Robust arbitrary pole placement with the extended Kautsky-Nichols-van Dooren parametric form

Robert Schmid, Thang Nguyen and Lorenzo Ntogramatzidis

***Abstract*— We consider the classic problem of pole placement by state feedback. Our recent work [1] offered an eigenstructure assignment algorithm to obtain a novel parametric form for the pole-placing gain matrix to deliver any set of desired closed-loop eigenvalues, with any desired multiplicities. The method was adapted from the classic eigenstructure assignment algorithm of Kautsky, Nichols and van Dooren [2]. In this paper we employ this parametric formula to introduce an unconstrained nonlinear optimisation algorithm to obtain a gain matrix that delivers any desired pole placement with optimal robustness.**

I. Introduction

We consider the classic problem of repeated pole placement for linear time-invariant (LTI) systems in state space form

$$\dot{x}(t) = A\,x(t) + B\,u(t), \tag{1}$$

where, for all $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input, and A and B are appropriate dimensional constant matrices. We assume that B has full column-rank, and that the pair (A,B) is reachable. We let $\mathscr{L} = \{\lambda_1,\ldots,\lambda_\nu\}$ be a self-conjugate set of $\nu \le n$ complex numbers, with associated algebraic multiplicities $\mathscr{M} = \{m_1,\ldots,m_\nu\}$ satisfying $m_1+\cdots+m_\nu = n$. The problem of arbitrary *exact pole placement (EPP) by state feedback* is that of finding a real gain matrix F such that the closed-loop matrix $A+BF$ has eigenvalues given by the set $\mathscr{L}$ with multiplicities given by $\mathscr{M}$, i.e., F satisfies the equation

$$(A+BF)X = X\Lambda, \tag{2}$$

where Λ is a $n\times n$ Jordan matrix obtained from the eigenvalues of $\mathscr{L}$, including multiplicities, and X is a matrix of closed-loop eigenvectors (or generalised eigenvectors) of unit length. The matrix Λ can be expressed in the Jordan (complex) canonical form

$$\Lambda = \text{blkdiag}\{J(\lambda_1), J(\lambda_2), \cdots, J(\lambda_\nu)\} \tag{3}$$

where each $J(\lambda_i)$ is a Jordan matrix for λ_i of order m_i, and may be composed of up to g_i mini-blocks

$$J(\lambda_i) = \text{blkdiag}\{J_1(\lambda_i), J_2(\lambda_i), \cdots, J_{g_i}(\lambda_i)\} \tag{4}$$

where $g_i \le m$. We use $\mathscr{P} \stackrel{\text{def}}{=} \{p_{i,k}\,|\,1\le i\le \nu, 1\le k\le g_i\}$ to denote the orders of the Jordan mini-blocks $J_k(\lambda_i)$ that comprise $J(\lambda_i)$. It is well-known that when (A,B) is a reachable pair, arbitrary multiplicities of the closed-loop eigenvalues can be assigned by state feedback, but the possible orders of the associated Jordan structures are constrained by the system *controllability indices* (or *Kronecker invariants*) [3]. If $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{P}$ satisfy the conditions of the Rosenbrock theorem, we say that the triple $(\mathscr{L},\mathscr{M},\mathscr{P})$ defines an *admissible Jordan structure* for (A,B).

Numerous parametric formulae for the set of gain matrices that deliver the desired pole placement have appeared in the literature in the past three decades. In [2], a method for obtaining suitable F was introduced involving a QR-factorisation for B and a Sylvester equation for X, which requires Λ in (2) to be a diagonal matrix. In particular this means that the desired multiplicities must satisfy $m_i \le m$ for all $i \in \{1,\ldots,\nu\}$. Both the widely-used MATLAB® routine `place.m` and the MATHEMATICA® routine `KNVD` are based on the algorithm proposed in [2]. In [4] this method was used to develop a parametric formula for X and F, in terms of a suitable parameter matrix.

Other parameterisations have been presented in the literature that do not impose a constraint on the multiplicity of the eigenvalues to be assigned. In [5] a procedure was given for obtaining the gain matrix by solving a Sylvester equation in terms of an $n\times m$ parameter matrix, provided the closed-loop eigenvalues do not coincide with the open-loop ones. In [6] a parametric formula was presented in terms of the inverses of the matrices $A-\lambda_i I_n$ (where I_n denotes the $n\times n$ identity matrix), which also requires that the closed-loop eigenvalues are be distinct from the open-loop ones.

More recently, in [7] the parametric formula of [2] was revisited for the case where Λ was any admissible Jordan matrix, and a parameterisations was obtained for the pole placing matrix F by using the eigenvector matrix X as a parameter. The case where $\mathscr{L}$ contains any desired closed-loop eigenvalues and multiplicities is also considered in [8], where a parametric form for F is presented in terms of the solution to a Sylvester equation, also using the eigenvector matrix X as a parameter. However, maximum generality in these parametric formulae has been achieved at the expense of efficiency. Where methods [2], [5], [6] all employ parameter matrices of dimension $m\times n$, the parameter matrices used in [7]-[8] have dimension $n\times n$.

In the recent paper [9], two of the present authors gave a novel parametric form for X and F based on the famous pole placement algorithm of Moore [10]. This parameterisation employed parameter matrices of dimension $m\times n$, but required Λ to be diagonal, and hence also assumed the

Robert Schmid is with the Department of Electrical and Electronic Engineering, University of Melbourne, Australia. E-mail: `rschmid@unimelb.edu.au`

Thang Nguyen is with the Department of Engineering, University of Exeter, UK. E-mail: `T.Nguyen-Tien@exeter.ac.uk`

Lorenzo Ntogramatzidis is with the Department of Mathematics and Statistics, Curtin University, Australia. E-mail: `L.Ntogramatzidis@curtin.edu.au`

closed-loop eigenvalues have multiplicities of at most m. Very recently the papers [11],[12] generalized this parametric form to accommodate arbitrary multiplicities; the method was based on the pole placement method of Klein and Moore [13]. The method avoids the need for matrix inversions, or the solution of Sylvester matrix equations. The principal merit of this approach was to obtain a parameterisation that combines the generality of [7] and [8] with the computational efficiency that comes from an $m\times n$ dimensional parameter matrix.

In [1] the present authors revisited the pole-placing feedback method of Kautsky, Nichols and van Dooren [2] and generalised it to obtain a parametric formula that can also accommodate arbitrary pole-placement, in terms an $m\times n$ dimensional parameter matrix. For a given real $m\times n$ parameter matrix K, we obtain the eigenvector matrix $X(K)$ and gain matrix $F(K)$ by building the Jordan chains of closed-loop generalised eigenvectors; the chains commence with the selection of eigenvectors from the kernel of certain matrix pencils. Thus the results of [1] neatly parallel the achievements of [11]-[12] in providing another novel parametric form to achieve pole placement with arbitrary multiplicities, while employing an $m\times n$-dimensional parameter matrix.

The virtue of having a comprehensive parametric formula for the matrices X and F that solve (2) is that they invite the consideration of optimal pole placement problems, in which one seeks a gain matrix that will deliver the desired closed-loop eigenstructure and also provide some other desirable features. One important such problem is the *minimum gain exact pole placement problem* (MGEPP), which involves solving the EPP problem and also obtaining the feedback matrix F that has the smallest gain, which is desirable in order reduce the control amplitude or energy used in achieving any desired closed-loop response. In the *robust exact pole placement problem* (REPP), we seek an F that solves the EPP problem and also renders the eigenvalues of $A+BF$ as insensitive to perturbations in A, B and F as possible. Numerous results [14] have appeared linking the sensitivity of the eigenvalues to various measures of the conditioning of X, the matrix of closed loop eigenvectors. A commonly used measure is the Frobenius condition number of X. For the case of diagonal Λ, there has been considerable literature on this problem. Papers addressing the REPP problem include [2], [4], [8], [9], [15], and [16].

Papers [11] and [17] employed a parametric formula based on the Klein-Moore method to consider the MGEPP and REPP problems, respectively. In [1] the present authors employed the parametric formula based on the Kautsky-Nichols-van Dooren method to consider the MGEPP problem. In this paper we complete this quartet of results by using the Kautsky-Nichols-van Dooren method to consider the REPP problem. Adopting a similar approach to these recent works, we introduce an unconstrained nonlinear optimisation problem that seeks the parameter matrix K that minimises the condition number of X with respect to the Frobenius norm. This optimisation problem is then addressed via gradient search methods.

As mentioned earlier, a novel feature of these two parametric forms is that they can accommodate arbitrary pole placement, including a possibly defective eigenstructure. To demonstrate the performance of our algorithm, we consider an example involving the assignment of deadbeat modes, and compare the performance against the methods of [8] and [17]. We see that the methods introduced in this paper are able to deliver the desired eigenstructure with superior robust conditioning to that of [8], and equivalent to that of [17].

We begin with some definitions and notation. We say that $\mathscr{L}$ is σ*-conformably ordered* if there exists an integer σ such that the first 2σ values of $\mathscr{L}$ are complex while the remaining are real, and for all odd $k\le 2\sigma$ we have $\lambda_{k+1}=\overline{\lambda}_k$. For example, the set $\mathscr{L}=\{10\,j,-10\,j,2+2\,j,2-2\,j,7\}$ is 2-conformably ordered. Notice that, since $\mathscr{L}$ is symmetric, we have $m_i=m_{i+1}$ for odd $i\le\sigma$. In the following we implicitly assume that an admissible Jordan structure $(\mathscr{L},\mathscr{M},\mathscr{P})$ is σ-*conformably ordered*, for some integer σ. For any matrix X we use $X(l)$ to denote the l-th column of X. The symbol 0_n represents the zero vector of length n, and I_n is the n-dimensional identity matrix.

Let X denote any complex matrix partitioned into submatrices $X=[X_1\ldots X_v]$ ordered such that any complex submatrices occur consecutively in complex conjugate pairs, and so that, for some integer s, the first $2s$ submatrices are complex while the remaining are real. We define a real matrix $\mathfrak{Re}\{X\}$ of the same dimension as X thus: if X_i and X_{i+1} are consecutive complex conjugate submatrices of X, then the corresponding submatrices of $\mathfrak{Re}\{X\}$ are $\frac{1}{2}(X_i+X_{i+1})$ and $\frac{1}{2j}(X_i-X_{i+1})$.

II. Pole placement methods

Since our pole-placing method employs the algorithm of [2] for the gain matrix F that solves the exact pole placement problem (2), for the case where Λ is a diagonal matrix, we first briefly review this classical earlier result.

Theorem 2.1: ([2, Theorem 3]) Given $\Lambda=\mathrm{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ and X non-singular, then there exists F, a solution to (2) if and only if

$$U_1^\top(AX-X\Lambda)=0, \tag{5}$$

where

$$B=[\,U_0\;\;U_1\,]\begin{bmatrix}Z\\0\end{bmatrix} \tag{6}$$

with $U=[\,U_0\;\;U_1\,]$ orthogonal and Z nonsingular. Then F is given by

$$F=Z^{-1}U_0^\top(X\Lambda X^{-1}-A) \tag{7}$$

We note that (6) uses a QR factorisation for B; Byers and Nash [4] pointed out that F may also be obtained from the singular value decomposition for B. Given $B=U\,S\,G^\top$, we let $U=[\,U_0\;\;U_1\,]$ and $S\,G^\top=\begin{bmatrix}Z\\0\end{bmatrix}$.

Corollary 2.1: ([2], Corollary 1) The eigenvector x_i of $A+BF$ corresponding to the assigned eigenvalue $\lambda_i\in\mathscr{L}$ must belong to the space

$$\mathscr{S}_i\stackrel{\text{def}}{=}\ker[U_1^\top(A-\lambda_i I_n)], \tag{8}$$

the null-space of $U_1^\top(A-\lambda_i I_n)$.

Byers and Nash used Corollary 2.1 to obtain a parametric form for the matrix of eigenvectors X satisfying (2), and then employed it to consider the REPP problem for the case of a diagonal Λ matrix. In [1] we adapted Corollary 2.1 to obtain a parametric form for X and F that can accommodate any admissible Jordan structure $(\mathscr{L},\mathscr{M},\mathscr{P})$ for (A,B), and we now briefly summarise this method.

We begin by noting that for each $i \in \{1,\dots,\nu\}$, each $\mathscr{S}_i$ in (8) has n rows and $n+m$ columns, and as the pair (A,B) is reachable, the dimension of $\mathscr{S}_i$ is equal to m. For each $i \in \{1,2,...,\nu\}$, we compute maximal rank matrices N_i and M_i satisfying

$$U_1^\top(A-\lambda_i I_n)N_i = 0, \quad U_1^\top(A-\lambda_i I_n)M_i = I_{n-m}. \tag{9}$$

Then N_i is a basis matrix for $\mathscr{S}_i$. It follows that, for each odd $i \le 2\sigma$, we have $N_{i+1} = \overline{N}_i$ because if $\lambda_{i+1} = \overline{\lambda}_i$.

For any σ-conformably ordered admissible Jordan structure $(\mathscr{L},\mathscr{M},\mathscr{P})$, we say that a $m \times n$ parameter matrix $K \stackrel{\text{def}}{=} \text{diag}\{K_1,\dots,K_\nu\}$ is *compatible* with $(\mathscr{L},\mathscr{M},\mathscr{P})$ if: (i) for each $1 \le i \le \nu$, K_i is a matrix of dimension $m \times m_i$; (ii) for all $1 \le i \le 2\sigma$, K_i is a complex matrix such that $K_i = \overline{K}_{i+1}$, for all odd $i \le 2\sigma$, and K_i is a real matrix for each $i \ge 2\sigma$; and (iii) each K_i matrix can be partitioned as

$$K_i = \left[\begin{array}{c|c|c|c} K_{i,1} & K_{i,2} & \dots & K_{i,g_i} \end{array}\right], \tag{10}$$

where, for $1 \le k \le g_i$, each $K_{i,k}$ has dimension $m \times p_{i,k}$.

In this section we develop our parametric form for the eigenvector matrix X and pole-placing gain matrix F that solve (2) for any admissible eigenstructure $(\mathscr{L},\mathscr{M},\mathscr{P})$. Our first task is to build a suitable eigenvector matrix. Given a compatible parameter matrix K for $(\mathscr{L},\mathscr{M},\mathscr{P})$, we build eigenvector chains as follows. For each pair $i \in \{1,\dots,\nu\}$ and $k \in \{1,\dots,g_i\}$, build vector chains of length $p_{i,k}$ as follows:

$$x_{i,k}(1) = N_i K_{i,k}(1), \tag{11}$$
$$x_{i,k}(2) = M_i U_1^\top x_{i,k}(1) + N_i K_{i,k}(2), \tag{12}$$
$$\vdots$$
$$x_{i,k}(p_{i,k}) = M_i U_1^\top x_{i,k}(p_{i,k}-1) + N_i K_{i,k}(p_{i,k}). \tag{13}$$

From these column vectors we construct the matrices

$$X_{i,k} \stackrel{\text{def}}{=} [x_{i,k}(1)|x_{i,k}(2)|\dots|x_{i,k}(p_{i,k})] \tag{14}$$
$$X_i \stackrel{\text{def}}{=} [X_{i,1}|X_{i,2}|\dots|X_{i,g_i}] \tag{15}$$
$$X_K \stackrel{\text{def}}{=} [X_1|X_2|\dots|X_\nu] \tag{16}$$

of dimensions $n \times p_{i,k}$, $n \times m_i$, and $n \times n$, respectively. Finally we obtain the feedback gain matrix

$$F_K \stackrel{\text{def}}{=} Z^{-1} U_0^\top (X_K \Lambda X_K^{-1} - A) \tag{17}$$

Given its origins in the classic paper [2], we refer to the parametric formulae (16) and (17) as the *extended Kautsky-Nichols-van Dooren parametric form* for X and F. The main result of [1] was the following:

Theorem 2.2: [1] Let $(\mathscr{L},\mathscr{M},\mathscr{P})$ be an admissible Jordan structure for (A,B) and K be a compatible parameter matrix. Then for almost all choices of K, the matrix X_K in (16) is invertible, i.e., X_K is invertible for every choice of K except those laying in a set of measure zero. The set of all real feedback matrices F_K such that the closed-loop matrix $A+BF_K$ has Jordan structure described by $(\mathscr{L},\mathscr{M},\mathscr{P})$ is parameterised in K by (17), where X_K is obtained with a parameter matrix K such that X_K is invertible.

III. Robust Optimal pole placement with the Kautsky-Nichols-van Dooren parametric form

We firstly note some classic results on eigenvalue sensitivity.

Theorem 3.1: [14, Theorem 4.4.2]

Let A and X be such that $A = XJX^{-1}$, where J is the Jordan form of A, and let $A' = A+H$. Then for each eigenvalue of A', there exists an eigenvalue λ of A such that

$$\frac{|\lambda-\lambda'|}{(1+|\lambda-\lambda'|)^{l-1}} \le \kappa_2(X)\|H\|_2 \tag{18}$$

where $\kappa_2(X) := \|X\|_2\|X^{-1}\|_2$ is the spectral condition number of X, and l is the size of the largest Jordan mini-block associated with λ.

The result indicates that the spectral condition number $\kappa_2(X)$ of the matrix X may be used a measure of the eigenvalue sensitivity of the matrix A. Since $\kappa_2(X)$ is non-differentiable, it is not amenable to optimisation via gradient search methods. The Frobenius condition number $\kappa_{fro}(X) = \|X\|_{fro}\|X^{-1}\|_{fro}$ is differentiable, and since $\kappa_2(X) \le \kappa_{fro}(X)$, many authors, including [4], [8], [16], have used this as their robustness measure.

We utilise the parametric form introduced in the previous section to consider the problem of minimising the Frobenius condition number of X. More precisely, we consider the unconstrained optimisation problem

$$(\mathscr{P}_1): \quad \min_K \|X_K\|_{fro}\|X_K^{-1}\|_{fro} \tag{19}$$

where X_K in (16) arises from any compatible parameter matrix K. As pointed out in [4], to minimise $\kappa_{fro}(X)$, for efficient computation we may instead consider the alternative objective function

$$(\mathscr{P}_2): \quad \min_K \|V_K\|_{fro}^2 + \|V_K^{-1}\|_{fro}^2, \tag{20}$$

with $V_K = \mathfrak{Re}\{X_K\}$. In order to determine the optimal input parameter matrix K that solves problem $(\mathscr{P}_2)$, we will exploit a gradient search employing the first and second order derivatives of $\|V_K\|_{fro}^2$ and $\|V_K^{-1}\|_{fro}^2$. From these expressions, the gradient and Hessian matrices are easily obtained, and unconstrained nonlinear optimisation methods can then be used to seek local minima. The details are given in the Appendix.

IV. Performance Comparison

In this section, we compare the algorithm presented in this paper with the methods given in [8] and [17].

Example 4.1: We consider the Example 2 in the Byers and Nash [4] collection of benchmark systems that have been investigated over the years by many authors [15], [16], [7], [8]. We use the state matrices A and B from that system, with $n=5$ states and $m=2$ inputs. Differing from [4], we seek to assign all the closed-loop eigenvalues to zero to obtain a deadbeat response, and thus we have $\mathscr{L}=\{0\}$ and $\mathscr{M}=\{5\}$. The controllability indices are $\{3,2\}$ and so we see that this can be achieved with two Jordan mini-blocks of dimensions three and two. Using the method of [8] to minimise the Frobenius condition number of the matrix of generalised eigenvectors X, we obtain

$$F_1=\begin{bmatrix}-46.83 & 201.32 & -430.03 & 357.41 & -102.63\\ -20.79 & 85.14 & -163.61 & 123.70 & -31.79\end{bmatrix}$$

yielding closed-loop generalized eigenvector matrix X_1 with $\kappa_{fro}(X_1)=54.4272$ and $\|F_1\|_{fro}=645.5096$.
Using the Method of [17], we obtain

$$F_2=\begin{bmatrix}-39.46 & 148.92 & -299.30 & 244.73 & -70.27\\ -15.12 & 44.86 & -63.10 & 37.07 & -6.92\end{bmatrix}$$

yielding closed-loop generalized eigenvector matrix X_2 with $\kappa_{fro}(X_2)=49.2575$, and $\|F_2\|_{fro}=430.5015$. Using the method given in this paper we obtained

$$F_3=\begin{bmatrix}-39.48 & 149.08 & -299.69 & 245.07 & -70.37\\ -15.14 & 44.98 & -63.40 & 37.33 & -6.99\end{bmatrix}$$

yielding closed-loop generalized eigenvector matrix X_3 with $\kappa_{fro}(X_3)=49.2575$ and $\|F_3\|_{fro}=431.1126$.
The results indicate that the robust pole placement methods given here obtained with the Kautsky-Nichols-van Dooren parametric formula can match the results given in [17] using the Klein-Moore parametric formula. Both methods were able to improve on the Frobenius conditioning of the generalised eigenvector matrix achieved by [8], and did so with considerably less gain.

V. Conclusion

We have revisited the arbitrary pole placement method of our recent paper [1] that can assign any desired eigenstructure with arbitrary multiplicities. The method has been applied to the problem of robust pole placement, and shown via an example to be capable of achieving comparable, and in some respects, superior robustness performance to other arbitrary pole placement methods from the recent literature. Future work will involve extensive numerical testing to see whether either of the two pole placement methods given here and in [1] enjoys any significant performance advantage over the other.

VI. Appendix

Here we consider the derivatives of V_K and V_K^{-1} in (20). We define

$$\chi_i \stackrel{\text{def}}{=} \begin{cases}\mathfrak{Re}\{K_i\} & i\in\{1,\ldots,2\sigma\}\ \text{odd},\\ \mathfrak{Im}\{K_{i-1}\} & i\in\{1,\ldots,2\sigma\}\ \text{even},\\ K_i & i\in\{2\sigma+1,\ldots,\nu\}.\end{cases}$$

Let

$$V_K=\mathfrak{Re}\{X_K\}. \tag{21}$$

Define $\chi_{i,k}(l,r)$ as the r-th entry of $\chi_{i,k}(l)$. We compute the derivative of $V_{p,q}$ with respect to $\chi_{i,k}$. We have

$$\frac{\partial V_{p,q}}{\partial \chi_{i,k}(l,r)}=0$$

for $p\in\{1,\ldots,2\sigma\}$ with $p\neq i, p\neq i+\sigma, p+\sigma\neq i$ and $p\in\{2\sigma+1,\ldots,\nu\}$ with $p\neq i$. Define

$$P(i,l)\stackrel{\text{def}}{=}\begin{cases}\{M_iU_1^\top\}^l N_i & \text{if } l\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

For each $i\in\{1,\ldots,\sigma\}$, $k\in\{1,\ldots,g_i\}$, $h,l\in\{1,\ldots,p_{i,k}\}$ and $r\in\{1,\ldots,m\}$ we find

$$\begin{aligned}\frac{\partial V_{i,k}(h)}{\partial\chi_{i,k}(l,r)}&=\mathfrak{Re}\{P(i,h-l)\}(r),\\ \frac{\partial V_{i+\sigma,k}(h)}{\partial\chi_{i,k}(l,r)}&=\mathfrak{Im}\{P(i,h-l)\}(r),\\ \frac{\partial V_{i,k}(h)}{\partial\chi_{i+\sigma,k}(l,r)}&=-\mathfrak{Im}\{P(i,h-l)\}(r),\\ \frac{\partial V_{i+\sigma,k}(h)}{\partial\chi_{i+\sigma,k}(l,r)}&=\mathfrak{Re}\{P(i,h-l)\}(r).\end{aligned}$$

For each $i\in\{2\sigma+1,\ldots,\nu\}$, $k\in\{1,\ldots,g_i\}$, $h,l\in\{1,\ldots,p_{i,k}\}$ and $r\in\{1,\ldots,m\}$ we have

$$\frac{\partial V_{i,k}(h)}{\partial\chi_{i,k}(l,r)}=P(i,h-l)(r).$$

Let $Y_K=V_K^{-1}$. Then,

$$f(K)=\|V_K\|_{fro}^2+\|Y_K\|_{fro}^2, \tag{22}$$

The derivatives of $\|V_K\|_{fro}^2$ and $\|Y_K\|_{fro}^2$ are given as

$$\frac{\partial\|V_K\|_{fro}^2}{\partial\chi_{i,k}(l,r)} = 2\,\text{trace}\left(V_K^\top\frac{\partial V_K}{\partial\chi_{i,k}(l,r)}\right),$$

$$\frac{\partial^2\|V_K\|_{fro}^2}{\partial\chi_{i_1,k_1}(l_1,r_1)\partial\chi_{i_2,k_2}(l_2,r_2)} =2\,\text{trace}\left(\frac{\partial V_K^\top}{\partial\chi_{i_1,k_1}(l_1,r_1)}\frac{\partial V_K}{\partial\chi i_2,k_2(l_2,r_2)}\right)$$

Using the well-known formula $\frac{\partial Y_K}{\partial\chi_{i,k}(l,r)}=-Y_K\frac{\partial V_K}{\partial\chi_{i,k}(l,r)}Y_K$, we compute

$$\frac{\partial\|Y_K\|_{fro}^2}{\partial\chi_{i,k}(l,r)} = 2\,\text{trace}\left(-Y_K^\top Y_K\frac{\partial V_K}{\partial\chi_{i,k}(l,r)}Y_K\right)$$

and

$$\begin{aligned}&\frac{\partial^2\|Y_K\|_{fro}^2}{\partial\chi_{i_1,k_1}(l_1,r_1)\partial\chi_{i_2,k_2}(l_2,r_2)}\\ &=2\,\text{trace}\Big(Y_K^\top\frac{\partial V_K^\top}{\partial\chi_{i_2,k_2}(l_2,r_2)}Y_K^\top Y_K\frac{\partial V_K}{\partial\chi_{i_1,k_1}(l_1,r_1)}Y_K\\ &+Y_K^\top Y_K\frac{\partial V_K}{\partial\chi_{i_2,k_2}(l_2,r_2)}Y_K\frac{\partial V_K}{\partial\chi_{i_1,k_1}(l_1,r_1)}Y_K\\ &+Y_K^\top Y_K\frac{\partial V_K}{\partial\chi_{i_1,k_1}(l_1,r_1)}Y_K\frac{\partial V_K}{\partial\chi_{i_2,k_2}(l_2,r_2)}Y_K\Big).\end{aligned}$$

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