

Universal estimate of the gradient for parabolic equations *

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Abstract

We suggest a modification of the estimate for weighted Sobolev norms of solutions of parabolic equations such that the matrix of the higher order coefficients is included into the weight for the gradient. More precisely, we found the upper limit estimate that can be achieved by variations of the zero order coefficient. As an example of applications, an asymptotic estimate was obtained for the gradient at initial time. The constant in the estimates is the same for all possible choices of the dimension, domain, time horizon, and the coefficients of the parabolic equation. As an another example of application, existence and regularity results are obtained for parabolic equations with time delay for the gradient.

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1 Introduction

We study Dirichlet boundary value problems for linear parabolic equations. The classical results give estimates for Sobolev norms of the solution via the L_2 -norm of the nonhomogeneous term (see, e.g., the second energy inequality (4.56) in Ladyzhenskaia (1985), Chapter III). We suggest a modification of this estimate for weighted Sobolev norms such that the matrix of the higher order coefficients is included into the weight for the gradient. We found the limit upper estimate that can be achieved by variations of the zero order coefficient. More precisely, we obtain estimates for $e^{-Kt}u(x, t)$ via L_2 -norm of $e^{-Kt}h(x, t)$, where $u(x, t)$ and $h(x, t)$ is the solution and the nonhomogeneous term respectively, and where $K > 0$ is being variable. As an example of applications, an asymptotic upper estimate was obtained for a weighted L_2 -norm of the

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gradient at initial time. The constants in these estimates are the same for all possible choices of the dimension, domain, time horizon, and the coefficients of the parabolic equations, i.e., these estimates can be called universal estimates. As an another example of applications, we establish solvability and regularity for special parabolic equations such that the gradient is included with time delay.

2 Definitions

Spaces and classes of functions.

We denote by $|\cdot|$ the Euclidean norm in \mathbf{R}^k and the Frobenius norm in $\mathbf{R}^{k \times m}$, and we denote by $\|\cdot\|_X$ the norm in a linear normed space X . We denote by $(\cdot, \cdot)_X$ the scalar product in a Hilbert space X . For a Banach space X , we denote by $C([a, b], X)$ the Banach space of continuous functions $x : [a, b] \rightarrow X$.

We are given an open domain $D \subseteq \mathbf{R}^n$ such that either $D = \mathbf{R}^n$ or D is bounded with C^2 -smooth boundary ∂D .

Let $T > 0$ be given, and let $Q \triangleq D \times (0, T)$.

We denote by $W_2^m(D)$ the Sobolev space of functions that belong to $L_2(D)$ together with the distributional derivatives up to the m th order, $m \geq 0$.

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq \overset{0}{W}_2^1(D)$ be the closure in the $W_2^1(D)$ -norm of the set of all smooth functions $u : D \rightarrow \mathbf{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W_2^2(D) \cap H^1$ be the space equipped with the norm of $W_2^2(D)$. The spaces H^k are Hilbert spaces, and H^k is a closed subspace of $W_2^k(D)$, $k = 1, 2$.

Let H^{-1} be the dual space to H^1 , with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u, w)_{H^0}$ over all $w \in H^1$ such that $\|w\|_{H^1} \leq 1$. H^{-1} is a Hilbert space.

We will write $(u, w)_{H^0}$ for $u \in H^{-1}$ and $w \in H^1$, meaning the obvious extension of the bilinear form from $u \in H^0$ and $w \in H^1$.

We denote by $\bar{\ell}_1$ the Lebesgue measure in \mathbf{R} , and we denote by $\bar{\mathcal{B}}_1$ the σ -algebra of Lebesgue sets in \mathbf{R}^1 .

For $k = -1, 0, 1, 2$, we introduce the spaces

$$X^k \triangleq L_2([0, T], \bar{\mathcal{B}}_1, \bar{\ell}_1; H^k), \quad \mathcal{C}^k \triangleq C([0, T]; H^k).$$

We introduce the spaces

$$Y^k \triangleq X^k \cap \mathcal{C}^{k-1}, \quad k \geq 0,$$

with the norm $\|u\|_{Y^k} \triangleq \|u\|_{X^k} + \|u\|_{C^{k-1}}$.

We use the notations

$$\nabla u \triangleq \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)^\top, \quad \nabla \cdot U = \sum_{i=1}^n \frac{\partial U_i}{\partial x_i}$$

for functions $u : \mathbf{R}^n \rightarrow \mathbf{R}$ and $U = (U_1, \dots, U_n)^\top : \mathbf{R}^n \rightarrow \mathbf{R}^n$. In addition, we use the notation

$$(U, V)_{H^0} = \sum_{i=1}^n (U_i, V_i)_{H^0}, \quad \|U\|_{H^0} = (U, U)_{H^0}^{1/2}$$

for functions $U, V : D \rightarrow \mathbf{R}^n$, where $U = (U_1, \dots, U_n)$ and $V = (V_1, \dots, V_n)$.

The boundary value problem

We consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u + h, & t \in (0, T), \\ u(x, 0) &= 0, & u(x, t)|_{x \in \partial D} = 0. \end{aligned} \tag{2.1}$$

Here $u = u(x, t)$, $(x, t) \in Q$, $h \in L_2(Q)$, and

$$\mathcal{A}y \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left(b_{ij}(x, t) \frac{\partial y}{\partial x_j}(x) \right) + \sum_{i=1}^n f_i(x, t) \frac{\partial y}{\partial x_i}(x) + \lambda(x, t)y(x), \tag{2.2}$$

where $b(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^{n \times n}$, $f(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$, and $\lambda(x, t) : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$, are bounded measurable functions, and where b_{ij}, f_i, x_i are the components of b, f , and x . The matrix $b = b^\top$ is symmetric.

To proceed further, we assume that Conditions 2.1-2.2 remain in force throughout this paper.

Condition 2.1 *There exists a constant $\delta > 0$ such that*

$$\xi^\top b(x, t) \xi \geq \delta |\xi|^2 \quad \forall \xi \in \mathbf{R}^n, (x, t) \in Q. \tag{2.3}$$

Condition 2.2 *The functions $b(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$, $f(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$, $\lambda(x, t) : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$, are differentiable in x , the function $b(x, t)$ is differentiable in t , and the corresponding derivatives are bounded.*

We introduce the sets of parameters

$$\begin{aligned} \mu &\triangleq (T, n, D, b(\cdot), f(\cdot), \lambda(\cdot)), \\ \mathcal{P} = \mathcal{P}(\mu) &\triangleq \left(T, n, D, \delta, \operatorname{ess\,sup}_{(x,t) \in Q} \left(|b(x, t)| + |f(x, t)| + |\lambda(x, t)| + \left| \frac{\partial b}{\partial x}(x, t) \right| + \left| \frac{\partial f}{\partial x}(x, t) \right| \right. \right. \\ &\quad \left. \left. + \left| \frac{\partial \lambda}{\partial x}(x, t) \right| + \left| \frac{\partial b}{\partial t}(x, t) \right| \right) \right). \end{aligned}$$

3 Special estimates for the solution

By the classical solvability results, there exists a unique solution $u \in Y^2$ of problem (2.1) for any $h \in L_2(Q)$ (see, e.g., Ladyzhenskaia (1985), Chapter III, §§4-5). Moreover, the second energy inequality gives that, for any $K \in \mathbf{R}$ and $M \geq 0$, there exists a constant $\tilde{C}(K, M, \mathcal{P}) > 0$ such that

$$\begin{aligned} & e^{-2Kt}(\nabla u(\cdot, t), b(\cdot, t)\nabla u(\cdot, t))_{H^0} \\ & + M \left[e^{-2Kt}\|u(\cdot, t)\|_{H^0}^2 + \int_0^t e^{-2Ks}(\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} ds \right] \\ & \leq \tilde{C}(K, M, \mathcal{P}) \int_0^t e^{-2Ks}\|h(\cdot, s)\|_{H^0}^2 ds \quad \forall h \in L_2(Q), t \in (0, T], \end{aligned} \quad (3.1)$$

for any solution u of problem (2.1). This estimate follows immediately from inequality (4.56) from Ladyzhenskaia (1985), Chapter III, and from obvious estimates

$$e^{-2Kt}(\nabla w, b(\cdot, t)\nabla w)_{H^0} \leq c_1\|w\|_{H^1}^2, \quad \|w\|_{H^0}^2 \leq c_2e^{-2Kt}\|w\|_{H^0}^2 \quad \forall t \in [0, T],$$

for any $w \in H^1$ and $w \in H^0$ respectively, where $c_1 \triangleq \text{ess sup}_{x,t} |b(x, t)|$ and $c_2 \triangleq e^{2KT}$.

Let $C(K, M, \mathcal{P}) \triangleq \inf \tilde{C}(K, M, \mathcal{P})$, where the infimum is taken over all $\tilde{C}(K, M, \mathcal{P})$ such that (3.1) holds.

Theorem 3.1

$$\sup_{\mu, M \geq 0} \inf_{K \geq 0} C(K, M, \mathcal{P}(\mu)) \leq \frac{1}{2}.$$

Corollary 3.1 *For any μ and $\varepsilon > 0$, there exists $K = K(\varepsilon, \mathcal{P}(\mu)) \geq 0$ such that*

$$\begin{aligned} \sup_{s \in [0, t]} e^{-2Ks}(\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} & \leq \left(\frac{1}{2} + \varepsilon\right) \int_0^t e^{-2Ks}\|h(\cdot, s)\|_{H^0}^2 ds \\ & \quad \forall h \in L_2(Q), t \in (0, T], \end{aligned}$$

where u is the solution of problem (2.1) for the corresponding h .

Remark 3.1 *For the case of a non-divergent operator $\mathcal{A}u = b_{ij}u_{x_i x_j} + f_i u_{x_i} + \lambda u$, the differentiability of the coefficients in Condition 2.2 can be weakened using the approach from Dokuchaev (2005).*

4 Applications: asymptotic estimate at initial time

Let

$$X_c^0 \triangleq \left\{ h \in X^0 : \lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds = \|h(\cdot, 0)\|_{H^0}^2 \right\}.$$

The condition that $h \in X_c^0$ is not restrictive for $h \in X^0$; for instance, it holds if $t = 0$ is a Lebesgue point for $\|h(\cdot, t)\|_{H^0}^2$.

Theorem 4.1 *For any admissible set of parameters μ ,*

$$\overline{\lim}_{t \rightarrow 0^+} \sup_{h \in X_c^0} \frac{1}{t} \frac{(\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0}}{\|h(\cdot, 0)\|_{H^0}^2} \leq \frac{1}{2}, \quad (4.1)$$

where u is the solution of problem (2.1) for the corresponding h .

5 On the sharpness of the estimates

Theorem 5.1 *There exists a set of parameters $(n, D, b(\cdot), f(\cdot), \lambda(\cdot))$ such that, for any $T > 0$, $M \geq 0$,*

$$\inf_{K \geq 0} C(K, M, \mathcal{P}(\mu)) = \frac{1}{2}. \quad (5.1)$$

for $\mu = (T, n, D, b(\cdot), f(\cdot), \lambda(\cdot))$.

Theorem 5.2 *There exists a set of parameters $(n, D, b(\cdot), f(\cdot), \lambda(\cdot))$ such that*

$$\overline{\lim}_{t \rightarrow 0^+} \sup_{h \in X_c^0} \frac{1}{t} \frac{(\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0}}{\|h(\cdot, 0)\|_{H^0}^2} = \frac{1}{2}, \quad (5.2)$$

where u is the solution of problem (2.1) for the corresponding h .

6 Applications: parabolic equations with time delay

Theorem 3.1 can be also applied to analysis of parabolic equations with time delay. These equations have many applications, and they were intensively studied, including equations with delay operators of a general form defined on the path of past values (see, e.g., Bátkai and Piazzera (2001), Pao (1997), Poorkarimi and Wiener (1999), Stein *et al* (2005), and references here). We use Theorem 3.1 to obtain sufficient conditions of solvability in Y^2 for the special case when the first derivatives of solutions are affected by time delay represented by a general measurable function. As far as we know, this case was not covered in the existing literature.

Let $\tau(\cdot) : [0, T] \rightarrow \mathbf{R}$ be a given measurable function such that $\tau(t) \in [0, t]$. Non-monotonic, piecewise constant, or nowhere continuous functions $\tau(\cdot)$ are not excluded.

For functions $u : Q \rightarrow \mathbf{R}$, we introduce the following operator

$$Bu \triangleq \beta(x, \tau(t))^\top \nabla u(x, \tau(t)) + \bar{\beta}(x, \tau(t)) u(x, \tau(t)). \quad (6.1)$$

We assume that the functions $\beta : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}^n$ and $\bar{\beta} : \mathbf{R}^n \times [0, T] \rightarrow \mathbf{R}$ are bounded and measurable.

Let us consider the following boundary value problem in Q :

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u + Bu + h, & t \in (0, T), \\ u(x, 0) &= 0, & u(x, t)|_{x \in \partial D} = 0. \end{aligned} \quad (6.2)$$

Here \mathcal{A} is such as defined above.

Theorem 6.1 *Assume that there exists a constant $\delta_1 > 0$ such that*

$$T|\xi^\top \beta(x, t)|^2 \leq (2 - \delta_1)\xi^\top b(x, t)\xi \quad \forall \xi \in \mathbf{R}^n, (x, t) \in D \times [0, T]. \quad (6.3)$$

Then there exists a unique solution $u \in Y^2$ of problem (6.2) for any $h \in L_2(Q)$, and $\|u\|_{Y^2} \leq c\|h\|_{L_2(Q)}$, where c is a constant that depends only on \mathcal{P} , δ_1 , and $\sup_{x,t} |\bar{\beta}(x, t)|$.

Theorem 6.2 *Assume that there exists $\theta \in [0, T)$ such that $\tau(t) = 0$ for $t < \theta$. Assume that the function $\tau(\cdot) : [\theta, T] \rightarrow \mathbf{R}$ is non-decreasing and absolutely continuous, and $\delta_* \triangleq \text{ess sup}_{t \in [\theta, T]} \left| \frac{d\tau}{dt}(t) \right|^{-1} < +\infty$. Then there exists a unique solution $u \in Y^2$ of problem (6.2) for any $h \in L_2(Q)$, and $\|u\|_{Y^2} \leq c_*\|h\|_{L_2(Q)}$, where c_* is a constant that depends only on \mathcal{P} , δ_* , $\sup_{x,t} |\beta(x, t)|$, and $\sup_{x,t} |\bar{\beta}(x, t)|$.*

7 Proofs

Lemma 7.1 *For any admissible μ and any $\varepsilon > 0$, $M > 0$, there exists $\tilde{K} = \tilde{K}(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$ such that*

$$\begin{aligned} (\nabla u(\cdot, t), b(\cdot, t)\nabla u(\cdot, t))_{H^0} + M \left(\|u(\cdot, t)\|_{H^0}^2 + \int_0^t (\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} ds \right) \\ \leq \left(\frac{1}{2} + \varepsilon \right) \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds \end{aligned} \quad (7.1)$$

for all $K \geq \tilde{K}(\varepsilon, M, \mathcal{P})$, $t \in (0, T]$, and $h \in L_2(Q)$, where $u \in Y^2$ is the solution of the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{A}u - Ku + h, \quad t \in (0, T), \\ u(x, 0) &= 0, \quad u(x, t)|_{x \in \partial D} = 0. \end{aligned} \quad (7.2)$$

Uniqueness and existence of solution $u \in Y^2$ of problem (7.2) follows from the classical results (see, e.g., Ladyzhenskaia (1985), Chapter III).

Proof of Lemma 7.1. Clearly, $\mathcal{A} = \mathcal{A}_s + \mathcal{A}_r$, where

$$\mathcal{A}_s u = \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left(b_{ij} \frac{\partial u}{\partial x_j} \right) = \nabla \cdot (b \nabla u), \quad \mathcal{A}_r u = \sum_{i=1}^n f_i \frac{\partial u}{\partial x_i} + \lambda u.$$

Assume that the function $h(\cdot, t) : D \rightarrow \mathbf{R}$ is differentiable and has a compact support inside D for all t . We have that

$$\begin{aligned} & (\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0} - (\nabla u(\cdot, 0), b(\cdot, 0) \nabla u(\cdot, 0))_{H^0} \\ &= 2 \int_0^t \left(\nabla u, b \nabla \frac{\partial u}{\partial s} \right)_{H^0} ds + \int_0^t \left(\nabla u, \frac{\partial b}{\partial s} \nabla u \right)_{H^0} ds \\ &= 2 \int_0^t (\nabla u, b \nabla (\mathcal{A}u - Ku + h))_{H^0} ds + \int_0^t \left(\nabla u, \frac{\partial b}{\partial s} \nabla u \right)_{H^0} ds \\ &= 2 \int_0^t (\nabla u, b \nabla (\nabla \cdot (b \nabla u)))_{H^0} ds + 2 \int_0^t (\nabla u, b \nabla \mathcal{A}_r u)_{H^0} ds \\ &\quad - 2K \int_0^t (\nabla u, b \nabla u)_{H^0} ds + 2 \int_0^t (\nabla u, b \nabla h)_{H^0} ds + \int_0^t \left(\nabla u, \frac{\partial b}{\partial s} \nabla u \right)_{H^0} ds. \end{aligned} \quad (7.3)$$

Let an arbitrary $\varepsilon_0 > 0$ be given. We have that in under the integrals in (7.3),

$$\left(\nabla u, \frac{\partial b}{\partial t} \nabla u \right)_{H^0} \leq \text{ess sup}_{x,t} \left| \frac{\partial b}{\partial t}(x, t) \right| \|\nabla u\|_{H^0}^2 \leq c'_\delta (\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0}, \quad (7.4)$$

where $c'_\delta = c'_\delta(\mathcal{P})$ is a constant that depends on \mathcal{P} only. Further,

$$2 (\nabla u, b \nabla h)_{H^0} = -2 (\nabla \cdot (b \nabla u), h)_{H^0} \leq \frac{2}{1 + 2\varepsilon_0} (\nabla u, b \nabla u)_{H^0}^2 + \left(\frac{1}{2} + \varepsilon_0 \right) \|h\|_{H^0}^2, \quad (7.5)$$

Let a bounded measurable function $v(x, t) : Q \rightarrow \mathbf{R}$ be such that $b = v^\top v$. We have that

$$2 (\nabla u, b \nabla \mathcal{A}_r u)_{H^0} = 2 (v \nabla u, v \nabla \mathcal{A}_r u)_{H^0} \leq \varepsilon_1^{-1} \|v \nabla u\|_{H^0}^2 + \varepsilon_1 \|v \nabla \mathcal{A}_r u\|_{H^0}^2 \quad \forall \varepsilon_1 > 0.$$

It follows that

$$2 (\nabla u, b \nabla \mathcal{A}_r u)_{H^0} \leq \varepsilon_1^{-1} (\nabla u, b \nabla u)_{H^0} + c_1 \varepsilon_1 \|u\|_{H^2}^2,$$

where $c_1 = c_1(\mathcal{P})$ is a constant that depends on \mathcal{P} only. By the second fundamental inequality, there exists a constant $c_* = c_*(\mathcal{P}) > 0$ such that

$$\int_0^t \|u(\cdot, s)\|_{H^2}^2 ds \leq c_* \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds. \quad (7.6)$$

(See, e.g., estimate (4.56) from Ladyzhenskaia (1985), Chapter III). By Lemma 5.3 from Dokuchaev (2005), one can choose the same constant c_* for all $t \in [0, T]$, $K > 0$. Hence

$$2 \int_0^t (\nabla u, b\nabla \mathcal{A}_r u)_{H^0} ds \leq \varepsilon_1^{-1} \int_0^t (\nabla u, b\nabla u)_{H^0} ds + \varepsilon_0 \int_0^t \|h\|_{H^0}^2 ds, \quad (7.7)$$

where $\varepsilon_1 > 0$ is such that $c_1 c_* \varepsilon_1 = \varepsilon_0$.

By Lemma 5.2 from Dokuchaev (2005), p. 357, it follows that there exists $K_1 = K_1(\varepsilon_0, M, \mathcal{P}) > 0$ such that if $K > K_1$, then

$$M \sup_{s \in [0, t]} \|u(\cdot, s)\|_{H^0}^2 \leq \varepsilon_0 \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds. \quad (7.8)$$

Assume now that $D = \mathbf{R}^n$. In that case, we have immediately that

$$\begin{aligned} 2 (\nabla u, b\nabla(\nabla \cdot (b\nabla u)))_{H^0} &= 2 (b\nabla u, \nabla(\nabla \cdot (b\nabla u)))_{H^0} = -2 (\nabla \cdot (b\nabla u), \nabla \cdot (b\nabla u))_{H^0} \\ &= -2 \|\nabla \cdot (b\nabla u)\|_{H^0}^2. \end{aligned} \quad (7.9)$$

The third equality here was obtained using integration by parts.

By (7.3)-(7.8), it follows that if $K > K_1$ and $2K > \varepsilon_1^{-1} + c'_\delta + M$, then

$$\begin{aligned} &(\nabla u(\cdot, t), b(\cdot, t)\nabla u(\cdot, t))_{H^0} + M \left(\|u(\cdot, t)\|_{H^0}^2 + \int_0^t (\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} ds \right) \\ &\leq \left(\frac{2}{1 + 2\varepsilon_0} - 2 \right) \int_0^t \|\nabla \cdot (b\nabla u)\|_{H^0}^2 ds + (\varepsilon_1^{-1} + c'_\delta + M - 2K) \int_0^t \|v\nabla u\|_{H^0}^2 ds \\ &+ \left(\frac{1}{2} + 3\varepsilon_0 \right) \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds \leq \left(\frac{1}{2} + 3\varepsilon_0 \right) \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds. \end{aligned} \quad (7.10)$$

Then the proof of Lemma 7.1 follows for the case when $D = \mathbf{R}^n$, since (7.10) holds for all h from a set that is dense in $L_2(Q)$.

To complete the proof of Lemma 7.1, we need to cover the case when $D \neq \mathbf{R}^n$. From now and up to the end of the proof of this lemma, we assume that $D \neq \mathbf{R}^n$. In that case, (7.9) does not hold, since the integration by parts used for the second inequality in (7.9) is not applicable anymore. To replace (7.9), we are going to show that there exists a constant $C = C(\mathcal{P}) > 0$ such that, for an arbitrarily $\varepsilon_2 > 0$,

$$(\nabla u, b\nabla(\nabla \cdot (b\nabla u)))_{H^0} \leq -\|\nabla \cdot (b\nabla u)\|_{H^0}^2 + \varepsilon_2 \|u\|_{H^2}^2 + C\varepsilon_2^{-1} \|u\|_{H^1}^2. \quad (7.11)$$

Assume that (7.11) holds. Since D is bounded, we have that $\|u\|_{H^1}^2 \leq c_\delta (\nabla u, b\nabla u)_{H^0}^2$ for some constant $c_\delta = c_\delta(\mathcal{P})$. In addition, (7.5) is still valid, since we assumed that $h(\cdot, t)$ have support inside D . Similarly to (7.10), we obtain that if $2K > \varepsilon_1^{-1} + c'_\delta + Cc_\delta\varepsilon_2^{-1} + M$ and $K > K_1$, then

$$\begin{aligned} & (\nabla u(\cdot, t), b(\cdot, t)\nabla u(\cdot, t))_{H^0} + M \left[\|u(\cdot, t)\|_{H^0}^2 + \int_0^t (\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} ds \right] \\ & \leq \left[\frac{2}{1+2\varepsilon_0} - 2 \right] \int_0^t \|\nabla \cdot (b\nabla u)\|_{H^0}^2 ds + [\varepsilon_1^{-1} + c'_\delta + M - 2K] \int_0^t (\nabla u(\cdot, s), b(\cdot, s)\nabla u(\cdot, s))_{H^0} ds \\ & + \left(\frac{1}{2} + 3\varepsilon_0 \right) \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds + 2\varepsilon_2 \int_0^t \|u(\cdot, s)\|_{H^2}^2 ds + 2C\varepsilon_2^{-1} \int_0^t \|u(\cdot, s)\|_{H^1}^2 ds \\ & \leq \left(\frac{1}{2} + 3\varepsilon_0 + 2c_*\varepsilon_2 \right) \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds, \end{aligned}$$

where c_* is the constant from (7.6). Then the proof of Lemma 7.1 follows provided that (7.11) holds. Therefore, it suffices to prove (7.11) for $D \neq \mathbf{R}^n$.

Let us prove (7.11). This estimate can be rewritten as

$$(w, \nabla(\nabla \cdot w))_{H^0} \leq -((\nabla \cdot w), (\nabla \cdot w))_{H^0} + \varepsilon_2 \|u\|_{H^2}^2 + C\varepsilon_2^{-1} \|u\|_{H^0}^2,$$

where $w = b\nabla u$. We have

$$(w, \nabla(\nabla \cdot w))_{H^0} = -((\nabla \cdot w), (\nabla \cdot w))_{H^0} + \sum_{i=1}^n \int_{\partial D} \widehat{J}_i dz. \quad (7.12)$$

Here $\widehat{J}_i = w(\nabla \cdot w) \cos(\mathbf{n}, e_i)$, where $\mathbf{n} = \mathbf{n}(s)$ is the outward pointing normal to the surface ∂D at the point $s \in \partial D$, and $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ is the k th basis vector in the Euclidean space \mathbf{R}^n . We have that

$$\widehat{J}_i = \sum_{j,k,m=1}^n \alpha_{ijkm} J_{ijkm} + \sum_{j,k,m=1}^n \alpha'_{ijkm} J'_{ijkm},$$

where

$$J_{ijkm} = \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_m} \cos(\mathbf{n}, e_i), \quad J'_{ijk} = \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \cos(\mathbf{n}, e_i),$$

and where $\alpha_{ijkm}, \alpha'_{ijk}$ are some bounded functions.

Let us estimate $\int_{\partial D} \widehat{J}_i dz$. We mostly follow the approach from Ladyzhenskaya and Ural'tseva (1968), Section 3.8. Let $x^0 = \{x_i^0\}_{i=1}^n \in \partial D$ be an arbitrary point. In its neighborhood, we introduce local Cartesian coordinates $y_m = \sum_{k=1}^n c_{mk}(x_k - x_k^0)$ such that the axis y_n is directed along the outward normal $\mathbf{n} = \mathbf{n}(x_0)$ and $\{c_{mk}\}$ is an orthogonal matrix.

Let $y_n = \psi(y_1, \dots, y_{n-1})$ be an equation determining the surface ∂D in a neighborhood of the origin. By the properties of the surface ∂D , the first order and second order derivatives of the function ψ are bounded. Since $\{c_{mk}\}$ is an orthogonal matrix, we have $x_k - x_k^0 = \sum_{m=1}^n c_{km} y_m$. Therefore, $\cos(\mathbf{n}, e_m) = c_{nm}$, $m = 1, \dots, n$. Then

$$J_{ijkm} = \sum_{l=1}^n c_{jl} \frac{\partial u}{\partial y_l} \sum_{p,q=1}^n c_{pk} c_{qm} \frac{\partial^2 u}{\partial y_p \partial y_q} c_{ni}, \quad J'_{ijk} = \sum_{l=1}^n c_{jl} \frac{\partial u}{\partial y_l} \sum_{p=1}^n c_{pk} \frac{\partial u}{\partial y_p} c_{ni}.$$

The boundary condition $u(x, t)|_{x \in \partial D} = 0$ can be rewritten as

$$u(y_1, \dots, y_{n-1}, \psi(y_1, \dots, y_{n-1}), t) = 0$$

identically with respect to y_1, \dots, y_{n-1} near the point $y_1 = \dots = y_{n-1} = 0$. Let us differentiate this identity with respect to y_p and y_q , $p, q = 1, \dots, n-1$, and take into account that

$$\frac{\partial \psi}{\partial y_p} = 0, \quad p = 1, \dots, n-1.$$

at x_0 . Then

$$\frac{\partial u}{\partial y_p} = 0, \quad \frac{\partial^2 u}{\partial y_p \partial y_q} = -\frac{\partial u}{\partial y_n} \frac{\partial^2 \psi}{\partial y_p \partial y_q} = -\frac{\partial u}{\partial \mathbf{n}} \frac{\partial^2 \psi}{\partial y_p \partial y_q}, \quad p, q = 1, \dots, n-1.$$

Hence

$$\begin{aligned} \int_{\partial D} \widehat{J}_i(z, s) dz &= \int_{\partial D} \sum_{j,k,m=1}^n \alpha_{ijkm} J_{ijkm}(z, s) dz + \sum_{j,k,m=1}^n \int_{\partial D} \alpha'_{ijkm} J'_{ijkm} \alpha_{ijkm}(z, s) dz \\ &\leq \widehat{c}_1 \int_{\partial D} \left| \frac{\partial u}{\partial \mathbf{n}} \right|^2 dz \leq \varepsilon_2 \sum_{i,j=1}^n \int_D \left| \frac{\partial^2 u}{\partial x_i \partial x_j} (x, s) \right|^2 dx + \widehat{c}_2 (1 + \varepsilon_2^{-1}) \|u(\cdot, s)\|_{H^1}^2 \quad \forall \varepsilon_2 > 0 \end{aligned} \quad (7.13)$$

for some constants $\widehat{c}_i = \widehat{c}_i(\mathcal{P})$. The last estimate follows from the estimate (2.38) from Ladyzhenskaya and Ural'tseva (1968), Chapter II. By (7.12) and (7.13), it follows (7.11). This completes the proof of Lemma 7.1. \square

Proof of Theorem 3.1. Clearly, $u(x, t) = e^{Kt} u_K(x, t)$, where u is the solution of problem (2.1) and u_K is the solution of (7.2) for the nonhomogeneous term $e^{-Kt} h(x, t)$. Therefore, Theorem 3.1 follows immediately from Lemma 7.1. \square

Corollary 3.1 follows immediately from Theorem 3.1.

Proof of Theorem 4.1. Let $\varepsilon > 0$ be given. By Corollary 3.1, there exists $K(\varepsilon) = K(\varepsilon, \mathcal{P}(\mu))$ such that

$$e^{-2K(\varepsilon)t} (\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0} \leq \left(\frac{1}{2} + \varepsilon \right) \int_0^t e^{-2K(\varepsilon)s} \|h(\cdot, s)\|_{H^0}^2 ds \quad \forall t \in (0, T), \quad h \in X^0. \quad (7.14)$$

Let $p(h, t) \triangleq \frac{1}{t} \int_0^t \|h(\cdot, s)\|_{H^0}^2 ds$ and $q(u, t) \triangleq (\nabla u(\cdot, t), b(\cdot, t) \nabla u(\cdot, t))_{H^0}$. It follows that

$$\sup_{h \in X^0} \left(\frac{q(u, t)}{tp(h, t)} - \frac{1 - e^{-2K(\varepsilon)t}}{tp(h, t)} q(u, t) \right) \leq \left(\frac{1}{2} + \varepsilon \right) \quad \forall t \in (0, T).$$

Hence

$$\sup_{h \in X^0} \frac{1}{tp(h, t)} q(u, t) \leq \left(\frac{1}{2} + \varepsilon \right) + \sup_{h \in X^0} \frac{1 - e^{-2K(\varepsilon)t}}{tp(h, t)} q(u, t). \quad \forall t \in (0, T).$$

By (7.14),

$$q(u, t) \leq e^{2K(\varepsilon)t} \left(\frac{1}{2} + \varepsilon \right) tp(h, t) \quad \forall t \in (0, T), \quad h \in X^0.$$

Hence

$$\sup_{h \in X^0} \frac{1 - e^{-2K(\varepsilon)t}}{tp(h, t)} q(u, t) \rightarrow 0 \quad \text{as } t \rightarrow 0+ \quad \forall \varepsilon > 0.$$

If $h \in X_c^0$, then $p(h, t) \rightarrow \|h(\cdot, 0)\|_{H^0}^2$ as $t \rightarrow 0+$. It follows that

$$\overline{\lim}_{t \rightarrow 0+} \sup_{h \in X^c} \frac{q(u, t)}{t \|h(\cdot, 0)\|_{H^0}^2} \leq \left(\frac{1}{2} + \varepsilon \right)$$

for any $\varepsilon > 0$. Then (4.1) follows. This completes the proof of Theorem 4.1. \square

Proof of Theorem 5.1. Repeat that $u(x, t) = e^{Kt} u_K(x, t)$, where u is the solution of problem (2.1) and u_K is the solution of (7.2) for $h_K(x, t) = e^{-Kt} h(x, t)$. Therefore, it suffices to find n , D , b , f , λ such that

$$\begin{aligned} \forall T > 0, c > 0, K > 0 \quad \exists h \in L_2(Q) : \\ (\nabla u(\cdot, T), b(\cdot, T) \nabla u(\cdot, T))_{H^0} \geq \left(\frac{1}{2} - c \right) \int_0^T \|h(\cdot, t)\|_{H^0}^2 dt, \end{aligned} \quad (7.15)$$

where u is the solution of problem (7.2).

Let us show that (7.15) holds for

$$n = 1, \quad D = (-\pi, \pi), \quad b(x, t) \equiv 1, \quad f(x, t) \equiv 0, \quad \lambda(x, t) \equiv 0. \quad (7.16)$$

In this case, (7.2) has the form

$$u'_t = u''_{xx} - Ku + h, \quad u(x, 0) \equiv 0, \quad u(x, t)|_{x \in \partial D} = 0, \quad (7.17)$$

Let

$$\gamma = m^2 + K, \quad h_m(x, t) \triangleq \sin(mx) e^{\gamma t}, \quad (7.18)$$

where $m = 1, 2, 3, \dots$. It can be verified immediately that the solution of the boundary value problem is

$$u(x, t) = \sin(mx) \int_0^t e^{-\gamma(t-s)+\gamma s} ds = \sin(mx) e^{-\gamma t} \int_0^t e^{2\gamma s} ds = \sin(mx) e^{-\gamma t} \frac{e^{2\gamma t} - 1}{2\gamma}.$$

Hence

$$\|u'_x(\cdot, T)\|_{H^0}^2 = m^2 \|\cos(mx)\|_{H^0}^2 e^{-2\gamma T} \left(\frac{e^{2\gamma} - 1}{2\gamma} \right)^2 = m^2 \pi e^{-2\gamma T} \frac{(e^{2\gamma T} - 1)^2}{4\gamma^2},$$

and

$$\int_0^T \|h(\cdot, t)\|_{H^0}^2 dt = \|\sin(mx)\|_{H^0}^2 \int_0^T e^{2\gamma t} dt = \pi \frac{e^{2\gamma T} - 1}{2\gamma}.$$

It follows that

$$\|u'_x(\cdot, T)\|_{H^0}^2 \left(\int_0^T \|h(\cdot, t)\|_{H^0}^2 dt \right)^{-1} = \frac{m^2}{2\gamma} e^{-2\gamma T} (e^{2\gamma T} - 1) = \frac{m^2}{2\gamma} (1 - e^{-2\gamma T}) \rightarrow \frac{1}{2} \quad (7.19)$$

as $\gamma \rightarrow +\infty$. In particular, it holds if K is fixed and $m \rightarrow +\infty$. It follows that (5.1) holds. This completes the proof of Theorem 5.1. \square

Proof of Theorem 5.2. Let the parameters of the equation be defined by (7.16). Consider a sequence $\{T_i\}$ such that $T_i \rightarrow 0+$ as $i \rightarrow +\infty$. Let $h = h_m$ be defined by (7.18) for an increasing sequence of integers $m = m_i$ such that $m_i > T_i^{-1}$. In that case, (7.19) holds since $\gamma T \rightarrow +\infty$. Then (5.2) holds. \square

Proof of Theorem 6.1. For $K > 0$, introduce operators

$$B_K u \triangleq e^{K(\tau(t)-t)} \left(\beta(x, \tau(t))^\top \nabla u(x, \tau(t)) + \bar{\beta}(x, \tau(t)) u(x, \tau(t)) \right). \quad (7.20)$$

Note that $u \in Y^2$ is the solution of the problem (6.2) if and only if $u_K(x, t) = e^{-Kt} u(x, t)$ is the solution of the problem

$$\begin{aligned} \frac{\partial u_K}{\partial t} &= \mathcal{A}u_K - Ku_K + B_K u_K + h_K, & t \in (0, T), \\ u_K(x, 0) &= 0, \quad u_K(x, t)|_{x \in \partial D} = 0, \end{aligned} \quad (7.21)$$

where $h_K(x, t) = e^{-Kt} h(x, t)$. In addition,

$$\|u\|_{Y^2} \leq e^{KT} \|u_K\|_{Y^2}, \quad \|h_K\|_{L_2(Q)} \leq \|h\|_{L_2(Q)}.$$

Therefore, uniqueness and solvability in Y^2 of problem (6.2) follows from existence of $K > 0$ such that problem (7.21) has an unique solution in Y^2 . Let us show that this K can be found.

We introduce operators $F_K : L_2(Q) \rightarrow Y^2$ such that $u = F_K h$ is the solution of problem (7.2).

Let $g \in L_2(Q)$ be such that

$$g = h + B_K w, \quad \text{where } w = F_K g. \quad (7.22)$$

It can be rewritten as $g = h + R_K g$, or

$$g - R_K g = h, \quad (7.23)$$

where the operator $R_K : L_2(Q) \rightarrow L_2(Q)$ is defined as

$$R_K = B_K F_K.$$

In that case, $u_K \triangleq F_K g \in Y^2$ is the solution of (7.21).

Let us show that there exists $K > 0$ such that

$$\|R_K\| < 1. \quad (7.24)$$

Let $w = F_K h$. By Theorem 3.1 reformulated as Lemma 7.1, for any $\varepsilon > 0$, $M > 0$, there exists $K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$ such that

$$\begin{aligned} \sup_{t \in [0, T]} \left((\nabla w(\cdot, t), b(\cdot, t) \nabla w(\cdot, t))_{H^0} + M \|w(\cdot, t)\|_{H^0}^2 + M \int_0^t (\nabla w(\cdot, s), b(\cdot, s) \nabla w(\cdot, s))_{H^0} ds \right) \\ \leq \left(\frac{1}{2} + \varepsilon \right) \|h\|_{L_2(Q)}^2 \quad \forall h \in L_2(Q). \end{aligned} \quad (7.25)$$

Let $\widehat{w}(x, t) \triangleq w(x, \tau(t))$, $\widehat{\beta}(x, t) \triangleq \beta(x, \tau(t))$, and $\widehat{b}(x, t) \triangleq b(x, \tau(t))$. By the definitions,

$$\|R_K h\|_{L_2(Q)} = \|B_K w_K\|_{L_2(Q)} \leq \|\widehat{\beta}^\top \nabla \widehat{w}\|_{L_2(Q)} + C_\beta \|\widehat{w}\|_{L_2(Q)}, \quad (7.26)$$

where $C_\beta \triangleq \sup_{x, t} |\widehat{\beta}(x, t)|$. Clearly,

$$\|\widehat{\beta}^\top \nabla \widehat{w}\|_{L_2(Q)}^2 \leq T \sup_{t \in [0, T]} \|\widehat{\beta}(\cdot, t)^\top \nabla \widehat{w}(\cdot, t)\|_{H^0}^2 \leq T \sup_{t \in [0, T]} \|\beta(\cdot, t)^\top \nabla w(\cdot, t)\|_{H^0}^2, \quad (7.27)$$

$$\|\widehat{w}\|_{L_2(Q)}^2 \leq T \sup_{t \in [0, T]} \|\widehat{w}(\cdot, t)\|_{H^0}^2 \leq T \sup_{t \in [0, T]} \|w(\cdot, t)\|_{H^0}^2. \quad (7.28)$$

Further, let $K > 0$, $M > 0$, and $\varepsilon > 0$, be such that (7.25) is satisfied and

$$\delta_2 < 1, \quad \delta_2^2 + \delta_3^2 + 2\delta_2\delta_3 < 1, \quad (7.29)$$

where

$$\delta_2 \triangleq \sqrt{(2 - \delta_1) \left(\frac{1}{2} + \varepsilon \right)}, \quad \delta_3 \triangleq C_\beta \sqrt{TM^{-1} \left(\frac{1}{2} + \varepsilon \right)}. \quad (7.30)$$

By (7.25), (7.27)-(7.30), and (6.3), it follows that

$$\|\widehat{\beta}^\top \nabla \widehat{w}\|_{L_2(Q)}^2 \leq T \sup_{t \in [0, T]} \|\beta(\cdot, t)^\top \nabla w(\cdot, t)\|_{H^0}^2 \leq (2 - \delta_1) \sup_{t \in [0, T]} (\nabla w, b \nabla w)_{H^0} \leq \delta_2^2 \|h\|_{L_2(Q)}^2.$$

In addition, we have that

$$C_\beta^2 \|\widehat{w}\|_{L_2(Q)}^2 \leq T C_\beta^2 \sup_{t \in [0, T]} \|w(\cdot, t)\|_{H^0}^2 \leq T C_\beta^2 M^{-1} \left(\frac{1}{2} + \varepsilon \right) \|h\|_{L_2(Q)}^2 \leq \delta_3^2 \|h\|_{L_2(Q)}^2.$$

By (7.26), it follows that

$$\|R_K h\|_{L_2(Q)}^2 \leq \left(\|\widehat{\beta}^\top \nabla \widehat{w}\|_{L_2(Q)} + C_\beta \|\widehat{w}\|_{L_2(Q)} \right)^2 \leq (\delta_2^2 + 2\delta_2\delta_3 + \delta_3^2) \|h\|_{L_2(Q)}^2.$$

By (7.29), it follows that (7.24) holds, where the norm of the operator $R_K : L_2(Q) \rightarrow L_2(Q)$ is considered. It follows that the operator $(I - R_K)^{-1} : L_2(Q) \rightarrow L_2(Q)$ is continuous. By (7.22)-(7.23), $u_K = F_K(I - R_K)^{-1}h_K$ is the solution of problem (7.21) for $h_K \in L_2(Q)$.

The choice of K , M , and ε , depends on \mathcal{P} , δ_1 , and $\sup_{x,t} |\bar{\beta}(x,t)|$ only. Hence δ_2 and δ_3 depends on these parameters only. It follows that the norm of the operator $(I - R_K)^{-1} : L_2(Q) \rightarrow L_2(Q)$ can be estimated from above by a constant that depends only on these parameters. This proves the estimate for the solution stated in Theorem 6.1. This completes the proof of this theorem. \square

Proof of Theorem 6.2 is based again on (7.25). It is similar to the proof of Theorem 6.1, with a minor modification: instead of (7.27), we use that

$$\begin{aligned} & \int_0^T \|\beta(\cdot, \tau(t))^\top \nabla u(\cdot, \tau(t))\|_{H^0}^2 dt = \int_\theta^T \|\beta(\cdot, \tau(t))^\top \nabla u(\cdot, \tau(t))\|_{H^0}^2 dt \\ & = \int_\theta^T \|\beta(\cdot, \tau(t))^\top \nabla u(\cdot, \tau(t))\|_{H^0}^2 \left(\frac{d\tau(t)}{dt} \right)^{-1} d\tau(t) \leq \delta_* \int_{\tau(\theta)}^{\tau(T)} \|\beta(\cdot, s)^\top \nabla u(\cdot, s)\|_{H^0}^2 ds \\ & \leq c \int_0^T (\nabla u(\cdot, s), b(\cdot, s) \nabla u(\cdot, s))_{H^0} ds, \end{aligned}$$

where c is a constant that depends on δ , δ_* , and $\sup_{x,t} |\beta(x,t)|$. \square

It can be seen from the proofs that the approach used for Theorems 6.1-6.2 can be extended on more general delay operators represented by integrals accumulating the past values.

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