

# New views of the spherical Bouguer gravity anomaly

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## **Abstract**

This paper presents a number of new concepts concerning the gravity anomaly. First, it identifies a distinct difference between a surface [2-D] gravity anomaly (the difference between actual gravity on one surface and normal gravity on another surface) and a solid [3-D] gravity anomaly defined in the fundamental gravimetric equation. Second, it introduces the “no topography” gravity anomaly (which turns out to be the complete spherical Bouguer anomaly) as a means to generate a quantity that is smooth, thus suitable for gridding, and harmonic, thus suitable for downward continuation. It is understood that the possibility of downward continuing a smooth gravity anomaly would simplify the task of computing an accurate geoid. It is also shown that the planar Bouguer anomaly is not harmonic, and thus cannot be downward continued.

**Keywords:** gravity anomaly, geoid, topography, Bouguer correction

## **1. Introduction**

There are two different types of the Bouguer gravity anomaly that are based on distinctly different conceptual models: the planar Bouguer anomaly and the spherical Bouguer anomaly (e.g., Quershi, 1976; Ervin, 1977; Chapin, 1996; Karl, 1971; LaFehr, 1991, 1998; Talwani, 1998; Smith, 2001; Vaníček, Novák and Martinec, 2001; Novák *et al.* 2002). The planar version uses an infinitely extending plate of thickness equal to the orthometric height of the topography at the point of interest to remove the gravitational

effect of the topography, whereas the spherical version uses a spherical shell of thickness equal to the orthometric height of the topography at the point of interest. If used alone, these conceptual models yield simple (spherical and planar) Bouguer anomalies. In order to model the gravitational attraction of the ‘roughness’ of the topography residual to the Bouguer plate or shell, then “terrain corrections” must be [algebraically] added to produce the spherical complete Bouguer anomaly and planar complete Bouguer anomaly, respectively.

In this paper, we attempt to answer three recently posed questions:

- 1) Is the complete spherical Bouguer gravity anomaly the same as the ‘standard’ (and more widely used) complete planar Bouguer gravity anomaly?
- 2) How can either the complete spherical or complete planar Bouguer gravity anomaly be continued downward from the surface of the Earth to the geoid?
- 3) Is either the complete spherical or the complete planar Bouguer gravity anomaly harmonic above the geoid?

In order to answer these and related questions, we have returned to the very basic principles and definitions of the theory of the Earth’s gravity field. This led to the need to introduce a distinction between “solid” (i.e., defined in the 3-D sense) and “surface” (i.e., defined in the 2-D sense) gravity anomalies. From this, it will be demonstrated that the spherical complete Bouguer anomaly is, as should be expected, very different from the planar complete Bouguer anomaly. It will also be shown that while the spherical complete Bouguer anomaly is indeed harmonic (i.e., satisfies Laplace’s equation) above the geoid, the planar complete Bouguer anomaly is not harmonic. Finally, while the

spherical complete Bouguer anomaly can be continued from the surface of the Earth down to the geoid using Poisson's approach, the planar complete Bouguer anomaly cannot.

## 2. Basic Concepts, Principles and Definitions

Let us begin by reviewing the most basic concepts needed in the studies of the gravity field. The fundamental quantity to study is the actual [Earth's] gravity potential  $W(\mathbf{r})$ , where  $\mathbf{r}$  is the position vector of a point in some geocentric coordinate system, and it is very often represented (in the first approximation) by the normal gravity potential  $U(\mathbf{r})$ . The difference between  $W(\mathbf{r})$  and  $U(\mathbf{r})$  *at the same point* is the well-known disturbing potential  $T(\mathbf{r})$ :

$$\forall \mathbf{r} : W(\mathbf{r}) - U(\mathbf{r}) = T(\mathbf{r}) \quad . \quad (1)$$

Since the Somigliana-Pizzetti normal gravity field used in geodesy (see below) represents the actual gravity field quite closely, the disturbing potential  $T(\mathbf{r})$  is about six orders of magnitude smaller than the actual potential  $W(\mathbf{r})$ .

Assuming that the mass of the reference ellipsoid that generates the normal gravity field is the same as the mass of the Earth, then the disturbing potential defined by means of the

difference between the actual and normal potentials can be expressed as a difference of two corresponding spherical harmonic series

$$\begin{aligned} \forall r \geq r_0(\phi) \cap R, \Omega \in \Omega_0 : T(r, \Omega) = \\ = \frac{GM}{r} \sum_{i=1}^{\infty} \left(\frac{R}{r}\right)^i \sum_{j=0}^i \left\{ \left[ J_{ij}(R) - \left(\frac{a}{R}\right)^i J_{ij}^N \delta_{j0} \right] Y_{ij}^c(\Omega) + K_{ij}(R) Y_{ij}^s(\Omega) \right\}, \end{aligned} \quad (2)$$

where  $\Omega$  denotes the horizontal position in latitude and longitude  $(\phi, \lambda)$ ,  $\Omega_0$  is the total solid angle,  $r$  is an arbitrary geocentric distance,  $r_0(\phi)$  is the geocentric distance to the surface of the reference ellipsoid (and, as such, it is a function of latitude  $\phi$ ),  $a$  is the major semi-axis of the geocentric reference ellipsoid and  $R$  is the mean radius of the Earth.

In Eq. (2), the spherical harmonic coefficients of the normal gravity potential,  $J_{ij}^N$ , referred to a sphere of radius  $a$ , are different from zero only for  $i = 2, 4, 6$  and  $8$  (beyond which they are negligible), and  $J_{ij}(R)$  and  $K_{ij}(R)$  are evaluated on the surface of the sphere of radius  $R$ . Note also that the series in Eq. (2) excludes the zero-degree term: this is because above we have assumed that the normal potential uses the correct value for the mass of the Earth. If this is not the case, then a zero-degree correction has to be added to the disturbing potential. We also note that the disturbing potential  $T(\mathbf{r})$  is harmonic everywhere outside the [Brillouin] sphere of radius  $R^B$  that contains all the Earth's masses and also outside the reference ellipsoid. The latter condition (i.e., that  $T(\mathbf{r})$  be outside the reference ellipsoid) does not have to be taken too seriously however, because the distribution of “normal” masses within the reference ellipsoid, according to Somigliana-

Pizzetti theory (Somigliana, 1929), does not have to be specified. The same holds true for normal gravity  $\gamma(r, \phi)$ , but we shall still use the quantifier  $r > r_0(\phi)$  systematically in the sequel.

When the corresponding normal (regular) and disturbed actual equipotential surfaces are defined by the same value of potential (normal and actual), their vertical separation  $Z(r, \Omega)$  is very closely related to the difference of the two potentials,  $T(r, \Omega)$ , i.e., the disturbing potential defined by Eq. (1), evaluated at an arbitrary point  $\mathbf{r} = (r, \Omega)$ . This important relation between physical and geometrical entities was first formulated by Bruns (1878); for a point  $r_g(\Omega)$  on the geoid it reads:

$$\forall \Omega \in \Omega_0 : N(\Omega) \approx \frac{T[r_g(\Omega)]}{\gamma_0(\phi)}, \quad (3)$$

where  $r = r_g(\Omega)$  denotes the geocentric radius vector of the geoid, which is a function of direction  $\Omega$  because the geoid is an undulating surface,  $\gamma_0(\phi)$  is normal gravity evaluated on the geometrical surface of the reference ellipsoid and thus

$$\forall \Omega \in \Omega_0 : N(\Omega) = Z[r_g(\Omega)]. \quad (4)$$

Equation (3) is the famous Bruns formula, and it is valid only when the normal potential  $U(\mathbf{r})$  is selected so that its value  $U_0$  on the [equipotential] surface of the reference ellipsoid is equal to the value  $W_0$  of the actual potential  $W(\mathbf{r})$  on the geoid. It will be assumed throughout the sequel that this is always the case; otherwise the generalised

Brun's formula results (e.g., Heiskanen and Moritz, 1967, sect 2-19). Note that this generalisation is different to the generalisation presented below.

As will be shown below, Brun's formula is accurate to better than  $1.5 \cdot 10^{-7} \text{ [m}^{-1}] N^2$ : the inaccuracy stems from the derivation of the formula, which reads as follows:

$$\begin{aligned} \forall \Omega \in \Omega_0 : T[r_g(\Omega)] = W_0 - U[r_g(\Omega)] = U_0 - U[r_g(\Omega)] = \\ = - \frac{\partial U(r, \phi)}{\partial n} \Big|_{r=r_0(\phi)} N(\Omega) - \frac{1}{2} \frac{\partial^2 U(r, \phi)}{\partial n^2} \Big|_{r=r_0(\phi)} N^2(\Omega) - \dots \end{aligned} \quad (5)$$

The above accuracy estimate is arrived at by realizing that the second derivative is equal to the vertical gradient of normal gravity at the ellipsoid, which equals  $\sim 0.3086 \text{ mGal/m}$ , and the higher derivatives are smaller still. Since the largest geoid height (in absolute value) is about 100 m, we will consider this inaccuracy, which may reach up to 1.5 Gal.m in the disturbing potential  $T$  (and equivalently, 1.5 mm in the geoid height  $N$ ) negligible (cf. Vaníček and Martinec, 1994) in our investigations here.

When a derivation similar to Eq. (3) is applied at an arbitrary point  $(r, \Omega)$ , we obtain a generalisation of Brun's formula (not to be confused with the generalisation for different values of  $U_0$  and  $W_0$ ; Heiskanen and Moritz, 1967), as follows:

$$\forall r \geq r_0(\phi), \Omega \in \Omega_0 : Z(r, \Omega) \approx \frac{T(r, \Omega)}{\gamma[r - Z(r, \Omega), \phi]}, \quad (6)$$

where  $Z$ , once again, is the vertical displacement of the corresponding equipotential surfaces  $W(\mathbf{r}) = \text{const.}$  and  $U(\mathbf{r}) = \text{const.}$  (taken here as the same values) that can be taken for all purposes along the direction  $\Omega$  as the displacement will always be quite small.

As an illustration, in Molodenskij's theory (Molodenskij, Eremeev and Yurkina, 1962), when  $\mathbf{r}$  is taken to describe the topographic surface of the Earth,  $Z$  correspondingly describes the depth of the telluroid beneath the Earth's surface, which is simply the height anomaly (i.e.,  $\zeta = Z$ ) reckoned along the ellipsoidal normal plumbline. In our approach, however, if  $\mathbf{r}$  is taken to describe the geoid,  $Z$  now describes the depth/height of the reference ellipsoid beneath/above the geoid, which is simply the geoid height (i.e.,  $N = Z$ ) measured along the Earth's plumbline.

Strictly speaking, Eq. (6) should be solved in an iterative fashion. However, the numerical difference between  $\gamma[r-Z(r,\Omega),\phi]$  and  $\gamma(r,\phi)$  for  $|Z| < 100$  m is at most 30.86 mGal (as alluded to in the previous paragraph), and we can see that carrying out only the first iteration will give a sufficiently accurate result. Therefore, Eq. (3) can be re-written as

$$\forall r \geq r_0(\phi), \Omega \in \Omega_0 : Z(r,\Omega) \approx \frac{T(r,\Omega)}{\gamma(r,\phi)}. \quad (7)$$

It is important to note that in the above derivations, while seemingly basic, the key difference from the more "standard" treatment of gravity anomalies is that everything is



considered at an arbitrary point in space (rather than on the geoid or at the Earth's surface) at which normal gravity is evaluated.

Some people (e.g., Hackney and Featherstone, 2003) feel nowadays that it may be more natural to use gravity disturbance  $\delta g(r, \Omega)$  - for the definition see (Heiskanen and Moritz, 1967; Eq. (2-142)) – rather than the gravity anomaly, arguing that it is now possible to measure the position  $(r, \Omega)$  of the gravity observation by one of the space-geodetic techniques, such as GPS. That is, of course, true, but by far the vast majority of gravity observations were collected in the pre-GPS age and these are the measurements that are used predominantly in gravity-field interpretation as well as in geoid computations.

### **3. Two generic definitions of the gravity anomaly**

Let us now turn to the definition of the gravity anomaly, where the situation becomes somewhat more complicated than for the above cases of the disturbing potential. Importantly, there are two subtly different definitions of the gravity anomaly that appear to be used interchangeably in practice. In the sequel, an attempt is made to introduce more generic definitions of the gravity anomaly, which are shown to degenerate into the more commonly used variants.

#### *3.1 The gravity anomaly from gravity (acceleration) observations*

Following the above arguments presented for the disturbing potential, the generic gravity anomaly is computed from the magnitude of observed gravity at  $(r, \Omega)$  and normal gravity computed at  $(r-Z(r, \Omega), \phi)$ ; this gives:

$$\forall r \geq r_0(\phi), \Omega \in \Omega_0 : \Delta g(r, \Omega) = g(r, \Omega) - \gamma[r - Z(r, \Omega), \phi], \quad (8)$$

which degenerates into the gravity anomaly given by Heiskanen and Moritz (1967, Eq. 2-139) when normal gravity is evaluated on the surface of the reference ellipsoid and subtracted from actual gravity evaluated (i.e., downward-continued from the surface measurement) on the geoid  $r_g$ . Alternatively, the generic gravity anomaly given by Eq. (8), can be evaluated at the surface of the Earth,  $r_t$ , by subtracting from  $g(r_t)$  the value of normal gravity  $\gamma(r_t - Z(r_t, \Omega), \phi)$  obtained by upward-continuing the normal gravity from the reference ellipsoid, which is a trivial and numerically stable procedure.

This generic gravity anomaly (Eq. 8) is directly related to gravity and it is simple to evaluate once gravity  $g(r, \Omega)$  at point  $(r, \Omega)$  is known, say from an observation. Clearly, Eq. (8) or the degenerate case contains neither any intrinsic requirement, nor any intrinsic information about the behaviour of  $\Delta g(r, \Omega)$  in the  $r$  direction. Accordingly, we shall call it a surface gravity anomaly, as it relates to the “surface”  $r(\Omega)$  on which  $g[r(\Omega)]$  is given. Here we follow, somewhat loosely, the precedent set by Molodenskij *et al.* (1962).

The surface gravity anomaly (Eq. 8 and its degenerate) is naturally the preferred form of gravity anomaly definition used in practice, because it is simple to calculate from the

gravity observation given its location. It allows the practitioner to simply convert gravity observed at point  $(r, \Omega)$  to a gravity anomaly referred to the same point  $(r, \Omega)$  as long as s/he knows how to evaluate the displacement  $Z(r, \Omega)$ . As such, one can simply calculate the surface gravity anomaly at any point  $(r, \Omega)$  above the reference ellipsoid, or perhaps, even a little below the reference ellipsoid  $r_0(\Omega)$  [as explained earlier], wherever actual gravity  $g(r, \Omega)$  is known/observed.

There are, however, some less fortunate consequences of accepting this particular definition of the gravity anomaly: it does present a problem if we are interested in evaluating gravity anomaly, say, on the geoid, beneath the topographical masses (e.g., for geoid determination by Stokes's formula). The definition by Eq. (8) and its degenerate form are mute as far as gravity anomaly values at points where gravity  $g(r, \Omega)$  is not known. In other words, Eq. (8) does not help us to determine/define the gravity anomaly at points where actual gravity is not already known/observed. We would have to know how to upward- or downward-continue the actual/observed gravity 'g' to the desired point. This cannot be done in a meaningful way unless one knows the physical law(s) that governs the behaviour of 'g' along the radius  $r$ .

One can certainly compute the value of normal gravity everywhere on, above and even a little below the surface of the reference ellipsoid, which is what Eq. (8) taken on the geoid calls for. However, what about the value of actual gravity  $g[r_g(\Omega)]$  on the geoid? Generations of geodesists have applied various vertical gradients of [model] gravity (see, for instance, Vaníček and Krakiwsky (1986) for an overview) to obtain what is usually

interpreted as “actual gravity on the geoid”. Taking a more rigorous view, however, this approximate approach is questionable; this will be discussed later.

### 3.2 The gravity anomaly from the “fundamental gravimetric equation”

Again following the earlier basic definitions for the disturbing potential, the second definition of the gravity anomaly uses the disturbing potential  $T(r, \Omega)$  as its starting point, which gives (Vaníček *et al.*, 1999):

$$\forall r \geq r_0(\phi), \Omega \in \Omega_0 : \Delta g(r, \Omega) \approx -\frac{\partial T(r, \Omega)}{\partial n} + \gamma[r - Z(r, \Omega), \phi]^{-1} \frac{\partial \gamma(r, \phi)}{\partial n} T(r, \Omega). \quad (9)$$

Here, the direction  $n$  in which the partial derivatives are evaluated is the direction perpendicular to the normal equipotential surface  $U(r, \phi) = U(r, \Omega) = \text{const.}$  that passes through the point  $(r, \Omega)$  and points upwards ( taken on the reference ellipsoid, this would be in the direction of the ellipsoidal normal). It was shown by Vaníček *et al.* (1999, Eq. 11) that the derivative  $\partial T(r, \Omega) / \partial n$  used in Eq.(9), instead of the correct derivative  $\partial T(r, \Omega) / \partial H$ , may cause a maximum error of up to 10  $\mu\text{Gal}$ , which we consider negligible here. Equation (9) requires that the *disturbing potential*  $T(r, \Omega)$ , *implied by the gravity anomaly, actually exists, which is not a trivial requirement that is automatically satisfied.* This is clearly a very different assumption to that used in Eq. (8), which immediately suggests that the two definitions may not be equivalent. We shall discuss this point later.

Next, it is important to note that  $Z(r, \Omega)$  is embedded in Eq. (9). Accordingly, it degenerates into the more recognised, yet unique, fundamental gravimetric equation at the geoid given in, for instance, Heiskanen and Moritz (1967, Eq. 2-148).

The definition of the gravity anomaly in Eq. (9) intrinsically requires some knowledge of the behaviour of  $T(r, \Omega)$  with depth/height, and would be mostly used when we work with a 3-D mathematical model of the disturbing potential  $T(r, \Omega)$ . This important contrast between Eqs. (8) and (9) makes the conceptual situation somewhat analogous with the distinction between surface and solid spherical harmonics. As such, we will call the gravity anomaly in Eq. (9) the solid gravity anomaly.

### 3.3 Preliminary assessment of the equivalence of Eqs. (8) and (9)

Let us now show that Eq. (8) can be derived from Eq. (9), i.e., is a special case of Eq. (9).

Substituting for  $T(r, \Omega)$  in the partial derivative from Eq. (1) gives

$$\forall r > r_0(\phi), \Omega \in \Omega_0 : \Delta g(r, \Omega) = -\frac{\partial W(r, \Omega)}{\partial n} + \frac{\partial U(r, \phi)}{\partial n} + \gamma[r - Z(r, \Omega), \phi]^{-1} \frac{\partial \gamma(r, \phi)}{\partial n} T(r, \Omega). \quad (10)$$

Realizing that

$$\forall r, \Omega \in \Omega_0 : \frac{\partial W(r, \Omega)}{\partial n} \approx -g(r, \Omega), \quad \forall r \geq r_0(\phi), \Omega \in \Omega_0 : \frac{\partial U(r, \phi)}{\partial n} = -\gamma(r, \phi) \quad (11)$$

(note the approximate equality in Eq. (10); the effect of this approximation was already discussed above and declared negligible) and using Eq. (6),

$$\forall r > r_0(\phi), \Omega \in \Omega_0 : \frac{T(r, \Omega)}{\gamma[r - Z(r, \Omega), \phi]} \frac{\partial \gamma(r, \phi)}{\partial n} = Z(r, \Omega) \frac{\partial \gamma(r, \phi)}{\partial n} = -\gamma[r - Z(r, \Omega), \phi] + \gamma(r, \phi), \quad (12)$$

Equation (9) can, to a very good accuracy, be rewritten as Eq. (8). To quantify this accuracy let us first note that the only approximations used in the above proof are those of the first of Eq. (11), which is negligible, and the approximation used in Bruns's formula (Eq. 3). The error in the approximation in Bruns's formula was shown above to be negligible and, as such, does not have to be accounted for here either.

Now, we want to replace the derivative  $\partial T(r, \Omega) / \partial n$  in the first term of Eq. (11) by a more convenient radial derivative  $\partial T(r, \Omega) / \partial r$ . The relation between the two derivatives can be written as, see e.g., (Jekeli, 1981; Vaníček *et al.*, 1999, Eq. 9)

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : \frac{\partial T(r, \Omega)}{\partial n} \approx \frac{\partial T(r, \Omega)}{\partial r} + \varepsilon_{\delta g}(r, \Omega), \quad (13)$$

where

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : \varepsilon_{\delta g}(r, \Omega) \approx g(r, \Omega) \beta(\phi) \xi(r, \Omega), \quad (14)$$

in which  $\beta(\phi)$  is the angle between the normal to the ellipsoid  $n$  and the radius vector  $r$ , and  $\xi(r, \Omega)$  is the meridional component of the deflection of the vertical. In Vaníček *et al.* (1999), the quantity  $\varepsilon_{\delta g}(r, \Omega)$  was called “ellipsoidal correction to the gravity disturbance” and it was shown that it may reach up to 0.5 mGal, and therefore it must be accounted for in the most accurate computations. We repeat here that this is **not** the case for either the error in Bruns’s formula or the error caused by the approximation in the first term of Eq.(11) – these two errors can be considered negligible at the present level of accuracy.

At this point, let us introduce another useful and often-used approximation in Eq. (9). It is called the “spherical approximation” and is given by Cruz (1985, Eqs. 2-4 and 2-22) and Martinec (1998, Eq. 1.40) as:

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : \gamma(r, \phi)^{-1} \frac{\partial \gamma(r, \phi)}{\partial n} T(r, \Omega) = -\frac{2}{r} T(r, \Omega) + \varepsilon_n(r, \Omega), \quad (15)$$

where  $\varepsilon_n(r, \Omega)$ , the “ellipsoidal correction for the spherical approximation” is

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : \varepsilon_n(r, \Omega) \approx e^2 (2 - 3 \sin^2 \phi) T(r, \Omega) / r_g, \quad (16)$$

in which  $e$  is the (first) numerical eccentricity of the reference ellipsoid.

#### 4. Compatibility of the ‘surface’ and ‘solid’ definitions of the gravity anomaly

We may now proceed by posing a rather obvious question: “Can the surface gravity anomaly (Eq. 8) be used as a 3-D quantity, i.e., as the solid gravity anomaly (Eq. 9)?” Surely, this can be achieved immediately provided that the “actual gravity”  $g(r, \Omega) \equiv g(\mathbf{r})$  is known in the 3-D region  $\mathbf{D}_3 \subset \mathbf{R}_3$  of interest. However, we were not able to derive Eq. (9) from Eq. (8) for an arbitrarily varying gravity in  $\mathbf{D}_3 \subset \mathbf{R}_3$ . Therefore, it appears, under such a heuristic investigation, that the surface gravity anomaly is, in some sense, more restricted than the solid variety, and we shall endeavour to demonstrate this fact a little more rigorously later. Until then, we shall use both Eqs. (8) and (9) side-by-side as two equivalent alternatives - even though Eq. (9) is 3-D while Eq. (8) is 2-D - as has been the custom in geodesy.

To discuss this question further, let us return to the simple example of the observed surface gravity  $g[r_t(\Omega)]$ , where  $r_t(\Omega)$  is the geocentric radius-vector of the topographic surface known from observations such as spirit-levelling and some *a priori* geoid model, or from GPS data, and the surface gravity anomalies being desired on the geoid, specified by its geocentric radius vector  $r_g(\Omega)$ . We thus need the gravity values  $g[r_g(\Omega)]$  on the geoid. Perhaps, we can compute them from a Taylor series, similar to that used for the continuation of normal gravity, i.e.,

$$\begin{aligned} \forall r_t(\Omega) > r_g(\Omega), \Omega \in \Omega_0 : g[r_t(\Omega)] = \\ = g[r_g(\Omega)] + \frac{\partial g(r, \Omega)}{\partial H} \Big|_{r=r_g(\Omega)} H(\Omega) + \frac{\partial^2 g(r, \Omega)}{\partial H^2} \Big|_{r=r_g(\Omega)} H^2(\Omega) / 2 + \dots \quad , \quad (17) \end{aligned}$$



where  $H$  is the distance (i.e., the orthometric height) measured along the plumbline, and  $g[r_t(\Omega)]$  on the Earth's surface are both known quantities. Equation (17) assumes that  $g(r, \Omega)$  has - on the geoid - all derivatives with respect to  $H$ , or equivalently, that the function  $g(r, \Omega)$  is analytical between the geoid and the Earth's surface. However, this assumption is certainly not automatically satisfied. As a matter of fact, we can be assured that real gravity between the geoid and the Earth's surface *is not an analytical function* of  $H$  because of the presence of the [discontinuous] topographic mass density distribution in the topography.

Alternatively, the Taylor series can be written, using the derivatives evaluated at the Earth's surface (instead of the geoid), as

$$\begin{aligned} \forall r_g(\Omega) < r_t(\Omega), \Omega \in \Omega_0 : g[r_g(\Omega)] = \\ = g[r_t(\Omega)] - \frac{\partial g(r, \Omega)}{\partial H} \Big|_{r=r_t(\Omega)} H(\Omega) + \frac{\partial^2 g(r, \Omega)}{\partial H^2} \Big|_{r=r_t(\Omega)} H^2(\Omega) / 2 - \dots, \end{aligned} \quad (18)$$

assuming, once more, that all the derivatives of  $g(r, \Omega)$  with respect to  $H$  exist at the Earth's surface. However, this assumption runs into the same problem for  $g(r, \Omega)$  of not being analytical between the geoid and the Earth's surface. Accordingly, this alternative (Eq. 18) is not necessarily any better than the first (Eq. 17).

Since  $g(r, \Omega) \equiv g(\mathbf{r})$  depends on the actual distribution of mass density  $\rho(\mathbf{r})$  within the Earth, even the derivatives of  $g(r, \Omega)$  with respect to  $H$ , needed in Eqs. (17) and (18), must be functions of the mass density  $\rho(\mathbf{r})$  within the Earth. It was shown (also by Bruns, 1878) that the first derivative of gravity inside the topographic masses is equal to:

$$\forall r \geq r_g(\Omega), \Omega \in \Omega_0 : \frac{\partial g(r, \Omega)}{\partial H} = -2g(r, \Omega)J(r, \Omega) + 4\pi G\rho(r, \Omega) - 2\omega^2, \quad (19)$$

where the symbol  $J(r, \Omega)$  denotes the [unknown] mean curvature of the equipotential surface  $W(r, \Omega) = \text{const.}$  that passes through the point  $(r, \Omega)$  of interest. Most geodesists do not consider the second and higher derivatives in Eqs. (17) and (18) worth evaluating because within the range of heights  $H$  used in terrestrial investigations (0 km to ~9 km) they are thought to contribute much less to the final result than the first derivative does. Generally, in addition to the troubles with the first derivative pointed out above, even the higher derivatives may not exist: this can be seen rather clearly in the case of the second derivative evaluated at the points where the density  $\rho(r, \Omega)$  is discontinuous in the  $H$ -direction. This is particularly likely because the Earth's geological structure is generally stratified.

As far as we can see, the only way of making the “corresponding” surface and solid gravity anomalies consistent with each other is to use Eq. (9) as a partial differential equation (of first order) for the determination of the [unknown] 3-D function  $T(r, \Omega)$  within the region  $D_3$  of interest, and the “corresponding” surface gravity anomaly  $\Delta g(r, \Omega)$  as the 2-D boundary-value on the boundary  $\Sigma$  of  $D_3$ . This would, of course, be

generally very difficult, nay impossible, except for some very trivial (and unrealistic) cases of  $D_3$ . It seems to us that if anyone wishes to show that a specific surface gravity anomaly can be converted to its solid counterpart, the onus of proving that it is possible must be on him/her. In keeping with a rigorous adherence to physics, any argument that one or another surface gravity anomaly can be upward- or downward-continued, would have to be accompanied by a proof that the underlying disturbing potential that satisfies the above requirement does indeed exist.

However, there is a rather obvious way of enforcing the “correspondence”, which goes in the “opposite direction”: it starts with the description of a physically meaningful disturbing potential  $T(r, \Omega)$  of desired properties which would generate the solid gravity anomaly through Eq. (9). So generated, the solid gravity anomaly would then be required to equal to the desired surface gravity anomaly at the defining surface. As shown earlier, if a gravity anomaly is of a solid kind, it is automatically also of a surface kind, but not the other way round (that is, Eq. (8) can be derived from Eq. (9) but not vice versa!).

The challenge in this approach, of course, is to formulate the correct mathematical expression for the meaningful disturbing potential  $T(r, \Omega)$  of the kind that we desire. For example, what is the disturbing potential that, through Eq. (9), generates the “solid free-air gravity anomaly”? What are the disturbing potentials that generate the “solid Poincaré-Prey”, “Bouguer”, and “Faye gravity anomalies”? To make sure that the disturbing potential exists, we introduce the appropriate model gravity fields (e.g., free-air, Poincaré-Prey, Bouguer, Faye, etc.), consisting of the density distribution, gravity

potential, disturbing potential, gravity, gravity anomaly, gravity disturbance, co-geoid height, orthometric height, etc., as being defined in a separate geometrical space.

Such a new space, different from the space where the actual gravity field is studied, must allow us to apply the laws of Newtonian physics needed for the study. This is to avoid operations such as “removing” some part of gravity field, or “restoring” some other part of gravity field; operations that do not make much physical sense. Most of the authors of this paper have been using this concept (of defining specific spaces for specific gravity fields) for about a decade to describe Helmert’s model gravity field and the operations performed on that model field, with several positive results. The interested reader is referred to Vaniček *et al.* (1999) and the papers cited therein, as well as other related papers in the *Journal of Geodesy*.

The remainder of this paper will now focus on operations in a space devoid of topography. This will be called the “no-topography (NT) space”, stemming from Bouguer’s original idea, which was however applied to a planar model as opposed to our more physically realistic spherical model. The motivation for this new approach is that Bouguer gravity anomalies are regarded as being smooth and thus more suitable for interpolation and, if harmonic, more suitable even for downward continuation. It is important to note that the mathematical modelling of the masses external to the geoid is a demanding task as the density of topographical masses is rather poorly known. Nevertheless, such modelling is needed for transforming all the desired quantities from

the real space to the NT-space. This situation is, of course, identical to the situation faced every day by users of planar Bouguer anomalies.

Our calculations (Vaniček and Martinec, 1994) have shown that assuming a constant density of  $2,670 \text{ kg/m}^3$  does not cause enough of an error in the topographical model to affect the Helmert gravity field too much. Inclusion of realistic lateral variations of density further improves the performance of the topographical model. On the other hand, the effect of a faulty topographical model on the NT-gravity field will certainly be larger because the NT-field departs from the real field much more than the Helmert field does. This effect should be investigated by anyone who intends to use the NT-anomaly by itself, rather than as an intermediate quantity between the real and Helmert anomalies, as we do in our geoid-specific applications.

## **5. “No Topography” space: a reinterpretation of Bouguer’s aim**

In this section and in the next, we shall reiterate some relations, which may be known to a significant proportion of the readership. We consider this reiteration to be reasonable because we are after a reinterpretation of a very well known and very long used concept. Thus we wish to bring out some subtleties of these known relations that should make the reinterpretation more palatable.

The Helmert space, as mentioned above, is used in the solution to the Stokes-Helmert boundary-value problem (e.g., Vaniček and Martinec, 1994); it is fundamentally characterised by the Earth's mass-density distribution external to the geoid, where the (topographical and atmospheric) masses above the geoid are condensed onto (or below) the geoid in the form of a layer of infinitesimal thickness. Let us now try to formulate an analogous model gravity field generated by the Earth completely devoid of topography and atmosphere.

The effect of the atmosphere cannot be neglected, but it can be treated in the same manner as the effect of topography. Since it would be clumsy and time consuming to show the parallel treatment of atmospheric attraction whenever the treatment of topography is discussed, we shall omit the atmosphere in this paper. We shall assume that the reader can complete the argument by taking the atmosphere into account in a parallel way to the topography, which is a simple task.

To begin with, we note that “the Earth without topography” is nothing else but the geoid, taken as a solid body with the actual (real) distribution of density within it. However, the co-geoid has also to be introduced here because the removal of the topographic and atmospheric masses significantly changes the potential of the Earth and hence the geoid computed from it. The corresponding [primary] indirect effect of removing the topography is thus very large and must be accounted for at some stage or another. However, the main motivations for using the “no topography” space is to generate a harmonic field (i.e., that satisfies Laplace's equation) that is also smooth and is thus

suitable for gridding, interpolation and downward continuation. Accordingly, it is important to note that we acknowledge that the topographical effect must be properly dealt with in some other appropriate way, if the geoid is to be computed.

Denoting the model gravity potential generated by the Earth with no topography by  $W^{NT}$ , we can write the following equation:

$$\begin{aligned} \forall r \geq r_g(\Omega), \Omega \in \Omega_0 : W^{NT}(r, \Omega) &= W(r, \Omega) - G \int_{Topo} \rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega' = \\ &= W(r, \Omega) - V^t(r, \Omega) \end{aligned} \quad (20)$$

where the second term on the right-hand-side,  $V^t(r, \Omega)$ , is the Newtonian integral for the gravitational potential generated by the topographical masses. In Eq. (20),  $G$  stands for Newton's gravitational constant,  $\rho(r, \Omega)$  stands for the topographical density at  $(r, \Omega)$  inside the topography (or atmosphere), and  $L$  is the 3-D-Euclidian distance between points  $(r, \Omega)$  and  $(r', \Omega')$ . Disregarding the atmospheric attraction, as before, the disturbing potential  $T^{NT}(r, \Omega) = W^{NT}(r, \Omega) - U(r, \Omega)$  becomes harmonic everywhere above the geoid, i.e., everywhere in the region of validity of Eq. (20).

It is easy to show that the Newton integral can be written as a sum of the potential  $V^B(r, \Omega)$  of a Bouguer shell of thickness  $r_i(\Omega) - r_g(\Omega) = H(\Omega)$  (see Eq. (24) below) and density  $\rho(r, \Omega)$ , and the potential  $V^R(r, \Omega)$  of the terrain, which has been called the "roughness term" by Martinec (1998) among others. To show this separation, it is sufficient to re-write the integral in Eq. (20) as:

$$\begin{aligned}
G \int_{Topo} \rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega' &= G \iint_{\Omega_0} \int_{r_g(\Omega)}^{r_t(\Omega)} \rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega' + \\
&G \iint_{\Omega_0} \int_{r(\Omega)}^{r_t(\Omega')} \rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega'
\end{aligned} \tag{21}$$

Approximating the geoid  $r_g(\Omega)$  by a sphere of radius  $R = \sqrt[3]{a^2 b}$ , where  $a$  and  $b$  are the major and minor semi-axes of the reference ellipsoid  $U(r, \Omega) = W_0$  (cf. Vaníček and Krakiwsky, 1986), the first integral on the right-hand-side of Eq. (21) can be rewritten as

$$\begin{aligned}
\forall r \geq r_g(\Omega), \Omega \in \Omega_0 : V^B(r, \Omega) &= G \iint_{\Omega_0} \int_{r_g(\Omega)}^{r_t(\Omega)} \rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') r'^2 dr' d\Omega' \approx \\
&\approx GR^2 \rho_0 \iint_{\Omega_0} \int_R^{R+H(\Omega)} L^{-1}(r, \Omega, r', \Omega') dr' d\Omega' + GR^2 \iint_{\Omega_0} \int_R^{R+H(\Omega)} \delta\rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') dr' d\Omega'
\end{aligned} \tag{22}$$

where  $\rho_0$  is the mean topographical density (usually taken as  $2,670 \text{ kg/m}^3$ ),  $\delta\rho(r, \Omega) = \rho(r, \Omega) - \rho_0$  is the “anomalous” topographical density, and  $H(\Omega)$  is the orthometric height of point  $r_t(\Omega)$ . This “spherical” approximation implicit in Eq. (22) should not be understood as an adoption of the sphere as the model for the geoid. Rather, it is an approximation that simplifies the computation of the disturbing potential  $T^{NT}$ , which is still defined in a rather obvious way



$$\begin{aligned} \forall r > r_0(\Omega), \Omega \in \Omega_0 : T^{NT}(r, \Omega) &= W^{NT}(r, \Omega) - U(r, \phi) = \\ &= W(r, \Omega) - V^B(r, \Omega) - V^R(r, \Omega) - U(r, \phi) \end{aligned} \quad (23)$$

The adoption of the spherical approximation results in the reduction of computational relative accuracy of the order of the geometrical flattening of the reference ellipsoid (i.e., of the order of  $3 \cdot 10^{-3}$ ). In deriving Eq. (22), we have also assumed that, to an accuracy of at most a few millimetres, the following approximately equality:

$$\forall \Omega \in \Omega_0 : r_t(\Omega) \approx r_g(\Omega) + H(\Omega) \quad (24)$$

is sufficiently accurate (i.e., assuming that the deflection of the vertical is small over the orthometric heights encountered on the Earth). Let us now evaluate the disturbing potential  $T^{NT}(r, \Omega)$  in the NT-space. To do this, we have to evaluate the two potentials  $V^B(r, \Omega)$  and  $V^R(r, \Omega)$  and subtract them from the actual disturbing potential  $T(r, \Omega)$  given by Eq. (1). Wichiencharoen (1982) shows that the potential of spherical Bouguer shell of constant density  $\rho_0$  (the first integral on the right-hand-side of Eq. (22)) can, for all  $\Omega \in \Omega_0$  be written as

$$V^B(r, \Omega) = \begin{cases} \frac{4\pi G \rho_0}{r} [R^2 H(\Omega) + R H^2(\Omega) + \frac{1}{3} H^3(\Omega)], & r \geq R + H(\Omega), \\ 2\pi G \rho_0 \left\{ [R + H(\Omega)]^2 - \frac{2}{3} \frac{R^3}{r} - \frac{1}{3} r^2 \right\}, & R \leq r \leq R + H(\Omega), \\ 4\pi G \rho_0 [R H(\Omega) + \frac{1}{2} H^2(\Omega)], & r \leq R. \end{cases} \quad (25)$$

The potential  $V^R(r, \Omega)$  of the terrain in a spherical approximation is given by the second integral in Eq. (21) as

$$\begin{aligned} \forall r > r_g(\Omega), \Omega \in \Omega_0 : V^R(r, \Omega) \approx \\ \approx GR^2 \rho_0 \iint_{\Omega_0} \int_{R+H(\Omega)}^{R+H(\Omega')} L^{-1}(r, \Omega, r', \Omega') dr' d\Omega' + GR^2 \iint_{\Omega_0} \int_{R+H(\Omega)}^{R+H(\Omega')} \delta\rho(r', \Omega') L^{-1}(r, \Omega, r', \Omega') dr' d\Omega' \cdot \end{aligned} \quad (26)$$

The first integral in Eq. (26) can be written as a surface integral (cf. Martinec, 1998, Eq. (3.52); Sjöberg, 2000, Eqs. 9-11). To evaluate this potential, however, one has to know the topographical heights  $H(\Omega)$  over the whole surface of the Earth and the anomalous topographical density  $\delta\rho(r, \Omega)$  inside the topography over the whole Earth.

## 6. Gravity on the surface of the Earth in the NT-space

It is of primary interest at this point to evaluate the NT disturbing potential  $T^{NT}(r, \Omega)$  at the point  $(r, \Omega) = r_t(\Omega) \approx [R+H(\Omega), \Omega]$  on the surface of the Earth, denoted in the sequel simply as  $H(\Omega)$ . The potential of the spherical Bouguer shell  $V^B(r, \Omega)$  on the Earth's surface, cf. Eq. (25), becomes:

$$\forall \Omega \in \Omega_0 : V^B[H(\Omega)] = 4\pi G \rho_0 H(\Omega) \frac{R^2 + RH(\Omega) + H^2(\Omega)/3}{R + H(\Omega)}, \quad (27)$$

where  $R$  is the inner radius of the spherical Bouguer shell equal, as before, to the mean radius of the Earth. Equation (27) can then be rewritten as

$$\forall \Omega \in \Omega_0 : V^B [H(\Omega)] = 4\pi G \rho_0 R H(\Omega) \{1 + H(\Omega)/R + [H(\Omega)/R]^2 / 3\} [1 - H(\Omega)/R + \dots] \quad (28)$$

to a relative accuracy better than  $10^{-7}$ . Given this approximation, the potential of the Bouguer shell at the Earth's surface (as given by Eq. (28)), can be finally written as

$$\forall \Omega \in \Omega_0 : V^B [H(\Omega)] \approx 4\pi G \rho_0 R H(\Omega), \quad (29)$$

which is an expression already derived by Vaníček, Novák and Martinec (2001) and others. The potential of terrain -  $V^R(r, \Omega)$  in spherical approximation - at a point  $H(\Omega)$  on the surface of the Earth in the NT-space is given, of course, by Eq. (26).

Let us now turn to the evaluation of gravity  $g^{NT}(r, \Omega)$  in the NT-space. The gravity  $g^{NT}(r, \Omega)$  must satisfy the “standard” definition, namely:

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : g^{NT}(r, \Omega) = - \frac{\partial W^{NT}(r, \Omega)}{\partial H^{NT}}, \quad (30)$$

where  $H^{NT}$  is in the direction of the plumbline in the NT-space. Substituting for  $W^{NT}(r, \Omega)$  from Eq. (23), we get

$$\begin{aligned} \forall r > r_g(\Omega), \Omega \in \Omega_0 : g^{NT}(r, \Omega) &= -\frac{\partial W(r, \Omega)}{\partial H^{NT}} + \frac{\partial V^t(r, \Omega)}{\partial H^{NT}} = \\ &= g(r, \Omega) \frac{\partial H}{\partial H^{NT}} + \frac{\partial V^B(r, \Omega)}{\partial H^{NT}} + \frac{\partial V^R(r, \Omega)}{\partial H^{NT}} \end{aligned} \quad (31)$$

where the two additive terms  $\partial V^B(r, \Omega)/\partial H^{NT}$  and  $\partial V^R(r, \Omega)/\partial H^{NT}$  represent the transformation of gravity from the actual-space to the NT-space at all the points  $(r, \Omega)$  above the geoid (i.e., they represent the two parts of the topographical attraction contribution to actual gravity).

Following the approach used by Vaniček *et al.* (1999), the partial derivative  $\partial H/\partial H^{NT}$  can be shown to equal

$$\frac{\partial H}{\partial H^{NT}} \approx 1 - \frac{(\theta^{NT})^2}{2} \quad , \quad (32)$$

where  $\theta^{NT}(r, \Omega)$  is the deflection of the plumbline in the NT space. This is likely to be one order of magnitude larger than its counterpart  $\theta(r, \Omega)$  in the real space, and can no longer be neglected. Let us denote the product  $(\theta^{NT}(r, \Omega))^2 g(r, \Omega)/2$  by  $\varepsilon^{NT}(r, \Omega)$ . This yields

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : g^{NT}(r, \Omega) = g(r, \Omega) - \varepsilon^{NT}(r, \Omega) + \frac{\partial V^t(r, \Omega)}{\partial H^{NT}} \quad . \quad (33)$$

Further, the partial derivative  $\partial V^t(r, \Omega)/\partial H^{NT}$  can be expressed as

$$\frac{\partial V^t(r, \Omega)}{\partial H^{NT}} = \frac{\partial V^t(r, \Omega)}{\partial r} \frac{\partial r}{\partial H^{NT}} \quad , \quad (34)$$

and again applying the quoted technique (*ibid.*), we obtain

$$\frac{\partial r}{\partial H^{NT}} \approx 1 - \frac{\beta^2(\varphi) + 2\beta(\varphi)\xi^{NT}(r, \Omega) + (\theta^{NT}(r, \Omega))^2}{2}, \quad (35)$$

where  $\beta(\varphi)$  was discussed after Eq.(14) above. Denoting

$$\frac{\partial V^t(r, \Omega)}{\partial r} \frac{\beta^2(\varphi) + 2\beta(\varphi)\xi^{NT}(r, \Omega) + (\theta^{NT}(r, \Omega))^2}{2} = \varepsilon^r(r, \Omega) \quad (36)$$

we can rewrite Eq.(31) as follows

$$\begin{aligned} \forall r > r_g(\Omega), \Omega \in \Omega_0 : g^{NT}(r, \Omega) &= -\frac{\partial W(r, \Omega)}{\partial H^{NT}} + \frac{\partial V^t(r, \Omega)}{\partial H^{NT}} = \\ &= g(r, \Omega) - \varepsilon^{NT}(r, \Omega) + \frac{\partial V^t(r, \Omega)}{\partial r} - \varepsilon^r(r, \Omega) = g(r, \Omega) + \frac{\partial V^t(r, \Omega)}{\partial r} - \varepsilon_{NT}(r, \Omega) \end{aligned}, \quad (37)$$

where  $\varepsilon_{NT}(r, \Omega)$  is given by

$$\begin{aligned} \varepsilon_{NT}(r, \Omega) &= \varepsilon^{NT}(r, \Omega) + \varepsilon^r(r, \Omega) = \\ &= g(r, \Omega) \frac{(\theta^{NT}(r, \Omega))^2}{2} + \frac{\partial V^t(r, \Omega)}{\partial r} \frac{\beta^2(\varphi) + 2\beta(\varphi)\xi^{NT}(r, \Omega) + (\theta^{NT}(r, \Omega))^2}{2}. \end{aligned} \quad (38)$$

Finally, realising that  $-\partial V^t(r, \Omega)/\partial r$  is the attraction of topography, i.e.,  $g^t(r, \Omega)$ , we get

$$\varepsilon_{NT}(r, \Omega) = g^{NT}(r, \Omega) \frac{(\theta^{NT}(r, \Omega))^2}{2} + g^t(r, \Omega) \frac{\beta^2(\varphi) + 2\beta(\varphi)\xi^{NT}(r, \Omega)}{2}. \quad (39)$$

Here, we can assume that the deflection of the vertical  $\theta^{NT}(r, \Omega)$  will not be larger than the angle  $\beta(\varphi)$ , while  $g^{NT}(r, \Omega)$  will always be several orders of magnitude larger than  $g^t(r, \Omega)$ . Under this assumption, the first term on the right-hand-side will be the dominant term; it may be as much as one order of magnitude larger than  $\varepsilon_{\delta g}(r, \Omega)$ , the ellipsoidal correction to gravity disturbance (cf. Eq. (14)). We can thus conclude that the two additive terms  $\partial V^B(r, \Omega)/\partial H^{NT}$  and  $\partial V^R(r, \Omega)/\partial H^{NT}$  can be replaced by  $\partial V^B(r, \Omega)/\partial r$  and  $\partial V^R(r, \Omega)/\partial r$  without an appreciable deterioration in accuracy.

To be able to refer to these two “corrections”, let us denote for simplicity, the first (Bouguer shell) term in Eq. (31) by  $TopoC^B(\rho;r,\Omega)$  and the second term by  $TopoC^R(\rho;r,\Omega)$ . We shall then refer to the anomalous density effect on these two topographic corrections as  $TopoC^B(\delta\rho;r,\Omega)$  and  $TopoC^R(\delta\rho;r,\Omega)$ . For a constant topographic density  $\rho_0$ , the sum of the two terms,  $TopoC^B(\rho_0;r,\Omega)$  and  $TopoC^R(\rho_0;r,\Omega)$ , can be expressed as a surface integral (cf. Sjöberg, 2000, Eqs. 28a and 28b).

The two last terms in Eq. (31) are obtained from Eqs. (25) and (26) by differentiation with respect to  $H^{NT}$ . For a constant topographical density, we derive the first term,  $TopoC^B(\rho_0;r,\Omega)$ , as

$$\frac{\partial V^B(r,\Omega)}{\delta H} \approx \frac{\partial V^B(r,\Omega)}{\partial r} = \begin{cases} \frac{-4\pi G\rho_0}{r^2} [R^2 H(\Omega) + RH^2(\Omega) + \frac{1}{3}H^3(\Omega)], & r \geq R + H(\Omega), \\ 2\pi G\rho_0 (\frac{2}{3}\frac{R^3}{r^2} - \frac{2}{3}r), & R \leq r \leq R + H(\Omega), \\ 0, & r \leq R. \end{cases} \quad (40)$$

which is valid for all  $\Omega \in \Omega_0$ , whereas the error introduced by taking the derivative with respect to the radius  $r$  is negligible, as stated above. The second term,  $TopoC^R(\rho_0;r,\Omega)$ , is given by

$$\begin{aligned}
& \forall r > r_g(\Omega), \Omega \in \Omega_0 : \partial V^R(r, \Omega) / \partial r \approx \\
& \approx GR^2 \rho_0 \iint_{\Omega_0} \int_{R+H(\Omega)}^{R+H(\Omega')} \frac{\partial L^{-1}(r, \Omega, r', \Omega')}{\partial r} dr' d\Omega' + \\
& + GR^2 \iint_{\Omega_0} \int_{R+H(\Omega)}^{R+H(\Omega')} \delta\rho(r', \Omega') \frac{\partial L^{-1}(r, \Omega, r', \Omega')}{\partial r} dr' d\Omega'
\end{aligned} \tag{41}$$

The correction due to anomalous topographical density  $\delta\rho(r, \Omega)$ ,  $TopoC^B(\delta\rho; r, \Omega)$  and  $TopoC^R(\delta\rho; r, \Omega)$ , on the two topographical corrections, has been investigated by other authors (e.g., Huang *et al.*, 2001). They found that the portion  $TopoC^B(\delta\rho; r, \Omega)$ , due to the Bouguer shell part of topographical correction is appreciable and should be taken into account. This correction can be computed from Eq. (22) after a radial derivative of the second term on the right-hand-side has been taken. The portion  $TopoC^R(\delta\rho; r, \Omega)$  that corresponds to the terrain correction is described by the second integral in Eq. (41). It is typically one order of magnitude smaller than  $TopoC^B(\delta\rho; r, \Omega)$ , and is thus often neglected all together. For simplicity, we shall thus also neglect it in the reminder of this paper.

To study the behaviour of the gravitational acceleration induced by the Bouguer spherical shell (Eq. 40), let us write the independent argument  $r'$  as  $R + H'$ . Then we get, for the space above the geoid, and for all  $\Omega \in \Omega_0$ :

$$\frac{\partial V^B(r, \Omega)}{\partial H} \approx \begin{cases} -4\pi G \rho_0 H(\Omega) \left[ 1 - \frac{H'}{R} + \frac{H(\Omega) - H'}{R} + \frac{\frac{1}{3} H^2(\Omega) - 2H(\Omega)H' + 3H'^2}{R^2} - \dots \right], & H' \geq H(\Omega), \\ -4\pi G \rho_0 H' \left[ 1 - \frac{H'}{R} + \frac{4}{3} \frac{H'^2}{R^2} - \dots \right], & 0 \leq H' \leq H(\Omega) \end{cases} \quad (42)$$

Let us, for a moment, focus our attention again on the surface of the Earth. For a point  $(r, \Omega)$  at the surface of the Earth, i.e., for  $H' = H(\Omega)$ , we get in particular

$$\forall \Omega \in \Omega_0 : \frac{\partial V^B(r, \Omega)}{\partial H} \Big|_{r=R+H(\Omega)} \approx -4\pi G \rho_0 H(\Omega) [1 - H(\Omega)/R + \dots], \quad (43)$$

which, to a relative accuracy better than  $1.4 \cdot 10^{-3}$ , can be approximated by (Vaniček *et al.* 2001)

$$\forall \Omega \in \Omega_0 : \frac{\partial V^B(r, \Omega)}{\partial H} \Big|_{r=R+H(\Omega)} \approx -4\pi G \rho_0 H(\Omega) . \quad (44)$$

The correction due to the terrain attraction  $\partial V^R(r, \Omega)/\partial r$  (Eq. (41)) at the surface of the Earth, i.e.,  $TopoC^R[\rho_0; H(\Omega)]$ , is given approximately by

$$\forall \Omega \in \Omega_0 : \partial V^R(r, \Omega) / \partial r \Big|_{r=R+H(\Omega)} \approx GR^2 \rho_0 \iint_{\Omega_0} \int_{R+H(\Omega)}^{R+H(\Omega')} \frac{\partial L^{-1}(r, \Omega, r', \Omega')}{\partial r} dr' d\Omega' , \quad (45)$$



which is nothing else but the spherical terrain correction to surface gravity, as discussed by Novák *et al.* (2001). For brevity, we shall call this correction  $TC^S[\rho_0; r_i(\Omega)]$ , or  $TC^S[\rho_0; H(\Omega)]$ .

Now, substituting all the above-derived or cited expressions back into Eq. (31), we obtain the final equation for the gravity in the NT-space *on the surface of the Earth*. Still using the spherical model of topography and denoting the corresponding gravity by  $g^{NT;S}[H(\Omega)]$  we have:

$$\begin{aligned} \forall \Omega \in \Omega_0 : g^{NT;S}[H(\Omega)] \approx & g[H(\Omega)] - \varepsilon_{HT}[H(\Omega)] - 4\pi G\rho_0 H(\Omega) + \\ & + TC^S[\rho_0; H(\Omega)] + TopoC^B[\delta\rho; H(\Omega)] \end{aligned} \quad (46)$$

We note that the last three terms in Eq. (46) represent the transformation of *surface* gravity  $g[H(\Omega)]$  from the real space to the NT-space.

Next, we can define the NT-gravity anomaly of the surface kind, on the Earth surface, by means of Eq. (8), as

$$\forall \Omega \in \Omega_0 : \Delta g^{NT;S}[H(\Omega)] = g^{NT;S}[H(\Omega)] - \gamma\{H(\Omega) - Z^{NT}[H(\Omega)], \phi\}, \quad (47)$$

where  $Z^{NT}[H(\Omega)]$  is the vertical separation between the equipotential surface  $W^{NT}(r, \Omega) = W^{NT}[H(\Omega)]$  and the normal equipotential surface  $U(r, \Omega) = W^{NT}[H(\Omega)]$  of the same potential. We note that the separation  $Z^{NT}$  in the NT-space is much larger than the

corresponding separation  $Z^H$  in the Helmert-space. The  $Z^{NT}$  may reach several hundreds of metres.

The NT-gravity anomaly defined in Eq. (47) can finally be rewritten, by means of Eq.(46), as

$$\forall \Omega \in \Omega_0 : \Delta g^{NT:S} [H(\Omega)] = g[H(\Omega)] - \gamma \{H(\Omega) - Z^{NT} [H(\Omega)], \phi\} - 4\pi G \rho_0 H(\Omega) + TC^S [\rho_0; H(\Omega)] - \varepsilon_{NT} [H(\Omega)] + TopoC^B [\delta\rho; H(\Omega)] \quad (48)$$

We shall now try to sort out the relation between our  $\Delta g^{NT:S} [H(\Omega)]$  gravity anomaly and the spherical Bouguer gravity anomaly  $\Delta g^{CB:S} [H(\Omega)]$

## 7. The relationship between NT-anomalies and Bouguer anomalies

An inspection of Eqs. (46) and (47) convinces us that the NT-anomaly  $\Delta g^{NT:S} [H(\Omega)]$  is nothing else but the complete spherical Bouguer anomaly. We shall simply call this gravity anomaly the spherical complete Bouguer anomaly, denoting it by  $\Delta g^{CB:S} [H(\Omega)]$  by introducing a new notation to reflect this situation. Given the use of more simplistic models by some authors, it seems to make sense to also define a simple (incomplete) spherical Bouguer anomaly (i.e., only considering the gravitational attraction of the spherical Bouguer shell without the terrain correction). The resulting  $\Delta g^{B:S} [H(\Omega)]$  would be defined by Eq. (47), where in the expression for the  $g^{B:S} [H(\Omega)]$ , the last term on the right-hand-side of Eq. (46) (i.e., the terrain correction) is omitted.

The planar complete Bouguer gravity  $g^{CB:P}[H(\Omega)]$  at the Earth's surface is defined by Heiskanen and Moritz (1967, Eq. 3-18) as

$$\forall \Omega \in \Omega_0 : g^{CB:P}[H(\Omega)] = g[H(\Omega)] - 2\pi G\rho_0 H(\Omega) + TC^P[\rho_0; H(\Omega)]. \quad (49)$$

We note that this represents a different interpretation of Bouguer gravity from the one used by Vaníček *et al.* [2001]: here it is assumed that the gravity in Eq. (49) is referred to the Earth's surface, while in the cited publication the interpretation was that Eq. (49) defines Bouguer gravity on the geoid.

Naturally, it is then instructive to consider also what difference there is between the spherical and the planar Bouguer anomalies. Comparison of Eq. (49) with Eq. (46) gives the difference we seek:

$$\begin{aligned} \forall \Omega \in \Omega_0 : g^{CB:P}[H(\Omega)] - g^{CB:S}[H(\Omega)] \approx \varepsilon_{NT}[H(\Omega)] + 2\pi G\rho_0 H(\Omega) - \\ - TopoC^B[\delta\rho; H(\Omega)] + TC^P[H(\Omega)] - TC^S[\rho_0; H(\Omega)] \end{aligned} \quad (50)$$

We were not able to derive any simple formula for the difference between the spherical terrain correction  $TC^S[\rho_0, H(\Omega)]$  and the planar terrain correction  $TC^P[H(\Omega)]$ . However, from some numerical experiments, it appears as if the value of the difference of the two terrain corrections tends to work against the “planar Bouguer plate term” ( $2\pi G\rho_0 H(\Omega)$ ) so that the difference described by Eq. (50) tends to be relatively small [Véronneau, personal communication, May, 2002]. This difference can be seen in Figure 1 for a

rugged part of the Canadian Rocky Mountains, which has been used in many of our previous studies.

We also wish to point out the discussions between La Fehr (1997) and Talwani (1998), in which it was argued that there is some level of equivalence between the planar and spherical Bouguer anomalies for certain conditions of height and spatial distance.

Another more involved discussion on the level of equivalence can be found in Moritz (1968) and Moritz (1990). We do not wish to review these discussions, as it has not been our intention to deal with other than the standard planar case of complete Bouguer anomaly as it is often used in practice.

We also wish to point out that even though the term  $TopoC^B(\delta\rho;r,\Omega)$  must be taken into account, it tends to be rather small. The correction is also often applied in the computations of the planar variety of the Bouguer anomaly and we shall thus not quote it when comparing the two varieties of the Bouguer anomaly. It should be also noted that it would make no physical sense to compare the two versions (spherical and planar) of incomplete Bouguer anomalies. Their difference is clearly very large, being equal to the “planar Bouguer term”, as one can easily glean from Eq. (50).

To further study the difference between the two versions of the complete Bouguer anomaly, let us have a deeper look at the definition of the spherical complete Bouguer anomaly of the surface kind given by Eq. (47). We can re-state it as follows

$$\begin{aligned} \forall \Omega \in \Omega_0 : \Delta g^{CB;S}[H(\Omega)] = g[H(\Omega)] - \gamma[H(\Omega)] - \gamma^{-1}\{H(\Omega) - Z^{NT}[H(\Omega)], \phi\} \frac{\partial \gamma(r, \phi)}{\partial n} \Big|_{r=R+H(\Omega)} T^{NT}[H(\Omega)] - \\ - \varepsilon_{NT}(r, \Omega) + \frac{\partial V^B(r, \Omega)}{\partial H} \Big|_{r=R+H(\Omega)} + \frac{\partial V^R(r, \Omega)}{\partial H} \Big|_{r=R+H(\Omega)} \end{aligned} \quad , \quad (50)$$

which can be further re-written as

$$\begin{aligned} \forall \Omega \in \Omega_0 : \Delta g^{CB;S}[H(\Omega)] \approx g[H(\Omega)] - \gamma_0(\phi) - \frac{\partial \gamma(r, \phi)}{\partial n} \Big|_{r=R+H(\Omega)} H(\Omega) - \\ - \frac{\partial^2 \gamma(r, \phi)}{\partial n^2} \Big|_{r=R+H(\Omega)} \frac{H^2(\Omega)}{2} - 4\pi G \rho_0 H(\Omega) + TC^S[\rho_0; H(\Omega)] - \\ - \varepsilon_{NT}(r, \Omega) + \frac{2}{R} \{V^B[H(\Omega)] + V^R[H(\Omega)]\} + TopoC^B[\delta\rho; H(\Omega)] + \end{aligned} \quad (51)$$

Taking into account Eq. (8), as a generic definition of the surface gravity anomaly

$\Delta g[H(\Omega)]$  on the surface of the Earth, we can finally re-write Eq. (47) as

$$\begin{aligned} \forall \Omega \in \Omega_0 : \Delta g^{CB;S}[H(\Omega)] = \Delta g[H(\Omega)] - 4\pi G \rho_0 H(\Omega) + TC^S[\rho_0; H(\Omega)] - \\ - \varepsilon_{NT}[H(\Omega)] + \frac{2}{R} \{V^B[H(\Omega)] + V^R[H(\Omega)]\} + TopoC^B[\delta\rho; H(\Omega)] + \end{aligned} \quad (52)$$

It is interesting to compare Eq. (52) with Eq. (46) for gravity on the Earth's surface in the NT-space. The main difference, besides the implicit presence of normal gravity in Eq. (52) and the correction  $\varepsilon_{NT}$  for the oblique derivative, is in the fifth term on the right-hand-side of Eq. (52), which is nothing else but the *secondary indirect topographical (and by association also the much smaller atmospheric) effect on spherical complete*

*Bouguer anomaly on the Earth's surface* ( $SITE^{NT}$ ). This secondary indirect effect is quite large (several tens, even hundreds, of mGal), compared to the secondary indirect topographical effect ( $SITE^H$ ) on Helmert's gravity anomaly (cf. Vaníček *et al.* 1999, Eq. A9) – also see Figure 2. While  $SITE^H$  is negligible for all practical applications,  $SITE^{NT}$  must be taken into account. Let us just mention in passing that the planar variety of the  $SITE^{NT}$  is not defined, i.e., does not exist, as the potential  $V^B[H(\Omega)]$  is infinite; hence we do not use the  $S$  in the superscript of the spherical case. The presence of the secondary indirect topographical (and atmospheric) effect constitutes the main difference between the two models.

For completeness, let us finally show also the difference between the spherical complete Bouguer anomaly on the Earth's surface  $\Delta g^{CB:S} [H(\Omega)]$  and the planar complete Bouguer anomaly on the Earth's surface  $\Delta g^{CB:P} [H(\Omega)]$ . This difference is a sum of the difference between the corresponding gravity values  $g^{CB:S} [H(\Omega)]$  and  $g^{CB:P} [H(\Omega)]$  (see Figure 1) and the difference due to the presence of the secondary indirect effect (see Figure 2), and the sum is shown in Figure 3. It is clear that this difference is very significant, both from the point of view of magnitude as well as wavelength. This reflects the fact that the two models of topography are very different; consequently, the resulting anomalies describe two very different gravity fields.

## **8. Harmonicity of spherical complete Bouguer anomalies**

The final component of this paper aims to show that the spherical complete Bouguer anomalies are harmonic, and thus suited to downward continuation. As the disturbing gravity potential  $T^{NT}(r, \Omega)$  in the NT-space is harmonic in the now ‘empty’ (up to the error introduced by inexact topographical model) space above the geoid, so is the product  $r^*g^{NT:S}[H(\Omega)]$  (cf. Heiskanen and Moritz 1967; Eq. 2-155). As its change with depth  $r$  is defined by its harmonicity, the surface gravity anomaly  $\Delta g^{CB:S}[H(\Omega)]$  can be converted to solid gravity anomaly, for example, by rewriting Eq. (47) for the whole space above the geoid:

$$\forall r > r_g(\Omega), \Omega \in \Omega_0 : \Delta g^{CB:S}(r, \Omega) = g^{NT:S}(r, \Omega) - \gamma[r - Z^{NT}(r, \Omega), \phi], \quad (53)$$

which definition assures that the product  $r^*\Delta g^{CB:S}(r, \Omega)$  (where  $\Delta g^{CB:S}(r, \Omega)$  is given by Eq. 52), is also harmonic everywhere above the geoid.

Since  $\Delta g^{CB:S}(r, \Omega)$  has been shown to be a solid gravity anomaly, we may now also define it by means of Eq. (9), in the NT-space, to get:

$$\begin{aligned} \forall r > r_g(\Omega), \Omega \in \Omega_0 : \Delta g^{CB:S}(r, \Omega) &= -\frac{\partial T^{NT}(r, \Omega)}{\partial n} + \gamma[r - Z^{NT}(r, \Omega), \phi]^{-1} \frac{\partial \gamma(r, \phi)}{\partial n} \Big|_r T^{NT}(r, \Omega) = \\ &= -\frac{\partial T^{NT}(r, \Omega)}{\partial r} - \frac{2}{r} T^{NT}(r, \Omega) - \varepsilon_{\delta g} - \varepsilon_n - \varepsilon_{NT} \end{aligned}, \quad 54$$

where  $T^{NT}$  is defined in Eq. (23). On the other hand, the planar variety of the surface complete Bouguer anomaly (on the Earth’s surface) based on Eq. (49), i.e., the “standard”

complete Bouguer anomaly, cannot be simply converted to the solid form. For this anomaly to be convertible into a solid form, it would have to be ascertained that a disturbing potential  $T^{NT,P}(r,\Omega)$ , that generates  $\Delta g^{CB;P}(r,\Omega)$  according to Eq. (9), exists.

We were not able to find such a disturbing potential  $T^{NT,P}(r,\Omega)$ .

We thus feel that it is a reasonably safe assertion that the solid form of the “standard” (planar) complete Bouguer anomaly in the sense defined in this paper does not exist, and thus the standard complete Bouguer anomaly cannot be continued downward to the geoid in a physically meaningful manner. If people wish to use the surface form of planar complete Bouguer anomaly and use a continuation law of their own choosing they are entitled to do it. The question then arises as to the physical interpretation of such “solid” anomaly.

## **9. Summary, discussion and conclusions**

To analyse the properties of Bouguer gravity anomaly, we started by assuming, as always, that the two definitions of gravity anomalies used routinely in geodesy, i.e., Eqs. (8) and (9), are really equivalent. We soon discovered that this really is not the case. We were thus driven to distinguishing between the two definitions and to introducing the distinction between anomalies defined by Eq. (9) – “solid anomalies”, defined in 3-D sense – and those defined by Eq. (8) – “surface anomalies”, defined in 2-D sense. We proceeded to demonstrate that a solid gravity anomaly is automatically also a surface



anomaly but not *vice versa*. A surface anomaly may have a natural solid extension, but it may not. Individual cases would have to be investigated separately.

The next thing we investigated was the question whether or not the complete Bouguer anomaly, is a solid anomaly or just a surface anomaly. To answer this question, we had to distinguish between the Bouguer anomaly computed by means of spherical model of topography and that computed by means of planar model. While our initial suspicion was that the spherical and planar varieties were practically the same, it soon became clear that they were not. The difference between the two models mainly arises from the fact that the spherical variety contains the “secondary indirect topographical effect (SITE)”, which in the case of complete Bouguer anomaly is rather large (as compared to Helmert’s anomaly). This effect cannot be evaluated for the planar variety. Nevertheless, even the differences between the two topographical effects (Bouguer shell or plate plus the terrain corrections of the appropriate kind) are significant tending towards the (planar) Bouguer plate reduction.

Another of our initial beliefs was that the planar complete Bouguer anomalies could be harmonic above the geoid. While it is even intuitively clear that the spherical variety is indeed harmonic above the geoid, we were not able to confirm our initial belief about the “standard” (planar) Bouguer anomaly. From the geometry of the topographical models used, it should have been rather obvious that the field constructed by means of the planar model is not harmonic in the NT-space.

Finally, the spherical complete Bouguer anomaly field should be significantly smoother than the Helmert anomaly field, as pointed out by Novak [Personal communication, May 2002]. This will make it more convenient than the Helmert anomaly to downward-continue it from the Earth's surface to the geoid, as well as making it more suited to gridding and interpolation. The possibility of using the spherical Bouguer anomaly instead of the Helmert anomaly for the downward continuation and then converting it to the Helmert anomaly on the co-geoid (to avoid the very large indirect effect in the NT-space) within the Stokes-Helmert computation scheme will be investigated in the near future.

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