Higher Order Weak Epiderivatives and Applications to Duality and Optimality Conditions[1]

C. R. Chen and S. J. Li[2]
College of Mathematics and Science,
Chongqing University, Chongqing, 400030, China
E-mails: chencr1981@163.com and lisj@cqu.edu.cn

K. L. Teo
Department of Mathematics and Statistics,
Curtin University of Technology, G.P.O. Box U1987, Perth, WA 6845, Australia
E-mail: K.L.Teo@curtin.edu.au

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Abstract: In this paper, the notions of higher order weak contingent epiderivative and higher order weak adjacent epiderivative for a set-valued map are defined. By virtue of higher order weak adjacent (contingent) epiderivatives and Henig efficiency, we introduce a higher order Mond-Weir type dual problem and a higher order Wolfe type dual problem for a constrained set-valued optimization problem (SOP) and discuss the corresponding weak duality, strong duality and converse duality properties. We also establish higher order Kuhn-Tucker type necessary and sufficient optimality conditions for (SOP).

Keywords: Higher order weak adjacent (contingent) epiderivative; Higher order duality; Higher order optimality conditions; Henig efficiency; Set-valued optimization.

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[2]Corresponding author.

1 Introduction

In the last two decades, various notions of derivatives for set-valued maps have been proposed and used for the formulation of optimality conditions and duality in set-valued optimization. With the concept of contingent derivative for a set-valued map (see [1]), Corley [2] investigated optimality conditions for set-valued optimization problems. But it turns out that necessary and sufficient optimality conditions do not coincide under standard assumptions. Jahn and Rauh [3] introduced the contingent epiderivative of a set-valued map and then obtained unified necessary and sufficient optimality conditions. The essential differences between the definitions of the contingent derivative and the contingent epiderivative are that the graph is replaced by the epigraph and the derivative is single-valued. Subsequently, Chen and Jahn [4] (see also Bednarczuk and Song [5]) introduced the concept of generalized contingent epiderivative in terms of minimizers of projection of the contingent cone to epigraph of a set-valued map. In general, since the epigraph of a set-valued map has nicer properties than its graph, it is useful to employ the epiderivatives in set-valued optimization. As to other concepts of epiderivatives for set-valued maps and applications to optimality conditions, one can refer to [6–9] and references therein. Recently, Jahn et al. [10] introduced second-order contingent epiderivative and generalized contingent epiderivative for set-valued maps and obtained some second-order optimality conditions based on these concepts. Very recently, Lalitha and Arora [11] introduced a notion of weak Clarke epiderivative for a set-valued map by using the concept of Clarke tangent cone and established optimality conditions for a constrained set-valued optimization problem in terms of weak Clarke epiderivative. On the other hand, various kinds of differentiable type dual problems for set-valued optimization problems, such as Mond-Weir type and Wolfe type dual problems have been investigated, for example, see [12–14] and so on.

To the best of our knowledge, there are only a few papers deal with higher order optimality conditions and duality of set-valued optimization problems by using higher order derivatives and epiderivatives. Since higher order tangent sets, in general, are not cones and convex sets, there are some difficulties in studying higher order optimality conditions and duality for set-valued optimization problems by virtue of the higher order derivatives or epiderivatives introduced by the higher order tangent sets. Very recently, Li et al. [15] studied some properties of higher order tangent sets and higher order derivatives introduced in [1] and then obtained higher order necessary and sufficient optimality conditions for set-valued optimization problems in terms of the higher order derivatives. By using these higher order derivatives, they [16] also discussed higher order Mond-Weir duality for a set-valued optimization problem based on weak efficiency. Li and Chen [17] introduced higher order generalized contingent epiderivative and higher order generalized adjacent epiderivative of set-valued maps. Higher order Fritz John type necessary and sufficient optimality conditions for Henig efficient solutions to a constrained set-valued optimization problem were obtained by employing the higher order generalized epiderivatives.

Motivated by the work reported in [11, 15–17], we introduce the concepts of higher order weak contingent epiderivative and higher order weak adjacent epiderivative for set-valued maps. Based on higher order weak adjacent (contingent) epiderivatives and Henig efficiency, we investigate higher order Mond-Weir type duality, higher order Wolfe type duality and higher order Kuhn-Tucker type optimality conditions to a constrained set-valued optimization problem (SOP).

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and their properties used in the paper. In Section 3, we define the higher order weak contingent epiderivative and adjacent epiderivative, and discuss the existence and other useful properties. In Sections 4 and 5, we introduce a higher order Mond-Weir type dual problem and a higher order Wolfe type dual problem to (SOP) by virtue of higher order weak adjacent (contingent) epiderivatives and discuss the corresponding weak duality, strong duality and converse duality properties, respectively. In Section 6, we establish higher order Kuhn-Tucker type necessary and sufficient optimality conditions of (SOP).

2 Preliminaries and Higher Order Tangent Sets

Throughout this paper, let X, Y and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with nonempty interiors $\text{int}C$ and $\text{int}D$, respectively. Let Y^* be the topological dual space of Y, S be a nonempty subset of X and $F : S \to 2^Y$ and $G : S \to 2^Z$ be two given set-valued maps. The domain, the graph and the epigraph of F are defined respectively by: $\text{dom}(F) = \{x \in S \mid F(x) \neq \emptyset\}$, $\text{graph}(F) = \{(x, y) \in X \times Y \mid x \in S,\ y \in F(x)\}$, $\text{epi}(F) = \{(x, y) \in X \times Y \mid x \in S,\ y \in F(x) + C\}$. The map F is said to be C-convex on a convex set S, if for any $x_1, x_2 \in S$ and $\lambda \in [0, 1]$, $\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C$.

It is well known that if F is C-convex on S, then $\text{epi}(F)$ is a convex subset in $X \times Y$.

Let C^* be the dual cone of C, defined by $C^* = \{\lambda \in Y^* \mid \lambda(y) \geq 0, \forall\, y \in C\}$. Denote the quasi-interior of C^* by $C^\sharp$, i.e., $C^\sharp = \{\lambda \in Y^* \mid \lambda(y) > 0, \forall\, y \in C \setminus \{0_Y\}\}$. Let M be a nonempty set in Y. Denote the closure of M by $\text{cl}(M)$ and the interior of M by $\text{int}(M)$. The cone hull of M is defined by $\text{cone}(M) = \{ty \mid t \geq 0,\, y \in M\}$. A nonempty convex subset B of the convex cone C is called a base of C, if $C = \text{cone}(B)$ and $0_Y \notin \text{cl}(B)$. It follows from Lemma 3.3 of [18] that $C^\sharp \neq \emptyset$ if and only if C has a base. Suppose that C has a base B. Denote

$$C_\varepsilon(B) = \text{cone}(B + \varepsilon U) \quad \text{for all } \; 0 < \varepsilon < \delta,$$

where $\delta = \inf\{\|b\| \mid b \in B\}$ and U is the closed unit ball of Y. It follows from [7] that, $\delta > 0$, $\text{cl}(\text{int}\, C_\varepsilon(B))$ is a closed convex pointed cone, and $C \setminus \{0_Y\} \subset \text{int}\, C_\varepsilon(B)$ for all $0 < \varepsilon < \delta$. Denote

$$C^\triangle(B) = \{f \in C^* \mid \inf\{f(b) : \; b \in B\} > 0\}.$$

By the separation theorem, $C^\triangle(B) \neq \emptyset$ (see [7]). Obviously, $C^\triangle(B) \subset C^\sharp$.

Lemma 2.1 *([7])*

(i) *For any* $\varepsilon \in (0, \delta)$, $C_\varepsilon(B)^* \setminus \{0_{Y^*}\} \subset C^\triangle(B)$.

(ii) *For any* $f \in C^\triangle(B)$, *there exists* $0 < \varepsilon < \delta$ *with* $f \in C_\varepsilon(B)^* \setminus \{0_{Y^*}\}$.

Definition 2.1 *([16])* $F : X \to 2^Y$ *is called pseudo-Lipschitzian at* $(x_0, y_0) \in graph(F)$, *if there exist* $M > 0$ *and neighborhoods* V *of* x_0 *and* W *of* y_0 *such that* $F(x_1) \cap W \subset F(x_2) + M\|x_1 - x_2\| B_Y$, $\forall\, x_1, x_2 \in V$, *where* B_Y *denotes the unit ball of the origin in* Y.

Let m be a positive integer, X be a normed space supplied with a distance d and K be a subset of X. We denote by $d(x, K) = \inf_{y \in K} d(x, y)$ the distance from x to K, where we set $d(x, \emptyset) = +\infty$. Now we recall the definitions of the higher order tangent sets.

Definition 2.2 *([1]) Let* $x \in K \subset X$ *and* $v_1, \cdots, v_{m-1}$ *be elements of* X.

(i) *The set*

$$\begin{aligned} T_K^{(m)}(x, v_1, \cdots, v_{m-1}) &= Limsup_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1} v_{m-1}}{h^m} \\ &= \{y \in X \mid \liminf_{h \to 0^+} d(y, \frac{K - x - hv_1 - \cdots - h^{m-1} v_{m-1}}{h^m}) = 0\} \end{aligned}$$

is called the m^{th}*-order contingent set of* K *at* $(x, v_1, \cdots, v_{m-1})$.

(ii) *The set*

$$T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1}) = \mathit{Liminf}_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}$$
$$= \{y \in X \mid \lim_{h \to 0^+} d(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}) = 0\}$$

is called the m^{th}-order adjacent set of K at $(x, v_1, \cdots, v_{m-1})$.

Remark 2.1 *([1])*

(a) *The following inclusion relation holds:*

$$T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1}) \subset T_K^{(m)}(x, v_1, \cdots, v_{m-1})$$
$$\subset cl(\bigcup_{h>0} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}).$$

(b) *Both tangent sets $T_K^{(m)}(x, v_1, \cdots, v_{m-1})$ and $T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1})$ are closed.*

From Propositions 3.1 and 3.2 of [15], we have the following results.

Proposition 2.1 *If K is convex, then $T_K^{\flat(m)}(x_0, v_1, \cdots, v_{m-1})$ is convex.*

Proposition 2.2 *If K is a convex subset and $v_1, v_2, \cdots, v_{m-1} \in K$, then*

$$T_K^{\flat(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0) = T_K^{(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0)$$
$$= cl(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}).$$

3 Higher Order Weak Epiderivatives

Definition 3.1 *Let $H \subset Y$ and $intC \neq \emptyset$. An element $\bar{y} \in H$ is said to be a minimal point (resp. weakly minimal point) of H if $H \cap (\bar{y} - C) = \{\bar{y}\}$ (resp. $H \cap (\bar{y} - intC) = \emptyset$). The set of all minimal points (resp. weakly minimal points) of H is denoted by $Min_C\, H$ (resp. $WMin_C\, H$).*

Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map. We first recall the definitions of higher order generalized contingent epiderivative and adjacent epiderivative introduced by Li and Chen [17].

Definition 3.2 **(i)** *The m^{th}-order generalized contingent epiderivative $D_g^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in graph(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by*

$$D_g^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$
$$= Min_C\{y \in Y \mid (x, y) \in T_{epi(F)}^{(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}.$$

(ii) *The m^{th}-order generalized adjacent epiderivative $D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in graph(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by*

$$D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$
$$= Min_C\{y \in Y \mid (x, y) \in T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}.$$

Now we introduce the following higher order weak contingent and adjacent epiderivatives in terms of weak efficiency.

Definition 3.3 **(i)** *The m^{th}-order weak contingent epiderivative $D_w^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in graph(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by*

$$D_w^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$
$$= WMin_C\{y \in Y \mid (x, y) \in T_{epi(F)}^{(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}.$$

(ii) *The m^{th}-order weak adjacent epiderivative $D_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in graph(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by*

$$D_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$
$$= WMin_C\{y \in Y \mid (x, y) \in T_{epi(F)}^{\flat(m)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}.$$

Remark that Jahn and Khan [8] have introduced the notion of first-order weak contingent epiderivative of set-valued maps. It is obvious that for all $x \in X$,

$$D_g^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \subset D_w^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$

and

$$D_g^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \subset D_w^{\flat(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x).$$

Now we give the following examples to explain various epiderivatives.

Example 3.1 *Let $F : R \to 2^{R^2}$ be a set-valued map given by*

$$F(x) = \{(y_1, y_2) \in R^2 \mid y_1 \geq x^4, y_2 \geq x^2\}$$

and $C = R^2_+$. Let $(x_0, y_0) = (0, (0, 0)) \in graph(F)$ and $(u_1, v_1) = (1, (0, 0))$. Then, $T_{epi(F)}(x_0, y_0) = R \times R^2_+$ and $T^{(2)}_{epi(F)}(x_0, y_0, u_1, v_1) = R \times (R_+ \times [1, +\infty))$. Hence, for all $x \in R$, we have

$$\begin{aligned}
&D_g F(x_0, y_0)(x) = \{(0, 0)\},\\
&D_w F(x_0, y_0)(x) = \{(y_1, 0) \mid y_1 \geq 0\} \cup \{(0, y_2) \mid y_2 \geq 0\},\\
&D^{(2)}_g F(x_0, y_0, u_1, v_1)(x) = \{(0, 1)\},\\
&D^{(2)}_w F(x_0, y_0, u_1, v_1)(x) = \{(y_1, 1) \mid y_1 \geq 0\} \cup \{(0, y_2) \mid y_2 \geq 1\}.
\end{aligned}$$

Example 3.2 *Let $F : R \to 2^{R^2}$ be a set-valued map given by*

$$F(x) = \{(y_1, y_2) \in R^2 \mid (y_1 - 1)y_2 \leq 0\}$$

and $C = R^2_+$. Let $(x_0, y_0) = (0, (1, 0)) \in graph(F)$ and $(u_1, v_1) = (1, (0, 0))$. Then, $T_{epi(F)}(x_0, y_0) = epi(F) - (x_0, y_0) = R \times (R^2 \backslash intR^2_-) = T^{(2)}_{epi(F)}(x_0, y_0, u_1, v_1)$. Hence, for all $x \in R$, we have

$$\begin{aligned}
&D_g F(x_0, y_0)(x) = D^{(2)}_g F(x_0, y_0, u_1, v_1)(x) = \emptyset,\\
&D_w F(x_0, y_0)(x) = D^{(2)}_w F(x_0, y_0, u_1, v_1)(x) = \{(y_1, 0) \mid y_1 \leq 0\} \cup \{(0, y_2) \mid y_2 \leq 0\}.
\end{aligned}$$

Definition 3.4 *([11, 19])*

(i) *The cone C is called Daniell, if any decreasing sequence in Y having a lower bound converges to its infimum.*

(ii) *A subset H of Y is said to be minorized, if there is a $y \in Y$ so that $H \subset \{y\} + C$.*

(iii) *The weak domination property (resp. domination property) is said to hold for a subset H of Y if $H \subset WMin_C H + intC \cup \{0_Y\}$ (resp. $H \subset Min_C H + C$).*

Now we give an existence theorem of $D^{(m)}_w F$ and $D^{\flat(m)}_w F$.

Theorem 3.1 *Let C be a closed pointed convex cone and let C be Daniell.*

(i) *Suppose that the set $P_0(x) := \{y \in Y \mid (x, y) \in T^{(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}$ is minorized for every $x \in domP_0$. Then $D^{(m)}_w F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$ exists for all $x \in domP_0$.*

(ii) *Suppose that the set* $P(x){:=}\{y \in Y \mid (x,y) \in T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})\}$ *is minorized for every* $x \in domP$*. Then* $D^{\flat(m)}_w F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$ *exists for all* $x \in domP$*.*

Proof. It follows from Remark 2.1(b) that the m^{th}-order contingent set (resp. m^{th}-order adjacent set) is closed. Thus for every $x \in \mathrm{dom}P_0$ (resp. $x \in \mathrm{dom}P$), $P_0(x)$ (resp. $P(x)$) is minorized and closed. From the existence theorem of minimal points (see [19]), $\mathrm{Min}_C P_0(x)$ (resp. $\mathrm{Min}_C P(x)$) is nonempty. Whence, $D^{(m)}_w F$ (resp. $D^{\flat(m)}_w F$) is well defined. □

Now we give the following crucial proposition.

Proposition 3.1 *Let* F *be* C*-convex on a nonempty convex subset* $E \subset X$*. Let* $(x_0, y_0) \in$ *graph*(F) *and* $(u_i, v_i) \in$ *epi*(F)*,* $i = 1, \cdots, m-1$*. If the set* $P(x - x_0){:=}\{y \in Y \mid (x - x_0, y) \in T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ *fulfills the weak domination property for all* $x \in E$*, then for all* $x \in E$*,*

$$F(x) - y_0 \subset D^{\flat(m)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Proof. The proof follows on the lines of Proposition 3.1 in [17] by replacing m^{th}-order generalized adjacent epiderivative by m^{th}-order weak adjacent epiderivative and domination property by weak domination property. □

Corollary 3.1 *Let* F *be* C*-convex on a nonempty convex subset* $E \subset X$*. Let* $(x_0, y_0) \in$ *graph*(F) *and* $(u_i, v_i) \in$ *epi*(F)*,* $i = 1, \cdots, m-1$*. If the set* $P_0(x - x_0){:=}\{y \in Y \mid (x - x_0, y) \in T^{(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ *fulfills the weak domination property for all* $x \in E$*, then for all* $x \in E$*,*

$$F(x) - y_0 \subset D^{(m)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C.$$

Proof. Since F is C-convex and $(u_i, v_i) \in \mathrm{epi}(F), i = 1, \cdots, m-1$, by Proposition 2.2, we get that $T^{(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0) = T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)$. Thus, it follows from Proposition 3.1 that the conclusion holds. □

4 Higher Order Mond-Weir Type Duality

In this section, we introduce a higher order Mond-Weir type dual problem for a constrained set-valued optimization problem by virtue of higher order weak adjacent epiderivatives and discuss the weak duality, strong duality and converse duality properties.

Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be two set-valued maps. Consider the following constrained set-valued optimization problem (SOP):

$$\begin{array}{ll} \min & F(x) \\ \text{s.t.} & x \in X,\ G(x) \cap (-D) \neq \emptyset. \end{array}$$

Set $A = \{x \in X \mid G(x) \cap (-D) \neq \emptyset\}$ and $F(A) = \bigcup\{F(x) \mid x \in A\}$. The notation $(F, G)(x)$ is used to denote $F(x) \times G(x)$.

A point $(x_0, y_0) \in X \times Y$ is called a feasible solution of (SOP) if $x_0 \in A$ and $y_0 \in F(x_0)$.

In the sequel, suppose that C has a base B, $\text{int}D \neq \emptyset$ and $\delta = \inf\{\|b\| \mid b \in B\}$.

Definition 4.1 *([17]) A feasible solution (x_0, y_0) is called a Henig minimal solution of (SOP) if for some $0 < \varepsilon < \delta$, $(F(A) - y_0) \cap (-intC_\varepsilon(B)) = \emptyset$.*

Suppose that $(u_i, v_i) \in \text{epi}(F)$, $(u_i, w_i) \in \text{epi}(G)$, $i = 1, \cdots, m-1$ and $(\hat{x}, \hat{y}) \in \text{graph}(F)$. We introduce a Mond-Weir type dual problem (DSOP) of (SOP) as follows:

$$\begin{array}{lll} \max & \hat{y} & \\ \text{s.t.} & \lambda D_w^{\flat(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x) + & \\ & \mu D_w^{\flat(m)} G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})(x) \geq 0,\ x \in X, & (1) \\ & \mu(\hat{z}) \geq 0, & (2) \\ & \lambda \in C^{\triangle}(B), & (3) \\ & \mu \in D^*, & (4) \end{array}$$

where $\hat{z} \in G(\hat{x})$, and (1) means that $\lambda(y) + \mu(z) \geq 0$, for all $(y, z) \in D_w^{\flat(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x) \times D_w^{\flat(m)} G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})(x)$.

Throughout this paper, we assume that $\forall\, \lambda \in C^{\triangle}(B)$, $\mu \in D^*$, $\lambda\emptyset = \mu\emptyset = +\infty$. Hence, (1) holds naturally whenever $x \notin \text{dom}[D_w^{\flat(m)} F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})] \cap \text{dom}[D_w^{\flat(m)} G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})]$.

Let

$$H_D := \{\hat{y} \in F(\hat{x}) \mid (\hat{x}, \hat{y}, \hat{z}, \lambda, \mu) \text{ satisfies conditions } (1) - (4)\}.$$

A point $(x_0, y_0, z_0, \lambda_0, \mu_0)$ satisfying (1)-(4) is called a feasible solution of (DSOP).

Definition 4.2 *A feasible solution* $(x_0, y_0, z_0, \lambda_0, \mu_0)$ *is called a maximal solution of (DSOP) if* $(H_D - y_0) \cap (C \setminus \{0_Y\}) = \emptyset$.

Theorem 4.1 *(Weak duality) Suppose that* F *and* G *are* C*-convex and* D*-convex on* X*, respectively. Let* $(u_i, v_i) \in epi(F)$*,* $(u_i, w_i) \in epi(G)$*,* $i = 1, \cdots, m-1$*. Assume the feasible solution* (x_0, y_0) *of (SOP) and the feasible solution* $(\hat{x}, \hat{y}, \hat{z}, \lambda, \mu)$ *of (DSOP) satisfying that the sets* $P_F(x_0 - \hat{x}) := \{y \in Y \mid (x_0 - \hat{x}, y) \in T^{\flat(m)}_{epi(F)}(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})\}$ *and* $P_G(x_0 - \hat{x}) := \{z \in Z \mid (x_0 - \hat{x}, z) \in T^{\flat(m)}_{epi(G)}(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})\}$ *fulfill the weak domination property. Then* $\lambda(y_0) \geq \lambda(\hat{y})$.

Proof. It follows from Proposition 3.1 that

$$y_0 - \hat{y} \in D^{\flat(m)}_w F(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})(x_0 - \hat{x}) + C \tag{5}$$

and

$$G(x_0) - \hat{z} \subset D^{\flat(m)}_w G(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})(x_0 - \hat{x}) + D. \tag{6}$$

Since (x_0, y_0) is a feasible solution of (SOP), $G(x_0) \cap (-D) \neq \emptyset$. Take $z_0 \in G(x_0) \cap (-D)$. Then, by (2) and (4), we have that

$$\mu(z_0) - \mu(\hat{z}) \leq 0. \tag{7}$$

It follows from (1), (3), (4), (5) and (6) that $\lambda(y_0 - \hat{y}) + \mu(z_0 - \hat{z}) \geq 0$. Therefore, by (7), we get $\lambda(y_0) \geq \lambda(\hat{y})$. □

Lemma 4.1 *Let* $(x_0, y_0) \in graph(F)$ *and* $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D), i = 1, \cdots, m-1$*. If* (x_0, y_0) *is a Henig minimal solution of (SOP), then for some* $0 < \varepsilon < \delta$ *and for any* $z_0 \in G(x_0) \cap (-D)$*,*

$$[D^{\flat(m)}_w (F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X) + C \times D + (0_Y, z_0)] \cap -int(C_\varepsilon(B) \times D) = \emptyset.$$

Proof. If $D^{\flat(m)}_w (F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) = \emptyset$ for some $x \in X$, then the result holds trivially. So we suppose $x \in \Omega := \mathrm{dom}[D^{\flat(m)}_w (F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)]$.

Then the proof follows on the lines of Theorem 4.1 in [17] by replacing m^{th}-order generalized adjacent epiderivative by m^{th}-order weak adjacent epiderivative. □

Proposition 4.1 *Suppose that* $P(x) := \{(y,z) \in Y \times Z \mid (x,y,z) \in T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ *fulfills the weak domination property for all* $x \in X$ *and* $G + D$ *is pseudo-Lipschitzian at* (x_0, z_0)*, where* $x_0 \in X$*,* $y_0 \in F(x_0)$ *and* $z_0 \in G(x_0)$*. Then for all* $x \in X$*,*

$$\begin{aligned} & D^{\flat(m)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ & \quad \times D^{\flat(m)}_w G(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x) \\ \subset \ & D^{\flat(m)}_w (F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, \\ & \quad u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) + C \times D. \end{aligned}$$

Proof. If either $D^{\flat(m)}_w F(\cdot)(x)$ or $D^{\flat(m)}_w G(\cdot)(x)$ is empty, then the inclusion relation holds trivially. Suppose that

$$\begin{aligned} (y,z) \ \in \ & D^{\flat(m)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ & \times D^{\flat(m)}_w G(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x). \end{aligned}$$

It follows from the definition of the m^{th}-order weak adjacent epiderivative that

$$(x,y) \in T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)$$

and

$$(x,z) \in T^{\flat(m)}_{epi(G)}(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0).$$

Whence, for any $h_n \to 0^+$, there exists $(x_n, y_n) \to (x, y)$ such that

$$\begin{aligned} & y_0 + h_n(v_1 - y_0) + \cdots + h_n^{m-1}(v_{m-1} - y_0) + h_n^m y_n \\ & \in F(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + C, \end{aligned} \tag{8}$$

and there exists $(\bar{x}_n, \bar{z}_n) \to (x, z)$ such that

$$\begin{aligned} & z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \\ & \in G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m \bar{x}_n) + D. \end{aligned} \tag{9}$$

By the pseudo-Lipschitzian assumption, there exist $M > 0$, and neighborhoods $\mathcal{W}$ of z_0 and $\mathcal{N}$ of x_0 such that

$$(G(x_1) + D) \cap \mathcal{W} \subset G(x_2) + D + M\|x_1 - x_2\| B_Z, \quad \forall x_1, x_2 \in \mathcal{N}, \tag{10}$$

where B_Z denotes the unit ball of the origin in Z. Naturally, there exists $N > 0$ satisfying

$$x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n \in \mathcal{N}, \ \forall n \geq N,$$

and

$$z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \in \mathcal{W},\ \forall n \geq N. \tag{11}$$

It follows from (9)-(11) that $\forall n \geq N$,

$$\begin{aligned}
& z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m \bar{z}_n \\
& \in (G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m \bar{x}_n) + D) \cap \mathcal{W} \\
& \subset G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + D + h_n^m M||\bar{x}_n - x_n||B_Z.
\end{aligned}$$

Then, there exists $z_n \to z$ such that for any $n \geq N$,

$$\begin{aligned}
& z_0 + h_n(w_1 - z_0) + \cdots + h_n^{m-1}(w_{m-1} - z_0) + h_n^m z_n \\
& \in G(x_0 + h_n(u_1 - x_0) + \cdots + h_n^{m-1}(u_{m-1} - x_0) + h_n^m x_n) + D.
\end{aligned} \tag{12}$$

It follows from (8) and (12) that

$$(x, y, z) \in T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0),$$

i.e., $(y, z) \in P(x)$. Since $P(x)$ fulfills the weak domination property for all $x \in X$, we get $(y, z) \in D^{\flat(m)}_w(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) + C \times D$. □

Theorem 4.2 *(Strong duality) Suppose that (x_0, y_0) is a Henig minimal solution of (SOP) and the following conditions are satisfied:*

(i) *F is C-convex on X and G is D-convex on X;*

(ii) *$(u_i, v_i, w_i) \in epi(F, G)$ and $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D)$, $i = 1, \cdots, m-1$;*

(iii) *there exists $x' \in X$ such that $G(x') \cap (-intD) \neq \emptyset$;*

(vi) *$z_0 \in G(x_0) \cap (-D)$ and $G + D$ is pseudo-Lipschitzian at (x_0, z_0);*

(v) *$P(x) := \{(y, z) \in Y \times Z \mid (x, y, z) \in T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ fulfills the weak domination property for all $x \in X$ and $(0_Y, 0_Z) \in P(0_X)$;*

(iv) *the sets $P_F(x_0 - \hat{x}) := \{y \in Y \mid (x_0 - \hat{x}, y) \in T^{\flat(m)}_{epi(F)}(\hat{x}, \hat{y}, u_1 - \hat{x}, v_1 - \hat{y}, \cdots, u_{m-1} - \hat{x}, v_{m-1} - \hat{y})\}$ and $P_G(x_0 - \hat{x}) := \{z \in Z \mid (x_0 - \hat{x}, z) \in T^{\flat(m)}_{epi(G)}(\hat{x}, \hat{z}, u_1 - \hat{x}, w_1 - \hat{z}, \cdots, u_{m-1} - \hat{x}, w_{m-1} - \hat{z})\}$ fulfill the weak domination property for all $\hat{x} \in X$, where $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$.*

Then, there exist $\lambda \in C^{\triangle}(B)$ and $\mu \in D^*$ such that $(x_0, y_0, z_0, \lambda, \mu)$ is a maximal solution of (DSOP).

Proof. Define

$$\begin{aligned} M &= D_w^{\flat(m)}(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, \\ &\quad u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X) + C \times D + (0_Y, z_0). \end{aligned}$$

By the similar proof method for the convexity of M in Theorem 5.1 in [17], just replacing m^{th}-order generalized adjacent epiderivative by m^{th}-order weak adjacent epiderivative and domination property by weak domination property, we have that M is a convex set.

By Lemma 4.1, there exists $0 < \varepsilon < \delta$ such that

$$M \cap -\mathrm{int}(C_\varepsilon(B) \times D) = \emptyset.$$

By the separation theorem for convex sets, there exist $\lambda \in Y^*$ and $\mu \in Z^*$, not both zero functionals, and a real number γ such that

$$\lambda(\bar{y}) + \mu(\bar{z}) < \gamma \le \lambda(\tilde{y}) + \mu(\tilde{z}),\ \forall\, \bar{y} \in -\mathrm{int}C_\varepsilon(B), \bar{z} \in -\mathrm{int}D, (\tilde{y}, \tilde{z}) \in M. \tag{13}$$

It follows from $(\bar{y}, \bar{z}) \in -\mathrm{int}(C_\varepsilon(B) \times D)$ and (13) that

$$\lambda(\bar{y}) + \mu(\bar{z}) \le 0,\ \forall\, \bar{y} \in -\mathrm{int}C_\varepsilon(B), \bar{z} \in -\mathrm{int}D, \tag{14}$$

and

$$0 \le \lambda(\tilde{y}) + \mu(\tilde{z}),\ \forall\, (\tilde{y}, \tilde{z}) \in M. \tag{15}$$

Then, by (14), we have $\lambda(\bar{y}) \le 0$ for all $\bar{y} \in -\mathrm{int}C_\varepsilon(B)$, and $\mu(\bar{z}) \le 0$ for all $\bar{z} \in -\mathrm{int}D$. Thus, $\lambda \in C_\varepsilon(B)^*$ and $\mu \in D^*$. By Lemma 2.1(i), $\lambda \in C^{\triangle}(B) \cup \{0_{Y^*}\}$. Since $P(x)$ fulfills the weak domination property for all $x \in X$, hence

$$\begin{aligned} P(X) &\subset D_w^{\flat(m)}(F,G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, \\ &\quad u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X) + C \times D \\ &= M - (0_Y, z_0). \end{aligned}$$

It follows from $(0_Y, 0_Z) \in P(0_X)$ that $(0_Y, 0_Z) \in M - (0_Y, z_0)$, i.e., $(0_Y, z_0) \in M$. From (15), we have $\mu(z_0) \ge 0$. It follows from $z_0 \in -D$ and $\mu \in D^*$ that $\mu(z_0) \le 0$. Thus, $\mu(z_0) = 0$. Moreover, it follows from (15) that

$$\lambda(y) + \mu(z) \ge 0,$$

for all $(y,z) \in D_w^{\flat(m)}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,\cdots,u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)(X)+C\times D$.

Now we show that $\lambda \neq 0_{Y^*}$. By Proposition 3.1, for any $(y',z') \in (F,G)(A)$, we get

$$\begin{aligned}(y',z')-(y_0,z_0) \in\ & D_w^{\flat(m)}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,\cdots,\\ & u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)(x'-x_0)+C\times D,\ x'\in A.\end{aligned}$$

Hence, $\lambda(y'-y_0)+\mu(z'-z_0) \geq 0$. Moreover, $\lambda(y'-y_0)+\mu(z') \geq 0$. Suppose that $\lambda = 0_{Y^*}$. Then $\mu \neq 0_{Z^*}$ and hence for all $z' \in G(A)$, $\mu(z') \geq 0$. Since the generalized Slater's constraint qualification is satisfied, there exists $\hat{x} \in X$ such that $G(\hat{x}) \cap (-\text{int}D) \neq \emptyset$. This implies that there exists $\hat{z} \in G(\hat{x}) \cap (-\text{int}D)$. Since $\hat{z} \in -\text{int}D$ and $\mu \in D^*\backslash\{0_{Z^*}\}$, it follows that $\mu(\hat{z}) < 0$, which leads to a contradiction.

Consequently, in view of Proposition 4.1, we see that $(x_0,y_0,z_0,\lambda,\mu)$ is a feasible solution of (DSOP).

Finally, we prove that $(x_0,y_0,z_0,\lambda,\mu)$ is a maximal solution of (DSOP). Suppose that $(x_0,y_0,z_0,\lambda,\mu)$ is not a maximal solution of (DSOP). Then, there exists a feasible solution $(\hat{x},\hat{y},\hat{z},\lambda',\mu')$ such that $\hat{y}-y_0 \in C\setminus\{0_Y\}$. By $\lambda' \in C^{\triangle}(B) \subset C^{\sharp}$, we have

$$\lambda'(\hat{y}) > \lambda'(y_0). \tag{16}$$

Since (x_0,y_0) is a feasible solution of (SOP), by Theorem 4.1, we have that $\lambda'(y_0) \geq \lambda'(\hat{y})$, which contradicts (16). Thus, the proof is complete. □

Remark 4.1 *In [7], Gong et al. introduced the assumption (C): For any $\xi \in D^*\backslash\{0_{Z^*}\}$, there exists $x \in A = \{x \in X \mid G(x)\cap(-D) \neq \emptyset\}$ such that $\xi(G(x))\cap(-intR_+) \neq \emptyset$. This assumption is weaker than the assumption (iii) of Theorem 4.2, which is called the generalized Slater's constraint qualification (CQ, in short). It is easy to show that (CQ) can be weakened to the assumption (C) in Theorem 4.2 and in what follows (e.g., Theorems 5.2 and 6.1), respectively.*

Theorem 4.3 *(Converse duality) Suppose that there exist $x_0 \in X, y_0 \in F(x_0), z_0 \in G(x_0)\cap(-D)$, $\lambda \in C^{\triangle}(B)$ and $\mu \in D^*$ such that $(x_0,y_0,z_0,\lambda,\mu)$ is a feasible solution of (DSOP) and the following conditions are satisfied:*

(i) *F is C-convex on X and G is D-convex on X;*

(ii) *$(u_i,v_i,w_i) \in epi(F,G)$, $i=1,\cdots,m-1$;*

(iii) *the sets* $P_F(x-x_0) := \{y \in Y \mid (x-x_0, y) \in T^{\flat(m)}_{epi(F)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ *and* $P_G(x-x_0) := \{z \in Z \mid (x-x_0, z) \in T^{\flat(m)}_{epi(G)}(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)\}$ *fulfill the weak domination property for all* $x \in A$.

Then, (x_0, y_0) *is a Henig minimal solution of (SOP).*

Proof. Suppose that $x \in A$. Then, there exists $z \in G(x) \cap (-D)$. It follows from Proposition 3.1 that

$$z - z_0 \in D^{\flat(m)}_w G(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0) + D.$$

By (2), we have that $\mu(z_0) \geq 0$. It follows from $z_0 \in G(x_0) \cap (-D)$ that $\mu(z_0) \leq 0$. So $\mu(z_0) = 0$, and

$$\mu(z - z_0) = \mu(z) - \mu(z_0) = \mu(z) \leq 0. \tag{17}$$

Therefore, it follows from (1) and (17) that

$$\lambda D^{\flat(m)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \geq 0, \; x \in A. \tag{18}$$

From Proposition 3.1 and (18), we have

$$\lambda(F(A) - y_0) \geq 0.$$

Since $\lambda \in C^{\triangle}(B)$, by Lemma 2.1(ii), there exists $\varepsilon \in (0, \delta)$ such that $\lambda \in C_\varepsilon(B)^* \setminus \{0_{Y^*}\}$.

Suppose that the feasible solution (x_0, y_0) is not a Henig minimal solution of (SOP). Then for ε, $(F(A) - y_0) \cap (-\text{int}C_\varepsilon(B)) \neq \emptyset$. Whence, there exists $x' \in A$ and $y' \in F(x')$ such that $y' - y_0 \in -\text{int}C_\varepsilon(B)$. Hence, $\lambda(y' - y_0) < 0$, which yields a contradiction. Thus, (x_0, y_0) is a Henig minimal solution of (SOP) and this completes the proof. □

5 Higher Order Wolfe Type Duality

In this section, we introduce a higher order Wolfe type dual problem for (SOP) by virtue of higher order weak adjacent epiderivatives and discuss the weak duality, strong duality and converse duality properties.

Suppose that $(u_i, v_i) \in \text{epi}(F)$, $(u_i, w_i) \in \text{epi}(G)$, $i = 1, \cdots, m-1$ and $(\hat{x}, \hat{y}) \in \text{graph}(F)$, $(\hat{x}, \hat{z}) \in \text{graph}(G)$. We introduce a Wolfe type dual problem (WDSOP) of

(SOP) as follows:

$$\begin{aligned}
\max \quad & \Psi(\hat{x},\hat{y},\hat{z},\lambda^*,\mu^*) = \lambda^*(\hat{y}) + \mu^*(\hat{z}) \\
\text{s.t.} \quad & \lambda^* D_w^{\flat(m)} F(\hat{x},\hat{y},u_1-\hat{x},v_1-\hat{y},\cdots,u_{m-1}-\hat{x},v_{m-1}-\hat{y})(x) + \\
& \mu^* D_w^{\flat(m)} G(\hat{x},\hat{z},u_1-\hat{x},w_1-\hat{z},\cdots,u_{m-1}-\hat{x},w_{m-1}-\hat{z})(x) \geq 0,\ x \in X, \quad (19) \\
& \lambda^* \in C^{\triangle}(B), \quad (20) \\
& \mu^* \in D^*. \quad (21)
\end{aligned}$$

A point $(x_0, y_0, z_0, \lambda_0, \mu_0)$ satisfying (19)-(21) is called a feasible solution of (WDSOP).

Definition 5.1 *A feasible solution $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is called a optimal solution of (WD-SOP) if for any feasible solution (x, y, z, λ, μ), $\Psi(x_0, y_0, z_0, \lambda_0, \mu_0) \geq \Psi(x, y, z, \lambda, \mu)$.*

Theorem 5.1 *(Weak duality) Suppose that (x_0, y_0) is a feasible solution of (SOP) and $(\hat{x},\hat{y},\hat{z},\lambda^*,\mu^*)$ is a feasible solution of (WDSOP), which satisfy the conditions stated in Theorem 4.1. Then $\lambda^*(y_0) \geq \Psi(\hat{x},\hat{y},\hat{z},\lambda^*,\mu^*)$.*

Proof. By virtue of Proposition 3.1, we have that for any $z_0 \in G(x_0) \cap (-D)$, there exist $y_F \in D_w^{\flat(m)} F(\hat{x},\hat{y},u_1-\hat{x},v_1-\hat{y},\cdots,u_{m-1}-\hat{x},v_{m-1}-\hat{y})(x_0-\hat{x})$ and $z_G \in D_w^{\flat(m)} G(\hat{x},\hat{z},u_1-\hat{x},w_1-\hat{z},\cdots,u_{m-1}-\hat{x},w_{m-1}-\hat{z})(x_0-\hat{x})$ such that $(y_0-\hat{y})-y_F \in C$ and $(z_0-\hat{z})-z_G \in D$, respectively. Thus,

$$\lambda^*(y_0-\hat{y}) \geq \lambda^*(y_F) \geq -\mu^*(z_G) \geq -\mu^*(z_0-\hat{z}) \geq \mu^*(\hat{z}).$$

Hence, we get that $\lambda^*(y_0) \geq \lambda^*(\hat{y}) + \mu^*(\hat{z}) = \Psi(\hat{x},\hat{y},\hat{z},\lambda^*,\mu^*)$. □

Theorem 5.2 *(Strong duality) Suppose that (x_0, y_0), where $y_0 = 0_Y$ is a Henig minimal solution of (SOP) and the conditions in Theorem 4.2 are satisfied. Then, there exist $\lambda_0 \in C^{\triangle}(B)$ and $\mu_0 \in D^*$ such that $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is a optimal solution of (WDSOP).*

Proof. From the proof of Theorem 4.2, we see that there exist $\lambda_0 \in C^{\triangle}(B)$ and $\mu_0 \in D^*$ such that $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is a feasible solution of (WDSOP) and $\mu_0(z_0) = 0$. Therefore, $\lambda_0(y_0) = \Psi(x_0, y_0, z_0, \lambda_0, \mu_0)$.

Suppose that $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is not a optimal solution of (WDSOP). Then, there exists a feasible solution $(x', y', z', \lambda', \mu')$ of (WDSOP) such that $\Psi(x_0, y_0, z_0, \lambda_0, \mu_0) <$

$\Psi(x', y', z', \lambda', \mu')$. Since (x_0, y_0) is a feasible solution of (SOP), by Theorem 5.1, we have that $\lambda'(y_0) \geq \Psi(x', y', z', \lambda', \mu')$. Thus, we get that $\lambda_0(y_0) < \lambda'(y_0)$, i.e., $(\lambda_0 - \lambda')(y_0) < 0$. Since $y_0 = 0_Y$, a contradiction yields. Consequently, $(x_0, y_0, z_0, \lambda_0, \mu_0)$ is a optimal solution of (WDSOP). □

Theorem 5.3 *(Converse duality) Suppose that there exist* $x_0 \in X, y_0 \in F(x_0), z_0 \in G(x_0) \cap (-D)$, $\lambda_0 \in C^{\triangle}(B)$ *and* $\mu_0 \in D^*$ *such that* $(x_0, y_0, z_0, \lambda_0, \mu_0)$ *is a feasible solution of (WDSOP) and* $u_0(z_0) \geq 0$. *Moreover, suppose the conditions in Theorem 4.3 are satisfied. Then,* (x_0, y_0) *is a Henig minimal solution of (SOP).*

Proof. The proof is similar to Theorem 4.3. □

6 Higher Order Kuhn-Tucker Type Optimality Conditions

In this section, we discuss higher order Kuhn-Tucker type necessary and sufficient optimality conditions for (SOP).

Theorem 6.1 *(Necessary condition) Suppose that* (x_0, y_0) *is a Henig minimal solution of (SOP) and the conditions (i)-(iii) and (v) of Theorem 4.2 are satisfied. Then, for any* $z_0 \in G(x_0) \cap (-D)$, *there exist* $\lambda \in C^{\triangle}(B)$ *and* $\mu \in D^*$ *such that*

$$\mu(z_0) = 0 \quad \textit{and} \quad \lambda(y) + \mu(z) \geq 0, \tag{22}$$

for all $(y, z) \in D_w^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X)$.

Furthermore, if $G+D$ *is pseudo-Lipschitzian at* (x_0, z_0), *then (22) holds for all* $(y, z) \in D_w^{\flat(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(X) \times D_w^{\flat(m)}G(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(X)$.

Proof. In the proof process of Theorem 4.2, we have obtained that (22) holds for all $(y, z) \in D_w^{\flat(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X) + C \times D$. Then the conclusion follows readily. □

Remark that if the generalized Slater's constraint qualification is not satisfied in Theorem 6.1, then $\lambda \in C^{\triangle}(B) \cup \{0_{Y^*}\}$. Thus, we obtain the so-called higher order Fritz John type necessary optimality conditions for (SOP).

Now we give the following example to illustrate the Kuhn-Tucker type necessary optimality conditions with respect to the m^{th}-order weak contingent epiderivative (see Remark 6.1). Here we only take $m = 1, 2$ yet.

Example 6.1 *Suppose that $X = Y = Z = R$ and $C = D = R_+$. Let $F : X \to 2^Y$ and $G : X \to Z$ be given by $F(x) = \{y \in R \mid y \geq x^2\}$ and $G(x) = x - 1$, respectively. Naturally, F and G are R_+-convex on X, respectively, and the generalized Slater's constraint qualification is satisfied. Consider the corresponding constrained set-valued optimization problem (SOP). We have $A = \{x \in R \mid x - 1 \leq 0\} = (-\infty, 1]$ and $F(A) = \bigcup_{x\in(-\infty,1]} F(x) = [0, +\infty)$. Let $B = \{1\}$. Obviously, B is a base of C and hence $\delta = 1$. Let $(x_0, y_0) = (0, 0) \in graph(F)$. Since $(F(A) - y_0) \cap (-intC_\varepsilon(B)) = \emptyset$ for all $0 < \varepsilon < \delta$, hence (x_0, y_0) is a Henig minimal solution of (SOP).*

It follows from the definitions of F and G that

$$epi(F, G) = \{(x, (y, z)) \in R \times R^2 \mid y \geq x^2, z \geq x - 1\}.$$

Take $z_0 = -1 \in G(x_0) \cap (-R_+)$. Then, we have

$$T_{epi(F,G)}(x_0, y_0, z_0) = \{(x, (y, z)) \in R \times R^2 \mid y \geq 0, z \geq x\},$$

and

$$D_w(F, G)(x_0, y_0, z_0)(x) = \{(y, x) \in R^2 \mid y \geq 0\} \cup \{(0, z) \in R^2 \mid z \geq x\},\ x \in R.$$

It is easy to verify that $P(x) = \{(y, z) \in R^2 \mid (x, (y, z)) \in T_{epi(F,G)}(x_0, y_0, z_0)\} = \{(y, z) \in R^2 \mid y \geq 0, z \geq x\}$ fulfills the weak domination property for all $x \in R$ and $(0, 0) \in P(0)$. Then, the conditions of Theorem 6.1 are satisfied for $D_w(F, G)$. Take $\lambda > 0$ and $\mu = 0$. Thus, for any $(y, z) \in D_w(F, G)(x_0, y_0, z_0)(x)$ and $x \in R$, we have

$$\lambda(y) + \mu(z) = 0 \quad and \quad \mu(z_0) = 0. \tag{23}$$

Clearly, $G + R_+$ is pseudo-Lipschitzian at (x_0, z_0). Moreover,

$$D_w F(x_0, y_0)(x) = WMin_C\{y \in R \mid y \geq 0\} = \{0\},$$
$$D_w G(x_0, z_0)(x) = WMin_C\{z \in R \mid z \geq x\} = \{x\}.$$

Thus, (23) holds for all $(y, z) \in D_w F(x_0, y_0)(x) \times D_w G(x_0, z_0)(x)$ *and* $x \in R$*. So that the* 1^{th}*-order Kuhn-Tucker type necessary optimality condition holds.*

Take $u_1 = 0$*,* $v_1 = 0$ *and* $w_1 = -1/2 \in -R_+$*. Then, we have*

$$T^{(2)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0) = \{(x, (y, z)) \in R \times R^2 \mid y \geq 0\},$$

and

$$D^{(2)}_w(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)(x) = \{0\} \times R, \ x \in R.$$

Clearly, $P^{(2)}(x) = \{(y, z) \in R^2 \mid (x, (y, z)) \in T^{(2)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0)\} = \{(y, z) \in R^2 \mid y \geq 0\}$ *fulfills the weak domination property for all* $x \in R$ *and* $(0, 0) \in P^{(2)}(0)$*. Hence, the conditions of Theorem 6.1 are satisfied for* $D^{(2)}_w(F, G)$*. In addition,*

$$D^{(2)}_w F(x_0, y_0, u_1 - x_0, v_1 - y_0)(x) = WMin_C\{y \in R \mid y \geq 0\} = \{0\},$$
$$D^{(2)}_w G(x_0, z_0, u_1 - x_0, w_1 - z_0)(x) = WMin_C R = \emptyset.$$

Choose $\lambda > 0$ *and* $\mu = 0$*. We have that the* 2^{th}*-order Kuhn-Tucker type necessary optimality condition holds.*

Theorem 6.2 *(Sufficient condition) Suppose that the following conditions are satisfied:*

(i) F *is* C*-convex on* X *and* G *is* D*-convex on* X*;*

(ii) $(x_0, y_0) \in graph(F)$ *and* $(u_i, v_i, w_i) \in epi(F, G)$*,* $i = 1, \cdots, m - 1$*;*

(iii) *there exist* $z_0 \in G(x_0) \cap (-D)$*,* $\lambda \in C^{\triangle}(B)$ *and* $\mu \in D^*$ *such that*

$$\mu(z_0) = 0 \quad \text{and} \quad \lambda(y) + \mu(z) \geq 0,$$

for all $(y, z) \in D^{\flat(m)}_w(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x - x_0)$ *and* $x \in A$*;*

(iv) $P(x - x_0) := \{(y, z) \in Y \times Z \mid (x - x_0, y, z) \in T^{\flat(m)}_{epi(F,G)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ *fulfills the weak domination property for all* $x \in X$*.*

Then, (x_0, y_0) *is a Henig minimal solution of (SOP).*

Proof. Since $\lambda \in C^{\triangle}(B)$, by Lemma 2.1(ii), there exists $\varepsilon \in (0,\delta)$ such that $\lambda \in C_{\varepsilon}(B)^* \setminus \{0_{Y^*}\}$. Assume that $(F(A)-y_0)\cap(-\text{int}C_{\varepsilon}(B)) \neq \emptyset$. Then, there exist $x' \in A$ and $y' \in F(x')$ such that $y'-y_0 \in -\text{int}C_{\varepsilon}(B)$. Since $x' \in A$, there exists $z' \in G(x')\cap(-D)$. By the weak domination property for $P(x-x_0)$ and Proposition 3.1, we have

$$\begin{aligned}(y'-y_0, z'-z_0) \;\in\; & D_w^{\flat(m)}(F,G)(x_0,y_0,z_0,u_1-x_0,v_1-y_0,w_1-z_0,\\ & \cdots,u_{m-1}-x_0,v_{m-1}-y_0,w_{m-1}-z_0)(x'-x_0)+C\times D.\end{aligned}$$

Thus, there exist $\bar{c} \in C$ and $\bar{d} \in D$ such that

$$\lambda(y'-y_0-\bar{c})+\mu(z'-z_0-\bar{d}) \geq 0. \tag{24}$$

Since $y'-y_0 \in -\text{int}C_{\varepsilon}(B)$, then $y'-y_0-\bar{c} \in -\text{int}C_{\varepsilon}(B)-C = -\text{int}C_{\varepsilon}(B)$. It follows from $\lambda \in C_{\varepsilon}(B)^* \setminus \{0_{Y^*}\}$ that $\lambda(y'-y_0-\bar{c})<0$. Since $z' \in G(x')\cap(-D)$, $\mu(z_0)=0$ and $\mu \in D^*$, we have $\mu(z'-z_0-\bar{d}) = \mu(z')-\mu(\bar{d}) \leq 0$. Thus,

$$\lambda(y'-y_0-\bar{c})+\mu(z'-z_0-\bar{d}) < 0,$$

which contradicts (24). Then, $(F(A)-y_0)\cap(-\text{int}C_{\varepsilon}(B)) = \emptyset$. Thus, the feasible solution (x_0,y_0) is a Henig minimal solution of (SOP) and the proof is complete. □

Theorem 6.3 *(Sufficient condition) Suppose that the following conditions are satisfied:*

(i) *F is C-convex on X and G is D-convex on X;*

(ii) *$(x_0,y_0) \in graph(F)$ and $(u_i,v_i,w_i) \in epi(F,G)$, $i=1,\cdots,m-1$;*

(iii) *there exist $z_0 \in G(x_0)\cap(-D)$, $\lambda \in C^{\triangle}(B)$ and $\mu \in D^*$ such that*

$$\mu(z_0)=0 \quad \textit{and} \quad \lambda(y)+\mu(z) \geq 0,$$

for all $(y,z) \in D_w^{\flat(m)}F(x_0,y_0,u_1-x_0,v_1-y_0,\cdots,u_{m-1}-x_0,v_{m-1}-y_0)(x-x_0) \times D_w^{\flat(m)}G(x_0,z_0,u_1-x_0,w_1-z_0,\cdots,u_{m-1}-x_0,w_{m-1}-z_0)(x-x_0)$ and $x \in A$;

(iv) *the sets $P_F(x-x_0) := \{y \in Y \mid (x-x_0,y) \in T_{epi(F)}^{\flat(m)}(x_0,y_0,u_1-x_0,v_1-y_0,\cdots,u_{m-1}-x_0,v_{m-1}-y_0)\}$ and $P_G(x-x_0) := \{z \in Z \mid (x-x_0,z) \in T_{epi(G)}^{\flat(m)}(x_0,z_0,u_1-x_0,w_1-z_0,\cdots,u_{m-1}-x_0,w_{m-1}-z_0)\}$ fulfill the weak domination property for all $x \in A$.*

Then, (x_0,y_0) is a Henig minimal solution of (SOP).

Proof. The conclusion can be obtained similarly as in the proof of Theorem 4.3. □

Remark 6.1 *Because F and G are C-convex and D-convex on X, respectively, and $(u_i, v_i) \in epi(F)$, $(u_i, w_i) \in epi(G)$, $i = 1, \cdots, m-1$, it follows from Proposition 2.2 that the m^{th}-order contingent set coincides with the m^{th}-order adjacent set. Thus, if we use m^{th}-order weak contingent epiderivative instead of the m^{th}-order weak adjacent epiderivative in all theorems of Sections 4-6 , then, the corresponding duality results and optimality conditions for m^{th}-order weak contingent epiderivative still hold.*

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