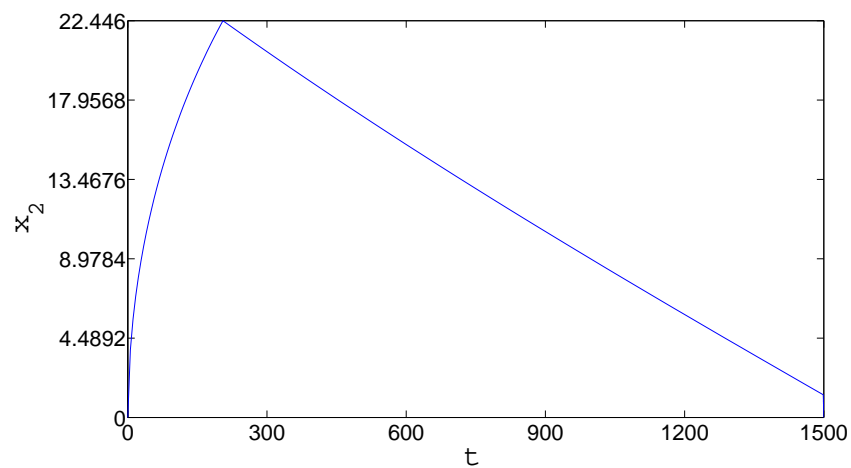


Fig. 2 The speed $x_2(t)$ against t

parametrization enhancing transform (CPET) directly with 6 switching points. It is worth noting that our method obtains the same result as in [6] when we increase the maximum number of switches to $N = 6$. More importantly, unlike CPET, our method ensures that the constraint on the maximum number of switches is always satisfied.

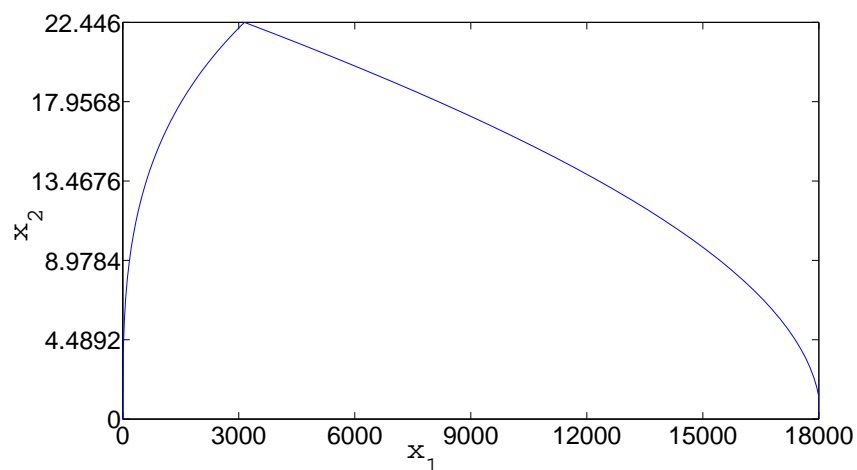


Fig. 3 The state space plot of x_2 against x_1

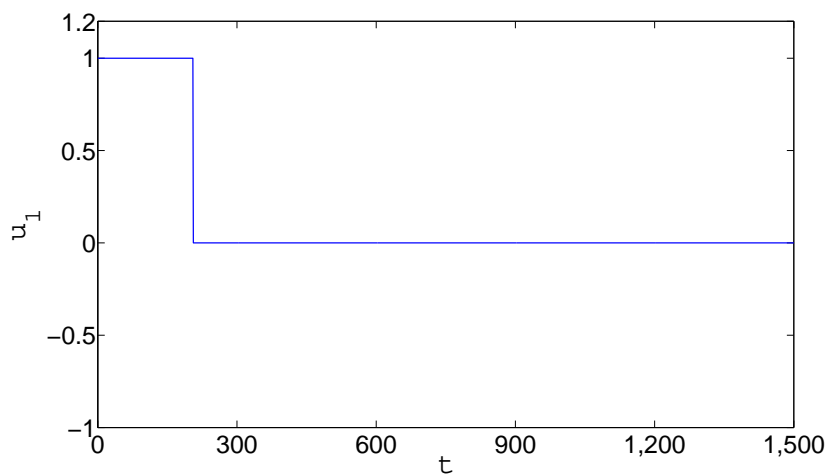


Fig. 4 The optimal control u_1 against t

4.2 Optimal Train Control on an Uneven Track

We now consider a more complicated train control problem [2, 7]. The dynamics for this problem are

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \varphi(x_2)u_1 + \zeta_2 u_2 + \rho(x_2) + \vartheta(x_1),\end{aligned}$$

where $x_1, x_2, u_1, u_2, \varphi(\cdot)$ and $\rho(\cdot)$ are as defined in Example 1, and $\xi_i, i = 1, \dots, 7$, are as defined in Table 1. The function $\vartheta(\cdot)$ is the gravitational ac-

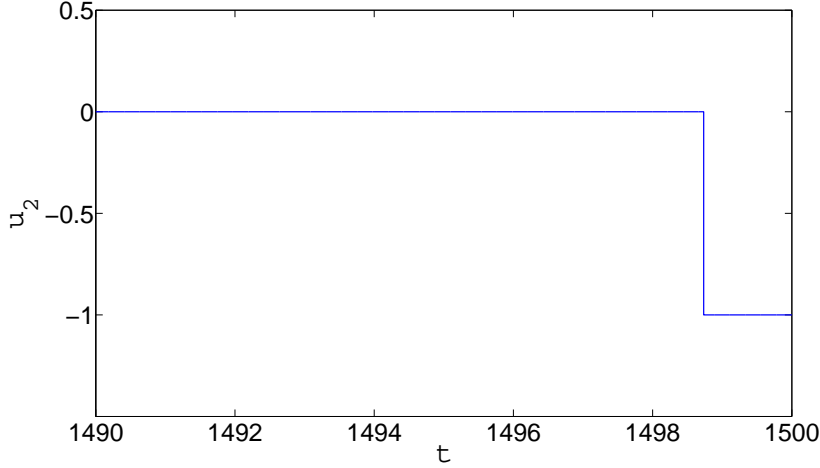


Fig. 5 The optimal control u_2 near the terminal time

celeration due to the non-constant gradient of the track given by:

$$\vartheta(x_1) = \begin{cases} 0, & \text{if } x_1 \leq 20000 - \zeta_8, \\ -0.05 \left\{ \frac{(x_1 - 20000)^2}{\zeta_8^2} + \frac{(x_1 - 20000)}{\zeta_8} + 1 \right\}, & \text{if } 20000 - \zeta_8 < x_1 \leq 20000, \\ -0.05 \left\{ -\frac{(x_1 - 20000)^2}{\zeta_8^2} + \frac{(x_1 - 20000)}{\zeta_8} + 1 \right\}, & \text{if } 20000 < x_1 \leq 20000 + \zeta_8, \\ -0.1, & \text{if } 20000 + \zeta_8 < x_1 \leq 25000 - \zeta_8, \\ -0.05 \left\{ -\frac{(x_1 - 25000)^2}{\zeta_8^2} - \frac{(x_1 - 25000)}{\zeta_8} + 1 \right\}, & \text{if } 25000 - \zeta_8 < x_1 \leq 25000, \\ -0.05 \left\{ \frac{(x_1 - 25000)^2}{\zeta_8^2} - \frac{(x_1 - 25000)}{\zeta_8} + 1 \right\}, & \text{if } 25000 < x_1 \leq 25000 + \zeta_8, \\ 0, & \text{if } x_1 > 25000 + \zeta_8, \end{cases}$$

where $\zeta_8 = 300$.

The initial and terminal states are

$$\mathbf{x}(0) = [0, 0]^T, \quad \mathbf{x}(2800) = [50000, 0]^T.$$

Again, we have a non-negativity constraint on x_2 to prevent the train from going backwards:

$$x_2(t) \geq 0, \quad t \in [0, 2800].$$

We also impose a speed limit on the train that decreases as the train moves further along the track:

$$0.0002x_1(t) + x_2(t) \leq 28, \quad t \in [0, 2800].$$

The control $\mathbf{u} = [u_1, u_2]^T$ is now restricted to the discrete set

$$\mathbf{U} = \{[1, 0]^T, [0, 0]^T, [0, -1]^T, [2, 0]^T\}.$$

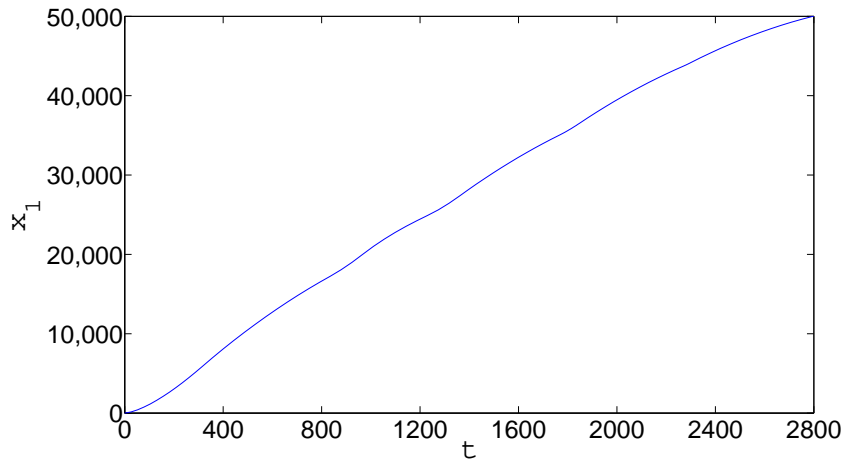


Fig. 6 The trajectory $x_1(t)$ against t

The objective is

$$\min J(\mathbf{u}) = \int_0^{2800} u_1(t) dt$$

Here, we assume that the maximum allowable number of switches is $N = 8$. Using our method, the problem is again solved by MISER 3.3. Figure 6 and Figure 7 show the optimal trajectory of x_1 and x_2 , respectively. Figure 8 and Figure 9 show the optimal controls u_1 and u_2 , respectively. Note that the optimal control does not assume the value $[2, 0]^T$. From Figure 10, we can see that the continuous inequality constraint is satisfied throughout the entire time horizon.

To solve this highly complex problem, we first used our method to determine the optimal switching sequence. After identifying the optimal switching sequence, we then applied CPET directly with the control sequence fixed to refine the switching times. The minimum fuel consumption is 937.42. This is better than the result obtained in [7], which uses the CPET directly with 18 switching points and has a larger fuel consumption of 938.63.

5 Conclusion

In this paper, a new computational method is proposed for solving optimal discrete-valued control problems. By introducing new controls and applying an equivalent transformation, the original problem becomes a standard optimal control problem subject to equality and inequality constraints. Then, an exact penalty method is employed to solve the transformed problem. Our numerical results for the train control problems in Section 4 show that this approach is superior to the direct application of the control parametrization

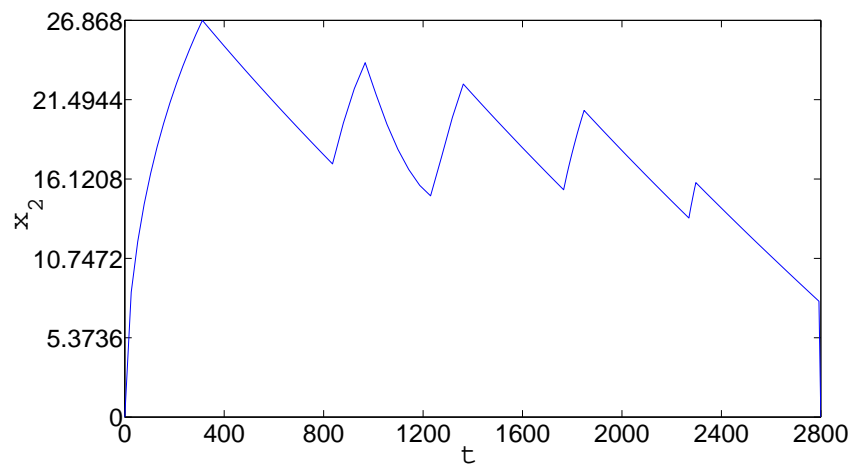


Fig. 7 The speed $x_2(t)$ against t

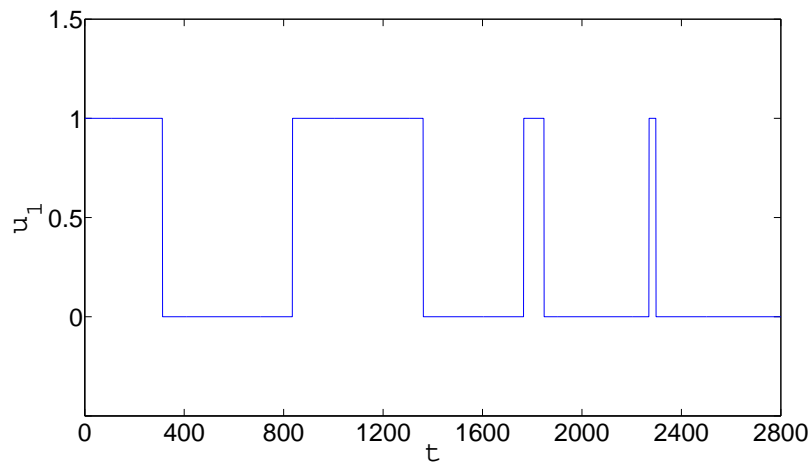


Fig. 8 The optimal control u_1 against t

enhancing transform, which involves introducing many artificial switches. Our optimal solutions require less switchings and always satisfy the constraint on the maximum allowable number of switchings.

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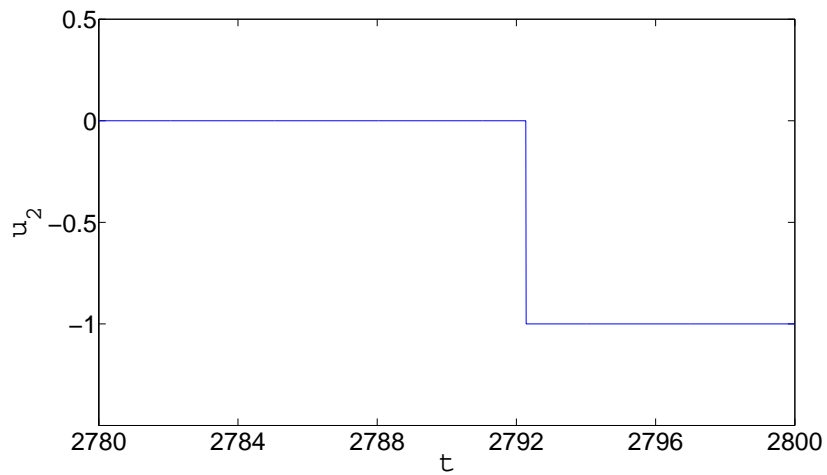


Fig. 9 The optimal control u_2 near the terminal time

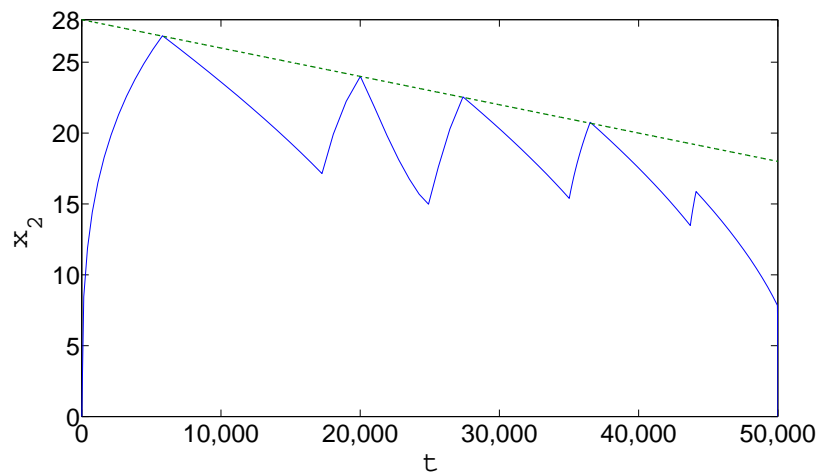


Fig. 10 The plot of speed limit constraint

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