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ROBUST EIGENSTRUCTURE ASSIGNMENT IN GEOMETRIC CONTROL THEORY*

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Abstract. In this paper we employ the Rosenbrock system matrix pencil for the computation of output-nulling subspaces of linear time-invariant systems which appear in the solution of a large number of control and estimation problems. We also consider the problem of finding friends of these output-nulling subspaces, i.e., the feedback matrices that render such subspaces invariant with respect to the closed-loop map and output-nulling with respect to the output map, and which at the same time deliver a robust closed-loop eigenstructure. We show that the methods presented in this paper offer considerably more robust eigenstructure assignment than the other commonly used methods and algorithms.

Key words. geometric control, controlled invariance, output-nulling subspaces, friends, Rosenbrock matrix pencil

AMS subject classifications. 93C05, 93B27, 93B52

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1. Introduction.

In the last 40 years, geometric control has played a fundamental role not only in the solution of important control and estimation problems—including disturbance decoupling, unknown-input observation, noninteracting and model matching control, fault detection and isolation, and LQ and H_2-optimal control problems, to name a few—but also in the understanding of several structural properties of both linear and nonlinear systems. At the same time, several polynomial and structural approaches have been introduced, offering a deeper insight and understanding of the properties of dynamical systems. Given the large number of contributions in this area, it would be impossible to quote even a fraction of the relevant references, and we consequently direct the interested reader to the comprehensive monographs [27], [4], [25], [7], which provide surveys of the extensive literature in this area.

The main subspaces that underpin the classic geometric theory of linear time-invariant (LTI) systems are the so-called controlled and conditioned invariant subspaces. The most important types of controlled invariant subspaces are usually referred to as *output-nulling subspaces*. These are subspaces of initial states of an LTI system for which a control function exists that maintains the entire state trajectory on that subspace and at the same time maintains the output at zero. For finite-dimensional systems over a field, such control laws can always be expressed as a static state feedback $u = F\,x$, where F is called a *friend* of the output-nulling subspace. In the control and estimation problems noted above, the solution requires the computation of a decoupling filter (which may be a controller or an observer, depending on the problem under consideration), which in turn typically requires

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the computation of a friend of the appropriate output-nulling (or its dual) subspace.

A related family of subspaces that also plays a pivotal role in control and estimation problems are the so-called reachability subspaces (often referred to as *controllability subspaces*). Moreover, the so-called stabilizability output-nulling subspaces are crucial in the solution of these problems with stability requirements.

This paper investigates several aspects related to the computation of basis matrices for these subspaces and the computation of their corresponding friends. Except for stabilizability and detectability subspaces, which require eigenspace computations, the traditional algorithms employed to compute the aforementioned subspaces are based on monotonic sequences of subspaces that converge in a finite number of steps (typically not greater than the system order) to the desired subspace. An alternative approach was taken by Moore and Laub in [13], who proposed an algorithm for the computation of the largest output-nulling reachability subspace that employs the Rosenbrock system matrix pencil. That paper made a number of restrictive assumptions, and perhaps for this reason the methods in [13] have been given only marginal attention. An approach via the special coordinate basis (SCB), which avoided the restrictive assumptions of [13], was given in [5], [6], [7].

From the perspective of the controller design, the computation of the friends of an output nulling subspace is equally as important as (if not more important than) the computation of a basis matrix for the output-nulling subspace itself, as it is employed in virtually all control and estimation problems for which a geometric solution is available. Indeed, when we consider problems such as disturbance decoupling (with unknown, measurable, and previewed signals) with state and measurable feedback, noninteraction, model matching, fault detection, unknown-input observation, and even H_2-optimal control and filtering problems (to name just a few examples), the computation of basis matrices for output-nulling and input-containing subspaces is crucial to determine necessary and sufficient solvability conditions. In the case the problem at hand is solvable, the computation of the associated friends provides the actual solution to the problem.

The computation of the friends was considered in [4] and is summarized in Appendix A. In the publicly available MATLAB *GA* toolbox,[1] the effesta.m routine is used for computing the friends. Similarly, the SCB method of [5] was incorporated into the computation of the friends in the MATLAB Linsyskit toolbox;[2] the atea.m routine is used for computing the friends and is described in [10]. However, it has been acknowledged by many authors [13], [8] that a major drawback of the applicability of the geometric approach is the lack of algorithms for the computation of friends that deliver a robust closed-loop eigenstructure, in which the closed-loop eigenvalues are rendered insensitive to perturbations in the state matrices. The classical methods for the computation of friends do not attempt to address the robustness aspects of the problem and leave unexploited all the degrees of freedom in the computation of the friend.

In this paper we add to this classical literature on the computation of basis matrices for these subspaces, and we also consider the problem of computing the associated friends in a robust manner. Taking inspiration from the pioneering work of

[1]The geometric approach toolbox *GA* for MATLAB is freely downloadable at www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm.

[2]The Linear Systems Toolkit is available on request from the first author of [7]; see http://vlab.ee.nus.edu.sg/∼bmchen/.

[13], we employ the Rosenbrock system matrix pencil to obtain an algorithm for the computation of a basis matrix for the largest output-nulling reachability subspace $\mathcal{R}^\star$ of an LTI system. Our method avoids the unnecessarily restrictive assumptions made in that paper and shows that computational methods based on the Rosenbrock matrix pencil can be used under the same general conditions as the subspace recursion methods of [4] and the SCB methods of [5]. Moving beyond [13], we next offer a method for the computation of a friend F that assigns any desired spectrum of the closed-loop mapping restricted to this subspace. For simplicity of exposition, in this paper we restrict our attention to the case of distinct eigenvalues and invariant zeros.

Our procedure parameterizes the friends that achieve the desired spectrum, and the parametric form is shown to be exhaustive of all the friends that deliver the specified closed-loop spectrum. Next, the parametric form is extended to accommodate all friends that also assign the free eigenvalues of the closed-loop that are *external* to $\mathcal{R}^\star$; see also [22]. A similar parametric form is proposed for the friends of the largest output-nulling subspace $\mathcal{V}^\star$, as well as for the friends of the largest stabilizability output-nulling subspace $\mathcal{V}_g^\star$. Finally we extend the procedure again and obtain a parametric form for all the friends that assign any desired inner and outer closed-loop spectrum.

The degrees of freedom associated with the computation of friends of an output-nulling subspace invites the formulation of optimization problems whose goal is to exploit the available freedom to address objectives such as minimum gain or improved robustness of the closed-loop eigenstructure. We propose a nonlinear unconstrained optimization problem to find a friend that minimizes the Frobenius condition number of the matrix of closed-loop eigenvectors, which is a commonly used robustness measure. Next, we propose a nonlinear unconstrained optimization problem that minimizes the Frobenius norm of the friend. We then show how these two optimization problems can be combined to minimize a weighted sum of the robustness and minimum gain measures, to be solved by gradient search methods.

Finally, we offer some performance comparisons of our method against those of [4] and [7]. We consider an example system and compare the robustness of the associated eigenstructure, the norms of the feedback gain matrices used, and the numerical accuracy of the pole placement delivered by each of these methods. We observe that the method introduced in this paper offers dramatically improved eigenvalue insensitivity with significantly smaller gain and vastly improved accuracy than the friends obtained from the *GA* toolbox and the Linear Systems Toolkit.

To further test the merits of our method against those of [4] and [7], we adopted a Monte Carlo–style approach in which 10,000 sets of sample systems were randomly generated, and the three methods were applied to each system to compute the friend that assigned a particular inner and outer eigenstructure of the closed loop with respect to the largest output-nulling reachability subspace. Comparisons were then made of the robustness, gain, and accuracy of the eigenstructure assignment of each method. In the vast majority of cases it was found that the methods offered here were able to deliver a more robust pole assignment with less gain and greater accuracy than both the alternatives.

Notation. Throughout this paper, the symbol $\{0\}$ will stand for the origin of a vector space. For convenience, a linear mapping between finite-dimensional spaces and a matrix representation with respect to a particular basis are not distinguished notationally. The image and the kernel of matrix A are denoted by im A and ker A,

respectively. The Moore–Penrose pseudoinverse of A is denoted by $A^\dagger$. When A is square, we denote by $\sigma(A)$ the spectrum of A. Given a linear map $A : \mathcal{X} \longrightarrow \mathcal{Y}$ and a subspace $\mathcal{S}$ of $\mathcal{Y}$, the symbol $A^{-1}\,\mathcal{S}$ stands for the inverse image of $\mathcal{S}$ with respect to the linear map A, i.e., $A^{-1}\,\mathcal{S} = \{x \in \mathcal{X} \,|\, A\,x \in \mathcal{S}\}$. If $\mathcal{J} \subseteq \mathcal{X}$, the restriction of the map A to $\mathcal{J}$ is denoted by $A\,|\mathcal{J}$. If $\mathcal{X} = \mathcal{Y}$ and $\mathcal{J}$ is A-invariant, the eigenvalues of A restricted to $\mathcal{J}$ are denoted by $\sigma\,(A\,|\mathcal{J})$. If $\mathcal{J}_1$ and $\mathcal{J}_2$ are A-invariant subspaces and $\mathcal{J}_1 \subseteq \mathcal{J}_2$, the mapping induced by A on the quotient space $\mathcal{J}_2/\mathcal{J}_1$ is denoted by $A\,|\mathcal{J}_2/\mathcal{J}_1$, and its spectrum is denoted by $\sigma\,(A\,|\mathcal{J}_2/\mathcal{J}_1)$. The symbol $\oplus$ stands for the direct sum of subspaces. The symbol $\uplus$ denotes union with any common elements repeated. Given a map $A : \mathcal{X} \longrightarrow \mathcal{X}$ and a subspace $\mathcal{B}$ of $\mathcal{X}$, we denote by $\langle A, \mathcal{B}\rangle$ the smallest A-invariant subspace of $\mathcal{X}$ containing $\mathcal{B}$. The symbol $\mathtt{i}$ stands for the imaginary unit, i.e., $\mathtt{i} = \sqrt{-1}$. The symbol $\bar{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{C}$. Finally, given a matrix M, we denote by M_i its ith row and by M^j its jth column, respectively.

2. Preliminaries. In what follows, whether the underlying system evolves in continuous or discrete time is irrelevant and, accordingly, the time index set of any signal is denoted by $\mathbb{T}$, on the understanding that this represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane $\mathbb{C}^-$ in the continuous time or the open unit disc $\mathbb{C}^\circ$ in the discrete time. Consider an LTI system Σ governed by

$$\Sigma : \begin{cases} \rho\, x(t) = A\, x(t) + B\, u(t), \quad x(0) = x_0, \\ y(t) = C\, x(t) + D\, u(t), \end{cases} \tag{2.1}$$

where for all $t \in \mathbb{T}$, $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} = \mathbb{R}^m$ is the control input, $y(t) \in \mathcal{Y} = \mathbb{R}^p$ is the output, and A, B, C, and D are appropriate dimensional constant real-valued matrices. The operator ρ denotes either the time derivative in the continuous time, i.e., $\rho x(t) = \dot{x}(t)$, or the unit time shift in the discrete time, i.e., $\rho x(t) = x(t+1)$. Let the system Σ described by (2.1) be identified with the quadruple (A, B, C, D). We assume with no loss of generality that all the columns of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ and all the rows of $[C \quad D]$ are linearly independent.[3]

We define the Rosenbrock system matrix pencil as

$$P_\Sigma(\lambda) \stackrel{\text{def}}{=} \begin{bmatrix} A - \lambda\, I & B \\ C & D \end{bmatrix}, \qquad \lambda \in \mathbb{C}; \tag{2.2}$$

see [17]. The invariant zeros of Σ are identified with the values of $\lambda \in \mathbb{C}$ for which the rank of $P_\Sigma(\lambda)$ is strictly smaller than its normal rank.[4] More precisely, the invariant zeros are the roots of the nonzero polynomials on the principal diagonal of the Smith form of $P_\Sigma(\lambda)$; see [1]. Given an invariant zero $\lambda = z \in \mathbb{C}$, the rank deficiency of $P_\Sigma(\lambda)$ at the value $\lambda = z$ is the geometric multiplicity of the invariant zero z and is equal

[3] If $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ has nontrivial kernel, a subspace $\mathcal{U}_0$ of the input space exists that does not influence the local state dynamics. By performing a suitable (orthogonal) change of basis in the input space, we may eliminate $\mathcal{U}_0$ and obtain an equivalent system for which this condition is satisfied. Likewise, if $[C \quad D]$ is not surjective, there are some outputs that result as linear combinations of the remaining ones, and these can be eliminated using a dual argument by performing a change of coordinates in the output space.

[4] The normal rank of a rational matrix $M(\lambda)$ is defined as $\operatorname{normrank} M(\lambda) \stackrel{\text{def}}{=} \max_{\lambda \in \mathbb{C}} \operatorname{rank} M(\lambda)$. The rank of $M(\lambda)$ is equal to its normal rank for all but finitely many $\lambda \in \mathbb{C}$.

to the number of elementary divisors (invariant polynomials) of $P_\Sigma(\lambda)$ associated with the complex frequency $\lambda = z$. The degree of the product of the elementary divisors of $P_\Sigma(\lambda)$ corresponding to the invariant zero z is the algebraic multiplicity of z; see [12].

Given $\lambda \in \mathbb{C}$, we use the symbol $N_\Sigma(\lambda)$ to denote a basis matrix for the null-space of $P_\Sigma(\lambda)$. We denote by d the dimension of the null-space of $P_\Sigma(\lambda)$—or, equivalently, the number of columns of $N_\Sigma(\lambda)$—when λ is not an invariant zero, and by d_z the dimension of the null-space of $P_\Sigma(z)$—or, equivalently, the number of columns of $N_\Sigma(z)$—when z is an invariant zero. Thus, if the set of invariant zeros of Σ is not empty, we have in general $d_z > d$. Clearly $d = n + m - \operatorname{normrank} P_\Sigma$.

For any matrix M with $n + m$ rows, we define the matrices $\overline{\pi}\{M\}$ and $\underline{\pi}\{M\}$ obtained by taking the upper n and lower m rows of M, respectively.

We recall that the *reachable subspace* from the origin is the smallest A-invariant subspace of $\mathcal{X}$ containing the image of B and is denoted by $\mathcal{R}_0 = \langle A, \operatorname{im} B\rangle$. An *output-nulling subspace* $\mathcal{V}$ of Σ is a subspace of $\mathcal{X}$ which satisfies the inclusion

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus \{0\}) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{2.3}$$

which is equivalent to the existence of a matrix $F \in \mathbb{R}^{m\times n}$ such that

$$(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(C + D\,F). \tag{2.4}$$

Any real matrix F satisfying (2.4) is referred to as a *friend* of $\mathcal{V}$. We denote by $\mathfrak{F}(\mathcal{V})$ the set of friends of $\mathcal{V}$. The set of output-nulling subspaces of Σ is closed under subspace addition.[5] The sum of all the output-nulling subspaces of Σ is the largest output-nulling subspace of Σ and is denoted by $\mathcal{V}^\star$. The subspace $\mathcal{V}^\star$ represents the set of all initial states of Σ for which a control input exists such that the corresponding output function is identically zero. Since we are considering finite-dimensional LTI systems over a field, such an input function can always be implemented as a static state feedback of the form $u(t) = F\,x(t)$, where $F \in \mathfrak{F}(\mathcal{V}^\star)$.

The so-called output-nulling reachability subspace on $\mathcal{V}^\star$, herein denoted $\mathcal{R}^\star$, is the smallest $(A+B\,F)$-invariant subspace of $\mathcal{X}$ containing the subspace $\mathcal{V}^\star \cap B$ ker D, where $F \in \mathfrak{F}(\mathcal{V}^\star)$, i.e.,

$$\mathcal{R}^\star = \langle A + B\,F, \mathcal{V}^\star \cap B \ker D\rangle, \qquad \text{where } F \in \mathfrak{F}(\mathcal{V}^\star). \tag{2.5}$$

Loosely speaking, $\mathcal{R}^\star$ represents the subspace of the state-space containing the states that are reachable from the origin with trajectories that correspond to identically zero output [25, Chapter 8].

Let $F \in \mathfrak{F}(\mathcal{V}^\star)$. The closed-loop spectrum can be partitioned as $\sigma(A + B\,F) = \sigma(A + B\,F \,|\, \mathcal{V}^\star) \uplus \sigma(A + B\,F \,|\, \mathcal{X}/\mathcal{V}^\star)$, where

- $\sigma(A+B\,F \,|\, \mathcal{V}^\star)$ is the spectrum of $A+B\,F$ restricted to $\mathcal{V}^\star$, and its elements are referred to as *inner eigenvalues* of the closed loop with respect to $\mathcal{V}^\star$. If $\sigma(A + B\,F \,|\, \mathcal{V}^\star) \subset \mathbb{C}_g$, we say that $\mathcal{V}^\star$ is *inner stabilizable*.
- $\sigma(A + B\,F \,|\, \mathcal{X}/\mathcal{V}^\star)$ is the spectrum of the mapping induced by $A + B\,F$ on the quotient space $\mathcal{X}/\mathcal{V}^\star$. Its elements are referred to as *outer eigenvalues* of the closed loop with respect to $\mathcal{V}^\star$. If $\sigma(A + B\,F \,|\, \mathcal{X}/\mathcal{V}^\star) \subset \mathbb{C}_g$, we say that $\mathcal{V}^\star$ is *outer stabilizable*.

[5] It is easy to see that the set of output-nulling subspaces of Σ is a modular upper semilattice with respect to the standard subspace addition + and with respect to the subspace inclusion $\subseteq$.

The eigenvalues of $A + B\,F$ restricted to $\mathcal{V}^\star$ can be further split into two disjoint sets: the eigenvalues of $A + B\,F$ restricted to $\mathcal{R}^\star$, i.e., $\sigma(A + B\,F|\mathcal{R}^\star)$, are all freely assignable[6] with a suitable choice of F in $\mathfrak{F}(\mathcal{V}^\star)$. The eigenvalues induced by the map $A + B\,F$ on the quotient space $\mathcal{V}^\star/\mathcal{R}^\star$, i.e., $\Gamma_{\text{in}} \stackrel{\text{def}}{=} \sigma\,(A + B\,F|\mathcal{V}^\star/\mathcal{R}^\star)$, are fixed for all the choices of F in $\mathfrak{F}(\mathcal{V}^\star)$. Thus, $\mathcal{V}^\star$ is inner stabilizable if and only if $\Gamma_{\text{in}} \subset \mathbb{C}_g$. Similarly, the eigenvalues $\sigma(A + B\,F|\mathcal{X}/\mathcal{V}^\star)$ are split into two sets: the eigenvalues $\sigma(A + B\,F|(\mathcal{V}^\star + \mathcal{R}_0)/\mathcal{V}^\star)$ are all freely assignable by a suitable choice of F in $\mathfrak{F}(\mathcal{V}^\star)$, whereas the eigenvalues in $\Gamma_{\text{out}} \stackrel{\text{def}}{=} \sigma(A + B\,F|\mathcal{X}/(\mathcal{V}^\star + \mathcal{R}_0))$ are fixed. Thus, $\mathcal{V}^\star$ is outer stabilizable if and only if $\Gamma_{\text{out}} \subset \mathbb{C}_g$. Hence, the set $\Gamma_{\text{in}} \uplus \Gamma_{\text{out}}$ does not depend on the choice of the friend F of $\mathcal{V}^\star$. The elements of Γ_{in} are the invariant zeros of Σ and are therefore also denoted by $\mathcal{Z}$. We also define $\mathcal{G} \stackrel{\text{def}}{=} \Gamma_{\text{out}}$.

3. Computation of $\mathcal{R}^\star$ and its associated friends.

3.1. Assignment of the inner eigenstructure of $\mathcal{R}^\star$. Given a set of h self-conjugate complex numbers $\mathcal{L} = \{\lambda_1, \ldots, \lambda_h\}$ containing exactly s complex conjugate pairs, we say that $\mathcal{L}$ is s-conformably ordered if $2\,s \le h$ and the first $2\,s$ values of $\mathcal{L}$ are complex, while the remaining are real, and for all odd $k \le 2\,s$ we have $\lambda_{k+1} = \bar{\lambda}_k$. For example, the sets $\mathcal{L}_1 = \{1+\mathrm{i}, 1-\mathrm{i}, 3, -4\}$, $\mathcal{L}_2 = \{10\,\mathrm{i}, -10\,\mathrm{i}, 2+2\,\mathrm{i}, 2-2\,\mathrm{i}, 7\}$, and $\mathcal{L}_3 = \{3, -1\}$ are respectively 1-, 2-, and 0-conformably ordered.

The following theorem presents a procedure for the computation of a basis matrix for $\mathcal{R}^\star$ and, simultaneously, for the parameterization of all the friends of $\mathcal{R}^\star$ that place the eigenvalues of the closed loop restricted to $\mathcal{R}^\star$ at arbitrary locations. This procedure aims at constructing a basis for $\mathcal{R}^\star$ starting from basis matrices $N_\Sigma(\lambda_i)$ of the null-spaces of the Rosenbrock matrix relative to an s-conformably ordered set $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$, where r is the dimension of $\mathcal{R}^\star$. The set $\mathcal{L}$ will result as the set of eigenvalues of the closed loop restricted to $\mathcal{R}^\star$. No generality is lost by assuming that for every odd $i \in \{1, \ldots, 2\,s\}$, the basis matrix $N_\Sigma(\lambda_{i+1})$ is constructed as $N_\Sigma(\lambda_{i+1}) = N_\Sigma(\overline{\lambda_i}) = \overline{N_\Sigma(\lambda_i)}$.

THEOREM 3.1 (parameterization of the friends of $\mathcal{R}^\star$). *Let $r = \dim \mathcal{R}^\star$. Let $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ be s-conformably ordered with elements all different from the invariant zeros of the system. Let $K \stackrel{\text{def}}{=} \operatorname{diag}\{k_1, \ldots, k_r\}$, where $k_i \in \mathbb{C}^d$ (recall that $d = n + m - \operatorname{normrank} P_\Sigma$) for each $i \in \{1, \ldots, 2\,s\}$, and for all odd $i \le 2\,s$ we have $\bar{k}_i = k_{i+1}$, and $k_i \in \mathbb{R}^d$ for $i \in \{2\,s+1, \ldots, r\}$. Let M_K be an $(n+m) \times r$ complex matrix given by*

$$M_K \stackrel{\text{def}}{=} \left[N_\Sigma(\lambda_1) \;\middle|\; N_\Sigma(\lambda_2) \;\middle|\; \ldots \;\middle|\; N_\Sigma(\lambda_r)\right] K \tag{3.1}$$

and let for all $i \in \{1, \ldots, r\}$

$$m_{K,i} \stackrel{\text{def}}{=} \begin{cases} \mathfrak{Re}\{M_K^i\} & \text{if } i \le 2\,s \text{ is odd,} \\ \mathfrak{Im}\{M_K^i\} & \text{if } i \le 2\,s \text{ is even,} \\ M_K^i & \text{if } i > 2\,s. \end{cases} \tag{3.2}$$

Finally, let

$$X_K \stackrel{\text{def}}{=} \overline{\pi}\,\{[m_{K,1}\;\; m_{K,2}\;\; \ldots\;\; m_{K,r}]\}, \tag{3.3}$$

$$Y_K \stackrel{\text{def}}{=} \underline{\pi}\,\{[m_{K,1}\;\; m_{K,2}\;\; \ldots\;\; m_{K,r}]\}. \tag{3.4}$$

[6] An assignable set of eigenvalues here is always intended to be a set of complex numbers which is mirrored with respect to the real axis.

The following statements hold:

- *The matrix X_K is generically full column-rank with respect to the parameter matrix $K = \operatorname{diag}\{k_1, \dots, k_r\}$, i.e.,* $\operatorname{rank} X_K = r$ *for every K except possibly for those lying in a set of Lebesgue measure zero.*
- *For all K such that* $\operatorname{rank} X_K = r$*, the identity $\mathcal{R}^\star = \operatorname{im} X_K$ holds.*
- *The set of all friends of $\mathcal{R}^\star$ such that $\sigma(A+B\,F \,|\, \mathcal{R}^\star) = \mathcal{L}$ is parameterized as*

$$F_K = Y_K\, X_K^\dagger, \tag{3.5}$$

where K is such that $\operatorname{rank} X_K = r$*.*

Proof. First, we show that the set of parameter matrices K such that $\operatorname{rank} X_K < r$ has Lebesgue measure zero. From [13, Proposition 4], every choice of a distinct complex conjugate set $\mathcal{L} = \{\lambda_1, \dots, \lambda_r\}$ which does not contain invariant zeros is such that the rank of $\overline{\pi}\{[N_\Sigma(\lambda_1)\ N_\Sigma(\lambda_2)\ \dots\ N_\Sigma(\lambda_r)]\}$ is equal to r. Thus, for almost all choices of K we have $\operatorname{rank} \overline{\pi}\,\{M_K\} = r$. To see this, let us partition $[N_\Sigma(\lambda_1)\ N_\Sigma(\lambda_2)\ \dots\ N_\Sigma(\lambda_r)]$ in (3.1) as $\left[\begin{smallmatrix}\Phi_\Sigma\\ \Psi_\Sigma\end{smallmatrix}\right]$, where Φ_Σ and Ψ_Σ have n and m rows, respectively. Since as mentioned $\operatorname{rank} \Phi_\Sigma = r$ from [13, Proposition 4], we can denote by $\{\Phi_\Sigma^{\beta_1}, \Phi_\Sigma^{\beta_2}, \dots, \Phi_\Sigma^{\beta_r}\}$ a basis for $\operatorname{im} \Phi_\Sigma$. If $\operatorname{rank}(\Phi_\Sigma\, K)$ is smaller than r, i.e., say, $r-1$, then the rank of the matrix $[\Phi_\Sigma^{\beta_1}\, k_{\beta_1}\ \ \Phi_\Sigma^{\beta_2}\, k_{\beta_2}\ \ \dots\ \ \Phi_\Sigma^{\beta_r}\, k_{\beta_r}]$ cannot be greater than $r-1$. This means that one column of such matrix is linearly dependent of all the remaining ones. For the sake of argument, assume this is the last column. This means that there exist coefficients $\alpha_1, \dots, \alpha_{r-1}$ not all equal to zero such that $\Phi_\Sigma^{\beta_r}\, k_{\beta_r} = \sum_{h=1}^{r-1} \alpha_h\, \Phi_\Sigma^{\beta_h}\, k_{\beta_h}$ which has a unique solution in k_{β_r}. This implies that $\operatorname{rank}(\Phi_\Sigma\, K) = r$ may fail only when $k_{\beta_r} = (\Phi_\Sigma^{\beta_r})^\dagger \sum_{h=1}^{r-1} \alpha_h\, \Phi_\Sigma^{\beta_h}\, k_{\beta_h}$, hence on an $(r-1)$-dimensional hyperplane in the r-dimensional parameter space.[7] The set of parameters that can potentially lead to a loss of rank in X_K is given by the union of a finite number of hyperplanes of dimension at most $r-1$ within the parameter space. This set therefore has empty interior and thus also zero Lebesgue measure.

We now prove the second and third points. Let K be such that $\operatorname{rank} \overline{\pi}\,\{M_K\} = r$, and let M_K be partitioned as $M_K = \left[\begin{smallmatrix} v_1' & v_2' & \dots & v_r' \\ w_1' & w_2' & \dots & w_r' \end{smallmatrix}\right]$ where for each $i \in \{1, \dots, r\}$, there hold

$$(A - \lambda_i\, I_n)\, v_i' + B\, w_i' = 0, \tag{3.6}$$

$$C\, v_i' + D\, w_i' = 0. \tag{3.7}$$

For odd $i \le 2\,s$, as $\lambda_i = \bar{\lambda}_{i+1}$ and $k_i = \bar{k}_{i+1}$, we have $\bar{v}_i' = v_{i+1}'$ and $\bar{w}_i' = w_{i+1}'$. Let $U \stackrel{\text{def}}{=} \frac{1}{2}\left[\begin{smallmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{smallmatrix}\right]$, and for each odd $i \le 2\,s$ let $[v_i \quad v_{i+1}] \stackrel{\text{def}}{=} [v_i' \quad v_{i+1}']\, U$ and $[w_i \quad w_{i+1}] \stackrel{\text{def}}{=} [w_i' \quad w_{i+1}']\, U$. Then, we have

$$v_i = \begin{cases} \frac{1}{2}\,(v_i' + v_{i+1}') & \text{if } i \le 2\,s \text{ is odd,} \\ \frac{1}{2\,\mathrm{i}}\,(v_i' - v_{i-1}') & \text{if } i \le 2\,s \text{ is even,} \\ v_i' & \text{if } i > 2\,s, \end{cases} \quad w_i = \begin{cases} \frac{1}{2}\,(w_i' + w_{i+1}') & \text{if } i \le 2\,s \text{ is odd,} \\ \frac{1}{2\,\mathrm{i}}\,(w_i' - w_{i-1}') & \text{if } i \le 2\,s \text{ is even,} \\ w_i' & \text{if } i > 2\,s, \end{cases}$$

which are real-valued. It follows that the matrices X_K and Y_K in (3.3)–(3.4) can be written as $X_K = [v_1\ \dots\ v_{2\,s} \,|\, v_{2\,s+1}\ \dots\ v_r]$ and $Y_K = [w_1\ \dots\ w_{2\,s} \,|\, w_{2\,s+1}\ \dots\ w_r]$.

[7] Notice that even with this choice of k_{β_r} we could still have $\operatorname{rank}(\Phi_\Sigma\, K) = r$ due to the contribution given by the remaining columns of Φ_Σ.

Since for this choice of K the rank of X_K is also equal to r, (3.5) is a solution of the linear equation $F_K X_K = Y_K$. This means that $F_K v_i' = w_i'$ for all $i \in \{1, \dots, r\}$. Moreover, $F_K [v_i \quad v_{i+1}] = [w_i \quad w_{i+1}]$ for all odd $i \in \{1, \dots, 2s\}$. From (3.6)–(3.7) we get $\left[\begin{smallmatrix} A+B F_K \\ C+D F_K \end{smallmatrix}\right] v_i' = \left[\begin{smallmatrix} v_i' \\ 0 \end{smallmatrix}\right] \lambda_i$. Since for all odd $i \leq 2s$ we have $v_{i+1}' = v_i + \mathbf{i} v_{i+1}$ and $v_i' = v_i - \mathbf{i} v_{i+1}$, for such i we obtain $\left[\begin{smallmatrix} A+BF_K \\ C+D F_K \end{smallmatrix}\right] v_i = \left[\begin{smallmatrix} v_i \, \mathfrak{Re}\{\lambda_i\} + v_{i+1} \, \mathfrak{Im}\{\lambda_i\} \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A+BF_K \\ C+D F_K \end{smallmatrix}\right] v_{i+1} = \left[\begin{smallmatrix} v_{i+1} \, \mathfrak{Re}\{\lambda_i\} - v_i \, \mathfrak{Im}\{\lambda_i\} \\ 0 \end{smallmatrix}\right]$. These two equations can be written together as

$$\begin{bmatrix} A + BF_K \\ C + D\,F_K \end{bmatrix} \begin{bmatrix} v_i & v_{i+1} \end{bmatrix} = \begin{bmatrix} v_i & v_{i+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{bmatrix}.$$

Thus, by defining $\Lambda_{i,i+1} = \left[\begin{smallmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{smallmatrix}\right]$ for all $i \in \{1, \dots, 2s-1\}$ and $\Lambda_i = \lambda_i$ for all $i \in \{2s+1, \dots, r\}$, we get

$$\begin{bmatrix} A + BF_K \\ C + D\,F_K \end{bmatrix} X_K = \begin{bmatrix} X_K \\ 0 \end{bmatrix} \Lambda,$$

where $\Lambda = \operatorname{diag}\{\Lambda_{1,2}, \Lambda_{3,4}, \dots, \Lambda_{2s-1,2s}, \Lambda_{2s+1}, \dots, \Lambda_r\}$. This equation says that (i) the columns of X_K form a basis for $\mathcal{R}^\star$; (ii) F_K is a friend of $\mathcal{R}^\star$; and (iii) the eigenvalues of $(A + B\,F_K)$ restricted to $\mathcal{R}^\star$ are the eigenvalues of Λ, i.e., the desired closed-loop eigenvalues $\mathcal{L}$.

It remains to be shown that this parameterization is exhaustive, i.e., the set of all friends of $\mathcal{R}^\star$ such that the eigenvalues of the closed loop restricted to $\mathcal{R}^\star$ are equal to $\mathcal{L}$ is parameterized in K as in (3.5). In other words, given $\mathcal{L}$ and a friend F of $\mathcal{R}^\star$ such that $(A + B\,F)\mathcal{R}^\star \subseteq \mathcal{R}^\star \subseteq \ker(C + D\,F)$ with $\sigma(A + B\,F \,|\, \mathcal{R}^\star) = \mathcal{L}$, we need to show that there exists K such that, building X_K and Y_K as in (3.3)–(3.4), there holds $F = Y_K X_K^\dagger$. First, notice that the set of friends F of $\mathcal{R}^\star$ such that $\sigma(A + B\,F \,|\, \mathcal{R}^\star) = \mathcal{L}$ is parameterized as the solutions of the linear equation $F\,R = -\Omega$, where Ω satisfies the linear equation $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] R = \left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right] \Lambda + \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] \Omega$ with a certain Λ such that $\sigma(\Lambda) = \mathcal{L}$ and where R is a basis matrix of $\mathcal{R}^\star$; see (7.2) in Appendix A. Let F be any of such friends of $\mathcal{R}^\star$. The associated matrix Λ is such that $\sigma(\Lambda) = \mathcal{L}$ satisfies $\left[\begin{smallmatrix} A+B\,F \\ C+D\,F \end{smallmatrix}\right] R = \left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right] \Lambda$. Consider a change of coordinates T that brings Λ into the Jordan real canonical form.[8] Let the blocks be ordered in such a way that the s complex conjugate pairs of eigenvalues are first. We can write

$$\begin{bmatrix} A + B\,F \\ C + D\,F \end{bmatrix} RT = \begin{bmatrix} R \\ 0 \end{bmatrix} T \underbrace{T^{-1} \Lambda T}_{\Lambda_J}, \tag{3.8}$$

where $\Lambda_J = \operatorname{diag}\{\Lambda_{1,2}, \Lambda_{3,4}, \dots, \Lambda_{2s-1,2s}, \Lambda_{2s+1}, \dots, \Lambda_r\}$ with $\Lambda_{i,i+1} = \left[\begin{smallmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{smallmatrix}\right]$ for all $i \in \{1, \dots, 2s-1\}$ and $\Lambda_i = \lambda_i$ for all $i \in \{2s+1, \dots, r\}$. Notice that RT is also a basis matrix for $\mathcal{R}^\star$. Let $X = RT$ and $Y = F\,RT$. We find

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} \Lambda_J. \tag{3.9}$$

[8] Since we are considering the case of distinct eigenvalues, all Jordan chains have unit length. The order of the Jordan mini-blocks associated with real and pairs of complex conjugate eigenvalues is one and two, respectively.

We can denote by $v_1, \dots, v_r$ the r columns of X and by $w_1, \dots, w_r$ the r columns of Y. Thus, there holds

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_i & v_{i+1} \\ w_i & w_{i+1} \end{bmatrix} = \begin{bmatrix} v_i & v_{i+1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{Re}\{\lambda_i\} & -\mathfrak{Im}\{\lambda_i\} \\ \mathfrak{Im}\{\lambda_i\} & \mathfrak{Re}\{\lambda_i\} \end{bmatrix},$$

where $i \leq 2\,s$ is odd and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} v_i \\ 0 \end{smallmatrix}\right] \lambda_i$ if $i > 2\,s$. By setting $v'_{i+1} = v_i + \mathrm{i}\, v_{i+1}$ and $v'_i = v_i - \mathrm{i}\, v_{i+1}$ with $i \leq 2\,s$ odd and $v'_i = v_i$ for $i > 2\,s$, and similarly for w'_{i+1} and w'_i, we find

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_i \\ w'_i \end{bmatrix} = \begin{bmatrix} v'_i \\ 0 \end{bmatrix} \lambda_i \quad \text{and} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_{i+1} \\ w'_{i+1} \end{bmatrix} = \begin{bmatrix} v'_{i+1} \\ 0 \end{bmatrix} \bar{\lambda}_i \tag{3.10}$$

for $i \in \{1, \dots, 2\,s\}$ with i odd, while for $i \in \{2\,s+1, \dots, r\}$ we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_i \\ w'_i \end{bmatrix} = \begin{bmatrix} v'_i \\ 0 \end{bmatrix} \lambda_i. \tag{3.11}$$

Hence, writing (3.10) and (3.11) together yields

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_1 & v'_2 & \dots & v'_r \\ w'_1 & w'_2 & \dots & w'_r \end{bmatrix} = \begin{bmatrix} \lambda_1\, v'_1 & \lambda_2\, v'_2 & \dots & \lambda_r\, v'_r \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

which implies that $\left[\begin{smallmatrix} v'_i \\ w'_i \end{smallmatrix}\right] \in \ker \left[\begin{smallmatrix} A-\lambda_i\, I & B \\ C & D \end{smallmatrix}\right]$ for each $i \in \{1, \dots, r\}$. Hence, a matrix K exists for which $X = X_K$ and $Y = Y_K$, where X_K and Y_K are given in (3.3)–(3.4). ∎

Example 3.1. Consider a quadruple (A, B, C, D), where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 4 \end{bmatrix}.$$

The only invariant zero of this system is $z = 0$. Using the standard algorithms of the geometric approach, it is easy to verify that $\mathcal{R}^\star$ is spanned by the first two canonical basis vectors of $\mathbb{R}^3$. Hence, $r = \dim \mathcal{R}^\star = 2$. Let us choose, for example, $\mathcal{L} = \{\lambda_1, \lambda_2\} = \{-2, -4\}$. Basis matrices for $\ker P_\Sigma(-2)$ and $\ker P_\Sigma(-4)$ are given, respectively, by $N_\Sigma(-2) = [5 \quad 4 \quad 0 \,|\, -10 \quad 0]^\top$ and $N_\Sigma(-4) = [7 \quad 8 \quad 0 \,|\, -28 \quad 0]^\top$. Thus, (3.1) becomes

$$M_K = \left[\begin{array}{c|c} 5 & 7 \\ 4 & 8 \\ 0 & 0 \\ \hline -10 & -28 \\ 0 & 0 \end{array}\right] K, \quad \text{where} \quad K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

By choosing, for example, $k_1 = k_2 = 1$, we find $X_K = \left[\begin{smallmatrix} 5 & 7 \\ 4 & 8 \\ 0 & 0 \end{smallmatrix}\right]$ and $Y_K = \left[\begin{smallmatrix} -10 & -28 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, as expected $\operatorname{im} X_K = \mathcal{R}^\star$, and $F_K = Y_K\, X_K^\dagger = \left[\begin{smallmatrix} 8/3 & -35/6 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is a friend of $\mathcal{R}^\star$ that delivers the desired closed-loop eigenstructure. Indeed, it can be immediately verified that $(A+B\,F_K)\,\mathcal{R}^\star \subseteq \mathcal{R}^\star \subseteq \ker(C+D\,F_K)$, and the eigenvalues of $(A+B\,F_K)$ restricted to $\mathcal{R}^\star$ are indeed $\{-2,-4\}$.

Example 3.2. Consider the following quadruple:

$$A = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

In this case, using the standard algorithms of the geometric approach, we see that $\mathcal{R}^\star$ is spanned by the first, third, and fourth canonical basis vectors of $\mathbb{R}^4$. Let $\mathcal{L} = \{\lambda_1, \lambda_2, \lambda_3\} = \{-1-\mathrm{i}, -1+\mathrm{i}, -2\}$. It is easy to see that with this choice of the closed-loop eigenvalues, (3.1) becomes

$$M_K = \left[\begin{array}{cc|cc|cc} 50 & 0 & 50 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 18\,\mathrm{i} & -2 & -18\,\mathrm{i} & -2 & 1 & 1 \\ 9-9\,\mathrm{i} & 1+\mathrm{i} & 9+9\,\mathrm{i} & 1-\mathrm{i} & -1 & -1 \\ \hline 84 & \mathrm{i} & 84 & -\mathrm{i} & 11 & -1 \\ 23-23\,\mathrm{i} & -3-3\,\mathrm{i} & 23+23\,\mathrm{i} & -3+3\,\mathrm{i} & 3 & 3 \end{array}\right] \left[\begin{array}{c|c|c} k_{11} & 0 & 0 \\ k_{12} & 0 & 0 \\ \hline 0 & \overline{k}_{11} & 0 \\ 0 & \overline{k}_{12} & 0 \\ \hline 0 & 0 & k_{31} \\ 0 & 0 & k_{32} \end{array}\right].$$

Choosing, for example, $k_{11} = 0$, $k_{12} = \mathrm{i}$, $k_{31} = 1$, and $k_{32} = 0$, we find $v_1' = [0 \quad 0 \quad -2\,\mathrm{i} \quad -1+\mathrm{i}]^\top$, $v_2' = [0 \quad 0 \quad 2\,\mathrm{i} \quad -1-\mathrm{i}]^\top$, $v_3' = [8 \quad 0 \quad 1 \quad -1]^\top$, $w_1' = [-1 \quad 3-3\,\mathrm{i}]^\top$, $w_2' = [-1 \quad 3+3\,\mathrm{i}]^\top$, $w_3' = [11 \quad 3]^\top$. We now compute

$$v_1 = \frac{1}{2}(v_1' + v_2') = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \qquad v_2 = \frac{1}{2\,\mathrm{i}}(v_2' - v_1') = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \qquad v_3 = v_3' = \begin{bmatrix} 8 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$w_1 = \frac{1}{2}(w_1' + w_2') = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \qquad w_2 = \frac{1}{2\,\mathrm{i}}(w_2' - w_1') = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \qquad w_3 = w_3' = \begin{bmatrix} 11 \\ 3 \end{bmatrix}.$$

Hence, $X_K = [v_1 \quad v_2 \quad v_3]$ and $Y_K = [w_1 \quad w_2 \quad w_3]$. We have $\operatorname{im} X_K = \mathcal{R}^\star$.Thus, by defining $\Lambda_{1,2} = \left[\begin{smallmatrix} \mathfrak{Re}\{\lambda_1\} & -\mathfrak{Im}\{\lambda_1\} \\ \mathfrak{Im}\{\lambda_1\} & \mathfrak{Re}\{\lambda_1\} \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ and $\Lambda_3 = \{\lambda_3\} = \{-2\}$, and with $F_K = Y_K\, X_K^\dagger = \left[\begin{smallmatrix} \frac{23}{16} & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & -3 \end{smallmatrix}\right]$ which in fact satisfies $\left[\begin{smallmatrix} A+B\,F_K \\ C+D\,F_K \end{smallmatrix}\right]\mathcal{R}^\star \subseteq \mathcal{R}^\star \oplus \{0\}$ with $\sigma(A + B\,F_K \,|\, \mathcal{R}^\star) = \{-1-\mathrm{i}, -1+\mathrm{i}, -2\}$, we find $\left[\begin{smallmatrix} A+B\,F_K \\ C+D\,F_K \end{smallmatrix}\right] X_K = \left[\begin{smallmatrix} X_K \\ 0 \end{smallmatrix}\right] \operatorname{diag}\{\Lambda_{1,2}, \Lambda_3\}$.

Remark 3.1. In Theorem 3.1 we assumed that $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ does not contain invariant zeros of the system. This requirement in inherited from [13, Proposition 4]. This fact seems to suggest that the parameterization offered in Theorem 3.1 is less complete than the one which follows from the classic approach, which is given in Appendix A, since the latter is not restricted to only delivering the friends such that

the eigenvalues of the closed loop restricted to $\mathcal{R}^\star$ are not coincident with invariant zeros. On the other hand, in the second part of the proof of Theorem 3.1 we showed that the parameterization (3.5) is exhaustive—and to prove that point we did not need to use the assumption on the absence of invariant zeros from within $\mathcal{L}$. Thus, for every invariant zero z_i of Σ, in the null-space of $P_\Sigma(z_i)$ there must exist at least a direction which is common to $\mathcal{R}^\star$, or else we would not be able to assign the corresponding zero as eigenvalue of the closed loop restricted to $\mathcal{R}^\star$. Thus, we have the following.

COROLLARY 3.2. *Let the set of invariant zeros of* Σ *be denoted by* $\mathcal{Z} = \{z_1, \dots, z_t\}$. *We have*

$$\mathcal{R}^\star \cap \operatorname{im} \overline{\pi}\{N_\Sigma(z_i)\} \neq \{0\} \qquad \forall\, i \in \{1, \dots, t\}. \tag{3.12}$$

A direct consequence of Corollary 3.2 is that if $\mathcal{L}$ contains one or more invariant zeros, for example, $\lambda_i = z \in \mathcal{Z}$, (3.1) becomes

$$M_K = \left[N_\Sigma(\lambda_1) \;\middle|\; \dots \;\middle|\; N_\Sigma(\lambda_i) \;\middle|\; \dots \;\middle|\; N_\Sigma(\lambda_r) \right] \operatorname{diag}\{k_1, \dots, k_i, \dots, k_r\}.$$

In view of (3.12) there exists a value $k_i \in \mathbb{C}^{d_z}$ such that for almost all choices of k_j, $j \in \{1, \dots, r\} \backslash \{i\}$, the rank of X_K is equal to r, $\operatorname{im} X_K = \mathcal{R}^\star$, and $F_K = Y_K\, X_K^\dagger$ is a friend of $\mathcal{R}^\star$ such that $\sigma(A + B\, F_K \,|\, \mathcal{R}^\star) = \{\lambda_1, \dots, \lambda_i, \dots, \lambda_r\}$. More specifically, we can chose a basis $N_\Sigma(z)$ to be partitioned as $[N'_\Sigma(z) \quad N''_\Sigma(z)]$, where $N'_\Sigma(z)$ is a basis for $\mathcal{R}^\star \cap \operatorname{im} \overline{\pi}\{N_\Sigma(z)\}$, and $k_i = \left[\begin{smallmatrix} k'_i \\ k''_i \end{smallmatrix}\right]$ is partitioned accordingly. Hence, there must hold $k''_i = 0$.

Example 3.3. Consider the system in Example 3.1. As mentioned, this system has an invariant zero at $z = 0$. We want to find the friend that assigns $\mathcal{L} = \{-2, 0\}$ as the eigenvalues of the closed loop restricted to $\mathcal{R}^\star$. Thus, we compute a basis of the null-space $P_\Sigma(0)$, which is spanned by the basis matrix $N_\Sigma(0) = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right]^\top$. Therefore, $\mathcal{R}^\star \cap \operatorname{im} \overline{\pi}\{N_\Sigma(0)\} = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$. In (3.1) we need to choose a value of the parameter matrix K that selects precisely this column vector. In the present case, we need

$$M_K = \left[\begin{array}{c|cc} 5 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \\ \hline -10 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] K, \quad \text{where} \quad K = \left[\begin{array}{c|c} k_1 & 0 \\ \hline 0 & k'_2 \\ 0 & k''_2 \end{array}\right],$$

with $k''_2 = 0$. Choosing, for example, $k_1 = 1$ and $k'_2 = 2$ yields $X_K = \left[\begin{smallmatrix} 5 & 2 \\ 4 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $Y_K = \left[\begin{smallmatrix} -10 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, which lead to $F_K = Y_K\, X_K^\dagger = \left[\begin{smallmatrix} 0 & -5/2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. Thus, $\operatorname{im} X_K = \mathcal{R}^\star$ and $\sigma(A + B\, F_K \,|\, \mathcal{R}^\star) = \{0, -2\}$.

Remark 3.2. The method presented in Theorem 3.1 can also be generalized to a set of closed-loop eigenvalues $\mathcal{L}$ with arbitrary multiplicity. The details on the case of repeated closed-loop eigenvalues will not be provided in this paper. However, we can notice that from Theorem 3.1 it follows that $d = \dim(\ker P_\Sigma(\lambda))$ represents the maximum number of Jordan mini-blocks of size 1 that can be obtained for a repeated closed-loop eigenvalue λ. Indeed, suppose $\lambda_1, \dots, \lambda_\nu$ are such that λ_1 has

multiplicity d and $d+\nu = r$, which means that the multiplicity of all other eigenvalues is one. Choosing k_1 to be a real $d \times d$ matrix (assuming for simplicity that λ_1 is real), while $k_2, \ldots, k_\nu$ are as described by Theorem 3.1, we can compute M_K, X_K, and Y_K using (3.1), (3.3), and (3.4), and $F_K = Y_K X_K^\dagger$ guarantees that $\sigma(A + B F \,|\, \mathcal{R}^\star) = \{\lambda_1, \ldots, \lambda_\nu\}$, where λ_1 has multiplicity equal to d. Notice that it may not be possible to assign a further eigenvalue with the same multiplicity, unless Jordan mini-blocks of order greater than one are allowed to be assigned in the closed-loop map. Indeed, it may very well happen that $\ker P_\Sigma(\lambda_1) \cap \ker P_\Sigma(\lambda_2) \neq \{0\}$. If this is the case, the largest multiplicity that we can assign to λ_2 with Jordan mini-blocks of order 1 is equal to $\dim(\ker P_\Sigma(\lambda_2)/(\ker P_\Sigma(\lambda_1) \cap \ker P_\Sigma(\lambda_2)))$, and so forth.

3.2. Algorithm for the computation of a basis matrix for $\mathcal{R}^\star$. The following algorithm provides a method for the computation of a basis of subspace $\mathcal{R}^\star$ of the system $\Sigma = (A, B, C, D)$ and also produces a friend F of $\mathcal{R}^\star$. We assume that $r = \dim \mathcal{R}^\star$ is not known a priori. For simplicity, assume $\mathcal{L} = \{\lambda_1, \ldots, \lambda_n\}$ to be any set of n distinct real numbers, all different from the invariant zeros of the system.

ALGORITHM 3.1.

1. Determine $e = \operatorname{rank} \overline{\pi}\{N_\Sigma(\lambda_1)\}$. If $e = 0$, then $\mathcal{R}^\star = \{0\}$ and $r = 0$. If $e \geq 1$, then continue as below.
2. Select a nonzero coefficient vector $k_1 \in \mathbb{R}^d$ and compute $s_1 = N_\Sigma(\lambda_1)\, k_1$, $v_1 = \overline{\pi}\{s_1\}$, $V_1 = v_1$, $w_1 = \underline{\pi}\{s_1\}$, and $W_1 = w_1$; then test the condition
$$1 < \operatorname{rank}[V_1 \mid \overline{\pi}\{N_\Sigma(\lambda_2)\}]. \tag{3.13}$$
If condition (3.13) fails, then set $i_{max} := 1$ and go to step 6.
3. While $2 \leq i \leq n$, successively obtain nonzero coefficient vectors $k_i \in \mathbb{R}^d$ such that
$$\begin{aligned} &s_i = N_\Sigma(\lambda_i) k_i, \quad v_i = \overline{\pi}\{s_i\}, \quad w_i = \underline{\pi}\{s_i\}, \\ &V_i = [V_{i-1} \,|\, v_i], \quad W_i = [W_{i-1} \,|\, w_i], \end{aligned} \tag{3.14}$$
and test the condition
$$i = \operatorname{rank} V_i. \tag{3.15}$$
If this condition fails, choose a different k_i to satisfy (3.15).
4. Test the condition
$$i < \operatorname{rank}[V_i \mid \overline{\pi}\{N_\Sigma(\lambda_{i+1})\}]. \tag{3.16}$$
5. For each i such that condition (3.16) holds, select a new coefficient vector k_{i+1}, evaluate (3.14), and check that V_{i+1} satisfies (3.15). Then test (3.16).
6. Let i_{max} be the first i such that (3.16) is false. Then, $\operatorname{rank} V_{i_{max}}$ is maximal and $r = i_{max}$. Denote $X := V_{i_{max}}$ and $Y := W_{i_{max}}$, and define an $m \times n$ real gain matrix F as $F = Y X^\dagger$. Thus, $\mathcal{R}^\star = \operatorname{im} X$ and F is a friend of X such that $\sigma(A + B F \,|\, \mathcal{R}^\star) = \{\lambda_1, \ldots, \lambda_r\}$.

3.3. Assignment of the complete eigenstructure of $\mathcal{R}^\star$. In the previous section, we showed how to construct a friend F of the subspace $\mathcal{R}^\star$ that arbitrarily assigns all the eigenvalues of the closed loop restricted to $\mathcal{R}^\star$. However, we also know

that the spectrum induced by the map $A + B\,F$ on the quotient space $(\mathcal{R}_0 + \mathcal{R}^\star)/\mathcal{R}^\star$ (where we recall that $\mathcal{R}_0 \stackrel{\text{def}}{=} \langle A, \operatorname{im} B\rangle$ is the classic reachable subspace from the origin) is assignable using a friend F. Since $\mathcal{R}^\star \subseteq \mathcal{R}_0$, these eigenvalues coincide with those induced by the map $A + B\,F$ on the quotient space $\mathcal{R}_0/\mathcal{R}^\star$; see, e.g., [22]. The following result shows how Theorem 3.1 can be adapted to this case.

THEOREM 3.3 (parameterization of friends of $\mathcal{R}^\star$ with complete spectrum assignment). *Let $r = \dim \mathcal{R}^\star$ and $r_0 = \dim \mathcal{R}_0$. Let $\mathcal{L}_{\text{in}} = \{\lambda_1, \dots, \lambda_r\}$ be s_{in}-conformably ordered with elements all different from the invariant zeros, and let $\mathcal{L}_{\text{out}} = \{\mu_{r+1}, \dots, \mu_{r_0}\}$ be s_{out}-conformably ordered with elements all different from the uncontrollable eigenvalues of the pair (A, B) with $\mathcal{L}_{\text{in}} \cap \mathcal{L}_{\text{out}} = \emptyset$. Let $K \stackrel{def}{=} \operatorname{diag}\{k_1, \dots, k_r\}$ be defined as in Theorem 3.1 for $\mathcal{L} = \mathcal{L}_{\text{in}}$. Moreover, let $K' \stackrel{def}{=} \operatorname{diag}\{k'_{r+1}, \dots, k'_{r_0}\}$, where $k'_i \in \mathbb{C}^m$ for each $i \in \{r+1, \dots, r+2\,s_{\text{out}}\}$, and for all odd $i - r \in \{1, \dots, 2\,s_{\text{out}} - 1\}$, we have $\bar{k'}_i = k'_{i+1}$, whereas $k'_i \in \mathbb{R}^m$ for $i \in \{r + 2\,s_{\text{out}} + 1, \dots, r_0\}$. Let $M_{K,K'}$ be an $(n+m) \times r_0$ complex matrix given by*

$$M_{K,K'} = \left[N_\Sigma(\lambda_1) \,\middle|\, \dots \,\middle|\, N_\Sigma(\lambda_r) \,\middle|\, S_\Sigma(\mu_{r+1}) \,\middle|\, \dots \,\middle|\, S_\Sigma(\mu_{r_0})\right] \operatorname{diag}\{K, K'\}, \tag{3.17}$$

where $S_\Sigma(\mu)$ represents a basis matrix for $\ker [A - \mu\, I_n \;\; B\,]$, and let $m_{K,K',i}$ be defined as

$$m_{K,K',i} = \begin{cases} \mathfrak{Re}\{M^i_{K,K'}\} & \text{if } i \le 2\,s_{\text{in}} \text{ is odd or if } i - r \in \{1, \dots, 2\,s_{\text{out}}\} \text{ is odd,} \\ \mathfrak{Im}\{M^i_{K,K'}\} & \text{if } i \le 2\,s_{\text{in}} \text{ is even or if } i - r \in \{1, \dots, 2\,s_{\text{out}}\} \text{ is even,} \\ M^i_{K,K'} & \text{if } i \in \{2\,s_{\text{in}} + 1, \dots, r\} \cup \{r + 2\,s_{\text{out}} + 1, \dots, r_0\}. \end{cases}$$

Let $X_{K,K'} = \overline{\pi}\{[m_{K,K',1} \;\; \dots \;\; m_{K,K',r_0}]\}$ and $Y_{K,K'} = \underline{\pi}\{[m_{K,K',1} \;\; \dots \;\; m_{K,K',r_0}]\}$.

For almost every choice of K and K' we have $\operatorname{rank} \overline{\pi}\{[m_{K,K',1} \;\; \dots \;\; m_{K,K',r}]\} = r$ and $\operatorname{rank} X_{K,K'} = r_0$. Moreover, the set of all friends of $\mathcal{R}^\star$ such that $\sigma(A + B\,F \,|\, \mathcal{R}^\star) = \mathcal{L}_{\text{in}}$ and $\sigma(A + B\,F \,|\, \mathcal{R}_0/\mathcal{R}^\star) = \mathcal{L}_{\text{out}}$ is parameterized in K and K' as

$$F_{K,K'} = Y_{K,K'}\, X^\dagger_{K,K'}, \tag{3.18}$$

where K and K' are such that $\operatorname{rank} X_{K,K'} = r_0$ (and therefore, for such K and K', the matrix $X_{K,K'}$ represents a basis for $\mathcal{R}_0$ adapted to $\mathcal{R}^\star$).

Proof. First, notice that when $\mu \in \mathbb{C}$ is not an uncontrollable eigenvalue, the dimension of the null-space of $S_\Sigma(\mu)$ is equal to m (while if μ is uncontrollable, such dimension is strictly greater than m). The argument in the proof of Theorem 3.1 shows that almost every choice of K and K' guarantees that the rank of $X_{K,K'}$ is r_0; see also [2, Lemma 2.4]. Thus, (3.18) is a solution of $F_{K,K'}\, X_{K,K'} = Y_{K,K'}$. Let K and K' be such that $\operatorname{rank} X_{K,K'} = r_0$, and let $M_{K,K'} = \left[\begin{smallmatrix} v'_1 & \dots & v'_r & v'_{r+1} & \dots & v'_{r_0} \\ w'_1 & \dots & w'_r & w'_{r+1} & \dots & w'_{r_0} \end{smallmatrix}\right]$, where for each $i \in \{1, \dots, r\}$, (3.6)–(3.7) hold, while for each $i \in \{r+1, \dots, r_0\}$ there holds $(A - \mu_i\, I_n)\, v'_i + B\, w'_i = 0$. Consider again the matrix $U = \frac{1}{2}\left[\begin{smallmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{smallmatrix}\right]$. For each odd $i \le 2\,s_{\text{in}}$ and if $i - r \in \{1, \dots, 2\,s_{\text{out}}\}$ is odd, let $[v_i \;\; v_{i+1}] = [v'_i \;\; v'_{i+1}]\, U$ and $[w_i \;\; w_{i+1}] = [w'_i \;\; w'_{i+1}]\, U$. Then, we have $F_{K,K'}[v_1 \;\; \dots \;\; v_r \,|\, v_{r+1} \;\; \dots \, v_{r_0}] = [w_1 \;\; \dots \;\; w_r \,|\, w_{r+1} \;\; \dots \, w_{r_0}]$, which implies $\left[\begin{smallmatrix} A + B\,F_{K,K'} \\ C + D\,F_{K,K'} \end{smallmatrix}\right] v_i = \left[\begin{smallmatrix} v_i \\ 0 \end{smallmatrix}\right] \lambda_i$ for all $i \in \{1, \dots, r\}$ and $(A + B\,F_{K,K'})\, v_i = \mu_i\, v_i$ for all $i \in \{r+1, \dots, r_0\}$. Thus

$$\begin{bmatrix} A + B\,F_{K,K'} \\ C + D\,F_{K,K'} \end{bmatrix} \left[v_1 \ \dots \ v_r \ \middle| \ v_{r+1} \ \dots \ v_{r_0} \right]$$

$$= \left[\begin{array}{ccc|ccc} v_1 & \dots & v_r & v_{r+1} & \dots & v_{r_0} \\ 0 & \dots & 0 & \star & \dots & \star \end{array}\right] \left[\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_r & & & \\ \hline & & & \mu_{r+1} & & \\ & 0 & & & \ddots & \\ & & & & & \mu_{r_0} \end{array}\right]$$

(where the diagonal structure of the matrix in the right-hand side assumes for simplicity that all the λ_i and μ_i are real). We prove that the parameterization (3.18)—for K and K' such that $\operatorname{rank} X_{K,K'} = r_0$—is exhaustive. Let F be a friend of $\mathcal{R}^\star$ such that $\sigma(A + B\,F \,|\, \mathcal{R}^\star) = \mathcal{L}_{\text{in}}$ and $\sigma(A + B\,F \,|\, \mathcal{R}_0/\mathcal{R}^\star) = \mathcal{L}_{\text{out}}$. Consider a basis matrix $R_0 = [\,R \quad R_c\,]$ of $\mathcal{R}_0$ adapted to $\mathcal{R}^\star$, i.e., such that R is a basis for $\mathcal{R}^\star$. Since $\mathcal{R}_0$ is $(A + B\,F)$-invariant, we can write

$$\begin{bmatrix} A + B\,F \\ C + D\,F \end{bmatrix} \begin{bmatrix} R & R_c \end{bmatrix} = \begin{bmatrix} R & R_c \\ 0 & \star \end{bmatrix} \begin{bmatrix} \Lambda_{\text{in}} & \Gamma \\ 0 & \Lambda_{\text{out}} \end{bmatrix} \tag{3.19}$$

for a certain matrix Γ, where $\sigma(\Lambda_{\text{in}}) = \mathcal{L}_{\text{in}}$ and $\sigma(\Lambda_{\text{out}}) = \mathcal{L}_{\text{out}}$. Let us consider a change of coordinates $T = \left[\begin{smallmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{smallmatrix}\right]$ partitioned conformably with $R_0 = [R \quad R_c]$ in which T_{11} is such that $T_{11}^{-1}\Lambda_{\text{in}}T_{11} = \Lambda_{\text{in}}^J$, T_{22} is such that $T_{22}^{-1}\Lambda_{\text{out}}T_{22} = \Lambda_{\text{out}}^J$, where Λ_{in}^J and Λ_{out}^J are in the real Jordan form with the same spectrum as Λ_{in} and Λ_{out}, respectively, and T_{12} satisfies the Lyapunov equation

$$\Lambda_{\text{in}}\,T_{12} - T_{12}\Lambda_{\text{out}}^J = -\Gamma\,T_{22},$$

which always admits a unique solution T_{12} since $\mathcal{L}_{\text{in}}$ and $\mathcal{L}_{\text{out}}$ are disjoint. It follows that

$$T^{-1} \begin{bmatrix} \Lambda_{\text{in}} & \star \\ 0 & \Lambda_{\text{out}} \end{bmatrix} T = \begin{bmatrix} \Lambda_{\text{in}}^J & 0 \\ 0 & \Lambda_{\text{out}}^J \end{bmatrix}.$$

In view of the special structure of T, the matrix $X = [R \quad R_c]\,T = [R\,T_{11} \quad \star]$ is still a basis matrix of $\mathcal{R}_0$ adapted to $\mathcal{R}^\star$. Defining $Y = F\,[R \quad R_c]\,T$, we can therefore rewrite (3.19) as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{bmatrix} = \begin{bmatrix} X_1\,\Lambda_{\text{in}}^J & X_2\,\Lambda_{\text{out}}^J \\ 0 & \star \end{bmatrix}, \tag{3.20}$$

where $X = [X_1 \quad X_2]$ and $Y = [Y_1 \quad Y_2]$ have been partitioned conformably with R_0. We denote by $v_1, \dots, v_r, v_{r+1}, \dots, v_{r_0}$ the r_0 columns of X and by $w_1, \dots, w_r$, $w_{r+1}, \dots, w_{r_0}$ the r_0 columns of Y. As already seen in Theorem 3.1, for $i \le r$ we can redefine vectors v_i' and w_i' such that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \left[\begin{smallmatrix} v_i' \\ w_i' \end{smallmatrix}\right] = \left[\begin{smallmatrix} \lambda_i\,v_i' \\ 0 \end{smallmatrix}\right]$, which implies that

$\left[\begin{smallmatrix} v_i' \\ w_i' \end{smallmatrix}\right] \in \ker \left[\begin{smallmatrix} A-\lambda_i I & B \\ C & D \end{smallmatrix}\right]$ for each $i \in \{1, \dots, r\}$. In the same way, from (3.20) we can define v_i' and w_i' for $i > r$ such that

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} v_i' \\ w_i' \end{bmatrix} = v_i' \, \mu_i \quad \forall i \in \{r+1, \dots, r_0\}.$$

Thus, $\left[\begin{smallmatrix} v_i' \\ w_i' \end{smallmatrix}\right] \in \ker S_\Sigma(\mu_i)$. This means that K and K' exist such that $X = \overline{\pi}\{M_{K,K'}\}$ and $Y = \underline{\pi}\{M_{K,K'}\}$, as required. This also implies that if $\mathcal{R}_0 \supset \mathcal{R}^\star$, for all $\mu \in \mathbb{C}$ there holds $\operatorname{im} \overline{\pi}\{S_\Sigma(\mu)\} \cap \mathcal{R}^\star \neq \{0\}$. ∎

Example 3.4. Consider the system in Example 3.1. Since the pair (A, B) is reachable, we can compute a friend F of $\mathcal{R}^\star$ by assigning a further eigenvalue of $(A + B\,F)$ which corresponds to the map induced by $A + B\,F$ on the quotient space $\mathcal{R}_0/\mathcal{R}^\star = \mathcal{X}/\mathcal{R}^\star$. Assume that $\mathcal{L}_{\text{in}} = \mathcal{L} = \{-2, -4\}$ and $\mathcal{L}_{\text{out}} = \{-6\}$. We have already computed $N_\Sigma(-2) = [5 \quad 4 \quad 0 \quad | \quad -10 \quad 0]^\top$ and $N_\Sigma(-4) = [7 \quad 8 \quad 0 \quad | \quad -28 \quad 0]^\top$. A basis matrix of $\ker[A - (-6)\, I_3 \quad B]$ is given, for example, by $S_\Sigma(-6) = \left[\begin{smallmatrix} 3 & 4 & 0 & | & -18 & 0 \\ 0 & 0 & -1 & | & 0 & 2 \end{smallmatrix}\right]^\top$. Thus, choosing for example $K = \operatorname{diag}\{k_1, k_2\} = \operatorname{diag}\{1, 1\}$ and $K' = k_3' = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ we get $X_{K,K'} = \operatorname{diag}\left\{\left[\begin{smallmatrix} 5 & 7 \\ 4 & 8 \end{smallmatrix}\right], -1\right\}$ and $Y_{K,K'} = \left[\begin{smallmatrix} -10 & -28 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$. Then, with $F_{K,K'} = Y_{K,K'}\, X_{K,K'}^\dagger = Y_{K,K'}\, X_{K,K'}^{-1} = \left[\begin{smallmatrix} 8/3 & -35/6 & 0 \\ 0 & 0 & -2 \end{smallmatrix}\right]$, we find $(A + B\,F_{K,K'})\,\mathcal{R}^\star \subseteq \mathcal{R}^\star \subseteq \ker(C + D\,F_{K,K'})$; moreover, the eigenvalues of $(A + B\,F_{K,K'})$ restricted to $\mathcal{R}^\star$ are $\{-2, -4\}$, while the eigenvalue induced by $(A + B\,F_{K,K'})$ on $\mathcal{R}_0/\mathcal{R}^\star$ is -6.

4. Computation of $\mathcal{V}^\star$ and the associated friends. We now address the problem of the computation of the largest output-nulling subspace $\mathcal{V}^\star$ of the system Σ and the computation of the friends that assign any desired eigenstructure.

THEOREM 4.1 (parameterization of the friends of $\mathcal{V}^\star$). *Let $r = \dim \mathcal{R}^\star$. Let all the invariant zeros of the system be distinct. Let $\mathcal{Z} = \{z_{r+1}, z_{r+2}, \dots, z_{r+t}\}$ be the s_z-conformably ordered set of invariant zeros of Σ. Let $\mathcal{L} = \{\lambda_1, \dots, \lambda_r\}$ be s-conformably ordered such that $\mathcal{L} \cap \mathcal{Z} = \emptyset$. Let $K \stackrel{def}{=} \operatorname{diag}\{k_1, \dots, k_r\}$ be defined as in Theorem* 3.1. *Let $H \stackrel{def}{=} \operatorname{diag}\{h_{r+1}, \dots, h_{r+t}\}$, where $h_i \in \mathbb{C}^{\dim(\ker P_\Sigma(z_i))}$ for each $i \in \{r+1, \dots, r+2\,s_z\}$, and for all odd $i - r \in \{1, \dots, 2\,s_z - 1\}$, we have $\bar{h}_i = h_{i+1}$, whereas $h_i \in \mathbb{R}^{\dim(\ker P_\Sigma(z_i))}$ for $i \in \{r+2\,s_z+1, \dots, r+t\}$. Let $M_{K,H}$ be a complex matrix given by*

$$M_{K,H} = \left[N_\Sigma(\lambda_1) \;\middle|\; \dots \;\middle|\; N_\Sigma(\lambda_r) \;\middle|\; N_\Sigma(z_{r+1}) \;\middle|\; N_\Sigma(z_{r+2}) \;\middle|\; \dots \;\middle|\; N_\Sigma(z_{r+t})\right] \operatorname{diag}\{K, H\}$$

and let for all $i \in \{1, \dots, r+t\}$

$$m_{K,H,i} = \begin{cases} \mathfrak{Re}\{M_{K,H}^i\} & \text{if } i \leq 2\,s \text{ is odd or if } i - r \in \{1, \dots, 2\,s_z\} \text{ is odd}, \\ \mathfrak{Im}\{M_{K,H}^i\} & \text{if } i \leq 2\,s \text{ is even or if } i - r \in \{1, \dots, 2\,s_z\} \text{ is even}, \\ M_{K,H}^i & \text{if } i \in \{2\,s+1, \dots, r\} \cup \{r+2\,s_z+1, \dots, r+t\}. \end{cases}$$

Finally, let

$$X_{K,H} = \overline{\pi}\,\{[m_{K,H,1} \;\; \dots \;\; m_{K,r} \;\; m_{K,H,r+1} \;\; \dots \;\; m_{K,H,r+t}]\}, \tag{4.1}$$

$$Y_{K,H} = \underline{\pi}\,\{[m_{K,H,1} \;\; \dots \;\; m_{K,H,r} \;\; m_{K,H,r+1} \;\; \dots \;\; m_{K,H,r+t}]\}. \tag{4.2}$$

For almost every choice of the parameter matrices $K = \operatorname{diag}\{k_1, \dots, k_r\}$ *and* $H = \operatorname{diag}\{h_{r+1}, \dots, h_{r+t}\}$ *we have* $\operatorname{rank} X_{K,H} = r + t$. *Moreover, the set of all friends of* $\mathcal{V}^\star$ *such that* $\sigma(A + B\,F \,|\, \mathcal{V}^\star) = \mathcal{L} \cup \mathcal{Z}$ *is parameterized in* K *and* H *as*

$$F_{K,H} = Y_{K,H}\, X_{K,H}^{\dagger}, \tag{4.3}$$

where K, H *are such that* $\operatorname{rank} X_{K,H} = r + t$ *(and therefore, for such* K *and* H*, the matrix* $X_{K,H}$ *represents a basis for* $\mathcal{V}^\star$ *adapted to* $\mathcal{R}^\star$*).*

Proof. By partitioning $[N_\Sigma(\lambda_1)| \dots |N_\Sigma(\lambda_r)|N_\Sigma(z_{r+1})| \dots |N_\Sigma(z_{r+t})]$ as $\left[\begin{smallmatrix}\Phi_\Sigma\\ \Psi_\Sigma\end{smallmatrix}\right]$ having n and m rows, respectively, then $\operatorname{rank}\Phi_\Sigma = r + t$ by virtue of [13, Proposition 5]. Thus, using the same argument employed in the proof of Theorem 3.1 with the obvious modifications, we see that for almost every choice of the parameter matrices K and H there holds $\operatorname{rank}\overline{\pi}\{M_{K,H}\} = r + t$. For such matrices H and K we can partition $M_{K,H}$ as $M_{K,H} = \left[\begin{smallmatrix} v'_1 & \dots & v'_r & v'_{r+1} & \dots & v'_{r+t}\\ w'_1 & \dots & w'_r & w'_{r+1} & \dots & w'_{r+t}\end{smallmatrix}\right]$, in which (3.6)–(3.7) hold for $i \in \{1, \dots, r\}$, while for $i \in \{r + 1, \dots, r + t\}$ there hold

$$(A - z_i\, I_n)\, v'_i + B\, w'_i = 0, \tag{4.4}$$

$$C\, v'_i + D\, w'_i = 0. \tag{4.5}$$

Thus, real-valued vectors v_i and w_i can be defined in the way indicated in the proof of Theorem 3.1 using $U = \frac{1}{2}\left[\begin{smallmatrix}1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{smallmatrix}\right]$ whenever λ_i, λ_{i+1} or z_i, z_{i+1} are complex conjugate pairs, so as to obtain $X_{K,H} = [v_1\ \dots\ v_{r+t}]$ and $Y_{K,H} = [w_1\ \dots\ w_{r+t}]$. Thus, defining $\Lambda \stackrel{\text{def}}{=} \operatorname{diag}\{\Lambda_{1,2}, \dots, \Lambda_{2\,s-1,2\,s}, \Lambda_{2\,s+1}, \dots, \Lambda_r, \Lambda_{r+1,r+2}, \dots, \Lambda_{r+2\,s_z-1,r+2\,s_z}, \Lambda_{r+2\,s_z+1}, \dots, \Lambda_{r+t}\}$, we get $\left[\begin{smallmatrix}A+B\,F_{K,H}\\ C+D\,F_{K,H}\end{smallmatrix}\right] X_{K,H} = \left[\begin{smallmatrix}X_{K,H}\\ 0\end{smallmatrix}\right]\Lambda$, which proves the result. In order to prove that the parameterization is exhaustive, consider a basis matrix V of $\mathcal{V}^\star$ adapted to $\mathcal{R}^\star$, so that it can be written as $V = [R \quad V_c]$, where R is a basis for $\mathcal{R}^\star$ for a certain V_c. Thus, the set of friends of $\mathcal{V}^\star$ such that $\sigma(A + B\,F|\mathcal{R}^\star) = \mathcal{L}$ and $\sigma(A + B\,F|\mathcal{V}^\star) = \mathcal{L} \cup \mathcal{Z}$ is parameterized by $F\,[R \quad V_c] = -[\Omega_1 \quad \Omega_2]$, where $[\Omega_1 \quad \Omega_2]$ satisfies $\left[\begin{smallmatrix}A\\ C\end{smallmatrix}\right][R \quad V_c] = \left[\begin{smallmatrix}R & V_c\\ 0 & 0\end{smallmatrix}\right]\Lambda + \left[\begin{smallmatrix}B\\ D\end{smallmatrix}\right][\Omega_1 \quad \Omega_2]$ with a certain Λ such that $\sigma(\Lambda) = \mathcal{L} \cup \mathcal{Z}$, and we can find an invertible matrix T such that $T^{-1}\,\Lambda\,T = \operatorname{diag}\{\Lambda_{\mathcal{L}}, \Lambda_{\mathcal{Z}}\}$, where both $\Lambda_{\mathcal{L}}$ and $\Lambda_{\mathcal{Z}}$ are in the real Jordan canonical form and $\sigma(A + B\,F|\mathcal{R}^\star) = \sigma(\Lambda_{\mathcal{L}}) = \mathcal{L}$ and $\sigma(A + B\,F|\mathcal{V}^\star/\mathcal{R}^\star) = \sigma(\Lambda_{\mathcal{Z}}) = \mathcal{Z}$. The rest of the proof carries over with obvious modifications from that of Theorem 3.1. ∎

Example 4.1. Consider again the system in Example 3.1. We want to compute a basis for $\mathcal{V}^\star$ and a friend of $\mathcal{V}^\star$ such that $\sigma(A + B\,F \,|\, \mathcal{R}^\star) = \{-2, -4\}$. Since this system has an invariant zero at the origin, this task can be accomplished with a friend such that $\sigma(A + B\,F \,|\, \mathcal{V}^\star) = \{-2, -4, 0\}$. We have already computed $N_\Sigma(-2) = [5 \quad 4 \quad 0 \quad | \quad -10 \quad 0]^\top$, $N_\Sigma(-4) = [7 \quad 8 \quad 0 \quad | \quad -28 \quad 0]^\top$ and $N_\Sigma(0) = \left[\begin{smallmatrix}1&0&0&1&0&0\\ 0&0&1&1&0&0\end{smallmatrix}\right]^\top$. Let

$$M_{K,H} = \left[\begin{array}{cc|cc} 5 & 7 & 1 & 0\\ 4 & 8 & 0 & 0\\ 0 & 0 & 0 & 1\\ \hline -10 & -28 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \left[\begin{array}{c|c|c} k_1 & 0 & 0\\ \hline 0 & k_2 & 0\\ \hline 0 & 0 & h_{31}\\ 0 & 0 & h_{32}\end{array}\right].$$

Choosing, for example, $k_1 = k_2 = 1$, $h_{31} = 0$ and $h_{32} = 1$, we find $X_{K,H} = \left[\begin{smallmatrix}5&7&0\\ 4&8&0\\ 0&0&1\end{smallmatrix}\right]$, $Y_{K,H} = \left[\begin{smallmatrix}-10&-28&0\\ 0&0&0\end{smallmatrix}\right]$, which yield $F_{K,H} = Y_{K,H}\, X_{K.H}^{\dagger} = \left[\begin{smallmatrix}8/3&-35/6&0\\ 0&0&0\end{smallmatrix}\right]$. Clearly, $(A +$

$B\,F_{K,H})\mathcal{V}^\star \subseteq \mathcal{V}^\star \subseteq \ker(C + D\,F_{K,H})$ and $\sigma(A + B\,F_{K,H}\,|\,\mathcal{V}^\star) = \sigma(A + B\,F_{K,H}) = \{0, -2, -4\}$ as required.

Remark 4.1. All the considerations of this section on the subspace $\mathcal{V}^\star$ can be straightforwardly adapted to the case of the largest output-nulling stabilizability subspace $\mathcal{V}_g^\star$. The only modification in the statement of Theorem 4.1 is that $\mathcal{Z}$ is the set of minimum-phase invariant zeros of the system.

5. Computation of friends with inner/output spectral assignment. We now show that it is always possible to parameterize all the friends that assign the inner and outer eigenstructure of $\mathcal{V}^\star$ (and therefore also of $\mathcal{R}^\star$) by means of the formula

$$F_K = Y_K\,X_K^{-1},$$

i.e., where this time X_K is square and invertible (for almost all choices of the parameter matrix K). This step is essential in the robust computation of friends. For simplicity of exposition, we assume that all the inner/outer eigenvalues are assigned and that all the invariant zeros and uncontrollable modes of the pair (A, B) are real and distinct. The complex conjugate case follows straightforwardly by applying the result in Theorem 3.3.

THEOREM 5.1 (parameterization of friends of $\mathcal{V}^\star$ with complete spectrum assignment). *Let $r = \dim \mathcal{R}^\star$, $\nu = \dim \mathcal{V}^\star$, and $q = \dim(\mathcal{V}^\star + \mathcal{R}_0)$. Let $\mathcal{L}_{\text{in}} = \{\lambda_1, \dots, \lambda_r\}$ be real. Let $\mathcal{Z} = \{z_{r+1}, \dots, z_\nu\}$ be the set of invariant zeros. Let $\mathcal{L}_{\text{out}} = \{\mu_{\nu+1}, \dots, \mu_q\}$ also be real. Finally, let $\mathcal{G} = \Gamma_{\text{out}} = \{\zeta_{q+1}, \dots, \zeta_n\}$. We assume $\mathcal{L}_{\text{in}} \cap \mathcal{Z} = \emptyset$, $\mathcal{L}_{\text{in}} \cap \mathcal{G} = \emptyset$, $\mathcal{L}_{\text{out}} \cap \mathcal{Z} = \emptyset$, and $\mathcal{L}_{\text{out}} \cap \mathcal{G} = \emptyset$. Define*

$$M_K = \big[N_\Sigma(\lambda_1)\ \dots\ N_\Sigma(\lambda_r)\ \big|\ N_\Sigma(z_{r+1})\ \dots\ N_\Sigma(z_\nu)\ \big|\ S_\Sigma(\mu_{\nu+1})\ \dots\ S_\Sigma(\mu_q)\ \big|\ S_\Sigma(\zeta_{q+1})\ \dots\ S_\Sigma(\zeta_n)\big]K,$$

where $K = \operatorname{diag}\{K_\lambda, K_z, K_\mu, K_\zeta\}$ and

- *$K_\lambda = \operatorname{diag}\{k_1^\lambda, \dots, k_r^\lambda\}$ with $k_i^\lambda \in \mathbb{R}^d$, and where $d = \dim(\ker P_\Sigma(\lambda))$ when λ is not an invariant zero;*
- *$K_z = \operatorname{diag}\{k_{r+1}^z, \dots, k_\nu^z\}$ with $k_i^z \in \mathbb{R}^{d_z}$, and $d_z = \dim(\ker P_\Sigma(z))$ when $z \in \mathcal{Z}$;*
- *$K_\mu = \operatorname{diag}\{k_{\nu+1}^\mu, \dots, k_q^\mu\}$ with $k_i^\mu \in \mathbb{R}^m$, since $m = \dim(\ker S_\Sigma(\mu))$ when μ is not in $\mathcal{G}$;*
- *$K_\zeta = \operatorname{diag}\{k_{q+1}^\zeta, \dots, k_n^\zeta\}$ with $k_i^\zeta \in \mathbb{R}^{m_\zeta}$, and where $m_\zeta = \dim(\ker S_\Sigma(\zeta))$ when $\zeta \in \Gamma_{\text{out}}$.*

Finally, define

$$X_K = \overline{\pi}\{M_K\} \in \mathbb{R}^{n\times n} \qquad \text{and} \qquad Y_K = \underline{\pi}\{M_K\} \in \mathbb{R}^{m\times n}. \tag{5.1}$$

For almost every choice of K, the matrix X_K is invertible, and the set of all friends of $\mathcal{V}^\star$ such that $\sigma(A + B\,F\,|\,\mathcal{R}^\star) = \mathcal{L}_{\text{in}}$, $\sigma(A + B\,F\,|\,\mathcal{V}^\star/\mathcal{R}^\star) = \mathcal{Z}$, and $\sigma(A + B\,F\,|\,(\mathcal{R}_0 + \mathcal{V}^\star)/\mathcal{V}^\star) = \mathcal{L}_{\text{out}}$ is parameterized in K as

$$F_K = Y_K\,X_K^{-1}, \tag{5.2}$$

where K is such that X_K is invertible. Moreover, for such K the first r columns of X_K are a basis for $\mathcal{R}^\star$, the first $\nu = r + t$ columns of X_K are a basis for $\mathcal{V}^\star$, and the first q are a basis for $\mathcal{V}^\star + \mathcal{R}_0$.

Proof. Let K be defined as above, and let the rank of $X_K = \overline{\pi}\{M_k\}$ be equal to n, so that X_K is invertible. Let us partition M_K as

$$M_K = \left[\begin{array}{ccc|ccc|ccc|ccc} v_1 & \dots & v_r & v_{r+1} & \dots & v_\nu & v_{\nu+1} & \dots & v_q & v_{q+1} & \dots & v_n \\ w_1 & \dots & w_r & w_{r+1} & \dots & w_\nu & w_{\nu+1} & \dots & w_q & w_{q+1} & \dots & w_n \end{array}\right].$$

By construction, we have

$$(A-\lambda_i I_n)\,v_i + B\,w_i = 0, \qquad (A-z_j I_n)\,v_j + B\,w_j = 0,$$
$$C\,v_i + D\,w_i = 0, \qquad C\,v_j + D\,w_j = 0$$

for $i \in \{1,\dots,r\}$ and $j \in \{r+1,\dots,\nu\}$, respectively, and

$$(A-\mu_i I_n)\,v_i + B\,w_i = 0, \qquad (A-\zeta_j I_n)\,v_j + B\,w_j = 0$$

for all $i \in \{\nu+1,\dots,q\}$ and $j \in \{q+1,\dots,n\}$, respectively. It follows that if K is such that $X_K = \overline{\pi}\{M_K\}$ is nonsingular, and we construct F_K as $F_K = Y_K\,X_K^{-1}$, where $Y_K = \underline{\pi}\{M_K\}$, we find

$$\begin{bmatrix} A+B\,F_K \\ C+D\,F_K \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc|ccc} v_1 & \dots & v_r & v_{r+1} & \dots & v_\nu & v_{\nu+1} & \dots & v_q & v_{q+1} & \dots & v_n \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc|ccc|ccc} v_1 & \dots & v_r & v_{r+1} & \dots & v_\nu & v_{\nu+1} & \dots & v_q & v_{q+1} & \dots & v_n \\ 0 & \dots & 0 & 0 & \dots & 0 & \star & \dots & \star & \star & \dots & \star \end{array}\right] L,$$

where $L = \operatorname{diag}\{\lambda_1,\dots,\lambda_r,z_{r+1},\dots,z_\nu,\mu_{\nu+1},\dots,\mu_q,\zeta_{q+1},\dots,\zeta_n\}$. Now we show that the parameterization is exhaustive. Let F be a friend of $\mathcal{V}^\star$ such that $\sigma(A+B\,F\,|\mathcal{V}^\star) = \mathcal{L}_{\rm in} \cup \mathcal{Z}$ and $\sigma(A+B\,F\,|\mathcal{X}/\mathcal{V}^\star) = \mathcal{L}_{\rm out} \cup \mathcal{G}$. Consider an $n \times n$ matrix $[R \quad V_c \quad V_0 \quad \Gamma]$, which is such that $\operatorname{im} R = \mathcal{R}^\star$, $\operatorname{im}[R \quad V_c] = \mathcal{V}^\star$, $\operatorname{im}[R \quad V_c \quad V_0] = \mathcal{V}^\star + \mathcal{R}_0$. Since F is also a friend of $\mathcal{R}^\star$, and since $\mathcal{V}^\star + \mathcal{R}_0$ is $(A+B\,F)$-invariant, we can write

$$\begin{bmatrix} A+B\,F \\ C+D\,F \end{bmatrix} \left[\begin{array}{cc|cc} R & V_c & V_0 & \Gamma \end{array}\right] = \left[\begin{array}{cc|cc} R & V_c & V_0 & \Gamma \\ 0 & 0 & \star & \star \end{array}\right] \begin{bmatrix} L_{\rm in} & L_1 & L_2 & L_3 \\ 0 & Z & L_4 & L_5 \\ 0 & 0 & L_{\rm out} & L_6 \\ 0 & 0 & 0 & G \end{bmatrix},$$

where $\sigma(L_{\rm in}) = \mathcal{L}_{\rm in}$, $\sigma(L_{\rm out}) = \mathcal{L}_{\rm out}$, $\sigma(Z) = \mathcal{Z}$, and $\sigma(G) = \mathcal{G}$. Let us now construct the change of coordinate matrix

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ 0 & T_{22} & T_{23} & T_{24} \\ 0 & 0 & T_{33} & T_{34} \\ 0 & 0 & 0 & T_{44} \end{bmatrix},$$

where T_{11}, T_{22}, T_{33}, and T_{44} bring $L_{\rm in}$, Z, $L_{\rm out}$, and G in diagonal form $L_{\rm in}^\Delta$, Z^Δ, $L_{\rm out}^\Delta$, and G^Δ, respectively, i.e., $T_{11}^{-1}\,L_{\rm in}\,T_{11} = L_{\rm in}^\Delta$, $T_{22}^{-1}\,Z\,T_{22} = Z^\Delta$, $T_{33}^{-1}\,L_{\rm out}\,T_{33} = L_{\rm out}^\Delta$, and $T_{44}^{-1}\,G\,T_{44} = L_{\rm out}^\Delta$. This is always possible because we are considering the case of real and distinct eigenvalues and invariant zeros. We then compute T_{12}, T_{23}, T_{13}, T_{34}, T_{24}, and T_{14} by solving in the right order the following Lyapunov equations:

$$L_{\rm in}\,T_{12} - T_{12}\,Z^\Delta = -L_1\,T_{22},$$
$$Z\,T_{23} - T_{23}\,L_{\rm out}^\Delta = -L_4\,T_{33},$$
$$L_{\rm out}\,T_{34} - T_{34}\,G^\Delta = -L_6\,T_{44},$$
$$L_{\rm in}\,T_{13} - T_{13}\,L_{\rm out}^\Delta = -L_1\,T_{23} - L_2\,T_{33},$$
$$Z\,T_{24} - T_{24}\,G^\Delta = -L_4\,T_{34} - L_5\,T_{44},$$
$$L_{\rm in}\,T_{14} - T_{14}\,G^\Delta = -L_1\,T_{24} - L_2\,T_{34} - L_3\,T_{44}.$$

Since $\mathcal{L}_{\rm in} \cap \mathcal{Z} = \emptyset$, $\mathcal{L}_{\rm in} \cap \mathcal{G} = \emptyset$, $\mathcal{L}_{\rm out} \cap \mathcal{Z} = \emptyset$, and $\mathcal{L}_{\rm out} \cap \mathcal{G} = \emptyset$, all these Lyapunov equations admit a unique solution. The matrix T thus constructed is such that

$$T^{-1} \begin{bmatrix} L_{\rm in} & L_1 & L_2 & L_3 \\ 0 & Z & L_4 & L_5 \\ 0 & 0 & L_{\rm out} & L_6 \\ 0 & 0 & 0 & G \end{bmatrix} T = \begin{bmatrix} L_{\rm in}^{\Delta} & 0 & 0 & 0 \\ 0 & Z^{\Delta} & 0 & 0 \\ 0 & 0 & L_{\rm out}^{\Delta} & 0 \\ 0 & 0 & 0 & G^{\Delta} \end{bmatrix}.$$

Let us now define $X \stackrel{\rm def}{=} [R \quad V_c \quad V_0 \quad \Gamma]\, T = [R\, T_{11} \quad \star \quad \star \quad \star]$ and $Y \stackrel{\rm def}{=} F\, X$. We find

$$\begin{bmatrix} A + B\, F \\ C + D\, F \end{bmatrix} X = X \cdot \operatorname{diag}\{L_{\rm in}^{\Delta}, Z^{\Delta}, L_{\rm out}^{\Delta}, G^{\Delta}\},$$

so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix} = \begin{bmatrix} X_1\, L_{\rm in}^{\Delta} & X_2\, Z^{\Delta} & X_3\, L_{\rm out}^{\Delta} & X_4\, G^{\Delta} \\ 0 & 0 & \star & \star \end{bmatrix},$$

where $X = [X_1 \quad X_2 \quad X_3 \quad X_4]$ and $Y = [Y_1 \quad Y_2 \quad Y_3 \quad Y_4]$ are partitioned conformably with $[R \quad V_c \quad V_0 \quad \Gamma]$. Let $v_1, \ldots, v_r$ denote the columns of X_1, $v_{r+1}, \ldots, v_\nu$ denote the columns of X_2, $v_{\nu+1}, \ldots, v_q$ denote the columns of X_3, and $v_{q+1}, \ldots, v_n$ denote the columns of X_4. Define the vectors w_i in a similar way as the columns of Y. Thus,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{cases} \begin{bmatrix} v_i \\ 0 \end{bmatrix} \lambda_i, & i \in \{1, \ldots, r\}, \\ \begin{bmatrix} v_i \\ 0 \end{bmatrix} z_i, & i \in \{r+1, \ldots, \nu\}, \end{cases}$$

and

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{cases} v_i\, \mu_i, & i \in \{\nu+1, \ldots, q\}, \\ v_i\, \zeta_i, & i \in \{q+1, \ldots, n\}. \end{cases}$$

As such, $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \ker P_\Sigma(\lambda_i)$ for $i \in \{1, \ldots, r\}$, $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \ker P_\Sigma(z_i)$ for $i \in \{r+1, \ldots, \nu\}$, $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \ker S_\Sigma(\mu_i)$ for $i \in \{\nu+1, \ldots, q\}$, and $\left[\begin{smallmatrix} v_i \\ w_i \end{smallmatrix}\right] \in \ker S_\Sigma(\zeta_i)$ for $i \in \{q+1, \ldots, n\}$. It follows that a matrix $K = \operatorname{diag}\{K_\lambda, K_z, K_\mu, K_\zeta\}$ exists for which X and Y are given by $X = \overline{\pi}\{M_K\}$ and $Y = \underline{\pi}\{M_K\}$.

We now show that for almost any choice of K, the matrix X_K is invertible. Observe that

$$\operatorname{rank} \overline{\pi}\{[N_\Sigma(\lambda_1) \ldots N_\Sigma(\lambda_r) | N_\Sigma(z_{r+1}) \ldots N_\Sigma(z_\nu) | S_\Sigma(\mu_{\nu+1}) \ldots S_\Sigma(\mu_q) | S_\Sigma(\zeta_{q+1}) \ldots S_\Sigma(\zeta_n)]\}$$

is equal to n. Indeed, if such rank was smaller than n, no parameter K would exist for which a feedback matrix F_K constructed as in (5.2) delivers the desired closed-loop eigenstructure. On the other hand, we showed that this parameterization is exhaustive, leading to a contradiction. Now, partitioning M_K as $\left[\begin{smallmatrix} \Phi_\Sigma \\ \Psi_\Sigma \end{smallmatrix}\right]$ and by following exactly the same argument of Theorem 3.1, we obtain that the matrix X_K is generically of full rank and is therefore generically invertible. ∎

6. The computation of friends for a robust eigenstructure with minimum gain. In this section we consider the problem of obtaining friends of $\mathcal{R}^\star$, $\mathcal{V}^\star$, and $\mathcal{V}_g^\star$ that also yield a robust closed-loop eigenstructure. For any square matrix M, it was shown in [26] that the sensitivity of the eigenvalue λ_i to perturbations in M can be measured by the condition number

$$c_i = \frac{\|y_i\|\|v_i\|}{|y_i^\top v_i|}, \tag{6.1}$$

where v_i and y_i are the right and left eigenvectors of λ_i, respectively. We use $c_\infty \stackrel{\text{def}}{=} \max_i c_i$ to denote the worst-case eigenvalue sensitivity. Furthermore, [11] linked the sensitivity of the eigenvalues to measures of the conditioning of the matrix V whose columns are comprised of the eigenvectors of M, in terms of the Euclidean and Frobenius norms,

$$c_\infty \le \kappa_2(V) \le \kappa_{\text{FRO}}(V), \tag{6.2}$$

where $\kappa_2(V) \stackrel{\text{def}}{=} \|V\|_2 \cdot \|V^{-1}\|_2$ and $\kappa_{\text{FRO}}(V) \stackrel{\text{def}}{=} \|V\|_{\text{FRO}} \cdot \|V^{-1}\|_{\text{FRO}}$ are the condition numbers of V with respect to the 2-norm and Frobenius norm, respectively.

For pairs (A, B), the problem of finding a gain matrix F that assigns a certain set of desired eigenvalues $\mathcal{L}$ to the matrix $A + BF$ and also minimizes these condition numbers is known as the *robust pole placement problem* and has an extensive literature. Notable contributions include [11], [6], [24], [16], and the recent paper [21]. An important related problem is the *minimum gain pole placement problem*, which seeks a gain matrix F that assigns a certain set of desired eigenvalues while also minimizing the norm of the gain matrix F; notable methods include [9], [23], and the recent [3].

In this paper we extend these classical pole placement problems to quadruples (A, B, C, D) and introduce the *robust friend computation* problem, which involves obtaining a friend of $\mathcal{R}^\star$, $\mathcal{V}^\star$, and $\mathcal{V}_g^\star$ that assigns a certain desired set of inner and outer closed-loop eigenvalues and also a robust closed-loop eigenstructure. We also introduce the *minimum gain friend computation* problem, which seeks a friend of $\mathcal{R}^\star$, $\mathcal{V}^\star$, and $\mathcal{V}_g^\star$ that assigns a certain desired set of inner and outer closed-loop eigenvalues, while minimizing the matrix gain of the friend. To date there have been no results for either of these problems.

For the robust friend problem, the upper bound on the eigenvalue sensitivity in (6.2) motivates us to consider the problem of minimizing the objective function

$$f_1(V) = \kappa_{\text{FRO}}(V), \tag{6.3}$$

which poses an unconstrained nonconvex optimization problem. Note that it is possible to reduce $\kappa_{\text{FRO}}(V)$ by suitably scaling the lengths of the column vectors of V. However, such scaling does not improve the eigenvalue conditioning in (6.1). Hence, we assume that the column vectors of V have been normalized. As pointed out in [6], for efficient computation we can study an alternative objective function

$$f_2(X) = \|X\|_{\text{FRO}}^2 + \|X^{-1}\|_{\text{FRO}}^2, \tag{6.4}$$

where X is a real matrix whose columns are obtained from those of V as follows: for the columns of V corresponding to real eigenvalues in $\mathcal{L}$, the columns of X are the same as those of V; for the columns of V corresponding to pairs of complex conjugate eigenvalues in $\mathcal{L}$, the corresponding real-valued columns of X are obtained using $U = \frac{1}{2}\left[\begin{smallmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{smallmatrix}\right]$, as indicated in the proof of Theorem 3.1.

For the minimum gain friend problem, we consider the problem of minimizing the objective function

$$g(F) = \|F\|_{\text{FRO}}^2, \tag{6.5}$$

which again presents an unconstrained nonconvex optimization problem. To simultaneously minimize both the eigenvalue conditioning and the matrix gain, we introduce the weighted objective function

$$f_3(X, F) = \alpha f_2(X) + (1 - \alpha) g(F), \tag{6.6}$$

where α is a weighting factor with $0 \leq \alpha \leq 1$. The parameterization of the friends given in Theorem 5.1 can be employed for the minimization of f_3. We may express the matrix X and the friend F in terms of a common input parameter matrix K, as in (5.1) and (5.2). Using these in (6.6), for any desired value of α, we may minimize f_3 via a gradient search employing the first and second order derivatives of $f_2(X_K)$ and $g(F_K)$; expressions for these were given in [18]. The result obtained will be a local minimum and hence contingent upon the initial condition (input parameter matrix K) used.

7. Numerical studies. In this section we examine the performance of the optimal pole placement methods introduced in section 6 and compare them against two alternative methods for the computation of the friends from the linear systems literature.

Example 7.1. Consider the following quadruple:

$$A = \begin{bmatrix} 0 & 6 & -4 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & -9 & -9 & -10 & 8 & 0 & 0 & 6 \\ 2 & 0 & 0 & 0 & 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 6 \\ 9 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & -3 & 0 \\ -3 & 0 & 0 & -10 & -3 & 0 & 8 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 9 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 7 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

In this example, we have $\mathcal{V}^\star = \mathcal{R}^\star$, $\dim \mathcal{R}^\star = 6$, and the pair (A, B) is reachable, so that $\mathcal{R}_0 = \mathcal{X}$. This system has no invariant zeros. We want to find a friend F of $\mathcal{R}^\star$ such that $\mathcal{L}_{\text{in}} = \{-1, -2, -3, -4, -5, -6\}$ and $\mathcal{L}_{\text{out}} = \{-7, -8\}$. Minimizing f_2 in (6.4) via a gradient search, we obtain the feedback matrix

$$F_0 = \begin{bmatrix} -0.1495 & -0.8175 & -0.3581 & 2.0241 & -0.4644 & -0.7285 & -0.5987 & -1.8265 \\ -1.0727 & -3.0008 & 1.0185 & -0.3810 & 0.8063 & -0.3769 & -0.7796 & 0.3626 \\ -0.1783 & -0.3409 & 0.8712 & -4.0825 & 0.9965 & 0.0659 & 1.3905 & 3.4818 \end{bmatrix}.$$

Using the routine effesta.m in the MATLAB toolbox *GA* [4], we find that a friend that accomplishes this task is given by

$$F_1 = \begin{bmatrix} -0.0648 & -3.3046 & -0.1467 & 0.0853 & 0.4753 & -0.7881 & -0.0953 & -1.0966 \\ -1.1223 & -1.5083 & 0.9430 & 0.8138 & 0.2027 & -0.3425 & -1.0706 & -0.1055 \\ -0.3199 & 3.9952 & 0.7620 & -0.5444 & -0.8419 & 0.1628 & 0.5686 & 2.0574 \end{bmatrix}.$$

In the Linear Systems Toolkit [7], the routine atea.m directly adopts the place.m pole placement algorithm in MATLAB for the subspace corresponding to $\mathcal{R}^\star$ and yields

$$F_2 = \begin{bmatrix} 3.3163 & -1.3615 & -1.1872 & -0.4456 & 0.8412 & -0.7835 & -0.0651 & -0.9233 \\ -2.9733 & -2.7000 & 1.6130 & 1.0320 & -0.1698 & -0.3660 & -0.9051 & -0.3408 \\ -5.3155 & 0.4780 & 2.8057 & -0.1256 & -2.2519 & 0.0498 & 1.4428 & 1.0926 \end{bmatrix}.$$

To compare these friends of $\mathcal{V}^\star$, we consider several performance measures. Computing the conditioning measure c_∞ in (6.1) arising from each friend, we observe that $c_\infty(F_0) = 61.7$, $c_\infty(F_1) = 624$, while $c_\infty(F_2) = 7144$, indicating that the method introduced in this paper gives reduced eigenvalue sensitivity by one and two orders of magnitude, respectively.

We also compare the norms of these gain matrices. We observe the values $\|F_0\|_2 = 6.42$, $\|F_1\|_2 = 5.20$, and $\|F_2\|_2 = 8.18$, indicating that the method described in this paper uses a somewhat higher gain than that of effesta.m but less than atea.m for this example. By considering the weighted robustness and gain minimization problem in (6.6) with, for example, $\alpha = 0.001$, we are able to obtain a matrix

$$F_3 = \begin{bmatrix} -0.0914 & -1.8620 & -0.8391 & 0.8569 & 0.0904 & -0.7584 & -0.2856 & -1.4403 \\ -1.1672 & -2.3583 & 1.3484 & 0.3826 & 0.4125 & -0.3600 & -0.9600 & 0.0853 \\ -0.5807 & 1.5595 & 1.9179 & -1.7291 & -0.2776 & 0.1125 & 0.8822 & 2.5786 \end{bmatrix},$$

yielding eigenvalue sensitivity $c_\infty(F_3) = 67.63$ and gain $\|F_3\|_2 = 4.94$ and thus offering improvement over F_1 and F_2 on both criteria.

Another performance consideration is the accuracy of the pole placement achieved by each method. We use the measure

$$\Delta(F) \stackrel{\text{def}}{=} \max\{|\text{eig}_i(A + BF) - \lambda_i| : \lambda_i \in \mathcal{L}\}, \tag{7.1}$$

which represents the largest absolute value difference between each eigenvalue of $A + BF$ and the corresponding λ_i in $\mathcal{L}$. In the present case we obtain $\Delta(F_0) = 3.20\times10^{-14}$, $\Delta(F_1) = 1.16 \times 10^{-12}$, $\Delta(F_2) = 2.34 \times 10^{-11}$, and $\Delta(F_3) = 3.74 \times 10^{-14}$. This result indicates that the method introduced in this paper can achieve more accurate pole placement, again by some orders of magnitude.

In order to probe more deeply into the performance delivered by the methods presented here with respect to the other available techniques, we constructed four Monte Carlo–like experiments. In our first two experiments, we generated 10,000 random triples (A, B, C). In Experiment 1, we chose $n = 5$ with $m = 4$ control inputs and $p = 3$ outputs, and in Experiment 2 we chose $n = 8$, $m = 3$, and $p = 1$. Every entry in each matrix of the triple was generated using the MATLAB command randn.m. In these two experiments, the feedthrough matrix was taken equal to zero.[9] Since generically when $D = 0$ the dimension of $\mathcal{R}^\star$ is equal to $n - p$, in Experiment 1 we chose $\{-1, -2\}$ to be the two eigenvalues of the closed-loop system restricted to $\mathcal{R}^\star$, and in Experiment 2 we chose $\{-1, -2, -3, -4, -5, -6, -7\}$. Moreover, since the system thus generated will be generically reachable, $\mathcal{R}_0/\mathcal{R}^\star$ will have dimension 3 in Experiment 1, which implies that we can assign three eigenvalues of $\sigma(A + B\,F|\mathcal{R}_0/\mathcal{R}^\star)$; we chose

[9] If the feedthrough D is generated in the same random fashion as the other three matrices A, B, C, the resulting $\mathcal{R}^\star$ generically coincides with the state-space, and the use of effesta.m and atea.m reduces to the mere use of the MATLAB routine place.m.

the values $\{-3, -4, -5\}$. In Experiment 2, $\mathcal{R}_0/\mathcal{R}^\star$ has dimension 1, which implies that we can assign one eigenvalue of $\sigma(A + B\,F|\mathcal{R}_0/\mathcal{R}^\star)$; we chose the value $\{-8\}$.

We denote the feedback matrix obtained using the methods described in this paper by F_0, and we use the symbol V_0 to denote the matrix of closed-loop eigenvectors. The gain matrix and eigenvector matrix obtained using effesta.m and atea.m are denoted, respectively, with F_1, V_1 and F_2, V_2. The results of these two experiments are shown in the first two columns of Table 1.

A consequence of generating our system matrices with the command randn.m is that, generically, all the entries in the matrices will be nonzero. This means that in such systems, the state, input, and output variables are directly dependent upon one another. This is unlikely to be the case in most real-world systems. Hence, we found it significant to also test our method in the case where the system matrices are sparse. Thus in Experiment 3 we generated 10,000 sample triples (A, B, C) with $n = 8$, $m = 3$, and $p = 1$. The entries of each matrix are integers between -20 and 20, but such that 75% of the entries were set to zero. The eigenvalues of $\sigma(A + B\,F\,|\,\mathcal{R}^\star)$ and $\sigma(A+B\,F|\mathcal{R}_0/\mathcal{R}^\star)$ were taken to be random values generated with the MATLAB command randn.m. The results of this experiment are shown in the third column of Table 1.

To consider both the robustness and the norm of the gain matrix, we considered the weighted robustness and gain minimization problem (6.6) using the value $\alpha = 0.0001$. Our Experiment 4 used the same 10,000 example systems chosen in Experiment 2, and the results are given in Table 2.

Finally, to gain a measure of the magnitude of the improvement offered by our method over the two alternatives, we introduced Experiments 5 and 6, in which we used the same 10,000 example systems chosen in Experiments 2 and 3, respectively. In this case we computed the percentage of systems in which our method provided better performance with respect to effesta.m and atea.m by at least one order of magnitude. The results are given in Table 3.

TABLE 1

	Experiment 1	Experiment 2	Experiment 3
$\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_1)$	100 %	100 %	100 %
$\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_2)$	91.95 %	99.92 %	99.91 %
$c_\infty(F_0) < c_\infty(F_1)$	99.90 %	99.83 %	99.91 %
$c_\infty(F_0) < c_\infty(F_2)$	90.19 %	99.66 %	99.69 %
$\Delta(F_0) < \Delta(F_1)$	76.87 %	87.39 %	87.17 %
$\Delta(F_0) < \Delta(F_2)$	76.87 %	87.39 %	87.17 %

TABLE 2

	Experiment 4
$\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_1)$	88.4 %
$\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_2)$	95.6 %
$c_\infty(F_0) < c_\infty(F_1)$	90.2 %
$c_\infty(F_0) < c_\infty(F_2)$	90.4 %
$\Delta(F_0) < \Delta(F_1)$	70.2 %
$\Delta(F_0) < \Delta(F_2)$	70.2 %
$\|F_0\|_{\text{FRO}} < \|F_1\|_{\text{FRO}}$	74.9 %
$\|F_0\|_{\text{FRO}} < \|F_2\|_{\text{FRO}}$	78.0 %

TABLE 3

	Experiment 5	Experiment 6
$10\,\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_1)$	3.57 %	70.38 %
$10\,\kappa_{\text{FRO}}(V_0) < \kappa_{\text{FRO}}(V_2)$	21.86 %	97.51 %
$10\,c_\infty(F_0) < c_\infty(F_1)$	7.09 %	72.68 %
$10\,c_\infty(F_0) < c_\infty(F_2)$	24.47 %	97.58 %
$10\,\Delta(F_0) < \Delta(F_1)$	16.93 %	58.74 %
$10\,\Delta(F_0) < \Delta(F_2)$	16.93 %	58.74 %

In all the six experiments, our method was able to offer, in the vast majority of cases, superior robust conditioning with reduced gain and greater accuracy than the other two methods surveyed. This superior performance can be explained as follows. There are many friends F_K in (5.2) that deliver eigenvectors lying within the appropriate subspaces. Implementing a gradient search to minimize f_3 in (6.6) with a suitable choice of α yields a friend with desirable robustness qualities, or minimum gain, or any desired combination of these two. By contrast, the methods of [4] and [7] do not attempt to make a robust selection or to minimize the gain of the friend.

Comparison of Experiments 1 and 2 indicates that this improvement was greater for the class of systems with dimensions $n = 8$, $m = 3$, and $p = 1$, rather than $n = 5$, $m = 4$, and $p = 1$. We note that both effesta.m and atea.m utilize the MATLAB place.m routine, which does attempt a robust choice of eigenvectors, using the heuristic methods of [11]. However, extensive testing in [21] showed that pole placement methods employing null-space techniques similar in spirit to the one given here can offer substantially improved robust conditioning, relative to the place.m routine and several other methods surveyed. The improvement was greater for systems in which m was small in relation to n, and this difference in the extent of the performance improvement has again been observed here in Experiments 1 and 2.

The results of Experiments 2 and 3 are almost identical, indicating that the use of sparse matrices did not lead to any significant change in the proportion of systems for which our methods were able to provide superior robustness performance. The results of Experiment 4 indicated that the weighted optimization problem of (6.6) with a suitably chosen value of α can simultaneously deliver improvements in robustness and gain over the effesta.m and atea.m methods. Again, this does not come as a surprise since the place.m routine employed by both effesta.m and atea.m has not been designed to minimize the matrix gain.

The results of Experiments 5 and 6 offer two interesting insights. These experiments attempt to gauge the magnitude of the improvement of our method over the alternatives, and we noted that large improvements were more frequently observed in relation to atea.m than effesta.m, suggesting that effesta.m is able to offer superior robustness performance than atea.m. The second notable difference observed was between systems with randomly generated (and hence nonsparse) entries in Experiment 5 and those with sparse entries in Experiment 6. The frequency of large improvements by our method over both effesta.m and atea.m was dramatically more prevalent in the case of systems with sparse matrices.

The improved accuracy of our method observed in all the experiments provides further numerical evidence for the observation noted in [21] that eigenstructure assignment methods employing null-space techniques in general provide superior accuracy in their pole placement than methods employing coordinate transformations or the solutions to Sylvester equations. Interestingly, Experiments 5 and 6 showed

that the magnitude of improvement in accuracy was much greater for sparse matrices than for nonsparse, suggesting that the effesta.m and atea.m routines experience computational difficulties with sparse matrices.

Concluding remarks. In this paper, we introduced a new parameterization of the friend matrices for the fundamental output-nulling subspaces $\mathcal{R}^\star$, $\mathcal{V}^\star$, and $\mathcal{V}_g^\star$ used in several decoupling, noninteracting, and tracking control problems. We exploited this result to obtain a procedure that delivers friends which robustly assign the free internal and external eigenstructure of the closed loop with respect to such subspaces. All the results presented in this paper can be dualized to input-containing subspaces, unobservability input-containing subspaces, and detectability input-containing subspaces.

We compared the method introduced in this paper against the two publicly available MATLAB toolboxes. In these examples our method for the computation of such subspaces showed dramatic improvement in reducing the eigenvalue sensitivity, while also using less matrix gain and achieving greater accuracy.

An important direction for future research is the application of these results to the design of linear state feedback control laws that yield a monotonic step response for an LTI MIMO system, as studied in [19] and [20], based on the computation of the Rosenbrock matrix.

Appendix A: Construction of friends. In this section we analyze how the friends of an output-nulling subspace can be computed so as to assign the free closed-loop eigenvalues. We begin by noticing that (2.3) is equivalent to the existence of two matrices Ξ and Ω such that

$$\begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} \Xi + \begin{bmatrix} B \\ D \end{bmatrix} \Omega, \tag{7.2}$$

where V is a basis matrix of $\mathcal{V}$. The set of solutions of (7.2) is parameterized in K_1 as

$$\begin{bmatrix} \Xi \\ \Omega \end{bmatrix} = \begin{bmatrix} V & B \\ 0 & D \end{bmatrix}^\dagger \begin{bmatrix} A \\ C \end{bmatrix} V + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} K_1, \tag{7.3}$$

where the columns of $\left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right]$ are a basis for the kernel of $\left[\begin{smallmatrix} V & B \\ 0 & D \end{smallmatrix}\right]$. On the other hand, (2.4) is equivalent to the existence of two matrices F and Λ such that

$$\begin{bmatrix} A + B\,F \\ C + D\,F \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} \Lambda, \tag{7.4}$$

and the eigenvalues of Λ are the eigenvalues of $A + B\,F$ restricted to $\mathcal{V}$.

It is easy to see that the set of all friends F of $\mathcal{V}$ are the solutions of the linear equation $\Omega = -F\,V$, where Ω is such that for a certain Ξ, (7.2) holds. Indeed, let (Ξ, Ω) be such that (7.2) holds. Then, by selecting F so that $\Omega = -F\,V$ holds, we get from (7.2) that $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] V + \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] F\,V = \left[\begin{smallmatrix} V \\ 0 \end{smallmatrix}\right] \Xi$, which says that (7.4) holds with $\Lambda = \Xi$. Now, consider F and Λ such that (7.4) holds. Then, clearly (7.2) holds with $\Xi = \Lambda$ and $\Omega = -F\,V$.

The set of solutions of the linear equation $\Omega = -F\,V$ can be written as

$$F = -\Omega\,(V^\top V)^{-1}\,V^\top + K_2\,H_2, \tag{7.5}$$

where $\ker H_2 = \mathcal{V}$ and K_2 is arbitrary. Thus we have identified two degrees of freedom in the construction of F, i.e., K_1 and K_2. In particular, K_1 affects only the inner eigenvalues of $\mathcal{V}$, whereas K_2 affects only the outer eigenvalues of $\mathcal{V}$. In other words, if we consider the change of coordinates $T = [\,V \quad V_c\,]$, where V_c is such that T is nonsingular, and let

$$T^{-1}(A + B\,F)T = \begin{bmatrix} L_1(K_1, K_2) & L_2(K_1, K_2) \\ 0 & L_3(K_1, K_2) \end{bmatrix},$$

then $L_1(K_1, K_2)$ does not depend on K_2, and $L_3(K_1, K_2)$ does not depend on K_1 (see also [15, p. 348]).

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