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Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions

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Abstract. In this paper, the existence and multiplicity of positive solutions to singular fractional differential system is investigated. Sufficient conditions which guarantee the existence of positive solutions are obtained, by using a well known fixed point theorem. An example is added to illustrate the results.

Keywords: Singular fractional differential system; Boundary condition including Stieltjes integrals; Positive solutions; Fixed point theorem

MSC: 26A33, 34B15, 34B18

1. Introduction

Fractional calculus has played a very significant role in engineering, science, economy, and many other fields. Recently, some works have been done to study the existence of solutions of nonlinear fractional differential equations (see[1-5]). In [3], El-Shahed considered the following nonlinear fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \ 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $a : (0, 1) \to [0, +\infty)$ is continuous with $\int_0^1 a(t)dt > 0$, and $f \in C([0, +\infty), [0, +\infty))$. He used the Krasnosel'skii fixed point theorem on cone expansion and compression to show the existence and non-existence of positive solutions for the above fractional boundary value problem.

Zhao et al. [5], by using the lower and upper solution method, Leggett-Williams fixed point theorem, Krasnosel'skii fixed point theorem and Leray-Schauder nonlinear alternative theorem, investigated the

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existence of positive solutions for the following boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$ is a real number, D_{0+}^{α} is the Riemann-Liouville fractional derivative, $f : [0,1] \times [0, \infty) \to [0, \infty)$ is continuous and $f(t, x)$ is nondecreasing with respect to x.

On the other hand, the study of differential systems is also important as this kind of systems occur in various problems of applied nature, we refer the readers to [6-12] and the reference therein for integer order systems, and [13-16] for fractional order systems. Recently, Goodrich [17] discussed a system of (continuous) fractional boundary value problems given by

$$\begin{cases} -D_{0+}^{\nu_1}y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \quad 0 < t < 1, \\ -D_{0+}^{\nu_2}y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \end{cases}$$

where $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in \mathbb{N}$, subject to the boundary conditions

$$\begin{cases} y_1^{(i)}(0) = y_2^{(i)}(0) = 0, & \text{for } 0 \leq i \leq n-2, \\ [D_{0+}^{\alpha}y_1(t)]_{t=1} = \phi_1(y), \ [D_{0+}^{\alpha}y_2(t)]_{t=1} = \phi_2(y), & \text{for } 1 \leq \alpha \leq n-2. \end{cases}$$

He obtained the existence of at least one positive solution by means of Krasnosel'skii fixed point theorem under the local boundary conditions ($\phi_1 = \phi_2 \equiv 0$) and the nonlocal boundary conditions ($\phi_1, \phi_2 \in C([0,1], (-\infty, +\infty))$). It should be noted that the nonlinearity in most of the previous works needs to be nonnegative to get the positive solutions [1-12,14-17].

Inspired by the work of the above papers and many known results in [18,19], we study the existence of positive solutions for the following singular differential system of fractional order

$$\begin{cases} -D_{0+}^{\alpha_i}y_i(t) = p_i(t) f_i(t, y_1(t), y_2(t)) - q_i(t), \quad 0 < t < 1, \ i = 1, 2, \\ y_i(0) = y_i'(0) = 0, \quad y_i'(1) = \lambda_i[y_i], \quad i = 1, 2, \end{cases} \tag{1.1}$$

where $2 < \alpha_i \leq 3$ are real numbers, $D_{0+}^{\alpha_i}$ are the standard Riemann-Liouville derivative, $f_i : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous, $q_i : (0,1) \to [0, +\infty) (i = 1, 2)$ are Lebesgue integrable. Here $\lambda_i[\cdot]$ $(i = 1, 2)$ are linear functionals on $C[0,1]$ given by

$$\lambda_i[y_i] = \int_0^1 y_i(t) dA_i(t), \quad i = 1, 2,$$

involving Stieltjes integrals with signed measures, that is, A_1, A_2 are suitable functions of bounded variation. A vector $(y_1, y_2) \in C[0,1] \times C[0,1]$ is said to be a positive solution of system (1.1) if and only if $D_{0+}^{\alpha_i}y_i(t) \in L(0,1) (i = 1, 2)$, (y_1, y_2) satisfies (1.1) and $y_1(t) \geq 0, y_2(t) > 0$ or $y_1(t) > 0, y_2(t) \geq 0$ for any $t \in (0,1)$.

The method we adopt, which has been widely used, is based on the ideas in [18]. The perturbed terms q_i $(i = 1, 2)$ are Lebesgue integrable and may be singular at some zero measures set of $[0,1]$, which implies the nonlinear terms may change sign. When the nonlinearity is allowed to take on both positive and negative values, such problems, e.g. system (1.1), are called semipositone problems in the literature. Semipositone problems have been studied by many authors using a variety of methods, see for example [18-23] and references therein. Meanwhile, $\lambda_1[\cdot]$ and $\lambda_2[\cdot]$ in (1.1) denote linear functionals on $C[0,1]$ involving Stieltjes integrals, this implies the case of boundary conditions (1.1) covers the multi-point boundary conditions and also integral boundary conditions in a single framework. For a comprehensive study of the case when there is a Stieltjes integral boundary condition at both ends, for the case of a differential equation of order two, see [24]. There are also other works for other order equations, see [19,25].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used later to prove our main results. In Section 3, we discuss the existence of positive solutions of the system (1.1). In Section 4, we give an example to illustrate the application of our main results.

2. Preliminaries and lemmas

For the convenience of the reader, we also present here some necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1. *The fractional integral of a function $u:(0,+\infty)\to R$ with order $\alpha>0$ is given by*

$$I_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2. *The fractional derivative of a continuous function $u:(0,+\infty)\to R$ with order $\alpha>0$ is given by*

$$D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n-1<\alpha\le n$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.1. *Let $\alpha>0$, $u(t)$ is integrable, then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}$$

where $c_i\in R$ $(i=1,2,\cdots,n)$, n is the smallest integer greater than or equal to α.

For $i=1,2,$ set

$$G_i(t,s)=\frac{1}{\Gamma(\alpha_i)}\begin{cases} t^{\alpha_i-1}(1-s)^{\alpha_i-2}, & 0\le t\le s\le 1,\\ t^{\alpha_i-1}(1-s)^{\alpha_i-2}-(t-s)^{\alpha_i-1}, & 0\le s\le t\le 1.\end{cases} \qquad (1)$$

Lemma 2.2. *The function $G_i(t,s)$ defined by (2.1) have the following properties:*

(1) $G_i(t,s)>0$, *for $t,s\in(0,1)$, $i=1,2$.*

(2) $\varrho_i(t)G_i(1,s) \le G_i(t,s) \le G_i(1,s)$, *for* $t,s \in [0,1]$.

(3) $\Gamma(\alpha_i)G_i(t,s) \le \varrho_i(t)$, *for* $t,s \in [0,1]$, *where* $\varrho_i(t) = t^{\alpha_i - 1}$, $i = 1,2$.

Proof. For the proof of (1) and (2) see [3]. The proof of (3) is clear, so we omit it. □

Lemma 2.3 (See [3]). *Given* $h(t) \in C(0,1) \cap L(0,1)$, *then the problem*

$$\begin{cases} D_{0+}^{\alpha_i} y_i(t) + h(t) = 0, \ 0 < t < 1, \ 2 < \alpha_i \le 3, \\ y_i(0) = y_i'(0) = 0, \quad y_i'(1) = 0, \quad i = 1,2, \end{cases} \tag{2}$$

has the unique solution

$$y_i(t) = \int_0^1 G_i(t,s)h(s)ds. \tag{3}$$

By Lemma 2.1, the unique solution of the problem

$$\begin{cases} D_{0+}^{\alpha_i} y_i(t) = 0, \quad 0 < t < 1, \ i = 1,2, \\ y_i(0) = y_i'(0) = 0, \quad y_i'(1) = 1 \end{cases}$$

is $\gamma_i(t) = \frac{t^{\alpha_i - 1}}{\alpha_i - 1}$ $(i = 1,2)$. As in [26], we see the Green function $(H_1(t,s), H_2(t,s))$ for the nonlocal system (1.1) is given by

$$H_i(t,s) = G_i(t,s) + \frac{\gamma_i(t)}{1 - \Lambda_i} \mathscr{G}_{A_i}(s), \quad i = 1,2, \tag{4}$$

where $\Lambda_i = \lambda_i[\gamma_i] \ne 1$, $\mathscr{G}_{A_i}(s) = \int_0^1 G_i(t,s)dA_i(t)$, $s \in [0,1]$ $(i = 1,2)$.

Lemma 2.4. *Let* $\Lambda_i \in [0,1)$ *and* $\mathscr{G}_{A_i}(s) \ge 0$ *for* $s \in [0,1]$ $(i = 1,2)$, *the functions defined by* (2.4) *satisfy:*

(1) $H_i(t,s) \ge G_i(t,s) > 0$, *for* $t,s \in (0,1)$, $i = 1,2$.

(2) $\varrho_i(t)G_i(1,s) \le H_i(t,s) \le \kappa_i G_i(1,s)$, *for* $t,s \in [0,1]$, $i = 1,2$.

(3) $\Gamma(\alpha_i)H_i(t,s) \le \kappa_i \varrho_i(t) \le \kappa_i$, *for* $t,s \in [0,1]$, *where*

$$\kappa_i = 1 + \frac{\lambda_i[1]}{1 - \Lambda_i}, \quad i = 1,2. \tag{5}$$

Proof. It is obvious that (1) and the left hand side of (2) hold. In the following, we will prove the right hand side of (2) and (3).

(i) By (2) of Lemma 2.2, since $1 < \alpha_i - 1 \le 2$, we have

$$\begin{aligned} H_i(t,s) &= G_i(t,s) + \frac{\gamma_i(t)}{1 - \Lambda_i} \int_0^1 G_i(t,s)dA_i(t) \\ &\le G_i(1,s) + \frac{G_i(1,s)}{(\alpha_i - 1)(1 - \Lambda_i)} \int_0^1 dA_i(t) \\ &\le \left(1 + \frac{\lambda_i[1]}{1 - \Lambda_i}\right) G_i(1,s) = \kappa_i G_i(1,s). \end{aligned}$$

(ii) By (3) of Lemma 2.2, we have

$$\begin{aligned} \Gamma(\alpha_i)H_i(t,s) &= \Gamma(\alpha_i)G_i(t,s) + \frac{\gamma_i(t)}{1 - \Lambda_i} \Gamma(\alpha_i)\mathscr{G}_{A_i}(s) \\ &\le t^{\alpha_i - 1} + \frac{t^{\alpha_i - 1}}{(\alpha_i - 1)(1 - \Lambda_i)} \int_0^1 \Gamma(\alpha_i)G_i(t,s)dA_i(t) \end{aligned}$$

$$\leq t^{\alpha_i-1}\left(1+\frac{1}{1-\Lambda_i}\int_0^1 t^{\alpha_i-1}dA_i(t)\right)$$
$$\leq \varrho_i(t)\left(1+\frac{\lambda_i[1]}{1-\Lambda_i}\right)=\kappa_i\varrho_i(t)\leq\kappa_i.$$

This completes the proof. □

For the convenience of presentation, we list here the hypotheses to be used later:

(H_1) $\Lambda_i\in[0,1)$ $(i=1,2)$, where $\Lambda_i=\lambda_i[\gamma_i]$ for $\gamma_i(t)=\frac{t^{\alpha_i-1}}{\alpha_i-1}$.

(H_2) A_i are functions of bounded variation, and $\mathscr{G}_{A_i}(s)\geq 0$ $(i=1,2)$, $s\in[0,1]$.

(H_3) $p_1,p_2\in C((0,1),[0,+\infty))$ and $q_1,q_2\in L^1([0,1],[0,+\infty))$ such that

$$0<\int_0^1 G_i(1,s)[p_i(s)+q_i(s)]ds<+\infty,\quad 0<\int_0^1 q_i(s)ds<\frac{\Gamma(\alpha_i)}{2\kappa_i^2},\quad i=1,2. \tag{6}$$

(H_4) $f_1,f_2:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are continuous, $p_1(t)f_1(t,y_1,y_2)\geq q_1(t)$, $\forall\ (t,y_1,y_2)\in[0,1]\times[0,1]\times[0,+\infty)$, $p_2(t)f_2(t,y_1,y_2)\geq q_2(t)$, $\forall\ (t,y_1,y_2)\in[0,1]\times[0,+\infty)\times[0,1]$.

Remark 2.1. It follows from (H_3) that there exists an interval $[\xi,\eta]\subset(0,1)$ such that

$$0<\int_\xi^\eta G_i(1,s)p_i(s)ds<+\infty,\quad i=1,2.$$

Lemma 2.5. *Assume that* $(H_1)-(H_3)$ *hold, then the boundary value problems*

$$\begin{cases} -D_{0+}^{\alpha_i}\omega_i(t)=2q_i(t),\quad 0<t<1,\\ \omega_i(0)=\omega_i'(0)=0,\quad \omega_i'(1)=\lambda_i[\omega_i],\quad i=1,2\end{cases}$$

have unique solution

$$\omega_i(t)=2\int_0^1 H_i(t,s)q_i(s)ds,\quad i=1,2, \tag{7}$$

which satisfy

$$\omega_i(t)\leq\frac{2\kappa_i\varrho_i(t)}{\Gamma(\alpha_i)}\int_0^1 q_i(s)ds,\quad t\in[0,1],\ i=1,2. \tag{8}$$

Proof. It follows from Lemma 2.4 and $(H_1)-(H_3)$ that (2.7)-(2.8) hold. □

Let $E=C[0,1]\times C[0,1]$, then E is a Banach space with the norm

$$\|(u,v)\|_1:=\|u\|+\|v\|,\quad \|u\|=\max_{0\leq t\leq 1}|u(t)|,\quad \|v\|=\max_{0\leq t\leq 1}|v(t)|$$

for any $(u,v)\in E$. Let

$$P=\left\{(u,v)\in E:u(t)\geq\kappa_1^{-1}\varrho_1(t)\|u\|,v(t)\geq\kappa_2^{-1}\varrho_2(t)\|v\|\text{ for }t\in[0,1]\right\},$$

then P is a cone of E.

Define a modified function $[z(t)]^+$ for any $z \in C[0,1]$ by

$$[z(t)]^+ = \begin{cases} z(t), & z(t) \geq 0, \\ 0, & z(t) < 0. \end{cases}$$

Next we consider the following singular nonlinear system:

$$\begin{cases} -D_{0+}^{\alpha_i} x_i(t) = p_i(t) f_i(t, [x_1(t) - \omega_1(t)]^+, [x_2(t) - \omega_2(t)]^+) + q_i(t), \quad 0 < t < 1, \\ x_i(0) = x_i'(0) = 0, \quad x_i'(1) = \lambda_i[x_i], \quad i = 1, 2. \end{cases} \tag{9}$$

Lemma 2.6. *If $(x_1, x_2) \in C[0,1] \times C[0,1]$ with $x_1(t) > \omega_1(t)$, $x_2(t) \geq \omega_2(t)$ or $x_1(t) \geq \omega_1(t)$, $x_2(t) > \omega_2(t)$ for any $t \in (0,1)$ is a positive solution of system* (2.9)*, then $(x_1 - \omega_1, x_2 - \omega_2)$ is a positive solution of singularly system* (1.1)*.*

Proof. In fact, if $(x_1, x_2) \in C[0,1] \times C[0,1]$ is a positive solution of system (2.9) such that $x_1(t) > \omega_1(t)$, $x_2(t) \geq \omega_2(t)$ or $x_1(t) \geq \omega_1(t)$, $x_2(t) > \omega_2(t)$ for any $t \in (0,1)$, then from (2.9) and the definition of $[\cdot]^+$, we have

$$\begin{cases} -D_{0+}^{\alpha_i} x_i(t) = p_i(t) f_i(t, x_1(t) - \omega_1(t), x_2(t) - \omega_2(t)) + q_i(t), \quad 0 < t < 1, \\ x_i(0) = x_i'(0) = 0, \quad x_i'(1) = \lambda_i[x_i], \quad i = 1, 2. \end{cases} \tag{10}$$

Let $y_i = x_i - \omega_i$ $(i = 1, 2)$, then $D_{0+}^{\alpha_i} y_i(t) = D_{0+}^{\alpha_i} x_i(t) - D_{0+}^{\alpha_i} \omega_i(t)$ $(i = 1, 2)$ for $t \in (0,1)$, which imply that

$$-D_{0+}^{\alpha_i} y_i(t) = -D_{0+}^{\alpha_i} x_i(t) + D_{0+}^{\alpha_i} \omega_i(t) = -D_{0+}^{\alpha_i} x_i(t) - 2q_i(t), \quad t \in (0,1), \ i = 1, 2.$$

Thus (2.10) becomes

$$\begin{cases} -D_{0+}^{\alpha_i} y_i(t) = p_i(t) f_i(t, y_1(t), y_2(t)) - q_i(t), \quad 0 < t < 1, \\ y_i(0) = y_i'(0) = 0, \quad y_i'(1) = \lambda_i[y_i], \quad i = 1, 2, \end{cases}$$

i.e., $(x_1 - \omega_1, x_2 - \omega_2)$ is a positive solution of singularly system (1.1). This proves Lemma 2.6. □

Define an operator $T : P \to P$ by

$$T(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2)),$$

where operators $T_1, T_2 : P \to C[0,1]$ are defined by

$$T_i(x_1, x_2)(t) = \int_0^1 H_i(t,s)[p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) + q_i(s)]ds, \quad i = 1, 2.$$

Clearly, if $(x_1, x_2) \in P$ is a fixed point of T, then (x_1, x_2) is a solution of system (2.9).

Lemma 2.7. *Assume that $(H_1) - (H_4)$ hold, then $T : P \to P$ is a completely continuous operator.*

Proof. For any $(x_1, x_2) \in P$, Lemma 2.4 implies that

$$\begin{aligned} \|T_i(x_1, x_2)\| &= \max_{0 \leq t \leq 1} \int_0^1 H_i(t,s)[p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) + q_i(s)]ds \\ &\leq \kappa_i \int_0^1 G_i(1,s)[p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) + q_i(s)]ds, \quad i = 1, 2. \end{aligned}$$

On the other hand, from Lemma 2.4, we also have

$$T_i(x_1,x_2)(t) = \int_0^1 H_i(t,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds$$
$$\geq \varrho_i(t)\int_0^1 G_i(1,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds, \quad i=1,2.$$

So

$$T_i(x_1,x_2)(t) \geq \kappa_i^{-1}\varrho_i(t)\|T_i(x_1,x_2)\|,\ t\in[0,1],\ i=1,2. \tag{11}$$

(2.11) yields that $T(P)\subset P$.

According to the Ascoli-Arzela theorem and the Lebesgue dominated convergence theorem, we can easily get that $T: P\to P$ is a completely continuous operator. □

Lemma 2.8 (Krasnosel'skii's theorem, see [27]). *Let E be a real Banach space, $P\subset E$ be a cone. Assume that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta\in\Omega_1, \overline{\Omega}_1\subset\Omega_2$, and $T: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ is a completely continuous operator such that either*

(1) $\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_2$, or

(2) $\|Tu\|\geq\|u\|$, $u\in P\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|$, $u\in P\cap\partial\Omega_2$.

Then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Main results

For convenience, we denote:

$$L_i = \kappa_i\int_0^1 G_i(1,s)[p_i(s)+q_i(s)]ds, \quad l_i=\varrho_i(\xi)\int_\xi^\eta G_i(1,s)p_i(s)ds, \quad i=1,2,$$
$$f_i^\infty = \lim_{\substack{y_1+y_2\to+\infty\\ y_1\geq0,y_2\geq0}}\max_{t\in[0,1]}\frac{f_i(t,y_1,y_2)}{y_1+y_2}, \quad f_{i\infty} = \lim_{\substack{y_1+y_2\to+\infty\\ y_1\geq0,y_2\geq0}}\min_{t\in[\xi,\eta]}\frac{f_i(t,y_1,y_2)}{y_1+y_2}, \quad i=1,2.$$

Theorem 3.1 *Assume that conditions $(H_1)-(H_4)$ are satisfied. Further assume that the following conditions hold:*

(C_1) There exists a constant

$$r_1 > \max\left\{2, 2L_1, 2L_2, \frac{4\kappa_1^2}{\Gamma(\alpha_1)}\int_0^1 q_1(s)ds, \frac{4\kappa_2^2}{\Gamma(\alpha_2)}\int_0^1 q_2(s)ds\right\} \tag{1}$$

such that for any $(t,y_1,y_2)\in[0,1]\times[0,r_1]\times[0,r_1]$,

$$f_i(t,y_1,y_2) < \frac{r_1}{2L_i}-1, \quad i=1,2.$$

(C_2) $f_{1\infty}=+\infty$ or $f_{2\infty}=+\infty$.

Then the system (1.1) has at least one positive solution.

Proof. Let $\Omega_1=\{(x_1,x_2)\in E: \|(x_1,x_2)\|_1<r_1\}$ and $\partial\Omega_1=\{(x_1,x_2)\in E:\|(x_1,x_2)\|_1=r_1\}$. Then for any $(x_1,x_2)\in P\cap\partial\Omega_1, s\in[0,1]$, we have

$$[x_i(s)-\omega_i(s)]^+\leq x_i(s)\leq\|x_i\|\leq r_1, \quad i=1,2.$$

It follows from (C_1) that

$$\begin{aligned}\|T_i(x_1,x_2)\| &= \max_{0\le t\le 1}\int_0^1 H_i(t,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds\\ &< \kappa_i\int_0^1 G_i(1,s)\left[p_i(s)\left(\frac{r_1}{2L_i}-1\right)+q_i(s)\right]ds\\ &\le \kappa_i\int_0^1 G_i(1,s)[p_i(s)+q_i(s)]ds\times\frac{r_1}{2L_i}\\ &= \frac{r_1}{2}=\frac{\|(x_1,x_2)\|_1}{2},\quad i=1,2.\end{aligned} \tag{2}$$

Consequently,

$$\|T(x_1,x_2)\|_1=\|T_1(x_1,x_2)\|+\|T_2(x_1,x_2)\|<\|(x_1,x_2)\|_1,\quad \text{for all } (x_1,x_2)\in P\cap\partial\Omega_1. \tag{3}$$

On the other hand, choose a real number $M>0$ big enough such that

$$\frac{1}{4}M\tau\min\{l_1,l_2\}>1,$$

where

$$\tau=\min\{\kappa_1^{-1}\varrho_1(\xi),\kappa_2^{-1}\varrho_2(\xi)\}. \tag{4}$$

By $f_{1\infty}=+\infty$ of (C_2), there exists $N>r_1$ such that, for any $x_1\ge 0$, $x_2\ge 0$ and $x_1+x_2\ge N$, for any $t\in[\xi,\eta]$, we have

$$f_1(t,x_1,x_2)\ge M(x_1+x_2). \tag{5}$$

Set $r_2=\max\{2r_1,4\tau^{-1}N\}$, then $r_2>r_1$.

Now let $\Omega_2=\{(x_1,x_2)\in E:\|(x_1,x_2)\|_1<r_2\}$ and $\partial\Omega_2=\{(x_1,x_2)\in E:\|(x_1,x_2)\|_1=r_2\}$. Then for any $(x_1,x_2)\in P\cap\partial\Omega_2$, there exists some component x_j $(1\le j\le 2)$ such that $\|x_j\|\ge\frac{r_2}{2}\ge r_1$. So for any $(x_1,x_2)\in P\cap\partial\Omega_2$, $t\in[\xi,\eta]$, by (2.8) and (3.1), we have

$$\begin{aligned}x_j(t)-\omega_j(t)&\ge x_j(t)-\frac{2\kappa_j\varrho_j(t)}{\Gamma(\alpha_j)}\int_0^1 q_j(s)ds\ge x_j(t)-\frac{2\kappa_j}{\Gamma(\alpha_j)}\int_0^1 q_j(s)ds\times\frac{\kappa_j x_j(t)}{r_1}\\ &\ge\frac{1}{2}x_j(t)\ge\frac{1}{2\kappa_j}\varrho_j(t)\|x_j\|\ge\frac{\varrho_j(\xi)r_2}{4\kappa_j}\ge\frac{1}{4}\tau r_2\ge N,\end{aligned}$$

and then

$$[x_1(t)-\omega_1(t)]^++[x_2(t)-\omega_2(t)]^+\ge[x_j(t)-\omega_j(t)]^+=x_j(t)-\omega_j(t)\ge\frac{1}{4}\tau r_2\ge N. \tag{6}$$

Thus for any $(x_1,x_2)\in P\cap\partial\Omega_2$, $t\in[\xi,\eta]$, by (3.5) and (3.6), we have

$$f_1(t,[x_1(t)-\omega_1(t)]^+,[x_2(t)-\omega_2(t)]^+)\ge M([x_1(t)-\omega_1(t)]^++[x_2(t)-\omega_2(t)]^+). \tag{7}$$

So for any $(x_1,x_2)\in P\cap\partial\Omega_2$, $t\in[\xi,\eta]$, by (3.6) and (3.7), we have

$$\begin{aligned}
T_1(x_1,x_2)(t) &= \int_0^1 H_1(t,s)[p_1(s)f_1(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_1(s)]ds\\
&\geq \int_0^1 G_1(t,s)p_1(s)f_1(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&\geq \varrho_1(t)\int_0^1 G_1(1,s)p_1(s)f_1(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&\geq \varrho_1(t)\int_\xi^\eta G_1(1,s)p_1(s)f_1(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&\geq \varrho_1(\xi)\int_\xi^\eta G_1(1,s)p_1(s)M([x_1(s)-\omega_1(s)]^+ +[x_2(s)-\omega_2(s)]^+)ds\\
&\geq \frac{M\tau r_2\varrho_1(\xi)}{4}\int_\xi^\eta G_1(1,s)p_1(s)ds\\
&\geq \frac{1}{4}M\tau\min\{l_1,l_2\}r_2\\
&> r_2=\|(x_1,x_2)\|_1.
\end{aligned}\tag{8}$$

Thus

$$\|T(x_1,x_2)\|_1\geq\|T_1(x_1,x_2)\|>\|(x_1,x_2)\|_1,\quad \text{for all } (x_1,x_2)\in P\cap\partial\Omega_2. \tag{9}$$

Obviously, if $f_{2\infty}=+\infty$ holds, (3.9) is still valid.

By (3.3), (3.9) and Lemma 2.8, T has a fixed point $(\widetilde{x}_1,\widetilde{x}_2)\in P\cap(\overline{\Omega}_2\backslash\Omega_1)$ such that $r_1\leq\|(\widetilde{x}_1,\widetilde{x}_2)\|_1\leq r_2$. Next we shall show $\widetilde{x}_1(t)>\omega_1(t)$, $\widetilde{x}_2(t)\geq\omega_2(t)$ (or $\widetilde{x}_1(t)\geq\omega_1(t)$, $\widetilde{x}_2(t)>\omega_2(t)$) for $t\in(0,1)$. For $\|(\widetilde{x}_1,\widetilde{x}_2)\|_1\geq r_1>2$, we shall divide the rather long proof into three cases: (i) $\|\widetilde{x}_1\|>1$, $\|\widetilde{x}_2\|>1$; (ii) $\|\widetilde{x}_1\|>1$, $\|\widetilde{x}_2\|\leq 1$; (iii) $\|\widetilde{x}_1\|\leq 1$, $\|\widetilde{x}_2\|>1$.

Case i. If $\|\widetilde{x}_1\|>1$, then from (2.6) and (2.8), we have

$$\widetilde{x}_1(t)\geq\kappa_1^{-1}\varrho_1(t)\|\widetilde{x}_1\|\geq\kappa_1^{-1}\cdot\frac{\Gamma(\alpha_1)\omega_1(t)}{2\kappa_1\int_0^1 q_1(s)ds}\cdot\|\widetilde{x}_1\|>\frac{\Gamma(\alpha_1)\omega_1(t)}{2\kappa_1^2\int_0^1 q_1(s)ds}\geq\omega_1(t),\quad t\in(0,1).$$

Similarly, from $\|\widetilde{x}_2\|>1$ we have $\widetilde{x}_2(t)>\omega_2(t)$, $t\in(0,1)$.

Case ii. If $\|\widetilde{x}_1\|>1$, similar to (i), we have $\widetilde{x}_1(t)>\omega_1(t)$, $t\in(0,1)$. If $\|\widetilde{x}_2\|\leq 1$, then $[\widetilde{x}_2(s)-\omega_2(s)]^+\leq\widetilde{x}_2(s)\leq\|\widetilde{x}_2\|\leq 1$. Set $J_1=\{t\in[0,1]:\widetilde{x}_2(t)\geq\omega_2(t)\}$, $J_2=\{t\in[0,1]:\widetilde{x}_2(t)<\omega_2(t)\}$. Obviously, $J_1\cup J_2=[0,1]$. Because $(\widetilde{x}_1,\widetilde{x}_2)$ is a solution of (2.9), we have

$$\begin{aligned}
\widetilde{x}_2(t) &= \int_0^1 H_2(t,s)[p_2(s)f_2(s,[\widetilde{x}_1(s)-\omega_1(s)]^+,[\widetilde{x}_2(s)-\omega_2(s)]^+)+q_2(s)]ds\\
&= \left(\int_{J_1}+\int_{J_2}\right)H_2(t,s)[p_2(s)f_2(s,[\widetilde{x}_1(s)-\omega_1(s)]^+,[\widetilde{x}_2(s)-\omega_2(s)]^+)+q_2(s)]ds.
\end{aligned}$$

As $t\in J_1$, $\widetilde{x}_1(t)>\omega_1(t)$, $\widetilde{x}_2(t)\geq\omega_2(t)$, then by the definition of $[\cdot]^+$, we have

$$\begin{aligned}
&\int_{J_1} H_2(t,s)[p_2(s)f_2(s,[\widetilde{x}_1(s)-\omega_1(s)]^+,[\widetilde{x}_2(s)-\omega_2(s)]^+)+q_2(s)]ds\\
&=\int_{J_1} H_2(t,s)[p_2(s)f_2(s,\widetilde{x}_1(s)-\omega_1(s),\widetilde{x}_2(s)-\omega_2(s))+q_2(s)]ds.
\end{aligned}$$

As $t \in J_2$, $\widetilde{x}_1(t) > \omega_1(t)$, $\widetilde{x}_2(t) < \omega_2(t)$, then by the definition of $[\cdot]^+$, we have

$$\begin{aligned}&\int_{J_2} H_2(t,s)[p_2(s)f_2(s,[\widetilde{x}_1(s)-\omega_1(s)]^+,[\widetilde{x}_2(s)-\omega_2(s)]^+)+q_2(s)]ds\\&=\int_{J_2} H_2(t,s)[p_2(s)f_2(s,\widetilde{x}_1(s)-\omega_1(s),0)+q_2(s)]ds.\end{aligned}$$

By assumption (H_4), we have

$$p_2(t)f_2(t,y_1,y_2)\geq q_2(t),\quad \forall\ (t,y_1,y_2)\in[0,1]\times[0,r_2]\times[0,1].$$

Then by the above discussion, we have

$$\begin{aligned}\widetilde{x}_2(t)&=\left(\int_{J_1}+\int_{J_2}\right)H_2(t,s)[p_2(s)f_2(s,[\widetilde{x}_1(s)-\omega_1(s)]^+,[\widetilde{x}_2(s)-\omega_2(s)]^+)+q_2(s)]ds\\&\geq 2\int_0^1 H_2(t,s)q_2(s)ds=\omega_2(t),\quad t\in[0,1].\end{aligned}$$

Then $\widetilde{x}_2(t)\geq\omega_2(t)$, $t\in[0,1]$.

Case iii. If $\|\widetilde{x}_1\|\leq 1$ and $\|\widetilde{x}_2\|>1$, similar to (ii), we have $\widetilde{x}_1(t)\geq\omega_1(t)$, $\widetilde{x}_2(t)>\omega_2(t)$, $t\in(0,1)$.

So by Lemma 2.6 we know that $(\widetilde{y}_1,\widetilde{y}_2)=(\widetilde{x}_1-\omega_1,\widetilde{x}_2-\omega_2)$ is the positive solution for the system (1.1). The proof is completed. □

Theorem 3.2 *Assume that conditions* $(H_1)-(H_4)$ *are satisfied. In addition, assume that the following conditions hold:*

(C_3) *There exists a constant*

$$R_0>\max\left\{1,\frac{4\kappa_1^2}{\Gamma(\alpha_1)}\int_0^1 q_1(s)ds,\frac{4\kappa_2^2}{\Gamma(\alpha_2)}\int_0^1 q_2(s)ds\right\}\tag{10}$$

such that

$$f_i(t,y_1,y_2)>\frac{R_0}{l_i},\qquad \textit{for any } t\in[\xi,\eta],\ \frac{1}{2}\tau R_0\leq y_1+y_2\leq 2R_0,\ i=1,2,$$

where $\kappa_i(i=1,2)$ *and* τ *are defined by* (2.5) *and* (3.4)*, respectively.*

(C_4) $f_i^\infty=0,\ i=1,2.$

Then the system (1.1) *has at least one positive solution.*

Proof. Let $R_1=2R_0$ and $\Omega_1=\{(x_1,x_2)\in E:\|(x_1,x_2)\|_1<R_1\}$. Then for any $(x_1,x_2)\in P\cap\partial\Omega_1$, there exists some component x_j $(1\leq j\leq 2)$ such that $\|x_j\|\geq R_0$. So for any $(x_1,x_2)\in P\cap\partial\Omega_1$, $t\in[\xi,\eta]$, by (2.8) and (3.10), we have

$$\begin{aligned}x_j(t)-\omega_j(t)&\geq x_j(t)-\frac{2\kappa_j\varrho_j(t)}{\Gamma(\alpha_j)}\int_0^1 q_j(s)ds\geq x_j(t)-\frac{2\kappa_j}{\Gamma(\alpha_j)}\int_0^1 q_j(s)ds\times\frac{\kappa_j x_j(t)}{R_0}\\&\geq\frac{1}{2}x_j(t)\geq\frac{1}{2\kappa_j}\varrho_j(t)\|x_j\|\geq\frac{\varrho_j(\xi)R_0}{2\kappa_j}\geq\frac{1}{2}\tau R_0>0,\end{aligned}\tag{11}$$

and

$$[x_i(t)-\omega_i(t)]^+\leq x_i(t)\leq\|x_i\|,\quad i=1,2.$$

So for any $(x_1,x_2)\in P\cap\partial\Omega_1, t\in[\xi,\eta]$, we have

$$\frac{1}{2}\tau R_0\le[x_1(t)-\omega_1(t)]^+ + [x_2(t)-\omega_2(t)]^+\le R_1=2R_0. \tag{12}$$

It follows from (C_3) and (3.12) that, for any $(x_1,x_2)\in P\cap\partial\Omega_1, t\in[\xi,\eta]$,

$$\begin{aligned}
T_i(x_1,x_2)(t)&=\int_0^1 H_i(t,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds\\
&\ge\int_0^1 G_i(t,s)p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&\ge\varrho_i(t)\int_0^1 G_i(1,s)p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&\ge\varrho_i(t)\int_\xi^\eta G_i(1,s)p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)ds\\
&>\varrho_i(\xi)\int_\xi^\eta G_i(1,s)p_i(s)ds\times\frac{R_0}{l_i}=R_0,\quad i=1,2.
\end{aligned} \tag{13}$$

This means that

$$\|T_i(x_1,x_2)\|>R_0=\frac{\|(x_1,x_2)\|_1}{2},\quad i=1,2.$$

Thus we get

$$\|T(x_1,x_2)\|_1>\|(x_1,x_2)\|_1,\qquad \text{for all } (x_1,x_2)\in P\cap\partial\Omega_1. \tag{14}$$

Next, let us choose $\varepsilon>0$ such that

$$2\varepsilon\kappa_i\int_0^1 G_i(1,s)p_i(s)ds<1,\quad i=1,2.$$

Then for the above ε, by (C_4), there exists $X_0>R_1>0$ such that, for any $x_1\ge0$, $x_2\ge0$ and $x_1+x_2>X_0$, for any $t\in[0,1]$, we have

$$f_i(t,x_1,x_2)\le\varepsilon(x_1+x_2),\ i=1,2.$$

Take

$$R_i^*=\frac{2M_iL_i+2\kappa_i\int_0^1G_i(1,s)q_i(s)ds}{1-2\varepsilon\kappa_i\int_0^1G_i(1,s)p_i(s)ds}+X_0,\quad i=1,2,$$

where $M_i=\max\{f_i(t,x_1,x_2)+1: t\in[0,1], x_1+x_2\le X_0\}(i=1,2)$. Let $R_2=\max\{R_1^*,R_2^*\}$, then $R_2>X_0>R_1$.

Now let $\Omega_2=\{(x_1,x_2)\in E:\|(x_1,x_2)\|_1<R_2\}$ and $\partial\Omega_2=\{(x_1,x_2)\in E:(x_1,x_2)\|_1=R_2\}$. Then for any $(x_1,x_2)\in P\cap\partial\Omega_2$, we have

$$\begin{aligned}
\|T_i(x_1,x_2)\|=&\max_{0\le t\le1}\int_0^1 H_i(t,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds\\
\le&\kappa_i\int_0^1 G_i(1,s)[p_i(s)f_i(s,[x_1(s)-\omega_1(s)]^+,[x_2(s)-\omega_2(s)]^+)+q_i(s)]ds\\
\le&\kappa_i\left(\max_{t\in[0,1],x_1+x_2\le X_0}f_i(t,x_1,x_2)+1\right)\int_0^1G_i(1,s)[p_i(s)+q_i(s)]ds\\
&+\kappa_i\int_0^1G_i(1,s)\Big[p_i(s)\varepsilon\left([x_1(s)-\omega_1(s)]^+ + [x_2(s)-\omega_2(s)]^+\right)+q_i(s)\Big]ds\\
\le&M_iL_i+\kappa_i\int_0^1G_i(1,s)\Big[p_i(s)\varepsilon(\|x_1\|+\|x_2\|)+q_i(s)\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M_iL_i + \kappa_i \int_0^1 G_i(1,s)q_i(s)ds + \varepsilon\kappa_i R_2 \int_0^1 G_i(1,s)p_i(s)ds \\
&< \left(\frac{1}{2} - \varepsilon\kappa_i \int_0^1 G_i(1,s)p_i(s)ds\right) R_i^* + \varepsilon\kappa_i R_2 \int_0^1 G_i(1,s)p_i(s)ds \\
&\leq \frac{R_2}{2} = \frac{\|(x_1,x_2)\|_1}{2}, \quad i=1,2.
\end{aligned} \tag{15}
$$

Thus

$$
\|T(x_1,x_2)\|_1 < \|(x_1,x_2)\|_1, \quad \text{for all } (x_1,x_2) \in P \cap \partial\Omega_2. \tag{16}
$$

By (3.14), (3.16) and Lemma 2.8, T has a fixed point $(\widehat{x}_1, \widehat{x}_2) \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that $R_1 \leq \|(\widehat{x}_1, \widehat{x}_2)\|_1 \leq R_2$. By the same method of Theorem 3.1, we can obtain

$$
\widehat{x}_1(t) > \omega_1(t), \quad \widehat{x}_2(t) \geq \omega_2(t), \quad t \in (0,1),
$$

or

$$
\widehat{x}_1(t) \geq \omega_1(t), \quad \widehat{x}_2(t) > \omega_2(t), \quad t \in (0,1).
$$

Then let $\widehat{y}_i = \widehat{x}_i - \omega_i$ $(i=1,2)$, by Lemma 2.6 we know that the system (1.1) has at least one positive solution $(\widehat{y}_1, \widehat{y}_2)$. This completes the proof of Theorem 3.2. □

Theorem 3.3 *Assume that conditions* $(H_1)-(H_4)$ *and* $(C_1),(C_4)$ *are satisfied. Further assume that the following condition holds:*
(C_5) *There exists a constant* $\widetilde{R}_0 > 2\tau^{-1}r_1$ *such that*

$$
f_i(t,y_1,y_2) > \frac{\widetilde{R}_0}{l_i}, \quad \text{for any } t \in [\xi,\eta],\ \frac{1}{2}\tau\widetilde{R}_0 \leq y_1+y_2 \leq 2\widetilde{R}_0,\ i=1,2.
$$

where $\kappa_i (i=1,2)$, r_1 *and* τ *are defined by* (2.5), (3.1) *and* (3.4), *respectively.*
Then the system (1.1) *has at least two positive solutions.*

Proof. Set $\Omega_1 = \{(x_1,x_2) \in E : \|(x_1,x_2)\|_1 < r_1\}$. From (C_1) and proceeding as in (3.2), we have

$$
\|T(x_1,x_2)\|_1 < \|(x_1,x_2)\|_1, \quad \text{for all } (x_1,x_2) \in P \cap \partial\Omega_1. \tag{17}
$$

On the other hand, let $R = 2\widetilde{R}_0$, $\Omega_2 = \{(x_1,x_2) \in E : \|(x_1,x_2)\|_1 < R\}$ and $\partial\Omega_2 = \{(x_1,x_2) \in E : \|(x_1,x_2)\|_1 = R\}$. Then for any $(x_1,x_2) \in P \cap \partial\Omega_2$, there exists some component x_j $(1 \leq j \leq 2)$ such that $\|x_j\| \geq \widetilde{R}_0$. So for any $(x_1,x_2) \in P \cap \partial\Omega_2$, $t \in [\xi,\eta]$, by (2.8), we have

$$
\begin{aligned}
x_j(t) - \omega_j(t) &\geq x_j(t) - \frac{2\kappa_j\varrho_j(t)}{\Gamma(\alpha_j)} \int_0^1 q_j(s)ds \geq x_j(t) - \frac{2\kappa_j}{\Gamma(\alpha_j)} \int_0^1 q_j(s)ds \times \frac{\kappa_j x_j(t)}{\widetilde{R}_0} \\
&\geq \frac{1}{2}x_j(t) \geq \frac{1}{2\kappa_j}\varrho_j(t)\|x_j\| \geq \frac{\varrho_j(\xi)\widetilde{R}_0}{2\kappa_j} \geq \frac{\tau}{2}\widetilde{R}_0 > 0,
\end{aligned}
$$

and

$$
[x_i(t) - \omega_i(t)]^+ \leq x_i(t) \leq \|x_i\|, \quad i=1,2.
$$

So for any $(x_1,x_2) \in P \cap \partial\Omega_2, t \in [\xi,\eta]$, we have

$$
\frac{\tau}{2}\widetilde{R}_0 \leq [x_1(t) - \omega_1(t)]^+ + [x_2(t) - \omega_2(t)]^+ \leq R = 2\widetilde{R}_0. \tag{18}
$$

By (C_5) and (3.18), for any $(x_1, x_2) \in P \cap \partial\Omega_2$, $t \in [\xi, \eta]$, we have

$$
\begin{aligned}
T_i(x_1, x_2)(t) &= \int_0^1 H_i(t,s)[p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) + q_i(s)] ds \\
&\geq \int_0^1 G_i(t,s) p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) ds \\
&\geq \varrho_i(t) \int_0^1 G_i(1,s) p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) ds \\
&\geq \varrho_i(t) \int_\xi^\eta G_i(1,s) p_i(s) f_i(s, [x_1(s) - \omega_1(s)]^+, [x_2(s) - \omega_2(s)]^+) ds \\
&> \varrho_i(\xi) \int_\xi^\eta G_i(1,s) p_i(s) ds \times \frac{\widetilde{R}_0}{l_i} = \widetilde{R}_0, \quad i = 1, 2,
\end{aligned}
$$

this yields that

$$
\|T_i(x_1, x_2)\| > \widetilde{R}_0 = \frac{\|(x_1, x_2)\|_1}{2}, \quad i = 1, 2.
$$

Thus we get

$$
\|T(x_1, x_2)\|_1 > \|(x_1, x_2)\|_1, \quad \text{for all } (x_1, x_2) \in P \cap \partial\Omega_2. \tag{19}
$$

Next, let us choose $\varepsilon > 0$ such that $2\varepsilon\kappa_i \int_0^1 G_i(1,s) p_i(s) ds < 1$ $(i = 1, 2)$. Then for the above ε, by (C_4), there exists $N > R > 0$ such that, for any $t \in [0, 1]$ and for any $x_1 \geq 0$, $x_2 \geq 0$ and $x_1 + x_2 > N$,

$$
f_i(t, x_1, x_2) \leq \varepsilon(x_1 + x_2), \quad i = 1, 2.
$$

Take

$$
R_i^* = \frac{2M_i L_i + 2\kappa_i \int_0^1 G_i(1,s) q_i(s) ds}{1 - 2\varepsilon\kappa_i \int_0^1 G_i(1,s) p_i(s) ds} + N, \quad i = 1, 2,
$$

where $M_i = \max\{f_i(t, x_1, x_2) + 1 : t \in [0, 1], x_1 + x_2 \leq N\}$ $(i = 1, 2)$. Let $R^* = \max\{R_1^*, R_2^*\}$, then $R^* > N > R$.

Now let $\Omega_3 = \{(x_1, x_2) \in E : \|(x_1, x_2)\|_1 < R^*\}$. Similar to (3.15), we have

$$
\|T(x_1, x_2)\|_1 < \|(x_1, x_2)\|_1, \quad \text{for all } (x_1, x_2) \in P \cap \partial\Omega_3. \tag{20}
$$

By (3.17), (3.19), (3.20) and Lemma 2.8, T has two fixed points $(\hat{x}_1, \hat{x}_2)$, $(\bar{x}_1, \bar{x}_2)$ in P and $r_1 < \|(\hat{x}_1, \hat{x}_2)\|_1 < R < \|(\bar{x}_1, \bar{x}_2)\|_1$. Let $\hat{y}_i = \hat{x}_i - \omega_i$, $\bar{y}_i = \bar{x}_i - \omega_i$ $(i = 1, 2)$. By arguments similar to Theorem 3.1, we can show that $(\hat{y}_1, \hat{y}_2)$ and $(\bar{y}_1, \bar{y}_2)$ are two positive solutions of the system (1.1). □

Theorem 3.4 *Assume that conditions $(H_1) - (H_4)$ and $(C_2), (C_3)$ are satisfied. In addition, assume that the following condition holds:*

(C_6) *There exists a constant $R > \max\left\{2R_0, 2L_1(1 + \frac{R_0}{l_1}), 2L_2(1 + \frac{R_0}{l_2})\right\}$ such that for any $(t, y_1, y_2) \in [0, 1] \times [0, R] \times [0, R]$,*

$$
f_i(t, y_1, y_2) < \frac{R}{2L_i} - 1, \quad i = 1, 2,
$$

where $\kappa_i (i = 1, 2)$ and R_0 are defined by (2.5) and (3.10), respectively.

Then the system (1.1) has at least two positive solutions.

Proof. Firstly, let $R_1 = 2R_0$ and $\Omega_1 = \{(x_1, x_2) \in E : \|(x_1, x_2)\|_1 < R_1\}$. From (C_3) and proceeding as in (3.11)-(3.13), we obtain

$$\|T(x_1, x_2)\|_1 > \|(x_1, x_2)\|_1, \quad \text{for all } (x_1, x_2) \in P \cap \partial\Omega_1. \tag{21}$$

Next, by (C_6), we have $R > R_1$ and $\frac{R}{2L_i} - 1 > \frac{R_0}{l_i} > 0 (i = 1, 2)$. Let $\Omega_2 = \{(x_1, x_2) \in E : \|(x_1, x_2)\|_1 < R\}$. Then for any $(x_1, x_2) \in P \cap \partial\Omega_2, s \in [0, 1]$, we have

$$[x_i(s) - \omega_i(s)]^+ \le x_i(s) \le \|x_i\| \le R, \quad i = 1, 2.$$

It follows from (C_6), proceeding as in (3.2), we have

$$\|T(x_1, x_2)\|_1 < \|(x_1, x_2)\|_1, \quad \text{for all } (x_1, x_2) \in P \cap \partial\Omega_2. \tag{22}$$

On the other hand, choose a real number $M > 0$ big enough such that $\frac{1}{4}M\tau \min\{l_1, l_2\} > 1$, where τ is defined by (3.4). From (C_2), there exists $N > R$ such that, for any $x_1 \ge 0$, $x_2 \ge 0$ and $x_1 + x_2 \ge N$, for any $t \in [\xi, \eta]$, there is (3.5) holds. Set $R^* = \max\left\{2R, 4\tau^{-1}N\right\}$, then $R^* > R > R_1$. Let $\Omega_3 = \{(x_1, x_2) \in E : \|(x_1, x_2)\|_1 < R^*\}$. Similar to the proof of (3.6), for any $(x_1, x_2) \in P \cap \partial\Omega_3$, $t \in [\xi, \eta]$, we have

$$f_1(t, [x_1(t) - \omega_1(t)]^+, [x_2(t) - \omega_2(t)]^+) \ge M([x_1(t) - \omega_1(t)]^+ + [x_2(t) - \omega_2(t)]^+). \tag{23}$$

Combing with (3.23) and proceeding as in (3.8), we have

$$\|T(x_1, x_2)\|_1 > \|(x_1, x_2)\|_1, \quad \text{for all } (x_1, x_2) \in P \cap \partial\Omega_3. \tag{24}$$

By (3.21), (3.22), (2.24) and Lemma 2.8, T has two fixed points $(\hat{x}_1, \hat{x}_2)$, $(\bar{x}_1, \bar{x}_2)$ in P and $R_1 < \|(\hat{x}_1, \hat{x}_2)\|_1 < R < \|(\bar{x}_1, \bar{x}_2)\|_1$. Let $\hat{y}_i = \hat{x}_i - \omega_i$, $\bar{y}_i = \bar{x}_i - \omega_i$ $(i = 1, 2)$. By arguments similar to Theorem 3.1, we can show that $(\hat{y}_1, \hat{y}_2)$ and $(\bar{y}_1, \bar{y}_2)$ are two positive solutions of the system (1.1). □

4. An Example

Example 4.1. Consider the following problem

$$\begin{cases} -D_{0+}^{\frac{5}{2}} y_1(t) = \dfrac{\sqrt{\pi}}{t\sqrt{(1-t)}} f_1(t, y_1, y_2) - \dfrac{\sqrt{\pi}}{48\sqrt{t(1-t)}}, \quad 0 < t < 1, \\ -D_{0+}^{\frac{9}{4}} y_2(t) = \dfrac{\Gamma(\frac{9}{4})}{t\sqrt[4]{(1-t)}} f_2(t, y_1, y_2) - \dfrac{\Gamma(\frac{9}{4})}{12\sqrt[4]{1-t}}, \quad 0 < t < 1, \\ y_1(0) = y_1'(0) = 0, \quad y_1'(1) = \dfrac{96}{97} y_1\left(\dfrac{1}{16}\right), \\ y_2(0) = y_2'(0) = 0, \quad y_2'(1) = \dfrac{40}{41} y_2\left(\dfrac{1}{16}\right). \end{cases} \tag{1}$$

Let

$$p_1(t) = \frac{\sqrt{\pi}}{t\sqrt{(1-t)}}, \qquad q_1(t) = \frac{\sqrt{\pi}}{48\sqrt{t(1-t)}},$$
$$p_2(t) = \frac{\Gamma(\frac{9}{4})}{t\sqrt[4]{1-t}}, \qquad q_2(t) = \frac{\Gamma(\frac{9}{4})}{12\sqrt[4]{1-t}}.$$

Take $[\frac{1}{16}, \frac{9}{16}] \subset (0,1)$, by direct calculation, we have

$$\begin{aligned}
&\varrho_1(t) = t^{\frac{3}{2}}, \quad \varrho_2(t) = t^{\frac{5}{4}}, \quad \gamma_1(t) = \frac{2}{3}t^{\frac{3}{2}}, \quad \gamma_2(t) = \frac{4}{5}t^{\frac{5}{4}}, \quad t \in [0,1],\\
&\Lambda_1 = \lambda_1[\gamma_1] = \int_0^1 \gamma_1(t)dA_1(t) = \frac{96}{97} \times \frac{2}{3} \cdot \left(\frac{1}{16}\right)^{\frac{3}{2}} = \frac{1}{97},\\
&\Lambda_2 = \lambda_2[\gamma_2] = \int_0^1 \gamma_2(t)dA_2(t) = \frac{40}{41} \times \frac{4}{5} \cdot \left(\frac{1}{16}\right)^{\frac{5}{4}} = \frac{1}{41},\\
&\mathscr{G}_{A_1}(s) = \frac{96}{97}G_1\left(\frac{1}{16}, s\right) \geq 0, \quad \mathscr{G}_{A_2}(s) = \frac{40}{41}G_2\left(\frac{1}{16}, s\right) \geq 0,\\
&\int_0^1 G_1(1,s)[p_1(s) + q_1(s)]ds = \int_0^1 \frac{s(1-s)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}[p_1(s) + q_1(s)]ds = \frac{73}{54},\\
&\int_0^1 G_2(1,s)[p_2(s) + q_2(s)]ds = \int_0^1 \frac{s(1-s)^{\frac{1}{4}}}{\Gamma(\frac{9}{4})}[p_2(s) + q_2(s)]ds = \frac{25}{24},\\
&\kappa_1 = 2, \quad \kappa_2 = 2, \quad L_1 = \frac{73}{27}, \quad L_2 = \frac{25}{12}, \quad l_1 = \frac{1}{96}, \quad l_2 = \frac{1}{64},\\
&\int_0^1 q_1(t)dt = \frac{\pi\sqrt{\pi}}{48} \approx 0.1160 < \frac{\Gamma(\alpha_1)}{2\kappa^2} = \frac{\Gamma(\frac{5}{2})}{8} \approx 0.1662,\\
&\int_0^1 q_2(t)dt = \frac{\Gamma(\frac{9}{4})}{9} \approx 0.1259 < \frac{\Gamma(\alpha_2)}{2\kappa^2} = \frac{\Gamma(\frac{9}{4})}{8} \approx 0.1416.
\end{aligned}$$

So conditions $(H_1) - (H_3)$ hold.

Next, in order to demonstrate the application of our main results obtained in section 3, we choose two different sets of functions $f_i(t, y_1, y_2)$ $(i = 1, 2)$ such that f_1 and f_2 satisfy the conditions of Theorem 3.1 and Theorem 3.4, respectively.

Case 1. Let $f_1(t, y_1, y_2) = \frac{1}{685}[(y_1 - 34)^2 + y_2^2]$, $f_2(t, y_1, y_2) = \frac{1}{685}[y_1^2 + (y_2 - 39)^2]$, $(t, y_1, y_2) \in [0,1] \times [0, +\infty) \times [0, +\infty)$. Obviously, $f_i(t, y_1, y_2)$ $(i = 1, 2)$ are continuous on $[0,1] \times [0, +\infty) \times [0, +\infty)$, and

$$\begin{aligned}
&p_1(t)f_1(t, y_1, y_2) \geq q_1(t), \quad (t, y_1, y_2) \in [0,1] \times [0,1] \times [0, +\infty),\\
&p_2(t)f_2(t, y_1, y_2) \geq q_2(t), \quad (t, y_1, y_2) \in [0,1] \times [0, +\infty) \times [0,1].
\end{aligned}$$

So condition (H_4) holds.

Take $r_1 = 73$, then $r_1 > \max\left\{2, 2L_1, 2L_2, \frac{4\kappa_1^2}{\Gamma(\alpha_1)}\int_0^1 q_1(s)ds, \frac{4\kappa_2^2}{\Gamma(\alpha_2)}\int_0^1 q_2(s)ds\right\}$. For any $(t, y_1, y_2) \in [0,1] \times [0,73] \times [0,73]$, we have

$$\begin{aligned}
&f_1(t, y_1, y_2) \leq \frac{1}{685} \times [(r_1 - 34)^2 + r_1^2] = 10 < \frac{r_1}{2L_1} - 1 = 12.5,\\
&f_2(t, y_1, y_2) \leq \frac{1}{685} \times [r_1^2 + (0 - 39)^2] = 10 < \frac{r_1}{2L_2} - 1 = 16.52.
\end{aligned}$$

In addition, we can easily to check that $f_{1\infty} = +\infty$, $f_{2\infty} = +\infty$, so conditions (C_1) and (C_2) of Theorem 3.1 are satisfied. Then by Theorem 3.1, the system (4.1) has at least one positive solution.

Case 2. Let $f_1(t, y_1, y_2) = [10^{-8} + g_1(y_1)] \times h_1(y_2)$, $f_2(t, y_1, y_2) = g_2(y_1) \times [10^{-8} + h_2(y_2)]$, where

$$g_1(y_1) = \begin{cases} 433, & 0 \leq y_1 \leq 128,\\ -\dfrac{1}{84}y_1 + \dfrac{9125}{21}, & 128 \leq y_1 \leq 36500,\\ (y_1 - 36500)^2, & y_1 \geq 36500, \end{cases} \qquad h_1(y_2) = \begin{cases} 15, & 0 \leq y_2 \leq 36500,\\ 15(y_2 - 36499)^2, & y_2 \geq 36500, \end{cases}$$

$$g_2(y_1)=\begin{cases}\dfrac{1}{9125}y_1+77, & 0\le y_1\le 36500,\\ (y_1-36491)^2, & y_1\ge 36500,\end{cases}\qquad h_2(y_2)=\begin{cases}\dfrac{1}{8}y_2+68, & 0\le y_2\le 128,\\ -\dfrac{1}{433}(y_2-36500), & 128\le y_2\le 36500,\\ (y_2-36500)^2, & y_2\ge 36500.\end{cases}$$

Obviously, $f_i(t,y_1,y_2)$ $(i=1,2)$ are continuous on $[0,1]\times[0,+\infty)\times[0,+\infty)$, and

$$p_1(t)f_1(t,y_1,y_2)\ge q_1(t),\quad (t,y_1,y_2)\in[0,1]\times[0,1]\times[0,+\infty),$$

$$p_2(t)f_2(t,y_1,y_2)\ge q_2(t),\quad (t,y_1,y_2)\in[0,1]\times[0,+\infty)\times[0,1].$$

So condition (H_4) holds.

Take $R_0=64$, then $R_0>\max\left\{1,\frac{4\kappa_1^2}{\Gamma(\alpha_1)}\int_0^1 q_1(s)ds,\frac{4\kappa_2^2}{\Gamma(\alpha_2)}\int_0^1 q_2(s)ds\right\}$, and for any $t\in[\frac{1}{16},\frac{9}{16}]$, $\frac{1}{4}=\frac{1}{2}\tau R_0\le y_1+y_2\le 2R_0=128$, we have

$$\begin{aligned}f_1(t,y_1,y_2)&=[10^{-8}+g_1(y_1)]\times h_1(y_2)>433\times 15=6495>\frac{R_0}{l_1}=6144,\\ f_2(t,y_1,y_2)&=g_2(y_1)\times[10^{-8}+h_2(y_2)]>\left(\frac{1}{9125}y_1+77\right)\times\left(\frac{1}{8}y_2+68\right)\ge 5236>\frac{R_0}{l_2}=4096.\end{aligned}$$

Choose $R=36500$, then $R>\max\left\{2R_0,2L_1(1+\frac{R_0}{l_1}),2L_2(1+\frac{R_0}{l_2})\right\}$, and for any $(t,y_1,y_2)\in[0,1]\times[0,36500]\times[0,36500]$, we have

$$\begin{aligned}f_1(t,y_1,y_2)&\le\left[1+\max_{0\le y_2\le 36500}g_1(y_1)\right]\times 15=434\times 15=6510<\frac{R}{2L_1}-1=6749,\\ f_2(t,y_1,y_2)&\le\left(\frac{1}{9125}y_1+77\right)\times\left[1+\max_{0\le y_2\le 36500}h_2(y_2)\right]\le 81\times 85=6885<\frac{R}{2L_2}-1=8759.\end{aligned}$$

In addition, it is not difficult to show that $f_{1\infty}=+\infty$ or $f_{2\infty}=+\infty$. So all conditions of Theorem 3.4 are satisfied. By Theorem 3.4, the system (4.1) has at least two positive solutions.

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