A Feasible Semismooth Newton Method for A Class of Stochastic Linear Complementarity Problems

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Abstract. In this paper we consider a class of stochastic linear complementarity problems (SLCPs) with finitely many realizations. We present a feasible semismooth Newton method for this class of SLCPs by reformulating it as a constrained system of semismooth equations. This method only solves one linear system of equations at each iteration and has nice global and local convergence properties. Preliminary numerical results show that this method may yield a solution with high safety for SLCPs.

Key Words. Stochastic linear complementarity problem, constrained semismooth equations, semismooth Newton method, convergence.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with $\Omega \subseteq \Re^m$. We suppose that the probability distribution $\mathcal{P}$ is known. The stochastic linear complementarity problem (SLCP) [2, 3, 10, 6, 7] is to find an $x \in \Re^n$ such that

$$x \geq 0,\ F(x,\omega) := M(\omega)x + q(\omega) \geq 0,\ x^T F(x,\omega) = 0, \tag{1.1}$$

where $M(\omega) \in \Re^{n\times n}$ and $q(\omega) \in \Re^n$ for $\omega \in \Omega$, are random matrices and vectors.

If Ω only contains a single realization, then (1.1) reduces to the following standard linear complementarity problem (LCP): Find a vector $x \in \Re^n$ such that

$$x \geq 0,\ Mx + q \geq 0,\ x^T(Mx + q) = 0, \tag{1.2}$$

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where $M \in \Re^{n\times n}$ and $q \in \Re^n$. We denote this problem as LCP(M, q). Over the last few decades, there have been numerous publications on LCP(M, q). An excellent survey of the research in this area prior to 1990 can be found in [5]. More recent work can be found in the recent book [9] by Facchinei and Pang.

Problem (1.1) has been studied in [2, 3, 8, 10, 6, 7]. In particular, Chen and Fukushima [2] have recently proposed an expected residual minimization (ERM) formulation which is to find a vector $x \in \Re^n_+$ that minimizes an *expected residual function*:

$$\min_{x\in\Re^n_+} E\left\{\|\Phi(x,\omega)\|^2\right\}, \tag{1.3}$$

where E stands for the expectation and the function Φ is defined as follows:

$$\Phi(x,\omega) := \begin{pmatrix} \phi(F_1(x,\omega), x_1) \\ \phi(F_2(x,\omega), x_2) \\ \vdots \\ \phi(F_n(x,\omega), x_n) \end{pmatrix}. \tag{1.4}$$

Here, $\phi : \Re^2 \to \Re$ is an NCP function which has the property

$$\phi(a,b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Lin and Fukushima [6] formulated (1.1) as a stochastic MPEC with recourse.

In this paper we consider the following class of stochastic linear complementarity problems in which Ω only has finitely many elements. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_m\}$. Find an $x \in \Re^n$ such that

$$x \geq 0,\ F(x,\omega_i) := M(\omega_i)x + q(\omega_i) \geq 0,\ x^T F(x,\omega_i) = 0,\ i = 1,2,...,m,\ m > 1. \tag{1.5}$$

In this paper we suppose

$$p_i = \mathcal{P}\{\omega_i \in \Omega\} > 0,\ i = 1,2,...,m.$$

Let $\bar{F}(x)$ be the expectation function of the random function $F(x,\omega)$. Then,

$$\bar{F}(x) = E[F(x,\omega)] = \bar{M}x + \bar{q},$$

where

$$\bar{M} = \sum_{i=1}^m p_i M(\omega_i) \text{ and } \bar{q} = \sum_{i=1}^m p_i q(\omega_i).$$

We call (1.5) a *monotone SLCP* if $\bar{M}$ is a positive semidefinite matrix. Clearly, the problem (1.5) is equivalent to (1.6) and (1.7).

$$x \geq 0,\ \bar{M}x + \bar{q} \geq 0,\ x^T(\bar{M}x + \bar{q}) = 0. \tag{1.6}$$

$$M(\omega_i)x + q(\omega_i) \geq 0,\ i = 1,2,...,m. \tag{1.7}$$

(1.6) is a standard linear complementarity problem (LCP) and we denote it as LCP($\bar{M}, \bar{q}$).

In this paper we present a feasible semismooth Newton method for (1.5). This method may be viewed as the feasible semismooth asymptotically Newton method [19] applied to a reformulation of (1.5) as a constrained system of semismooth equations. This reformulation is introduced in Section 2, where we also state some preliminary and mostly known results about this reformulation. The algorithm is stated in detail in Section 3, whereas numerical results are given in Section 4. We then conclude this paper with some final remarks in Section 5.

Some words about our notation. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm. For a continuously differentiable function $\Phi : \Re^n \to \Re^n$, we denote the Jacobian of Φ at $x \in \Re^n$ by $\Phi'(x)$. Let $F : \Re^n \to \Re^m$ be a locally Lipschitzian vector function. By Rademacher's theorem, F is differentiable almost everywhere. Let D_F denote the set of points where F is differentiable. Then the B-subdifferential of F at $x \in \Re^n$ is defined to be

$$\partial_B F(x) = \{ \lim_{\substack{x^k \to x \\ x^k \in D_F}} \nabla F(x_k)^T \}, \tag{1.8}$$

while Clarke's generalized Jacobian [4] of F at x is defined to be

$$\partial F(x) = \text{conv} \partial_B F(x). \tag{1.9}$$

Qi [16] and Sun and Han [18] introduced a generalized Jacobian ∂_C, defined by

$$\partial_C F(x) = \partial F_1(x) \times \cdots \times \partial F_m(x). \tag{1.10}$$

By (1.8) - (1.10), for any x,

$$\partial_B F(x) \subseteq \partial F(x) \subseteq \partial_C F(x). \tag{1.11}$$

F is called semismooth at x if F is directionally differentiable at x and for all $V \in \partial F(x+d)$ and $d \to 0$,

$$F'(x; d) = Vd + o(\|d\|); \tag{1.12}$$

F is called strongly semismooth at x if F is semismooth at x and for all $V \in \partial F(x+d)$ and $d \to 0$,

$$F'(x; d) = Vd + O(\|d\|^2); \tag{1.13}$$

F is called a (strongly) semismooth function if it is (strongly) semismooth everywhere. Here, $o(\|d\|)$ stands a vector function of d, satisfying

$$\lim_{d \to 0} \frac{o(\|d\|)}{\|d\|} = 0,$$

while $O(\|d\|^2)$ stands a vector function of d, satisfying

$$\|O(\|d\|^2)\| \leq c\|d\|^2$$

for all d satisfying $\|d\| \leq \delta$, and some $c > 0$ and $\delta > 0$.

2 A Constrained Semismooth Equation (CSE) Reformulation

In this section, we reformulate the problem (1.5) as a constrained system of semismooth equations and give some properties for this system of semismoth equations.

(1.6) is a standard linear complementarity problem (LCP) and it can be reformulated as a system of semismooth equations by an NCP function. Over the last decade, various NCP functions have been studied for solving complementarity problems [9, 17]. Two frequently used NCP functions are the "min" function

$$\phi_{\min}(a, b) = \min(a, b)$$

and the Fischer-Burmeister (FB) function [12]

$$\phi_{\mathrm{FB}}(a, b) = a + b - \sqrt{a^2 + b^2}.$$

Let

$$\phi_\alpha(a, b) = a + b - \sqrt{a^2 + b^2} + \alpha a_+ b_+, \alpha \geq 0, \tag{2.14}$$

where for any scalar c, $c_+ = \max\{0, c\}$. Obviously,

$$\phi_0(a, b) = \phi_{FB}(a, b).$$

For $\alpha > 0$, this function is an equivalent form of the penalized Fischer-Burmeister function [1]. It has been showed in [1] that this function has a strong result about the level set.

Proposition 2.1 *[1, 17] $\phi_\alpha(a, b)$ has the following properties:*

1. *$\phi_\alpha(a, b)$ is an NCP function.*
2. *$\phi_\alpha(a, b)$ is strongly semismooth on $\Re^2$.*
3. *$\phi_\alpha(a, b)^2$ is continuously differentiable on $\Re^2$.*

4. *The generalized gradient $\partial\phi_\alpha(a,b)$ of ϕ_α at a point $(a,b) \in \Re^2$ is equal to the set of all (v_a, v_b) such that*

$$(v_a, v_b) = \begin{cases} (1 - \frac{a}{\sqrt{a^2+b^2}}, 1 - \frac{b}{\sqrt{a^2+b^2}}) + \alpha(b_+\partial a_+, a_+\partial b_+) & \text{if } (a,b) \neq (0,0), \\ (1-\xi, 1-\eta) & \text{if } (a,b) = (0,0), \end{cases} \tag{2.15}$$

where (ξ, η) is any vector satisfying $\sqrt{\xi^2+\eta^2} \leq 1$ and

$$\partial c_+ = \begin{cases} 1 & \text{if } c > 0, \\ [0,1] & \text{if } c = 0, \\ 0 & \text{if } c < 0. \end{cases}$$

Let

$$\Phi_\alpha(x) := \begin{pmatrix} \phi_\alpha((\bar{M}x+\bar{q})_1, x_1) \\ \phi_\alpha((\bar{M}x+\bar{q})_2, x_2) \\ \vdots \\ \phi_\alpha((\bar{M}x+\bar{q})_n, x_n) \end{pmatrix}. \tag{2.16}$$

Then x is a solution of (1.6) if and only if $\Phi_\alpha(x) = 0$. Define

$$\Psi_\alpha(x) = \frac{1}{2}\|\Phi(x)\|^2.$$

Then solving (1.6) is equivalent to finding a global solution of the following minimization problem:

$$\min_{x\in\Re^n} \Psi_\alpha(x). \tag{2.17}$$

Proposition 2.2 *[1, 17] The function Ψ_α has the following properties:*

1. *$\Psi_\alpha(x)$ is continuously differentiable on $\Re^n$.*

2. *If LCP($\bar{M}, \bar{q}$) has a strictly feasible solution, then, for all $c \geq 0$, the level sets*

$$\mathcal{L}_\alpha(c) = \{x \in \Re^n : \Psi_\alpha(x) \leq c\}$$

are bounded.

By introducing a slack variable $y = [y_1^T, y_2^T, ..., y_m^T]^T \in \Re^{mn}$, where $y_i \in \Re^n, i = 1, 2, ..., m$, (1.6) and (1.7) can be written as the following system of nonsmooth equations with nonnegative constraints:

$$H(x, y) = 0, \ y \geq 0, \tag{2.18}$$

where

$$H(x,y) = \begin{pmatrix} \Phi_\alpha(x) \\ M(\omega_1)x + q(\omega_1) - y_1 \\ M(\omega_2)x + q(\omega_2) - y_2 \\ \vdots \\ M(\omega_m)x + q(\omega_m) - y_m \end{pmatrix}.$$

The system $H(x,y) = 0$ has $(m+1)n$ equations with $(m+1)n$ unknowns. Problem (1.5) may have no solution. If (1.5) has a solution, then solving (1.5) is equivalent to solving (2.18).

Proposition 2.3 *The function H is strongly semismooth.*

Proof. By Theorem 19 [13] and Proposition 2.1, the function H is strongly semismooth. ∎

Suppose that x^* is a solution of LCP$(\bar{M}, \bar{q})$. Define

$$\begin{aligned} \mathcal{I} &= \{i : x_i^* > 0, (\bar{M}x^* + \bar{q})_i = 0\}, \\ \mathcal{J} &= \{i : x_i^* = 0, (\bar{M}x^* + \bar{q})_i = 0\}, \\ \mathcal{K} &= \{i : x_i^* = 0, (\bar{M}x^* + \bar{q})_i > 0\}. \end{aligned}$$

LCP$(\bar{M}, \bar{q})$ is said to be R-regular at x^* if $\bar{M}_{\mathcal{II}}$ is nonsingular and its Schur complement $\bar{M}_{\mathcal{JJ}} - \bar{M}_{\mathcal{JI}}\bar{M}_{\mathcal{II}}^{-1}\bar{M}_{\mathcal{IJ}}$ is a P-matrix [9].

Proposition 2.4 *[1] Suppose that LCP$(\bar{M}, \bar{q})$ is R-regular at a solution x^*. Then all $V_\Phi \in \partial_C\Phi_\alpha(x^*) = \partial\Phi_{\alpha,1}(x^*) \times \partial\Phi_{\alpha,2}(x^*) \times \ldots \times \partial\Phi_{\alpha,n}(x^*)$ are nonsingular.*

For any $(x,y) \in \Re^{(m+1)n}$, we have that

$$\partial_C H(x,y) = \left\{ \begin{pmatrix} V_\Phi & 0 & 0 & \cdots & 0 \\ M(\omega_1) & -I & 0 & \cdots & 0 \\ M(\omega_2) & 0 & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M(\omega_m) & 0 & 0 & \cdots & -I \end{pmatrix} : V_\Phi \in \partial_C\Phi_\alpha(x) \right\},$$

where I is the $n \times n$ identity matrix. Hence, by Proposition 2.4, we have the following proposition.

Proposition 2.5 *Suppose (x^*, y^*) is a solution of (2.18) and LCP$(\bar{M}, \bar{q})$ is R-regular at x^*. Then all $V_H \in \partial_C H(x^*, y^*)$ are nonsingular.*

For any $x \in \Re^n$, an element of the C-subdifferential $\partial_C\Phi_\alpha(x)$ can be calculated as follows [1].

Algorithm 2.1 *(Procedure to calculate an element $V_\Phi \in \partial_C \Phi_\alpha(x)$)*

(S.0) Let $x \in \Re^n$ be given, and let $V_{\Phi,i}$ denote the ith row of a matrix $V_\Phi \in \Re^{n\times n}$.

(S.1) Set $S_1 = \{i : x_i = (\bar{M}x + \bar{q})_i = 0\}$ and $S_2 = \{i : x_i > 0, (\bar{M}x + \bar{q})_i > 0\}$.

(S.2) Set $c \in \Re^n$ such that $c_i = 0$ for $i \notin S_1$ and $c_i = 1$ for $i \in S_1$.

(S.3) For $i \in S_1$, set

$$V_{\Phi,i} = \left(1 - \frac{c_i}{\|(c_i, \bar{M}_i c)\|}\right) e_i^T + \left(1 - \frac{\bar{M}_i c}{\|(c_i, \bar{M}_i c)\|}\right) \bar{M}_i,$$

where $\bar{M}_i$ is the ith row of the matrix $\bar{M}$.

(S.4) For $i \in S_2$, set

$$V_{\Phi,i} = \left(1 - \frac{c_i}{\|(c_i, (\bar{M}x + \bar{q})_i)\|} + \alpha(\bar{M}x + \bar{q})_i\right) e_i^T + \left(1 - \frac{(\bar{M}x + \bar{q})_i}{\|(c_i, (\bar{M}x + \bar{q})_i)\|} + \alpha x_i\right) \bar{M}_i.$$

(S.5) For $i \notin S_1 \cup S_2$, set

$$V_{\Phi,i} = \left(1 - \frac{c_i}{\|(c_i, (\bar{M}x + \bar{q})_i)\|}\right) e_i^T + \left(1 - \frac{(\bar{M}x + \bar{q})_i}{\|(c_i, (\bar{M}x + \bar{q})_i)\|}\right) \bar{M}_i.$$

3 A Feasible Semismooth Newton Algorithm

Let $z = (x, y) \in \Re^{(m+1)n}$ and define a merit function of (2.18) by

$$\theta(z) = \frac{1}{2}\|H(z)\|^2.$$

If (1.5) has a solution, then solving (2.18) is equivalent to finding a global solution of the following minimization problem:

$$\begin{aligned} \min \quad & \theta(z) \\ s.t. \quad & z \geq 0. \end{aligned} \tag{3.19}$$

z is a stationary point of (3.19) if it satisfies

$$\Pi_Z[z - \nabla\theta(z)] - z = 0, \tag{3.20}$$

where $\Pi_Z(\cdot)$ is an orthogonal projection operator onto $Z = \{z \in \Re^{(m+1)n} : z \geq 0\}$. Clearly, (3.20) is equivalent to the following complementarity system:

$$z^T \nabla\theta(z) = 0, \nabla\theta(z) \geq 0, z \geq 0. \tag{3.21}$$

Proposition 3.1 *The merit function θ has the following properties:*

1. *$\theta(z)$ is continuously differentiable on $\Re^{(m+1)n}$ with $\nabla\theta(z) = V_H^T H(z)$ for any $V_H \in \partial_C H(z)$.*

2. *If LCP($\bar{M}, \bar{q}$) has a strictly feasible solution, then for all $c \geq 0$, the level sets*

$$\mathcal{L}(c) = \{z \in \Re^{(m+1)n} : \theta(z) \leq c\}$$

are bounded.

Proof. By using Proposition 2.2, this proposition holds. ∎

For some monotone SLCPs, a stationary point of (3.19) may not be a solution. See the following example.

Example 3.1. Let $n = 1$, $m = 2$, $\Omega = \{\omega_1, \omega_2\} = \{0, 1\}$, $M(\omega_1) = M(\omega_2) = 1$, $q(\omega_1) = 1$, $q(\omega_2) = -1$, and $p_i = \mathcal{P}\{\omega_i \in \Omega\} = 0.5,\ i = 1, 2$.

Clearly, Example 3.1 is a monotone SLCP and all points $x \geq 1$ are feasible, but this example has no solution. By simple computation, we have

$$H(x, y_1, y_2) = \begin{pmatrix} 2x + \sqrt{2}|x| + \alpha(x_+)^2 \\ x + 1 - y_1 \\ x - 1 - y_2 \end{pmatrix}$$

and $(0, 1, 0)$ is a stationary point of the following constrained programming problem:

$$\begin{array}{ll} \min & \dfrac{1}{2}\|H(x, y_1, y_2)\|^2 \\ s.t. & x \geq 0,\ y_1 \geq 0,\ y_2 \geq 0. \end{array}$$

However, $x = 0$ is not a solution of Example 3.1.

In the following proposition, we give some conditions under which a stationary point of (3.19) is a solution of (1.5).

Proposition 3.2 *For monotone problem (1.5), let $z^* = (x^*, y^*)$ be a stationary point of (3.19). If $M(\omega_i)x^* + q(\omega_i) - y_i^* = 0,\ i = 1, 2, ..., m$, then x^* is a solution.*

Proof. Suppose that $z^* = (x^*, y^*)$ be a stationary point of (3.19). If $M(\omega_i)x^* + q(\omega_i) - y_i^* = 0,\ i = 1, 2, ..., m$, then, by (3.21), we have x^* is the stationary point of the following problem:

$$\min\{\Psi_\alpha(x) : x \geq 0\}.$$

Similar to the proof of Theorem 3 [11], we have x^* is a solution of $\Psi_\alpha(x) = 0$. Thus x^* is a solution of (1.5). ∎

Now we state a feasible semismooth Newton algorithm for solving (3.19).

Algorithm 3.1

Step 0. Choose constants $\eta, \rho, \sigma \in (0,1)$, $p_1 > 0$ and $p_2 > 2$. Let $z^0 \in Z$ and set $k := 0$.

Step 1. Choose $V_H^k \in \partial_C H(z^k)$ and compute $\nabla\theta(z^k) = (V_H^k)^T H(z^k)$. If z^k is a stationary point, stop. Otherwise let

$$d_G^k = -\gamma_k \nabla\theta(z^k),$$

where

$$\gamma_k = \min\left\{1, \frac{\eta\theta(z^k)}{\|\nabla\theta(z^k)\|^2}\right\}$$

and go to Step 2.

Step 2. Compute d_N^k by solving the following linear system:

$$H(z^k) + V_H^k d = 0. \tag{3.22}$$

If (3.22) has no solution or

$$-\nabla\theta(z^k)^T d_N^k < p_1 \left\| d_N^k \right\|^{p_2},$$

then let $d_N^k := d_G^k$.

Step 3. Let m_k be the smallest nonnegative integer m satisfying

$$\theta(z^k + \bar{d}^k(\rho^m)) \le \theta(z^k) + \sigma\nabla\theta(z^k)^T \tilde{d}_G^k(\rho^m), \tag{3.23}$$

where for any $\lambda \in (0,1]$,

$$\bar{d}^k(\lambda) = \tau^*(\lambda)\tilde{d}_G^k(\lambda) + [1 - \tau^*(\lambda)]\tilde{d}_N^k(\lambda). \tag{3.24}$$

Here

$$\tilde{d}_G^k(\lambda) := \Pi_Z(z^k + \lambda d_G^k) - z^k, \quad \tilde{d}_N^k(\lambda) := \Pi_Z(z^k + \lambda d_N^k) - z^k, \tag{3.25}$$

$\tau^*(\lambda)$ is a solution of the following minimization problem:

$$\min_{\tau\in[0,1]} \frac{1}{2}\|H(z^k) + V_H^k[\tau\tilde{d}_G^k(\lambda) + (1-\tau)\tilde{d}_N^k(\lambda)]\|^2.$$

Let $\lambda_k = \rho^{m_k}$ and $z^{k+1} = z^k + \bar{d}^k(\lambda_k)$.

Step 4. Set $k := k + 1$ and go to Step 1.

Remark 1. (a) For a constrained system of semismooth equations, some methods have been proposed in [11, 14, 19]. Algorithm 3.1 was proposed in [19] to solve constrained system of semismooth equations. From [19], we have

$$\tau^*(\lambda) = \max\{0, \min\{1, \tau(\lambda)\}\}, \tag{3.26}$$

where $\tau(\lambda)$ is defined as

$$\tau(\lambda) = \begin{cases} 0, & \text{if } \tilde{d}_G^k(\lambda) - \tilde{d}_N^k(\lambda) = 0, \\ -\dfrac{[H(w^k) + V_H^k \tilde{d}_N^k(\lambda)]^T V_H^k [\tilde{d}_G^k(\lambda) - \tilde{d}_N^k(\lambda)]}{\|V_H^k[\tilde{d}_G^k(\lambda) - \tilde{d}_N^k(\lambda)]\|^2}, & \text{otherwise.} \end{cases}$$

(b) For Algorithm 3.1, the main computation work is in Step 2. The linear system in Step 2 can be solved as follows. Firstly, we solve the following linear system to get d_Φ^k:

$$V_\Phi^k d_\Phi^k + \Phi_\alpha(x^k) = 0. \tag{3.27}$$

Let $d_i^k = M(\omega_i)d_\Phi^k + M(\omega_i)x^k + q(\omega_i) - y_i^k, i = 1, 2, ..., m$. Then, $d_N^k = [(d_\Phi^k)^T, (y_1^k)^T, ..., (y_m^k)^T]^T$.

Note that (3.27) is an $n \times n$ linear system. If n is not large, then the linear system in Step 2 of Algorithm 3.1 can be solved easily.

The convergence properties of this method are summarized in the following theorem.

Theorem 3.1 *The following results holds for Algorithm 3.1.*

1. *The algorithm is well defined for a monotone SLCP (1.5). If the algorithm does not stop at a stationary point in finite steps, an infinite sequence $\{z^k\}$ is generated with $\{z^k\} \subset Z$.*

2. *Any accumulation point of the sequence $\{z^k\}$ is a stationary point of (3.19). In particular, if LCP($\bar{M}, \bar{q}$) has a strictly feasible solution, such an accumulation point exists.*

3. *Let $z^* = (x^*, y^*)$ be any accumulation point. If $M(\omega_i)x^* + q(\omega_i) - y_i^* = 0,\ i = 1, 2, ..., m$, then x^* is a solution of (1.5).*

4. *If, in addition, x^* is an R-regular solution of LCP($\bar{M}, \bar{q}$), then the whole sequence generated by Algorithm 3.1 converges to z^* and the rate of convergence is Q-quadratic.*

Proof. These results can be proved by following from Propositions 2.5, 3.1 and 3.2 and the same arguments used for Theorems 4.1 and 4.2 of [19]. Instead of using Qi's [15] theory on the B-subdifferential, however, we have to apply the local convergence results for the C-subdifferential, also due to Qi [16]. ∎

Remark 2. Algorithm 3.1 can produce a stationary point x^* of (3.19). If x^* is a global solution of (3.19), then x^* can be considered as a solution of (1.5) with the measure of $\theta(z)$ defined in (3.19). If x^* is not a global solution of (3.19), we do not know what the stationary point x^* of (3.19) means in SLCP. We leave it as our future research topic.

4 Numerical Results

In this section we report numerical results for the algorithm proposed in Section 3. We implemented the algorithm in MATLAB and tested it on some random generated examples. Now we give a procedure to generate a test problem of monotone SLCP (1.5). This procedure is from [3]. We suppose that $p^j = \mathcal{P}\{\omega_j \in \Omega\} = 1/m,\ j = 1, 2, ..., m.$

Procedure 1.

1. Randomly generate a vector $\hat{x} \in \Re^n$ that has $n_x(< n)$ elements are in $(0, c_1)$, $c_1 > 0$, and other elements are 0. Let $\mathcal{J} = \{i : \hat{x}_i > 0\}$.

2. Generate a diagonal matrix D whose elements are determined as

$$D_{ii} = \begin{cases} 1/\mu & i = 1 \\ \mu^{\lambda_i} & i = 2, 3, ..., n-1 \\ \mu & i = n, \end{cases}$$

   where $\mu > 0$ and $\lambda_i, i = 2, 3, ..., n-1$ are uniform variate in the interval $(-1, 1)$.

3. Generate a random orthogonal matrix $U \in \Re^{n\times n}$ by using the singular value decomposition of a random matrix, and let $\bar{M} = UDU^T$.

4. Generate m random matrices $B^j \in \Re^{n\times n}, j = 1, 2, ..., m$ whose elements are in the interval $(0, 1)$. Set

$$M^j = M(\omega_j) = \bar{M} + c_2(B^j - B^{m-j+1}), j = 1, 2, ..., m,$$

   where $c_2 > 0$.

5. For each $j = 1, 2, ..., m$, let n_I be the number of elements in the index set $\mathcal{I}_j = \{i : \hat{x}_i = 0, [M(\omega_j)\hat{x} + q(\omega_j)]_i > 0\}$ and n_K be the number of elements in the index set $\mathcal{K}_j = \{i : \hat{x}_i = 0, [M(\omega_j)\hat{x} + q(\omega_j)]_i = 0\}$. Set

$$q_i^j = [q(\omega_j)]_i = \begin{cases} (-M^j\hat{x})_i & i \in \mathcal{K}_j \\ (-M^j\hat{x} + c_3 v^j)_i & i \in \mathcal{J} \\ (-M^j\hat{x} + c_4 v^j)_i & i \in \mathcal{I}_j, \end{cases}$$

where $c_3, c_4 \geq 0$ and $v^j \in \Re^n$ is a random vector whose elements are in the interval $(0, 1)$.

Obviously, for the above random generated test problem, if $c_3 = 0$ then $\hat{x}$ is a solution. If $c_3 > 0$ then the test problem may have no solution.

Throughout our computational experiments, the parameters used in Algorithm 3.1 are as follows:

$$\alpha = 10, \eta = 0.9, \rho = 0.5, \sigma = 10^{-4}, p_1 = 10^{-10} \text{ and } p_2 = 2.1.$$

We terminated the iteration if

$$\theta(z^k) \leq 10^{-12} \text{ or } \|\nabla(z^k)\| \leq 10^{-10} \text{ or } k_{max} = 100.$$

We first tested Algorithm 3.1 on some random generated problems with $c_3 = 0$ by Procedure 1 with different parameters $(n, m, \mu, c_1, c_2, c_4)$ and starting points $x^0 = le$, where $l = 1, 10, 20, 30, 40, 50$ and e is the n-dimensional vector of ones. These random generated problems can be reformulated as an ERM problem (1.3) in which the expected residual function is defined by the penalized Fischer-Burmeister function defined in (2.14). The ERM problem is a constrained optimization problem, so it can be solved by using `fmincon` in the Matlab (version 7) tool box for constrained optimization. We also tested `fmincon` on these random generated problems. Since $c_3 = 0$, these random generated problems have a solution. We have observed that for each starting point Algorithm 3.1 can compute a solution in less than 20 iterations. However, `fmincon` only can find a local minimizer of the ERM problem (1.3).

Next, for each fixed (n, n_x, c_2, c_3) with $c_3 > 0$, we used Procedure 1 to generate some test problems with the following parameters:

$$c_1 = 20, \mu = 10, c_4 = 15, m = 100.$$

Since $c_3 > 0$, these test problem may have no solutions. For these test problems, a measure of optimality and feasibility can be defined as in [3] as follows:

$$\Gamma(x) = \sum_{i=1}^{m} [\| \min(0, M(\omega_i)x + q(\omega_i))\| + x^T(M(\omega_i)x + q(\omega_i))_+], \; x \in \Re^n_+. \tag{4.28}$$

Clearly, if x^* is a solution of the problem (1.5), then $\Gamma(x^*) = 0$. Let

$$Op(x) = \sum_{i=1}^{m} x^T(M(\omega_i)x + q(\omega_i))_+, \; x \in \Re^n_+ \tag{4.29}$$

and

$$Fe(x) = \sum_{i=1}^{m} \| \min(0, M(\omega_i)x + q(\omega_i))\|, \; x \in \Re^n_+. \tag{4.30}$$

Here, the function $Op(x)$ is a measure of optimality and the function $Fe(x)$ is a measure of feasibility. For these random generated test problems, we use `fmincon` to get solutions of their ERM formulations. Their CSE reformulations are solved by using Algorithm 3.1. The test results are summarized in Table 1. In Table 1, the random generated test problems are denoted as (n, n_x, c_2, c_3, le), where le means that the starting point is le for Algorithm 3.1 and `fmincon`. x^1 and x^2 are the computed solutions by `fmincon` and Algorithm 3.1, respectively.

The results reported in Table 1 show that as to the measure of optimality and feasibility $\Gamma(\cdot)$, x^2 dominates x^1 in most cases. As to the measure of optimality $Op(\cdot)$, x^1 dominates x^2 in most cases, whereas x^2 dominates x^1 in most cases as to the measure of feasibility $Fe(\cdot)$. From these results, we may conclude that the CSE formulation can yield a solution with a nice property in regard to the measure of feasibility, compared with the ERM formulation.

In many engineering and economic applications of SLCP, the inequality $F(x, \omega) = M(\omega)x + q(\omega) \geq 0$ describes the safety of the system, and the guarantee of safety is critically important. For these kinds of problems, the CSE formulation may yield a solution with high safety.

The numerical tests reported in the paper are very preliminary. Further experience with testing and with actual applications will be necessary and we leave it as our future research topic.

Problem	fmincon			Algorithm 3.1		
(n, n_x, c_2, c_3, le)	$Fe(x^1)$	$Op(x^1)$	$\Gamma(x^1)$	$Fe(x^2)$	$Op(x^2)$	$\Gamma(x^2)$
(30, 10, 20, 10, e)	6.17e4	4.87e2	6.22e4	3.09e1	2.75e4	2.75e4
(30, 10, 20, 10, 10e)	5.67e4	3.78e4	9.44e4	9.33e2	3.62e4	3.71e4
(30, 10, 20, 10, 20e)	6.17e4	6.28e2	6.23e4	3.09e1	2.75e4	2.75e4
(30, 10, 20, 10, 30e)	6.17e4	6.85e2	6.24e4	3.37e2	2.86e4	2.89e4
(30, 10, 20, 10, 40e)	6.17e4	6.85e2	6.24e4	6.06e2	3.07e4	3.13e4
(30, 10, 20, 10, 50e)	6.17e4	6.06e2	6.23e4	3.09e1	2.75e4	2.75e4
(30, 10, 10, 10, e)	2.73e4	3.95e2	2.77e4	4.97e1	2.25e4	2.26e4
(30, 10, 10, 10, 10e)	1.62e4	4.44e4	6.06e4	4.97e1	2.25e4	2.26e4
(30, 10, 10, 10, 20e)	2.74e4	3.27e2	2.77e4	4.97e1	2.25e4	2.26e4
(30, 10, 10, 10, 30e)	2.74e4	2.31e2	2.76e4	4.97e1	2.25e4	2.26e4
(30, 10, 10, 10, 40e)	2.74e4	3.27e2	2.77e4	4.97e1	2.25e4	2.26e4
(30, 10, 10, 10, 50e)	2.74e4	1.77e2	2.76e4	2.53e2	2.24e4	2.27e4
(30, 10, 10, 5, e)	3.13e4	3.82e2	3.17e4	2.77e1	1.41e4	1.41e4
(30, 10, 10, 5, 10e)	2.32e4	8.04e4	1.04e5	6.10e2	1.93e4	1.99e4
(30, 10, 10, 5, 20e)	3.13e4	3.82e2	3.17e4	2.69e2	1.53e4	1.55e4
(30, 10, 10, 5, 30e)	3.13e4	3.82e2	3.17e4	6.42e2	2.12e4	2.18e4
(30, 10, 10, 5, 40e)	3.13e4	1.88e2	3.15e4	2.77e1	1.41e4	1.41e4
(30, 10, 10, 5, 50e)	3.13e4	3.82e2	3.17e4	2.77e1	1.41e4	1.41e4
(30, 10, 0, 5, e)	3.52e2	4.32e0	3.57e2	4.04e1	5.46e3	5.50e3
(30, 10, 0, 5, 10e)	3.52e2	4.32e0	3.57e2	3.92e1	5.77e3	5.80e3
(30, 10, 0, 5, 20e)	3.52e2	4.32e0	3.57e2	4.04e1	5.46e3	5.50e3
(30, 10, 0, 5, 30e)	3.52e2	4.32e0	3.57e2	4.04e1	5.46e3	5.50e3
(30, 10, 0, 5, 40e)	1.88e3	1.13e5	1.15e5	4.26e1	5.77e3	5.82e3
(30, 10, 0, 5, 50e)	1.44e3	2.03e5	2.04e5	4.04e1	5.46e3	5.50e3

Table 1: Test results for Algorithm 3.1 and `fmincon`

Problem	fmincon			Algorithm 3.1		
(n, n_x, c_2, c_3, le)	$Fe(x^1)$	$Op(x^1)$	$\Gamma(x^1)$	$Fe(x^2)$	$Op(x^2)$	$\Gamma(x^2)$
(60, 20, 20, 10, e)	1.44e5	1.98e3	1.46e5	4.25e3	1.35e5	1.40e5
(60, 20, 20, 10, 10e)	1.19e5	4.86e5	6.05e5	3.15e3	1.17e5	1.20e5
(60, 20, 20, 10, 20e)	1.44e5	4.41e2	1.44e5	3.79e3	1.20e5	1.24e5
(60, 20, 20, 10, 30e)	1.44e5	2.08e2	1.44e5	6.07e3	1.75e5	1.82e5
(60, 20, 20, 10, 40e)	1.44e5	1.28e2	1.44e5	2.10e3	9.59e4	9.80e4
(60, 20, 20, 10, 50e)	1.44e5	9.15e2	1.45e5	9.15e3	2.48e5	2.57e5
(60, 20, 10, 10, e)	5.64e4	8.21e2	5.72e4	1.59e3	7.12e4	7.28e4
(60, 20, 10, 10, 10e)	7.04e4	3.29e6	3.36e6	9.31e2	5.58e4	5.67e4
(60, 20, 10, 10, 20e)	5.64e4	7.50e2	5.71e4	2.21e3	7.82e4	8.04e4
(60, 20, 10, 10, 30e)	5.63e4	8.76e2	5.72e4	2.50e3	8.30e4	8.55e4
(60, 20, 10, 10, 40e)	5.64e4	4.09e2	5.68e4	7.60e2	5.64e4	5.72e4
(60, 20, 10, 10, 50e)	5.64e4	7.92e1	5.65e4	8.60e2	5.68e4	5.77e4
(60, 20, 10, 5, e)	6.11e4	1.09e3	6.22e4	5.75e2	3.34e4	3.40e4
(60, 20, 10, 5, 10e)	7.52e2	3.38e4	3.46e4	1.39e3	4.73e4	4.87e4
(60, 20, 10, 5, 20e)	6.11e4	8.04e2	6.19e4	6.80e1	2.75e4	2.75e4
(60, 20, 10, 5, 30e)	6.12e4	6.56e2	6.18e4	7.64e2	3.73e4	3.80e4
(60, 20, 10, 5, 40e)	6.12e4	5.85e2	6.17e4	7.67e2	3.82e4	3.90e4
(60, 20, 10, 5, 50e)	6.12e4	2.68e2	6.14e4	1.03e3	4.31e4	4.42e4
(60, 20, 0, 5, e)	5.93e2	7.27e2	1.32e3	6.27e1	1.14e4	1.15e4
(60, 20, 0, 5, 10e)	5.44e2	2.12e1	5.65e2	6.60e1	1.91e4	1.91e4
(60, 20, 0, 5, 20e)	8.64e2	4.05e2	1.27e3	6.27e1	1.14e4	1.15e4
(60, 20, 0, 5, 30e)	5.60e2	2.97e1	5.90e2	6.27e1	1.14e4	1.15e4
(60, 20, 0, 5, 40e)	7.46e2	1.45e5	1.45e5	6.31e1	1.17e4	1.17e4
(60, 20, 0, 5, 50e)	6.11e2	5.34e1	6.65e2	6.27e1	1.14e4	1.15e4

Table 1 (continued): Test results for Algorithm 3.1 and `fmincon`

5 Conclusions

In this paper we have presented a feasible semismooth Newton method for a class of stochastic linear complementarity problems (SLCPs) with finitely many realizations by reformulating it as a constrained system of semismooth equations (CSE). Under some standard assumptions, this method has nice global and local superlinear convergence properties. Compared with the method given by Chen and Fukushima [2], our proposed method can produce solutions with high safety for SLCPs.

For (1.1) with Ω having infinitely many elements, a possible way to find an approximate solution of (1.1) is to solve the following deterministic problem: Find an $x \in \Re^n$ such that

$$x \geq 0, \; F(x, \omega_i) := M(\omega_i)x + q(\omega_i) \geq 0, \; x^T F(x, \omega_i) = 0, \; i \in \Omega_N, \tag{5.31}$$

where $\Omega_N = \{\omega_1, \omega_2, ..., \omega_N\}$ is a large sample of Ω. Clearly, (5.31) can be solved by the proposed method in this paper when n is not too large. An immediate concern is how to choose the sample Ω_N such that the solution of (5.31) approaches that of (1.1). We leave it as our future research topic.

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