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Existence of positive solutions for singular higher-order fractional differential equations with infinite-point boundary conditions

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Abstract

In this paper, we consider a class of singular fractional differential equations with infinite-point boundary conditions. The fractional orders are involved in the nonlinearity of the boundary value problem, and the nonlinearity is allowed to be singular with respect to not only the time variable but also to the space variable. Firstly, we give Green's function and establish its properties. Then, we utilize the sequential technique and regularization to investigate the existence of positive solutions. Finally, we give an example of application of our result.

MSC: 34A08; 34B10; 34B16; 34B18

Keywords: fractional differential equation; singular problem; infinite-point boundary conditions; positive solution; sequential techniques and regularization

1 Introduction

In this paper, we consider the following class of nonlinear singular fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0^+}^{\mu_1}u(t),D_{0^+}^{\mu_2}u(t),\ldots,D_{0^+}^{\mu_{n-2}}u(t))=0, \quad 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ u^{(n-2)}(1)=\sum_{j=1}^{\infty}\eta_j u(\xi_j), \end{cases} \tag{1.1}$$

where $\alpha,\mu_i\in\mathbb{R}_+^1=[0,+\infty)$, $n\in\mathbb{N}$ (the set of natural numbers), and $n-1<\alpha\leq n$, $n\geq 4$, $i-1<\mu_i\leq i$ $(i=1,2,\ldots,n-2)$, $f(t,x_1,x_2,\ldots,x_{n-1})$ may have singularity at $t=0,1$, $x_i=0$ $(i=1,2,\ldots,n-1)$, $0<\xi_1<\xi_2<\cdots<\xi_j<\cdots<1$, $\eta_j>0$ $(j=1,2,\ldots)$, $\sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1}<\Theta$, $\Theta=(\alpha-1)(\alpha-2)\cdots(\alpha-n+2)$, and $D_{0+}^{\alpha}u$, $D_{0^+}^{\mu_i}u$ $(i=1,2,\ldots,n-2)$ are the Riemann-Liouville derivatives.

Boundary value problems for nonlinear fractional differential equations arise from the studies of complex problems in many disciplinary areas such as fluid flows, electrical networks, rheology, biology chemical physics, and so on. Fractional-order models have been shown to be more accurate than integer-order models, and in applications of these models, it is important to theoretically establish conditions for the existence of positive solutions. In recent years, many authors investigated the existence of positive solutions for fractional equation boundary value problems (see [1–24] and the references therein), and a great deal of results have been developed for differential and integral boundary value problems. In [14], the author considered the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) + g(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{i=1}^{\infty} \alpha_j u(\xi_j), \quad 1 \le i \le n-2, \end{cases}$$

where $\alpha \in \mathbb{R}_+^1$, $n \in \mathbb{N}$, $n-1 < \alpha \le n$, $n \ge 3$, $\alpha_j \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1,2,\ldots$), and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, and $f \in C((0,1) \times (0,+\infty), \mathbb{R}_+^1)$ allows singularities with respect to both time ($t = 0,1$) and space variables ($u = 0$). The author established the existence and multiplicity of positive solutions. In [15], the authors investigated the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t),D_{0^+}^{\nu}u(t),D_{0^+}^{\mu}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases}$$

where $\alpha,\nu,\mu \in \mathbb{R}_+^1$, $3 < \alpha \le 4$, $0 < \nu \le 1$, $0 < \mu \le 1$ are real numbers, D_{0+}^{α}, $D_{0^+}^{\nu}$, and $D_{0^+}^{\mu}$ are the Riemann-Liouville fractional derivatives, $f(t,x,y,z)$ is a Carathéodory function singular at $x,y,z = 0$. The authors obtained the existence and multiplicity of positive solutions by means of Krasnoselskii's fixed point theorem. As there do not exists $0 < L < M$ such that (3.13) of [15] holds, the results of the multiple solutions are not correct in [15]. In [16], the authors investigated the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t),D_{0^+}^{\mu}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $\alpha,\mu \in \mathbb{R}_+^1$, $1 < \alpha \le 2$, $\mu > 0$ are real numbers, $\alpha - \mu \ge 1$, D_{0+}^{α} and $D_{0^+}^{\mu}$ are the Riemann-Liouville fractional derivatives, f is a Carathéodory function, and $f(t,x,y)$ is singular at $x = 0$. The authors obtained the existence of positive solutions by means of the Krasnoselskii's fixed point theorem. In [17], the author investigated the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t),u'(t),D_{0^+}^{\mu}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u'(0) = u'(1) = 0, \end{cases}$$

where $\alpha,\mu \in \mathbb{R}_+^1$, $2 < \alpha \le 3$, $0 < \mu \le 1$ are real numbers, D_{0+}^{α} and $D_{0^+}^{\mu}$ are the Riemann-Liouville fractional derivatives, f is a Carathéodory function, and $f(t,x,y,z)$ is singular at $x,y,z = 0$. The authors obtained the existence of positive solutions by means of the Krasnoselskii's fixed point theorem. In [18], the author investigated the singular problem

$$\begin{cases} D_{0+}^{\alpha}u(t) + q(t)f(t,u(t),u'(t),\ldots,u^{(n-2)}(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{cases}$$

where $\alpha \in \mathbb{R}_+^1$, $n-1 < \alpha \le n$, $n \ge 2$, the nonlinear function $f(t,x_1,x_2,\ldots,x_{n-1})$ may be singular at $x_i = 0$ ($i = 1,2,\ldots,n-1$), and $q(t)$ may be singular at $t = 0$. The existence results of positive solutions are obtained by a fixed point theorem for a mixed monotone operator.

Motivated by the results mentioned, in this paper, we utilize the sequential technique and regularization to investigate the existence of positive solutions of BVP (1.1), where $u \in C^{n-2}[0,1] \cap C^{n-1}(0,1)$ is said to be a positive solution of BVP (1.1) if and only if u satisfies (1.1) and $u(t) > 0$ for any $t \in (0,1]$. By using the sequential technique and regularization on a cone, some new existence results are obtained for the case where the nonlinearity is allowed to be singular with respect to both time and space variables. We emphasize here that our work presented in this paper has various new features. Firstly, we study singular nonlinear differential equation boundary value problems, that is, $f(t,x_1,x_2,\dots,x_{n-1})$ may have singularity at $t = 0,1$ and $x_i = 0$ $(i = 1,2,\dots,n-1)$, which leads to many difficulties in analysis. Secondly, compared with [15–17], we complete the proof without the need of imposing the third condition of the Carathéodory conditions, that is, the condition $|f(t,x_1,x_2,\dots,x_{n-1})| \le \varphi_H(t)$ is successfully removed, and, at the same time, the condition

$$\lim_{x\to\infty} \frac{h(x,x,\dots,x)}{x} = 0$$

is extended to

$$\limsup_{x\to\infty} \frac{h(x,x,\dots,x)}{x} = \lambda < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1}.$$

Thirdly, a special cone in a special space is established to overcome the difficulties caused by the singularity. Finally, values at infinite points are involved in the boundary conditions, and the fractional orders are involved in the nonlinearity of the boundary value problem (1.1).

For convenience, we list some conditions to be used throughout the paper.

(H_0) f satisfies the local Carathéodory condition on $[0,1] \times (0,\infty)^{n-1}$ if
  (1) $f(t,x_1,x_2,\dots,x_{n-1}) : [0,1] \to \mathbb{R}^1_+$ is measurable for all $(x_1,x_2,\dots,x_{n-1}) \in (0,+\infty)^{n-1}$;
  (2) $f(t,x_1,x_2,\dots,x_{n-1}) : (0,+\infty)^{n-1} \to \mathbb{R}^1_+$ is continuous for a.e. $t \in [0,1]$.

(H_1) There exists a constant $C > 0$ such that, for a.e. $t \in [0,1]$ and for any $(x_1,x_2,\dots,x_{n-1}) \in (0,\infty)^{n-1}$,

$$f(t,x_1,x_2,\dots,x_{n-1}) \ge C. \tag{1.2}$$

(H_2) For all $(x_1,x_2,\dots,x_{n-1}) \in (0,\infty)^{n-1}$ and a.e. $t \in [0,1]$,

$$f(t,x_1,x_2,\dots,x_{n-1}) \le \beta(t)p(x_1,x_2,\dots,x_{n-1}) + \gamma(t)h(x_1,x_2,\dots,x_{n-1}),$$

where $\beta,\gamma \in L^1((0,1),(0,+\infty))$, $p \in C((0,\infty)^{n-1},\mathbb{R}^1_+)$ is nonincreasing with respect to all arguments, $h \in (\mathbb{R}^{n-1}_+,\mathbb{R}^1_+)$ is nondecreasing with respect to all arguments, and

$$\int_0^1 \beta(s)p\left(Ms^{\alpha-1}, \frac{(n-2-\mu_1)M}{n!}s^{n-1-\mu_1},\dots,\frac{(n-2-\mu_{n-2})M}{3!}s^{n-1-\mu_{n-2}}\right)ds < \infty,$$

$$M = \frac{C}{(\alpha-n+2)\Gamma(\alpha+1)}.$$

(H_3)

$$\limsup_{x\to\infty}\frac{h(x,x,\ldots,x)}{x}=\lambda<\frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1},$$

where $e=\frac{\Theta}{P(0)\Gamma(\alpha-n+2)}$.

The main result of this paper is as follows.

Theorem 1.1 *If* (H_0)-(H_3) *hold, then problem* (1.1) *has a positive solution* x, *and for* $t\in[0,1]$, *we have*

$$x(t)\geq Mt^{\alpha-1},\qquad D_{0^+}^{\mu_i}x(t)\geq\frac{(n-2-\mu_i)M}{(n-i+1)!}t^{n-1-\mu_i},\quad i=1,2,\ldots,n-2, \tag{1.3}$$

where M *is defined by* (H_2).

In order to overcome the singularity, we utilize the sequential technique and regularization to testify the existence of positive solutions for problem (1.1). Next, for any $m\in\mathbb{N}$, we define $X_m:\mathbb{R}^1\to\mathbb{R}^1$ and $f_m:[0,1]\times\mathbb{R}_+^{n-1}\to\mathbb{R}^1$ by the following formulas:

$$X_m(\tau)=\begin{cases}\tau & \text{if } \tau\geq\frac{1}{m},\\ \frac{1}{m} & \text{if } \tau<\frac{1}{m}\end{cases}$$

for all $(x_1,x_2,\ldots,x_{n-1})\in\mathbb{R}^{n-1}$ and a.e. $t\in[0,1]$,

$$f_m(t,x_1,x_2,\ldots,x_{n-1})=f\bigl(t,X_m(x_1),X_m(x_2),\ldots,X_m(x_{n-1})\bigr).$$

Then condition (H_1) gives that f_m satisfies the local Carathéodory condition on $[0,1]\times\mathbb{R}_+^{n-1}$ and $f_m(t,x_1,x_2,\ldots,x_{n-1})\geq C$ for a.e. $t\in[0,1]$ and all $(x_1,x_2,\ldots,x_{n-1})\in\mathbb{R}_+^{n-1}$. Condition (H_2) provides that

$$\begin{aligned}f_m(t,x_1,x_2,\ldots,x_{n-1})&\leq\beta(t)p\left(\max\left\{x_1,\frac{1}{m}\right\},\max\left\{x_2,\frac{1}{m}\right\},\ldots,\max\left\{x_{n-1},\frac{1}{m}\right\}\right)\\&\quad+\gamma(t)h\left(x_1+\frac{1}{m},x_2+\frac{1}{m},\ldots,x_{n-1}+\frac{1}{m}\right)\end{aligned} \tag{1.4}$$

for a.e. $t\in[0,1]$ and all $(x_1,x_2,\ldots,x_{n-1})\in\mathbb{R}_+^{n-1}$.

Next, we discuss the regular fractional differential equation

$$\begin{cases}D_{0+}^{\alpha}u(t)+f_m(t,u(t),D_{0^+}^{\mu_1}u(t),D_{0^+}^{\mu_2}u(t),\ldots,D_{0^+}^{\mu_{n-2}}u(t))=0, & 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & u^{(n-2)}(1)=\sum_{j=1}^{\infty}\eta_j u(\xi_j).\end{cases} \tag{1.5}$$

2 Preliminaries and lemmas

For convenience of the reader, we first present some basic definitions and lemmas. These definitions can be found in the recent literature such as [19, 20]. In this paper, $\|x\|_1=\int_0^1|x(t)|\,dt$ is the norm in $L^1[0,1]$, $\|x\|=\max\{|x(t)|:t\in[0,1]\}$ is the norm in $C[0,1]$, and

$$\|x\|_2=\max\bigl\{\|x\|,\bigl\|x'\bigr\|,\bigl\|x''\bigr\|,\ldots,\bigl\|x^{(n-2)}\bigr\|\bigr\}$$

is the norm in $E = C^{n-2}[0,1]$; $AC^k[0,1]$ ($k = 0,1,2,\ldots$) is the space of absolutely continuous functions having absolutely continuous kth-order derivatives on $[0,1]$.

Definition 2.1 [19, 20] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}^1$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 [19, 20] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to \mathbb{R}^1$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds$$

with $n = [\alpha]+1$, where $[\alpha]$ denotes the integer part of α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 [17] *We have*

$$I_{0^+}^{\alpha} : L^1[0,1] \to \begin{cases} L^1[0,1] & \textit{if } \alpha \in (0,1), \\ AC^{[\alpha]-1}[0,1] & \textit{if } \alpha \geq 1, \end{cases}$$

where $[\alpha]$ is the least integer greater than or equal to α, and $AC^0[0,1] = AC[0,1]$.

Lemma 2.2 [21] *If $x \in L^1[0,1]$ and $\alpha + \beta \geq 1$, then $(I_{0^+}^{\alpha} I_{0^+}^{\beta} x)(t) = (I_{0^+}^{\alpha+\beta} x)(t)$ for all $t \in [0,1]$, that is,*

$$\int_0^t (t-s)^{\alpha-1} \left(\int_0^t (s-\xi)^{\beta-1} x(\xi)\, d\xi\right) ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} x(s)\, ds.$$

Lemma 2.3 [22] *Suppose that $\alpha > 0$. If $x \in C(0,1]$ and $D_{0^+}^{\alpha} x \in L^1[0,1]$, then*

$$x(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) + \sum_{k=1}^{n} c_k t^{\alpha-k}$$

for $t \in (0,1]$, where $n = [\alpha]+1$ and $c_k \in \mathbb{R}^1$ ($k = 1,2,\ldots,n$).

Lemma 2.4 *Suppose that $i-1 < \mu_i \leq i$ ($i = 1,2,\ldots,n-2$) and $u \in C^{n-2}[0,1]$, $u^{(i)}(0) = 0$ ($i = 0,1,2,\ldots,n-3$). Then $D_{0^+}^{\mu_i} u \in C[0,1]$ ($i = 1,2,\ldots,n-2$), and*

$$D_{0^+}^{\mu_i} u(t) = \frac{1}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-\mu_i-3} u^{(n-2)}(s)\, ds, \quad i = 1,2,\ldots,n-2. \tag{2.1}$$

Proof By integration by parts we get

$$\int_0^t (t-s)^{i-1-\mu_i} u(s)\, ds = \frac{1}{i-\mu_i} \int_0^t (t-s)^{i-\mu_i} u'(s)\, ds,$$

$$
\begin{aligned}
&\int_0^t (t-s)^{i-\mu_i} u'(s)\,ds = \frac{1}{i+1-\mu_i}\int_0^t (t-s)^{i+1-\mu_i} u''(s)\,ds, \\
&\ldots, \\
&\int_0^t (t-s)^{i+n-3-\mu_i} u^{(n-3)}(s)\,ds = \frac{1}{i+n-2-\mu_i}\int_0^t (t-s)^{i+n-2-\mu_i} u^{(n-2)}(s)\,ds.
\end{aligned}
$$

So

$$
\begin{aligned}
D_{0^+}^{\mu_i} u(t) &= \frac{1}{\Gamma(i-\mu_i)}\left(\frac{d}{dt}\right)^i \int_0^t (t-s)^{i-1-\mu_i} u(s)\,ds \\
&= \frac{1}{\Gamma(i+1-\mu_i)}\left(\frac{d}{dt}\right)^i \int_0^t (t-s)^{i-\mu_i} u'(s)\,ds \\
&= \frac{1}{\Gamma(i+2-\mu_i)}\left(\frac{d}{dt}\right)^i \int_0^t (t-s)^{i+1-\mu_i} u''(s)\,ds \\
&= \frac{1}{\Gamma(i+n-2-\mu_i)}\left(\frac{d}{dt}\right)^i \int_0^t (t-s)^{i+n-3-\mu_i} u^{(n-2)}(s)\,ds \\
&= \frac{1}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-\mu_i-3} u^{(n-2)}(s)\,ds, \quad i=1,2,\ldots,n-2.
\end{aligned}
$$

Hence, we have

$$
D_{0^+}^{\mu_i} u(t) = \frac{1}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-\mu_i-3} u^{(n-2)}(s)\,ds, \quad i=1,2,\ldots,n-2.
$$

Further, by the continuity of

$$
\int_0^t (t-s)^{n-3-\mu_i} u^{(n-2)}(s)\,ds, \quad t\in[0,1], i=1,2,\ldots,n-2,
$$

we get that $D_{0^+}^{\mu_i} u(t)$ $(i=1,2,\ldots,n-2)$ is continuous on $[0,1]$. □

Lemma 2.5 *Given* $g\in C(0,1)\cap L^1(0,1)$,

$$
u(t) = \int_0^1 G(t,s)g(s)\,ds,
$$

is the unique solution in $C^{n-2}[0,1]\cap C^{n-1}(0,1)$ *of the equation*

$$
\begin{cases}
D_{0^+}^{\alpha} u(t) + g(t) = 0, \quad 0<t<1, \\
u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \\
u^{(n-2)}(1) = \sum_{j=1}^{\infty}\eta_j u(\xi_j),
\end{cases}
\tag{2.2}
$$

where

$$
G(t,s) = \frac{1}{P(0)\Gamma(\alpha)}
\begin{cases}
t^{\alpha-1}P(s)(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-1}, & 0\le s\le t\le 1, \\
t^{\alpha-1}P(s)(1-s)^{\alpha-n+1}, & 0\le t\le s\le 1
\end{cases}
\tag{2.3}
$$

and

$$P(s) = \Theta - \sum_{s\le\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^{n-2}, \quad \Theta = (\alpha-1)(\alpha-2)\cdots(\alpha-n+2).$$

Proof Applying Lemma 2.3, we can reduce (2.2) to the equivalent integral equation

$$u(t) = -I_{0^+}^{\alpha} g(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for $C_1, C_2, \ldots, C_n \in \mathbb{R}^1$. By Lemma 2.3 we have that

$$u(t) = -I_{0^+}^{\alpha} g(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

is a solution of (2.2) in $C(0,1]$. Since $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_2 = C_3 = \cdots = C_n = 0$, but $C_1 \neq 0$, and thus

$$u(t) = C_1 t^{\alpha-1} - I_{0^+}^{\alpha} g(t).$$

By Lemma 2.1 we have $I_{0^+}^{\alpha} g \in AC^{n-2}[0,1]$, so that

$$u(t) = -I_{0^+}^{\alpha} g(t) + C_1 t^{\alpha-1} \tag{2.4}$$

is a solution of (2.2) in the space $AC^{n-2}[0,1]$. Taking the derivative step by step for (2.4), we have

$$\begin{aligned} &u'(t) = C_1(\alpha-1)t^{\alpha-2} - I_{0^+}^{\alpha-1} g(t), \\ &u^{(i)}(t) = C_1(\alpha-1)(\alpha-2)\cdots(\alpha-i)t^{\alpha-i-1} - I_{0^+}^{\alpha-i} g(t), \quad i = 2,3,\ldots,n-2. \end{aligned}$$

On the other hand, the equality $u^{(n-2)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$, combined with

$$u^{(n-2)}(1) = C_1 \Theta - I_{0^+}^{\alpha-n+2} g(1),$$

gives

$$\begin{aligned} C_1 = {}& \int_0^1 \frac{(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+2)(\Theta - \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1})} g(s)\,ds \\ & - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j - s)^{\alpha-1}}{\Gamma(\alpha)(\Theta - \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1})} g(s)\,ds \\ = {}& \int_0^1 \frac{(1-s)^{\alpha-n+1} P(s)}{P(0)\Gamma(\alpha)} g(s)\,ds, \end{aligned}$$

where

$$P(s) = \Theta - \sum_{s\le\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^{n-2}, \quad \Theta = (\alpha-1)(\alpha-2)\cdots(\alpha-n+2).$$

Hence,

$$\begin{aligned} u(t) &= C_1 t^{\alpha-1} - I_{0^+}^{\alpha} g(t) \\ &= -\int_0^t \frac{P(0)(t-s)^{\alpha-1}}{P(0)\Gamma(\alpha)} g(s)\,ds + \int_0^1 \frac{P(s)(1-s)^{\alpha-n+1}t^{\alpha-1}}{P(0)\Gamma(\alpha)} g(s)\,ds \\ &= \int_0^1 G(t,s)g(s)\,ds, \end{aligned}$$

where

$$G(t,s) = \frac{1}{P(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}P(s)(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}P(s)(1-s)^{\alpha-n+1}, & 0 \le t \le s \le 1 \end{cases}$$

and

$$P(s) = \Theta - \sum_{s \le \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^{n-2}, \quad \Theta = (\alpha-1)(\alpha-2)\cdots(\alpha-n+2).$$

□

Lemma 2.6 *The Green function $G(t,s)$ defined in Lemma* 2.5 *has the following properties*:

(1)

$$\frac{\partial^j}{\partial t^j} G(t,s) \in C\bigl([0,1]\times[0,1]\bigr), \quad j = 0,1,2,\dots,n-2;$$

(2)

$$0 \le \frac{\partial^j}{\partial t^j} G(t,s) \le \frac{\Theta}{\Gamma(\alpha-j)P(0)}, \quad (t,s) \in [0,1]\times[0,1], j = 0,1,2,\dots,n-2;$$

(3)

$$\int_0^1 \frac{\partial^j}{\partial t^j} G(t,s)\,ds \ge \frac{t^{\alpha-j-1}}{(\alpha-n+2)\Gamma(\alpha-j+1)}, \quad t \in [0,1], j = 0,1,2,\dots,n-3$$

and

$$\int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,ds \ge \frac{t(1-t)}{\Gamma(\alpha-n+3)}, \quad t \in [0,1].$$

Proof By calculating the derivative we get

$$\frac{\partial^j}{\partial t^j} G(t,s) = \begin{cases} \frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-j-1}}{P(0)\Gamma(\alpha-j)}, & 0 \le s \le t \le 1, \\ \frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-j)}, & 0 \le t \le s \le 1 \end{cases} \tag{2.5}$$

for $j = 0,1,2,\dots,n-2$.

(1) For $n-1 < \alpha \le n$, $j \le n-2$, we have $\alpha - j > 1$. Hence, by (2.5) we have that $\frac{\partial^j}{\partial t^j} G(t,s)$ ($j = 0,1,2,\dots,n-2$) are continuous on $[0,1]\times[0,1]$, and so (1) holds.

(2) By direct calculation we get $P'(s) \ge 0$, $s \in [0,1]$. Thus, $P(s)$ is nondecreasing with respect to $s \in [0,1]$, and we easily get

$$P(s) = \Theta - \sum_{s \le \xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^{n-2} \le \Theta, \quad s \in [0,1].$$

On the other hand, $P(s)$ is nondecreasing on $[0,1]$, so we have

$$\begin{aligned} P(s) &= \Theta - \sum_{s\le\xi_j}\eta_j\left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1}(1-s)^{n-2} \\ &\ge P(0) = \Theta - \sum_{j=1}^{\infty}\eta_j\xi_j^{\alpha-1} > 0, \quad s\in[0,1]. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial^j}{\partial t^j}G(t,s) &\le \frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-j)} \\ &\le \frac{\Theta}{\Gamma(\alpha-j)P(0)}, \quad (t,s)\in[0,1]\times[0,1], j=0,1,2,\dots,n-2. \end{aligned}$$

Next, we will prove that

$$\frac{\partial^j}{\partial t^j}G(t,s)\ge 0, \quad (t,s)\in[0,1]\times[0,1], j=0,1,2,\dots,n-2.$$

Since $\frac{1-s}{t-s}$ is increasing with respect to s on $(0,t)$, we get that $\frac{1-s}{t-s} > \frac{1}{t}$. For $0\le s\le t\le 1$, we get

$$\begin{aligned} &P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-j-1} \\ &\quad\ge P(0)(t-s)^{\alpha-n+1}\left[t^{\alpha-j-1}\left(\frac{1-s}{t-s}\right)^{\alpha-n+1} - (t-s)^{n-2-j}\right] \\ &\quad\ge P(0)(t-s)^{\alpha-n+1}\left[t^{n-i-2} - (t-s)^{n-i-2}\right]\ge 0. \end{aligned}$$

On the other hand, for $0\le t\le s\le 1$, we get

$$\frac{\partial^j}{\partial t^j}G(t,s) = \frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-j)}\ge 0, \quad j=0,1,2,\dots,n-2.$$

Hence,

$$\frac{\partial^j}{\partial t^j}G(t,s)\ge 0, \quad (t,s)\in[0,1]\times[0,1], j=0,1,2,\dots,n-2,$$

so the proof of (2) is completed. We further prove (3).

(3) For $j=0,1,2,\dots,n-3$, $n-j\ge 3$, we have

$$\begin{aligned} \int_0^1\frac{\partial^j}{\partial t^j}G(t,s)\,ds &= \int_0^t\frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-j-1}}{P(0)\Gamma(\alpha-j)}\,ds \\ &\quad + \int_t^1\frac{P(s)t^{\alpha-j-1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-j)}\,ds \\ &\ge \frac{P(0)}{P(0)\Gamma(\alpha-j)}\left(t^{\alpha-j-1}\int_0^1(1-s)^{\alpha-n+1}\,ds - \int_0^t(t-s)^{\alpha-j-1}\,ds\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha - j)}\left(\frac{t^{\alpha-j-1}}{\alpha - n + 2} - \frac{t^{\alpha-j}}{\alpha - j}\right) \\
&= \frac{t^{\alpha-j-1}[\alpha - j - t(\alpha - n + 2)]}{(\alpha - n + 2)\Gamma(\alpha - j + 1)} \\
&\geq \frac{t^{\alpha-j-1}}{(\alpha - n + 2)\Gamma(\alpha - j + 1)}, \quad t \in [0,1].
\end{aligned} \tag{2.6}$$

Changing j of (2.5) by $n-2$, we have

$$\frac{\partial^{(n-2)}}{\partial t^{(n-2)}} G(t,s) = \begin{cases} \frac{P(s)t^{\alpha-n+1}(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-n+2)}, & 0 \leq s \leq t \leq 1, \\ \frac{P(s)t^{\alpha-n+1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha-n+2)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.7}$$

Hence, for $t \in [0,1]$, we have

$$\begin{aligned}
\int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\, ds &= \int_0^t \frac{P(s)t^{\alpha-n+1}(1-s)^{\alpha-n+1} - P(0)(t-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha - n + 2)}\, ds \\
&\quad + \int_t^1 \frac{P(s)t^{\alpha-n+1}(1-s)^{\alpha-n+1}}{P(0)\Gamma(\alpha - n + 2)}\, ds \\
&\geq \frac{P(0)}{P(0)\Gamma(\alpha - n + 2)}\left(t^{\alpha-n+1}\int_0^1 (1-s)^{\alpha-n+1}\, ds - \int_0^t (t-s)^{\alpha-n+1}\, ds\right) \\
&= \frac{1}{\Gamma(\alpha - n + 2)}\left(\frac{t^{\alpha-n+1}}{\alpha - n + 2} - \frac{t^{\alpha-n+2}}{\alpha - n + 2}\right) \\
&= \frac{t^{\alpha-n+1}(1-t)}{\Gamma(\alpha - n + 3)} \\
&\geq \frac{t(1-t)}{\Gamma(\alpha - n + 3)}.
\end{aligned}$$

It is clear that

$$\int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\, ds \geq \frac{t(1-t)}{\Gamma(\alpha - n + 3)}, \quad t \in [0,1].$$

The proof of Lemma 2.6 is completed. □

3 Auxiliary regular problem

Let $E = C^{n-2}[0,1]$ and define the cone K in E as

$$K = \{u \in E, u^{(i)}(0) = 0, u^{(i)}(t) \geq 0, t \in [0,1], i = 0,1,2,\dots,n-2\}.$$

By Lemma 2.4 and (2.1) we have

$$D_{0^+}^{\mu_i} u \in C[0,1], \qquad D_{0^+}^{\mu_i} u(t) \geq 0, \quad u \in K, t \in [0,1], i = 0,1,2,\dots,n-2. \tag{3.1}$$

For any $m \in \mathbb{N}$, define the operator $Q_m : K \to E$ as follows:

$$(Q_m u)(t) = \int_0^1 G(t,s) f_m\big(s, u(s), D_{0^+}^{\mu_1} u(s), D_{0^+}^{\mu_2} u(s), \dots, D_{0^+}^{\mu_{n-2}} u(s)\big)\, ds. \tag{3.2}$$

Lemma 3.1 *Let* (H_0)-(H_3) *hold. Then, for any* $m \in \mathbb{N}$, $Q_m : K \to K$ *is a completely continuous operator.*

Proof First, we show that $Q_m : K \to K$. Given $u \in K$, by Lemma 2.6 we get that

$$\frac{\partial^j}{\partial t^j} G(t,s), \quad j = 0,1,2,\ldots,n-2,$$

are nonnegative and continuous on $[0,1] \times [0,1]$ and $G(0,s) = 0$ for $s \in [0,1]$. So we have

$$\begin{aligned} &Q_m u \in C^{n-2}[0,1], \qquad (Q_m u)^{(j)}(0) = 0, \quad j = 0,1,2,\ldots,n-2, \\ &(Q_m u)^{(j)}(t) \geq 0, \quad t \in [0,1], j = 0,1,2,\ldots,n-2. \end{aligned}$$

As a result, $Q_m : K \to K$.

In order to prove that Q_m is a continuous operator, let $\{u_k\} \subset K$ be a convergent sequence. Suppose that $\lim_{k\to\infty} u_k = u \in K$. Then

$$\lim_{k\to\infty} u_k^{(j)}(t) = u^{(j)}(t), \quad j = 0,1,2,\ldots,n-2,$$

uniformly for $t \in [0,1]$. For $i-1 < \mu_i \leq i$ $(i = 1,2,\ldots,n-2)$ and $t \in [0,1]$, we get

$$\left|D_{0^+}^{\mu_i} u_k(t) - D_{0^+}^{\mu_i} u(t)\right| \leq \frac{\|u_k^{(n-2)} - u^{(n-2)}\|}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-\mu_i-3}\, ds \leq \frac{\|u_k^{(n-2)} - u^{(n-2)}\|}{\Gamma(n-1-\mu_i)},$$

so we get

$$\lim_{k\to\infty} D_{0^+}^{\mu_i} u_k(t) = D_{0^+}^{\mu_i} u(t), \quad i = 1,2,\ldots,n-2$$

uniformly for $t \in [0,1]$. Moreover, $\{u_k\} \subset K$ is a convergent sequence. There exists $r > 0$ such that $\|u_k\|_2 \leq r$ $(k \in \mathbb{N})$. Then $\|u_k^{(j)}\| \leq r$ $(j = 0,1,2,\ldots,n-2; k \in \mathbb{N})$. For $i-1 < \mu_i \leq i$ $(i = 1,2,\ldots,n-2)$, by (2.1), for any $t \in [0,1]$, we have

$$\begin{aligned} 0 \leq D_{0^+}^{\mu_i} u_k(t) &\leq \frac{\|u_k^{(n-2)}\|}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-\mu_i-3}\, ds \\ &\leq \frac{\|u_k^{(n-2)}\|}{\Gamma(n-1-\mu_i)} \leq \frac{r}{\Gamma(n-1-\mu_i)}, \quad i = 0,1,2,\ldots,n-2, k \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Let

$$\begin{aligned} &\rho_k(t) = f_m\bigl(t, u_k(t), D_{0^+}^{\mu_1} u_k(t), D_{0^+}^{\mu_2} u_k(t), \ldots, D_{0^+}^{\mu_{n-2}} u_k(t)\bigr), \\ &\rho(t) = f_m\bigl(t, u(t), D_{0^+}^{\mu_1} u(t), D_{0^+}^{\mu_2} u(t), \ldots, D_{0^+}^{\mu_{n-2}} u(t)\bigr), \quad t \in [0,1], k \in \mathbb{N}. \end{aligned} \tag{3.4}$$

For $s \in [0,1] \setminus \Gamma$, where $\mathrm{mes}(\Gamma) = 0$, $f_m(s, x_1, x_2, \ldots, x_{n-1})$ is continuous on $\mathbb{R}_+^{n-1}$ with respect to x_i, so $f_m(s, x_1, x_2, \ldots, x_{n-1})$ is uniformly continuous with respect to x_i on

$$[0,r] \times \left[0, \frac{r}{\Gamma(n-1-\mu_1)}\right] \times \cdots \times \left[0, \frac{r}{\Gamma(n-1-\mu_{n-2})}\right].$$

Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x_1^1, x_1^2 \in [0,r], x_2^1, x_2^2 \in [0, \frac{r}{\Gamma(n-1-\mu_1)}]$, $\ldots, x_{n-1}^1, x_{n-1}^2 \in [0, \frac{r}{\Gamma(n-1-\mu_{n-2})}]$, $|x_1^1 - x_1^2| < \delta, |x_2^1 - x_2^2| < \delta, \ldots, |x_{n-1}^1 - x_{n-1}^2| < \delta$, we have

$$\left|f_m\left(s, x_1^1, x_2^1, \ldots, x_{n-1}^1\right) - f_m\left(s, x_1^2, x_2^2, \ldots, x_{n-1}^2\right)\right| < \varepsilon. \tag{3.5}$$

Since $\|u_k - u\|_2 \to 0$, for the above $\delta > 0$, there exists $N \in \mathbb{N}$ such that, for $k > N$,

$$\begin{aligned}
&\left|u_k(t) - u(t)\right|, \left|D_{0^+}^{\mu_1} u_k(t) - D_{0^+}^{\mu_1} u(t)\right|, \ldots, \left|D_{0^+}^{\mu_{n-2}} u_k(t) - D_{0^+}^{\mu_{n-2}} u(t)\right| \\
&\quad \leq \|u_k - u\|_2 < \delta, \quad \forall t \in [0,1].
\end{aligned}$$

Therefore, for $k > N$, by (3.5) we have

$$\begin{aligned}
\left|\rho_k(s) - \rho(s)\right| \leq{}& \left|f_m\left(s, u_k(s), D_{0^+}^{\mu_1} u_k(s), D_{0^+}^{\mu_2} u_k(s), \ldots, D_{0^+}^{\mu_{n-2}} u_k(s)\right)\right. \\
&\left. - f_m\left(s, u(s), D_{0^+}^{\mu_1} u(s), D_{0^+}^{\mu_2} u(s), \ldots, D_{0^+}^{\mu_{n-2}} u(s)\right)\right| < \varepsilon.
\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\rho_k(s) \to \rho(s) \quad \text{for a.e. } s \in [0,1]. \tag{3.7}$$

By (1.4) we have

$$\begin{aligned}
0 &\leq \rho_k(t) = f_m\left(t, u_k(t), D_{0^+}^{\mu_1} u_k(t), D_{0^+}^{\mu_2} u_k(t), \ldots, D_{0^+}^{\mu_{n-2}} u_k(t)\right) \\
&\leq \beta(t) p\left(\max\left\{u_k(t), \frac{1}{m}\right\}, \max\left\{D_{0^+}^{\mu_1} u_k(t), \frac{1}{m}\right\}, \ldots, \max\left\{D_{0^+}^{\mu_{n-2}} u_k(t), \frac{1}{m}\right\}\right) \\
&\quad + \gamma(t) h\left(u_k(t) + \frac{1}{m}, D_{0^+}^{\mu_1} u_k(t) + \frac{1}{m}, \ldots, D_{0^+}^{\mu_{n-2}} u_k(t) + \frac{1}{m}\right) \\
&\leq \beta(t) p\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right) \\
&\quad + \gamma(t) h\left(r + 1, \frac{r}{\Gamma(n-1-\mu_1)} + 1, \ldots, \frac{r}{\Gamma(n-1-\mu_{n-2})} + 1\right) \\
&= \varphi_r(t), \quad k = 1, 2, \ldots.
\end{aligned} \tag{3.8}$$

By the integrability of $\beta(t)$, $\gamma(t)$ on $[0,1]$ we get that $\varphi_r \in L^1(0,1)$, and by (3.7) and (3.8) we have

$$\left|\rho_k(t) - \rho(t)\right| \leq 2\varphi_r(t) \quad \text{for a.e. } t \in [0,1], k = 1, 2, 3, \ldots.$$

It follows from the relations in Lemma 2.6 and the Lebesgue dominated convergence theorem that, for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\left|(Q_m u_k)^{(j)}(t) - (Q_m u)^{(j)}(t)\right| \\
&\quad = \left|\int_0^1 \frac{\partial^j}{\partial t^j} G(t,s)\left[f_m\left(s, u_k(s), D_{0^+}^{\mu_1} u_k(s), D_{0^+}^{\mu_2} u_k(s), \ldots, D_{0^+}^{\mu_{n-2}} u_k(s)\right)\right.\right. \\
&\qquad \left.\left. - f_m\left(s, u(s), D_{0^+}^{\mu_1} u(s), D_{0^+}^{\mu_2} u(s), \ldots, D_{0^+}^{\mu_{n-2}} u(s)\right)\right] ds\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\Theta}{P(0)\Gamma(\alpha-j)}\int_0^1 \left|\rho_k(s)-\rho(s)\right| ds \to 0,\\
&\quad k\to\infty, t\in[0,1], \quad j=0,1,2,\ldots,n-2.
\end{aligned}$$

Hence, for any $m\in\mathbb{N}$, we have

$$\lim_{k\to\infty}(Q_m u_k)^{(j)}(t) = (Q_m u)^{(j)}(t), \quad j=0,1,2,\ldots,n-2$$

uniformly for $t\in[0,1]$. Therefore, for any $m\in\mathbb{N}$, Q_m is a continuous operator.

Now, for any bounded set $D\subset K$, we need to prove that $\{Q_m(D)\}$ is relatively compact in E. In order to apply the Arzelà-Ascoli theorem, we have to prove that $\{Q_m(D)\}$ is bounded in E and that,for any $m\in\mathbb{N}$, $\{(Q_m(D))^{(n-2)}(t)\}$ is equicontinuous on $[0,1]$. By the boundedness of $D\subset K$ there exists a positive number $R>0$ such that

$$\left\|u^{(j)}\right\| \leq R, \quad \forall u\in D, j=0,1,2,\ldots,n-2.$$

Then (3.3) means that

$$\left\|D_{0^+}^{\mu_i}u\right\| \leq \frac{R}{\Gamma(n-1-\mu_i)}, \quad \forall u\in D, i=1,2,\ldots,n-2.$$

Put ρ as in (3.4) and $0\leq\rho(t)\leq\varphi_R(t)$. Then, for $u\in D$, we have

$$\begin{aligned}
0 &\leq (Q_m u)^{(j)}(t) = \int_0^1 \frac{\partial^j G(t,s)}{\partial t^j}\rho(s)\,ds\\
&\leq \frac{\Theta}{P(0)\Gamma(\alpha-j)}\int_0^1 \varphi_R(s)\,ds\\
&= \frac{\Theta\|\varphi_R\|_1}{P(0)\Gamma(\alpha-j)}, \quad j=0,1,\ldots,n-2,
\end{aligned}$$

which shows that, for any $m\in\mathbb{N}$, $\{Q_m(D)\}$ is bounded in E. Moreover, for $0\leq t_1\leq t_2\leq 1$ and $u\in D$, we have

$$\begin{aligned}
&\left|(Q_m u)^{(n-2)}(t_2)-(Q_m u)^{(n-2)}(t_1)\right|\\
&= \left|\int_0^1\left(\frac{\partial^{n-2}G(t_2,s)}{\partial t^{n-2}}-\frac{\partial^{n-2}G(t_1,s)}{\partial t^{n-2}}\right)f_m\bigl(s,u(s),D_{0^+}^{\mu_1}u(s),D_{0^+}^{\mu_2}u(s),\ldots,D_{0^+}^{\mu_{n-2}}u(s)\bigr)\,ds\right|\\
&= \left|\int_0^1\left(\frac{\partial^{n-2}G(t_2,s)}{\partial t^{n-2}}-\frac{\partial^{n-2}G(t_1,s)}{\partial t^{n-2}}\right)\rho(s)\,ds\right|\\
&\leq e\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)\int_0^1(1-s)^{\alpha-n+1}\rho(s)\,ds\\
&\quad+\frac{1}{\Gamma(\alpha-n+2)}\left|\int_0^{t_2}(t_2-s)^{\alpha-n+1}\rho(s)\,ds-\int_0^{t_1}(t_1-s)^{\alpha-n+1}\rho(s)\,ds\right|\\
&\leq \|\rho\|_1 e\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)+\frac{1}{\Gamma(\alpha-n+2)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-n+1}\rho(s)\,ds\right.\\
&\quad\left.+\int_0^{t_1}\bigl((t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}\bigr)\rho(s)\,ds\right|
\end{aligned}$$

$$\begin{aligned}
&\leq e\|\varphi_R\|_1\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)+\frac{1}{\Gamma(\alpha-n+2)}\biggl[(t_2-t_1)^{\alpha-n+1}\|\varphi_R\|_1\\
&\quad+\int_0^{t_1}\bigl((t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}\bigr)\varphi_R(s)\,ds\biggr]\\
&\leq e\|\varphi_R\|_1\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)+e\biggl[(t_2-t_1)^{\alpha-n+1}\|\varphi_R\|_1\\
&\quad+\int_0^{t_1}\bigl((t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}\bigr)\varphi_R(s)\,ds\biggr],
\end{aligned}$$

where e is from (H_3). Since $(t-s)^{\alpha-n+1}$ is uniformly continuous on $[0,1]\times[0,1]$ and $t^{\alpha-n+1}$ is uniformly continuous on $[0,1]$, for any $\varepsilon>0$, there exists $\delta>0$ such that, for $0\leq t_1\leq t_2\leq 1$, $t_2-t_1<\delta$, $0<s\leq t_1$,

$$\begin{aligned}
&t_2^{\alpha-n+1}-t_1^{\alpha-n+1}<\varepsilon,\\
&(t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}<\varepsilon.
\end{aligned}$$

Consequently, for all $u\in D$, $0\leq t_1\leq t_2\leq 1$, and $t_2-t_1<\min\{\delta,\sqrt[\alpha-n+1]{\varepsilon}\}$, we have the inequality

$$\bigl|(Q_m u)^{(n-2)}(t_2)-(Q_m u)^{(n-2)}(t_1)\bigr|\leq 3e\|\varphi_R\|_1\varepsilon.$$

Hence, for any $m\in\mathbb{N}$, $\{(Q_m(D))^{(n-2)}(t)\}$ is equicontinuous on $[0,1]$. Therefore, for any $m\in\mathbb{N}$, $Q_m:K\to K$ is a completely continuous operator. □

To prove the main results, we need the following well-known fixed point theorem.

Lemma 3.2 [20] *Let K be a positive cone in a Banach space E, Ω_1 and Ω_2 be two bounded open sets in E such that $\theta\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$, and $A:K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator, where θ denotes the zero element of E. Suppose that one of the following two conditions holds*:

(i) $\|Au\|\leq\|u\|$, $\forall u\in K\cap\partial\Omega_1$; $\|Au\|\geq\|u\|$, $\forall u\in K\cap\partial\Omega_2$;

(ii) $\|Au\|\geq\|u\|$, $\forall u\in K\cap\partial\Omega_1$; $\|Au\|\leq\|u\|$, $\forall u\in K\cap\partial\Omega_2$.

Then A has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Theorem 3.1 *Let* (H_0)-(H_3) *hold. Then problem* (1.5) *has a solution $x_m\in K$, and*

$$\begin{aligned}
&x_m(t)\geq Mt^{\alpha-1},\\
&D_{0^+}^{\mu_i}x_m(t)\geq\frac{(n-2-\mu_i)M}{(n-i+1)!}t^{n-1-\mu_i},\quad i=1,2,\ldots,n-2, t\in[0,1], m\in\mathbb{N},
\end{aligned}\tag{3.9}$$

where M is defined by (H_2).

Proof By Lemma 3.1, $Q_m:K\to K$ is a completely continuous operator. Then by (1.2) and Lemma 2.6 we have

$$(Q_m u)(t)\geq C\int_0^1 G(t,s)\,ds\geq Mt^{\alpha-1},\tag{3.10}$$

and hence, $\|Q_m u\| \geq M$ and $\|Q_m u\|_2 \geq M$ for $u \in K$. Let $\Omega_1 = \{u \in E : \|u\|_2 < M\}$. Then

$$\|(Q_m u)\|_2 \geq \|u\|_2 \quad \text{for } u \in K \cap \partial\Omega_1.$$

Let $W_m = p(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$. For any $u \in K$ and $t \in [0,1]$, by Lemma 2.6 and (1.4) we have

$$\begin{aligned} 0 &\leq (Q_m u)^{(i)}(t) \\ &\leq e\int_0^1 \left(\beta(t)W_m + \gamma(s)h\left(u(s) + \frac{1}{m}, D_{0^+}^{\mu_1}u(s) + \frac{1}{m}, D_{0^+}^{\mu_2}u(s) + \frac{1}{m}, \ldots,\right.\right. \\ &\quad \left.\left. D_{0^+}^{\mu_{n-2}}u(s) + \frac{1}{m}\right)\right) ds \\ &\leq e\left(\|\beta\|_1 W_m + h\left(\|u\| + \frac{1}{m}, \|D_{0^+}^{\mu_1}u\| + \frac{1}{m}, \|D_{0^+}^{\mu_2}u\| + \frac{1}{m}, \ldots,\right.\right. \\ &\quad \left.\left. \|D_{0^+}^{\mu_{n-2}}u\| + \frac{1}{m}\right)\|\gamma\|_1\right), \quad i = 0,1,2,\ldots,n-3,n-2, \end{aligned} \tag{3.11}$$

where e is from (H_3).

For any $u \in K$ such that $\|u\|_2 \leq S$ $(S > M)$, by (3.3) and (3.11) we have

$$\begin{aligned} \|Q_m u\|_2 &\leq e\left(\|\beta\|_1 W_m + h\left(\|u\|_2 + 1, \frac{\|u\|_2}{\Gamma(n-1-\mu_1)} + 1, \ldots,\right.\right. \\ &\quad \left.\left. \frac{\|u\|_2}{\Gamma(n-1-\mu_{n-2})} + 1\right)\|\gamma\|_1\right) \\ &\leq e\left(\|\beta\|_1 W_m + h\left(\frac{S}{\Gamma(n-1-\mu_{n-2})} + 1, \frac{S}{\Gamma(n-1-\mu_{n-2})} + 1, \ldots,\right.\right. \\ &\quad \left.\left. \frac{S}{\Gamma(n-1-\mu_{n-2})} + 1\right)\|\gamma\|_1\right). \end{aligned} \tag{3.12}$$

By (H_3), taking $\overline{\lambda} > 0$ such that

$$\limsup_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} = \lambda < \overline{\lambda} < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1},$$

we have that there exists $G > M + 1$ such that, for any $x > G$,

$$h(x,x,\ldots,x) < \overline{\lambda}x. \tag{3.13}$$

Taking

$$S > \max\left\{M + 1, (G-1)\Gamma(n-1-\mu_{n-2}), \frac{e\|\beta\|_1 W_m + e\overline{\lambda}\|\gamma\|_1}{1 - e\overline{\lambda}\|\gamma\|_1\Gamma^{-1}(n-1-\mu_{n-2})}\right\},$$

we have $\frac{S}{\Gamma(n-1-\mu_{n-2})} + 1 > G$ and $S > M$. Let $\Omega_2 = \{u \in E : \|u\|_2 < S\}$. Then, for any $u \in K \cap \partial\Omega_2$, by (3.12) and (3.13) we get

$$\begin{aligned} \|Q_m u\|_2 &\leq e\left(\|\beta\|_1 W_m + \frac{\overline{\lambda}S\|\gamma\|_1}{\Gamma(n-1-\mu_{n-2})} + \overline{\lambda}\|\gamma\|_1\right) \\ &= e\|\beta\|_1 W_m + e\overline{\lambda}\|\gamma\|_1 + \frac{e\overline{\lambda}\|\gamma\|_1}{\Gamma(n-1-\mu_{n-2})}S \leq S. \end{aligned}$$

Hence,

$$\|Q_m u\|_2 \leq \|u\|_2 \quad \text{for } u \in K \cap \partial\Omega_2. \tag{3.14}$$

By Lemma 3.2 we get that the operator Q_m has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and, as a result, x_m is a solution of problem (1.5). Since x_m is a solution of problem (1.5),

$$\begin{aligned} x_m(t) &= \int_0^1 G(t,s) f_m\bigl(s, x_m(s), D_{0^+}^{\mu_1} x_m(s), D_{0^+}^{\mu_2} x_m(s), \ldots, D_{0^+}^{\mu_{n-2}} x_m(s)\bigr)\,ds, \\ &\quad t \in [0,1], m \in \mathbb{N}, \end{aligned} \tag{3.15}$$

and x_m satisfies $x_m(t) \geq Mt^{\alpha-1}$. In addition, Lemma 2.6 and (1.2) imply

$$\begin{aligned} x_m^{(j)}(t) &\geq C \int_0^1 \frac{\partial^j}{\partial t^j} G(t,s)\,ds \\ &\geq \frac{Ct^{\alpha-j-1}}{(\alpha-n+2)\Gamma(\alpha-j+1)}, \quad t \in [0,1], m \in \mathbb{N}, j = 0,1,2,\ldots,n-3, \\ x_m^{(n-2)}(t) &\geq C \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s)\,ds \geq \frac{Ct(1-t)}{\Gamma(\alpha-n+3)}, \quad t \in [0,1], m \in \mathbb{N}. \end{aligned}$$

By (2.1) we get

$$\begin{aligned} D_{0^+}^{\mu_i} x_m(t) &= \frac{1}{\Gamma(n-2-\mu_i)} \int_0^t (t-s)^{n-3-\mu_i} x_m^{(n-2)}(s)\,ds \\ &\geq \frac{C}{\Gamma(n-2-\mu_i)\Gamma(\alpha-n+3)} \int_0^t (t-s)^{n-\mu_i-3} s(1-s)\,ds, \quad i = 1,2,\ldots,n-2. \end{aligned}$$

Further, since

$$\begin{aligned} \int_0^t (t-s)^{n-\mu_i-3} s(1-s)\,ds &= \frac{1}{(n-2-\mu_i)} \int_0^t (t-s)^{n-2-\mu_i}(1-2s)\,ds \\ &= \frac{1}{(n-2-\mu_i)} \left(\frac{t^{n-1-\mu_i}}{n-1-\mu_i} - \frac{2t^{n-\mu_i}}{(n-1-\mu_i)(n-\mu_i)} \right) \\ &= \frac{t^{n-1-\mu_i}}{(n-2-\mu_i)} \frac{n-\mu_i-2t}{(n-1-\mu_i)(n-\mu_i)} \\ &\geq \frac{t^{n-1-\mu_i}}{(n-2-\mu_i)} \frac{n-2-\mu_i}{(n-1-\mu_i)(n-\mu_i)} \\ &= \frac{t^{n-1-\mu_i}}{(n-1-\mu_i)(n-\mu_i)}, \quad i = 1,2,\ldots,n-2, \end{aligned}$$

we get

$$\begin{aligned} D_{0^+}^{\mu_i} x_m(t) &\geq \frac{C}{\Gamma(n-2-\mu_i)\Gamma(\alpha-n+3)} \frac{t^{n-1-\mu_i}}{(n-1-\mu_i)(n-\mu_i)} \\ &\geq \frac{C(n-2-\mu_i)}{\Gamma(n+1-\mu_i)\Gamma(\alpha-n+3)} t^{n-1-\mu_i}, \quad i = 1,2,\ldots,n-2. \end{aligned} \tag{3.16}$$

Furthermore,

$$\Gamma(n+1-\mu_i) < \Gamma\bigl(n-(i-2)\bigr) = (n-i+1)!, \quad i = 1,2,\ldots,n-2, \tag{3.17}$$

and then it follows from $x_m(t) \geq Mt^{\alpha-1}$, (3.16) and (3.17) that, for $t \in [0,1]$ and $m \in \mathbb{N}$, we have

$$x_m(t) \geq Mt^{\alpha-1}, \qquad D_{0^+}^{\mu_i} x_m(t) \geq \frac{(n-2-\mu_i)M}{(n-i+1)!} t^{n-1-\mu_i}, \quad i = 1,2,\ldots,n-2, \tag{3.18}$$

where M is defined by (H_2). □

In order to finish the main result, we also need the following lemma.

Lemma 3.3 *Let x_m be a solution of problem* (1.5). *If* (H_0)-(H_3) *hold, then the sequence* $\{x_m\}$ *is relatively compact in E.*

Proof For $t \in [0,1]$ and $m \in \mathbb{N}$, since p is nondecreasing, by (3.18) we have

$$\begin{aligned} &p\bigl(x_m(t), D_{0^+}^{\mu_1} x_m(t), \ldots, D_{0^+}^{\mu_{n-2}} x_m(t)\bigr) \\ &\quad \leq p\left(Mt^{\alpha-1}, \frac{(n-2-\mu_1)M}{n!} t^{n-1-\mu_1}, \ldots, \frac{(n-2-\mu_{n-2})M}{3!} t^{n-1-\mu_{n-2}}\right), \end{aligned} \tag{3.19}$$

and by Lemma 2.6, (1.4), (3.3), (3.15), and (3.19), for $t \in [0,1]$ and $m \in \mathbb{N}$, we get

$$\begin{aligned} 0 &\leq x_m^{(i)}(t) \\ &\leq \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) f_m\bigl(s, x_m(s), D_{0^+}^{\mu_1} x_m(s), \ldots, D_{0^+}^{\mu_{n-2}} x_m(s)\bigr)\, ds \\ &\leq e \int_0^1 \beta(t) p\left(Mt^{\alpha-1}, \frac{(n-2-\mu_1)M}{n!} t^{n-1-\mu_1}, \ldots, \frac{(n-2-\mu_{n-2})M}{3!} t^{n-1-\mu_{n-2}}\right) ds \\ &\quad + eh\left(\|x_m\|_2 + \frac{1}{m}, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_1)} + \frac{1}{m}, \ldots, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_{n-2})} + \frac{1}{m}\right) \int_0^1 \gamma(s)\, ds \\ &= e\left(\Upsilon + h\left(\|x_m\|_2 + 1, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_1)} + 1, \ldots, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_{n-2})} + 1\right) \|\gamma\|_1\right), \\ &\quad i = 0,1,2,\ldots,n-3,n-2, \end{aligned} \tag{3.20}$$

where

$$\Upsilon = \int_0^1 \beta(t) p\left(Mt^{\alpha-1}, \frac{(n-2-\mu_1)M}{n!} t^{n-1-\mu_1}, \ldots, \frac{(n-2-\mu_{n-2})M}{3!} t^{n-1-\mu_{n-2}}\right) ds.$$

By (H_2) and (3.20) we have

$$\begin{aligned} &\|x_m\|_2 \leq e\left(\Upsilon + h\left(\|x_m\|_2 + 1, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_1)} + 1, \ldots, \frac{\|x_m\|_2}{\Gamma(n-1-\mu_{n-2})} + 1\right) \|\gamma\|_1\, ds\right), \\ &\quad m \in \mathbb{N}, \end{aligned} \tag{3.21}$$

and $\Upsilon < \infty$.

By (H_3), taking $\lambda_1 > 0$ such that

$$\limsup_{x\to\infty} \frac{h(x,x,\dots,x)}{x} = \lambda < \lambda_1 < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1},$$

we have that there exists $A > M+1$ such that, for any $x > A$,

$$h(x,x,\dots,x) < \lambda_1 x. \tag{3.22}$$

In order to prove that $\{x_m\} \subset K$ is relatively compact in $E = C^{(n-2)}[0,1]$, we need to prove that $\{x_m\}$ is bounded in E and $\{x_m\}$ is equicontinuous on $[0,1]$. First, we prove that $\{x_m\}$ is bounded in E. If $\{x_m\}$ is unbounded, then there exists a subsequence $\{x_{m_j}\} \subset \{x_m\}$ such that $\|x_{m_j}\|_2 \to +\infty$, and then there exists j_0 such that

$$\|x_{m_{j_0}}\|_2 > \max\left\{M+1, (A-1)\Gamma(n-1-\mu_{n-2}), \frac{e\Upsilon + e\lambda_1\|\gamma\|_1}{1-e\lambda_1\|\gamma\|_1\Gamma^{-1}(n-1-\mu_{n-2})}\right\},$$

and then $\frac{\|x_{m_{j_0}}\|_2}{\Gamma(n-1-\mu_{n-2})} + 1 > A$, and by (3.21) and (3.22) we get

$$\begin{aligned}\|x_{m_{j_0}}\|_2 &\le e\left(\Upsilon + \frac{\lambda_1\|\gamma\|_1\|x_{m_{j_0}}\|_2}{\Gamma(n-1-\mu_{n-2})} + \lambda_1\|\gamma\|_1\right)\\ &= e\Upsilon + e\lambda_1\|\gamma\|_1 + \frac{b\lambda_1\|\gamma\|_1}{\Gamma(n-1-\mu_{n-2})}\|x_{m_{j_0}}\|_2\\ &< \|x_{m_{j_0}}\|_2.\end{aligned}$$

This is a contradiction, which means that $\{x_m\}$ is bounded in E. Next, we will prove that $\{x_m^{(n-2)}(t)\}$ is equicontinuous on $[0,1]$. Since $\{x_m\}$ is bounded in E, there exists $\Lambda > 0$ such that $\|x_m\| \le \Lambda$. Let

$$\begin{aligned}&V_1 = h\left(\Lambda+1, \frac{\Lambda}{\Gamma(n-1-\mu_1)}+1,\dots,\frac{\Lambda}{\Gamma(n-1-\mu_{n-2})}+1\right),\\ &\phi(t) = \beta(t)p\left(Mt^{\alpha-1}, \frac{(n-2-\mu_1)M}{n!}t^{n-1-\mu_1},\dots,\frac{(n-2-\mu_{n-2})M}{3!}t^{n-1-\mu_{n-2}}\right),\\ &\quad t\in(0,1].\end{aligned} \tag{3.23}$$

Then $\Upsilon = \int_0^1 \phi(t)\,dt$, and for any $m \in \mathbb{N}$ and a.e. $t \in [0,1]$,

$$f_m\bigl(t, x_m(t), D_{0^+}^{\mu_1}x_m(t), D_{0^+}^{\mu_2}x_m(t),\dots,D_{0^+}^{\mu_{n-2}}x_m(t)\bigr) \le \phi(t) + V_1\gamma(t). \tag{3.24}$$

Assume that $0 \le t_1 < t_2 \le 1$. Then by (3.24), for any $m \in \mathbb{N}$, we have

$$\begin{aligned}&\bigl|x_m^{(n-2)}(t_2) - x_m^{(n-2)}(t_1)\bigr|\\ &\quad= \left|\int_0^1 \left(\frac{\partial^{n-2}}{\partial t^{n-2}}G(t_2,s) - \frac{\partial^{n-2}}{\partial t^{n-2}}G(t_1,s)\right)\right.\\ &\qquad\left.\times f_m\bigl(s, x_m(s), D_{0^+}^{\mu_1}x_m(s), D_{0^+}^{\mu_2}x_m(t),\dots,D_{0^+}^{\mu_{n-2}}x_m(t)\bigr)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq e\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)\int_0^1\bigl(\phi(s)+V_1\gamma(s)\bigr)\,ds \\
&\quad+e\left|\int_0^{t_2}(t_2-s)^{\alpha-n+1}\bigl(\phi(s)+V_1\gamma(s)\bigr)\,ds-\int_0^{t_1}(t_1-s)^{\alpha-n+1}\bigl(\phi(s)+V_1\gamma(s)\bigr)\,ds\right| \\
&\leq e\bigl(\Upsilon+V_1\|\gamma\|_1\bigr)\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)+e\biggl[\int_{t_1}^{t_2}(t_2-s)^{\alpha-n+1}\bigl(\Upsilon+V_1\|\gamma\|_1\bigr)\,ds \\
&\quad+\int_0^{t_1}\bigl((t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}\bigr)\bigl(\Upsilon+V_1\|\gamma\|_1\bigr)\,ds\biggr] \\
&\leq e\bigl(\Upsilon+V_1\|\gamma\|_1\bigr)\bigl(t_2^{\alpha-n+1}-t_1^{\alpha-n+1}\bigr)+e\biggl[(t_2-t_1)^{\alpha-n+1}\bigl(\Upsilon+V_1\|\gamma\|_1\bigr) \\
&\quad+\int_0^{t_1}\bigl((t_2-s)^{\alpha-n+1}-(t_1-s)^{\alpha-n+1}\bigr)\bigl(\Upsilon+V_1\|\gamma\|_1\bigr)\,ds\biggr].
\end{aligned}
$$

Hence, we can prove that $\{x_m^{(n-2)}(t)|m=1,2,\ldots\}$ is equicontinuous on $[0,1]$. □

4 Proof of Theorem 1.1

Proof of Theorem 1.1 According to Theorem 3.1, we know that (1.5) has a solution $x_m\in K$ for any $m\in\mathbb{N}$. Moreover, Lemma 3.3 implies that $\{x_m\}$ is relatively compact in E and satisfies inequality (3.16) for $t\in[0,1]$ and $m\in\mathbb{N}$. The sequence $\{x_m\}$ has a subsequence converging to $x^\star\subset K$. Without loss of generality, we still assume that $\{x_m\}$ itself uniformly converges to $x^\star$. So $x^\star\in K$ satisfies the boundary conditions of (1.1), and according to (2.1), we get

$$
\lim_{m\to\infty}D_{0^+}^{\mu_i}x_m=D_{0^+}^{\mu_i}x^\star,\quad i=1,2,\ldots,n-2,
$$

in $C[0,1]$. Take the limit in (3.16) as $m\to\infty$. Then $x^\star$ satisfies (1.3). Moreover, for a.e. $t\in[0,1]$,

$$
\begin{aligned}
&\lim_{m\to\infty}f_m\bigl(t,x_m(t),D_{0^+}^{\mu_1}x_m(t),D_{0^+}^{\mu_2}x_m(t),\ldots,D_{0^+}^{\mu_{n-2}}x_m(t)\bigr) \\
&\quad=f\bigl(t,x^\star(t),D_{0^+}^{\mu_1}x^\star(t),D_{0^+}^{\mu_2}x^\star(t),\ldots,D_{0^+}^{\mu_{n-2}}x^\star(t)\bigr).
\end{aligned}
$$

By (3.3) we get

$$
\bigl\|D_{0^+}^{\mu_i}x_m\bigr\|\leq\frac{\Lambda}{\Gamma(n-1-\mu_i)},\quad m\in\mathbb{N},i=1,2,\ldots,n-2.
$$

Therefore, for all $m\in\mathbb{N}$ and a.e. $(t,s)\in[0,1]\times[0,1]$, we get

$$
\begin{aligned}
0&\leq G(t,s)f_m\bigl(s,x_m(s),D_{0^+}^{\mu_1}x_m(s),D_{0^+}^{\mu_2}x_m(s),\ldots,D_{0^+}^{\mu_{n-2}}x_m(s)\bigr) \\
&\leq\frac{\Theta}{P(0)\Gamma(\alpha)}\left(\phi(s)+h\left(\Lambda+1,\frac{\Lambda}{\Gamma(n-1-\mu_1)}+1,\ldots,\frac{\Lambda}{\Gamma(n-1-\mu_{n-2})}+1\right)\gamma(s)\right),
\end{aligned}
\tag{4.1}
$$

where ϕ is defined by (3.23). Taking $m\to\infty$ in (3.15) and combining with (4.1), we obtain

$$
x^\star(t)=\int_0^1G(t,s)f\bigl(s,x^\star(s),D_{0^+}^{\mu_1}x^\star(s),D_{0^+}^{\mu_2}x^\star(s),\ldots,D_{0^+}^{\mu_{n-2}}x^\star(s)\bigr)\,ds,\quad t\in[0,1],
$$

by the Lebesgue dominated convergence theorem. Hence, $x^\star$ is a positive solution of problem (1.1) and satisfies inequality (1.3). □

5 Example

We consider the following nonlinear singular fractional differential equation:

$$\begin{cases} D_{0^+}^{\frac{9}{2}}u(t) + \frac{1}{t}(u(t))^{-a} + \frac{1}{t}(D_{0^+}^{\frac{1}{2}}u(t))^{-b} + \frac{1}{t}(D_{0^+}^{\frac{3}{2}}u(t))^{-c} + \frac{1}{t}(D_{0^+}^{\frac{5}{2}}u(t))^{-d} + 7 \\ \quad + \frac{t^2}{100}(4.5u(t) + (D_{0^+}^{\frac{1}{2}}u(t))^{b_1} + (D_{0^+}^{\frac{3}{2}}u(t))^{c_1} + (D_{0^+}^{\frac{5}{2}}u(t))^{d_1}) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, \\ u'''(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases} \tag{5.1}$$

where $\alpha = \frac{9}{2}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{3}{2}$, $\mu_3 = \frac{5}{2}$, $\eta_j = \frac{14}{j^2}$, $\xi_j = \frac{1}{j^{\frac{9}{7}}}$, $a \in (0, \frac{2}{7})$, $b \in (0, \frac{1}{9})$, $c \in (0, \frac{2}{7})$, $d \in (0, \frac{2}{5})$, $b_1, c_1, d_1 \in (0,1)$. Letting $\gamma(t) = \frac{3+t^{\frac{1}{2}}}{2{,}100t^{\frac{1}{2}}}$ and $\beta(t) = \frac{1+t^{\frac{1}{2}}}{t^{\frac{1}{2}}}$, we easily get

$$\begin{aligned} &\sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1} = \sum_{j=1}^{\infty} \frac{14}{j^2}\left(\frac{1}{j^{\frac{9}{7}}}\right)^{\frac{7}{2}} \approx 12.8492 < \Theta = \frac{3}{2} \times \frac{5}{2} \times \frac{7}{2} = \frac{105}{8} = 13.1250, \\ &P(0) = \Theta - \sum_{j=1}^{\infty} \eta_j \xi_j^{\alpha-1} \approx 13.1250 - 12.8492 = 0.2758, \\ &\Gamma(n-1-\mu_{n-2}) = \Gamma(\alpha - n + 2) = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \approx \frac{1.7725}{2} = 0.8863, \\ &e = \frac{\Theta}{P(0)\Gamma(\alpha - n + 2)} \approx \frac{13.1250}{0.2758 \times 0.8863} = \frac{13.1250}{0.2444} \approx 53.7029, \\ &\frac{\Gamma(n-1-\mu_{n-2})}{b\|\gamma\|_1} \approx \frac{0.8863}{53.7029 \times 0.0033} \approx 5.0017. \end{aligned}$$

Then the function

$$f(t,x,y,z,w) = \frac{1}{t}x^{-a} + \frac{1}{t}y^{-b} + \frac{1}{t}z^{-c} + \frac{1}{t}w^{-d} + 7 + \frac{3 + t^{\frac{1}{2}}}{2{,}100t^{\frac{1}{2}}}\left(4.5x + y^{b_1} + z^{c_1} + w^{d_1}\right)$$

satisfies the hypotheses (H_0), (H_1), and (H_2) for

$$\begin{aligned} &p(x,y,z,w) = x^{-a} + y^{-b} + z^{-c} + w^{-d} + 7, \\ &\gamma(t) = \frac{3 + t^{\frac{1}{2}}}{2{,}100t^{\frac{1}{2}}}, \qquad \beta(t) = \frac{1 + t^{\frac{1}{2}}}{t^{\frac{1}{2}}}, \\ &h(x,y,z,w) = 4.5x + y^{b_1} + z^{c_1} + w^{d_1}, \end{aligned}$$

and

$$\begin{aligned} \limsup_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} &= \limsup_{x\to\infty} \frac{4.5x + x^{b_1} + x^{c_1} + x^{d_1}}{x} = 4.5000 \\ &= \lambda < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1} \approx 5.0017, \end{aligned}$$

so hypothesis (H_3) is also satisfied. Therefore, Theorem 1.1 guarantees that the fractional differential equation (5.1) has one positive solution u satisfying inequality (1.3) for $t \in [0,1]$, where $M = \frac{7}{(\alpha-4)\Gamma(\alpha+1)}$.

Remark 5.1 In the examples of [15], the index of the independent variables of h cannot be 1, but the index of the independent variables of h can be 1 in this paper because $\lim_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} = 0$ is replaced by $\limsup_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} = \lambda < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1}$.

6 Conclusions

In this paper, some existence results are obtained successfully for the boundary value problem (1.1) for the case where the nonlinearity is allowed to be singular with respect to not only the time variable but also the space variable and also the boundary conditions may involve infinite number of points. Compared with previous work [15–17], we complete the proof without imposing the third Carathéodory condition, that is, the condition $|f(t,x_1,x_2,\ldots,x_{n-1})| \le \varphi_H(t)$ is successfully removed, and, at the same time, the condition

$$\lim_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} = 0$$

is extended to

$$\limsup_{x\to\infty} \frac{h(x,x,\ldots,x)}{x} = \lambda < \frac{\Gamma(n-1-\mu_{n-2})}{e\|\gamma\|_1},$$

which leads to more general results. Moreover, the results of [15] seem to be wrong when $\lim_{x\to 0} \frac{h(x,x,\ldots,x)}{x} = 0$. So we have improved the result of [15–17]. The method we utilized for the analysis is the sequential technique and regularization, and the existence of positive solutions is obtained by the fixed point theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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