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Analysis of a new variational model for image multiplicative denoising

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Abstract

In this paper, we study the mathematical properties of a new variational model for image multiplicative noise removal. Some important properties of the model, including the lower semicontinuity, the differential property, the convergence and regularization property, are established for the first time. The existence and uniqueness of a solution for the problem as well as a comparison principle have also been established.

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1 Introduction

Consider the following variational problem:

$$\min_u E(u) := \min_u \left\{ \int_\Omega \phi(Du) + \lambda \int_\Omega \left(\log u + \frac{f}{u} \right) \right\}, \tag{1}$$

where $\Omega \subset R^N$ is an open bounded open set with Lipschitz-regular boundary $\partial\Omega$, λ is a constant, $f : \Omega \to R^+$ is a given function, ϕ is an even function from R^N to R having the linear growth

$$\phi(s) = \begin{cases} |s| \log(1 + |s|), & |s| < M, \\ b|s| - \frac{M^2}{1+M}, & |s| \geq M, \end{cases} \tag{2}$$

where $b = M/(1 + M) + \log(1 + M)$, M is a positive constant and its value is determined by the size of an image.

The total variation has been introduced in computer vision first by Rudin, Osher and Fatemi [1] as a regularizing criterion for solving inverse problems. It has proved to be very efficient for regularizing images without smoothing the boundaries of the objects. A variety of variational methods have been proposed for imaging processing over the last decades, and the main variational approaches devoted to multiplicative noises include the *RLO* model [2] proposed by Rudin *et al.*, the *AA* model [3] by Aubert and Aujol, and the *JY* model [4] by Jin and Yang.

Variational problem (1) is a multiplicative noise removal model. This model was first obtained via the MAP estimator [5]. It is specifically devoted to the denoising of images

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corrupted by a gamma noise. This model cannot yield good denoising effectiveness, but it can significantly avoid or reduce 'edge blur' and 'step effect'.

From a mathematical point of view, most of the existing variational models for denoising have been studied extensively [2–4, 6]. For example, for a very informative discussion of the use of total variation regularization in the field of image processing, see the introduction of [7]. In [6], Aubert and Aujol proved the existence of a minimizer for a variational model and studied the associated evolution problem, for which they derived existence and uniqueness results for the solution. Moreover, they proved the convergence of an implicit iterative scheme to compute the solution. Jin and Yang in [4] established the existence and uniqueness of a weak solution for the associated evolution equations of the *JY* model, and showed that the solution of the evolution equation converges weakly in *BV* and strongly in L^2 to the minimizer as $t \to \infty$. For model (1), the initial boundary value problem of the partial differential equation for the model is derived and discreted numerically. The experiments in [5] showed that the quality of the images restored by the model is excellent. However, in order to further study and apply model (1), further rigorous work is needed to investigate the mathematical properties of the model.

The main goal of this paper is thus to further study the mathematical properties of model (1). The study is conducted in the space of functions with bounded variations (BV). We first establish some important mathematical properties for the function ϕ in the model including the lower semicontinuity, the differential property, the convergence and regularization of its mollification ϕ^ϵ. Then it is easy to prove the existence and uniqueness of a solution for the model under appropriate assumptions. Furthermore, a comparison principle is obtained.

The paper is organized as follows. Some preliminaries are given in the next section. In Section 3, some properties about the function ϕ are obtained. Some new results for variational model (1) such as the existence, uniqueness and comparison results are derived in Section 4. Finally, a conclusion is given in Section 5.

2 Preliminaries

In this paper, we use the following classical distributional spaces. For the convenience of readers, we here recall some basic notations and facts, and for details we refer the readers to the works of Aubert-Kornprobst in [6].

Let $|E|$ denote the Lebesgue measure of a measurable set E in R^N, let $\mathcal{H}^{N-1}$ denote the Hausdorff measure of dimension $N-1$, and let $W^{1,p}(\Omega)$ and $L^p(\Omega)$ be the standard notations for the Sobolev and Lebesgue spaces, respectively, let $C_0^\infty(\Omega)$ be the set of functions in $C^\infty(\Omega)$ with compact support in Ω.

Definition 2.1 If for all functions $\varphi = (\varphi_1, \varphi_2) \in C_0^1(\Omega)^2$, $|\varphi|_{L^\infty(\Omega)}| \leq 1$, the formula

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega Du \cdot \varphi \, dx$$

holds, $Du = (D_1 u, D_2 u)$ is called the distribution gradient of u, u is called a bounded variation function, and the total variation of Du on Ω is defined as follows:

$$\int_\Omega |Du| \, dx := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi = (\varphi_1, \varphi_2) \in C_0^1(\Omega)^2, |\varphi|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Definition 2.2 The space of bounded variation functions, $BV(\Omega)$, is defined as follows:

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \int_\Omega |Du|\, dx < \infty \right\}.$$

It should be addressed here that the $BV(\Omega)$ endowed with the norm $\|u\|_{BV(\Omega)} = \int_\Omega |u|\, dx + \int_\Omega |Du|\, dx$ is a Banach space.

With regard to the compactness in $BV(\Omega)$, we state the following theorem.

Theorem 2.1 ([8]) *Assume that $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $BV(\Omega)$. Then there exists a subsequence $\{u_{n_j}\}_{j=1}^\infty$ and a function $u \in BV(\Omega)$ such that $u_{n_j} \to u$ as $j \to \infty$ in $L^1(\Omega)$.*

Definition 2.3 The approximate upper limit $u^+(x)$ and the approximate lower limit $u^-(x)$, respectively, are defined by

$$u^+(x) = \inf\left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|\{u > t\} \cap B(x, \rho)|}{\rho^2} = 0 \right\},$$

$$u^-(x) = \sup\left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{|\{u < t\} \cap B(x, \rho)|}{\rho^2} = 0 \right\},$$

where $B(x, \rho)$ is the ball of center x and radius ρ. When $u^+(x)$ is different from $u^-(x)$, we define the jump set S_u as follows:

$$S_u = \left\{ x \in \Omega : u^-(x) < u^+(x) \right\}.$$

After choosing a normal $n_u(x)$ ($x \in S_u$) pointing toward the largest value of u, we recall the following decompositions (see [9] for more details):

$$Du = \nabla u \cdot dx + D^s u, \tag{3}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure, $D^s u = C_u + (u^+ - u^-)n_u \mathcal{H}^{N-1}_{|S_u}$ is the singular part, C_u is the Cantor part, and $(u^+ - u^-)n_u \mathcal{H}^{N-1}_{|S_u}$ is the jump part, $\mathcal{H}^{N-1}_{|S_u}$ is the Hausdorff measure restricted to the set S_u.

Let η_ϵ be the usual mollifier with compact support $B(0, \epsilon)$ and $\int_{R^N} \eta_\epsilon(y)\, dy = 1$. If $g \in L^1_{\text{loc}}(\Omega)$, define its mollification $g^\epsilon := (g * \eta_\epsilon)$ in $\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$. That is,

$$g^\epsilon(x) = \int_\Omega g(y)\eta_\epsilon(x - y)\, dy = \int_{B(0,\epsilon)} g(x - y)\eta_\epsilon(y)\, dy \quad \text{for } x \in \Omega_\epsilon.$$

Using the standard properties of mollifiers, we have the following properties.

Theorem 2.2

(1) *If $1 \le p < \infty$ and $g \in L^p_{\text{loc}}(\Omega)$, then $g^\epsilon \in C^\infty(\Omega)$ and $g^\epsilon \to g$ in $L^p_{\text{loc}}(\Omega)$ and if $g \in L^p(\Omega)$, then $g^\epsilon \to g$ in $L^p(\Omega)$.*

(2) *If $A \le g(x) \le B$ for all x, then $A \le g^\epsilon(x) \le B$ for all $x \in \Omega_\epsilon$.*

(3) *If $h, g \in L^1(\Omega)$, then $\int_\Omega h^\epsilon g\, dx = \int_\Omega h g^\epsilon\, dx$.*

(4) *If $g \in C^1(\Omega)$, then $\frac{\partial g^\epsilon}{\partial x_i} = \left(\frac{\partial g}{\partial x_i}\right)^\epsilon$.*

Proof Using the standard properties of mollifiers, we can easily prove the above results.
□

3 Some properties about ϕ

By the definition of ϕ in (2), it is easy to see that ϕ belongs to $C^1(R^N)$ and ϕ is Lipschitz continuous with the gradient

$$\nabla\phi(s) = \begin{cases} [\log(1 + |s|) + \frac{|s|}{1+|s|}]\frac{s}{|s|}, & |s| \leq M, \\ b\frac{s}{|s|}, & |s| > M. \end{cases}$$

Furthermore, it can be proved that

$$\left|\phi(s_1) - \phi(s_2)\right| \leq b|s_1 - s_2|, \tag{4}$$

$$\left|\nabla\phi(s_1) - \nabla\phi(s_2)\right| \leq b|s_1 - s_2|. \tag{5}$$

Theorem 3.1 *If $\phi^\epsilon = \phi * \eta_\epsilon$, where η_ϵ is the mollifier, then*
  (1) $\phi^\epsilon \in C^\infty(R^N)$ *and ϕ^ϵ is convex.*
  (2) $\phi^\epsilon \to \phi$ *uniformly on compact subsets of R^N as $\epsilon \to 0$.*
  (3) $\nabla\phi^\epsilon(x) \cdot x \geq 0$ *for all $x \in R^N$.*

Proof (1) Since ϕ is continuous and convex, ϕ^ϵ is clearly smooth and convex in R^N (*cf.* Theorem 6 of Appendix C in [8]).
  (2) According to (4), for all $x \in R^N$, we have

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|\phi(y) - \phi(x)\right| dx = 0. \tag{6}$$

For a fixed point x, by the definition of ϕ^ϵ, we have from (6) that

$$\begin{aligned} &\left|\phi^\epsilon(x) - \phi(x)\right| \\ &= \left|\int_{B(x,\epsilon)} \eta^\epsilon(x - y)\left[\phi(y) - \phi(x)\right] dy\right| \\ &\leq \frac{1}{\epsilon^N} \int_{B(x,\epsilon)} \eta\left(\frac{x - y}{\epsilon}\right)\left|\phi(y) - \phi(x)\right| dy \\ &\leq \frac{C}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} \left|\phi(y) - \phi(x)\right| dy \to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

Now $\phi \in C(R^N)$ and $V \subset\subset R^N$, we choose $V \subset\subset W \subset\subset R^N$ and note that ϕ is uniformly continuous on W. Thus the limit (6) holds uniformly for $x \in V$. Consequently, the calculation above implies $\phi^\epsilon \to \phi$ uniformly on V.
  (3) Let $\epsilon < \frac{1}{4}$. The convexity of ϕ^ϵ implies that $\phi^\epsilon(0) - \phi^\epsilon(x) \geq \nabla\phi^\epsilon(x) \cdot (0 - x)$, so $\nabla\phi^\epsilon(x) \cdot x \geq \phi^\epsilon(x) - \phi^\epsilon(0)$. Further, we have

$$\begin{aligned} &\phi^\epsilon(x) - \phi^\epsilon(0) \\ &= \int_{B(0,\epsilon)} \phi(x - y)\eta_\epsilon(y) dy - \int_{B(0,\epsilon)} \phi(y)\eta_\epsilon(y) dy \end{aligned}$$

$$= \int_{B(0,\epsilon)\cap B(x,1)} \left[|x-y|\log\left(1+|x-y|\right) - |y|\log\left(1+|y|\right) \right]\eta_\epsilon(y)\,dy$$

$$+ \int_{B(0,\epsilon)\cap\{|x-y|\geq 1\}} \left[b|x-y| - \frac{1}{2} - |y|\log\left(1+|y|\right) \right]\eta_\epsilon(y)\,dy.$$

For the first term on the right-hand side, we consider two possible cases.

Case 1. $|x| \geq 2\epsilon$. Since $|y| \leq \epsilon$, it implies $|x-y| \geq \epsilon \geq |x|$. Noting that the function $\phi(s)$ is increasing on $(0,+\infty)$, we obtain that $|x-y|\log(1+|x-y|) - |y|\log(1+|y|) \geq 0$. So the first term on the right-hand side is nonnegative.

Case 2. $|x| < 2\epsilon$. Since $|y| \leq \epsilon < \frac{1}{3}$, $|x-y| \leq |x| + |y| \leq 2\epsilon + \epsilon \leq \frac{3}{4} < 1$, and so $B(0,\epsilon) \cap B(x,1) = B(0,\epsilon)$. In terms of the convexity of ϕ, we have

$$|x-y|\log\left(1+|x-y|\right) - |y|\log\left(1+|y|\right)$$

$$\geq \nabla\phi(y) \cdot \left((y-x) - y\right)$$

$$= \left(\log\left(1+|y|\right) + \frac{|y|}{1+|y|} \right)\frac{y}{|y|} \cdot (-x).$$

Noting that $\int_{B(0,\epsilon)} y\eta_\epsilon(y)\,dy = 0$, we get

$$\int_{B(0,\epsilon)\cap B(x,1)} \left[|x-y|\log\left(1+|x-y|\right) - |y|\log\left(1+|y|\right) \right]\eta_\epsilon(y)\,dy$$

$$\geq \int_{B(0,\epsilon)} \left[\left(\log\left(1+|y|\right) + \frac{|y|}{1+|y|} \right)\frac{y}{|y|} \cdot (-x) \right]\eta_\epsilon(y)\,dy = 0.$$

For the second term on the right-hand side, noting that $\epsilon < 1$, we have

$$\int_{B(0,\epsilon)\cap\{|x-y|\geq 1\}} \left[b|x-y| - \frac{1}{2} - |y|\log\left(1+|y|\right) \right]\eta_\epsilon(y)\,dy$$

$$\geq \int_{B(0,\epsilon)\cap\{|x-y|\geq 1\}} \left(b - \frac{1}{2} - \log 2 \right)\eta_\epsilon(y)\,dy = 0.$$

Therefore, $\phi^\epsilon(x) - \phi^\epsilon(0) \geq 0$. □

Remark Since $\phi : R \to R^+$ is convex, increasing on R^+ with linear growth at infinity, and $\phi^\infty(1) = \lim_{s\to\infty} \phi(s)/s = b$, for $u \in BV(\Omega)$, we can get the Lebesgue decomposition of the measure $\int_\Omega \phi(Du)\,dx$:

$$\int_\Omega \phi(Du)\,dx = \int_\Omega \phi(\nabla u)\,dx + b\int_\Omega |D^s u|$$

$$= \int_\Omega \phi(\nabla u)\,dx + b\int_{\Omega - S_u} |C_u| + b\int_{S_u} |u^+ - u^-|\,d\mathcal{H}^{N-1}.$$

Since $b|s| - M^2/(1+M) \leq \phi(s) \leq b|s|$, for all $s \in R^N$, we have

$$\int_\Omega |Du| - \frac{M^2}{1+M}|\Omega| \leq \int_\Omega \phi(Du) \leq \int_\Omega |Du|, \quad \forall u \in BV(\Omega). \tag{7}$$

The lower semicontinuity of the functional ϕ with respect to convergence in $L^1(\Omega)$, which is one of the most important properties of BV functions, is given in the next theorem below.

Theorem 3.2 *Assume that $\{u_i\}_{i=1}^{\infty} \subset BV(\Omega)$ and $u_i \to u$ in $L^1(\Omega)$ as $i \to \infty$, then*

$$\int_{\Omega} \phi(Du) \leq \liminf_{i \to \infty} \int_{\Omega} \phi(Du_i). \tag{8}$$

Proof Let $v \in C_0^1(\Omega)$ be such that $|v| \leq 1$. Then

$$\int_{\Omega} u \operatorname{div} v \, dx = \lim_{i \to \infty} \int_{\Omega} u_i \operatorname{div} v \, dx \leq \liminf_{i \to \infty} \int_{\Omega} |Du_i|.$$

Taking the supremum over all such v, we have

$$\int_{\Omega} |Du| \leq \liminf_{i \to \infty} \int_{\Omega} |Du_i|. \tag{9}$$

Thus (8) follows on combining (7) and (9). □

Theorem 3.3 (Regularization) *Suppose $u \in BV(\Omega)$. If u^{ϵ} are the mollified functions described in Section 2 (where u is extended to be 0 outside Ω if necessary), then*

$$\lim_{\epsilon \to 0} \int_{\Omega} \phi(Du^{\epsilon}) = \int_{\Omega} \phi(Du).$$

Proof Since $u^{\epsilon} \to u$ as $\epsilon \to 0$ in $L^1(\Omega)$ from Theorem 2.2, we have, by Theorem 3.2, the inequality

$$\int_{\Omega} \phi(Du) \leq \liminf_{\epsilon \to 0} \int_{\Omega} \phi(Du^{\epsilon}),$$

and so it remains only to prove a verse inequality.

For $u \in BV(\Omega)$, by the method of the proof in Theorem A.2 of [10], we can obtain

$$\int_{\Omega} \phi(Du) = \sup_{\substack{v \in C_0^1(\Omega) \\ |v| \leq 1}} - \int_{\Omega} \left(|v| \log(1 + |v|) + u \operatorname{div} v \right) dx. \tag{10}$$

Suppose $v \in C_0^1(\Omega)$ and $|v| \leq 1$, then, by Theorem 2.2,

$$\int_{\Omega} u^{\epsilon} \operatorname{div} v \, dx = \int_{\Omega} u(\operatorname{div} v)^{\epsilon} \, dx = \int_{\Omega} u \operatorname{div} v^{\epsilon} \, dx$$

and

$$\lim_{\epsilon \to 0} \int_{\Omega} |v^{\epsilon}| \log(1 + |v^{\epsilon}|) = \int_{\Omega} |v| \log(1 + |v|).$$

Thus

$$- \int_{\Omega} \left(|v| \log(1 + |v|) + u^{\epsilon} \operatorname{div} v \right) dx = \lim_{\epsilon \to 0} - \int_{\Omega} \left(|v^{\epsilon}| \log(1 + |v^{\epsilon}|) + u \operatorname{div} v^{\epsilon} \right) dx. \tag{11}$$

Since $|v| \leq 1$, by Theorem 2.2, we get that $|v^\epsilon| \leq 1$. On taking the supremum over all such v and noting (10), we get that

$$
\int_\Omega \phi(Du^\epsilon) = \lim_{\epsilon \to 0} - \int_\Omega \left(|v^\epsilon| \log(1 + |v^\epsilon|) + u \operatorname{div} v^\epsilon \right) dx
$$

$$
\leq \lim_{\epsilon \to 0} \sup_{|v^\epsilon| \leq 1} - \int_\Omega \left(|v^\epsilon| \log(1 + |v^\epsilon|) + u \operatorname{div} v^\epsilon \right) dx
$$

$$
= \int_\Omega \phi(Du).
$$

Thus

$$
\liminf_{\epsilon \to 0} \int_\Omega \phi(Du^\epsilon) \leq \int_\Omega \phi(Du). \qquad \square
$$

4 Some facts for the variational problem

Consider the variational problem. Assume that $f \in L^1(\Omega)$ with $0 < \inf_\Omega f \leq f \leq \sup_\Omega f < \infty$, find a function $\tilde{u} \in S(\Omega)$ such that

$$
E(\tilde{u}) = \min_{u \in S(\Omega)} \left\{ \int_\Omega \phi(Du) + \int_\Omega \left(\log u + \frac{f}{u} \right) \right\}, \tag{12}
$$

where $S(\Omega) = \{ u \in BV(\Omega) : \nabla u \in L^1(\Omega)^2 \}$.

For the existence of solutions for minimization problem (12), we have obtained the existence result in the BV space (see [5]), namely, for $f \in L^\infty(\Omega)$ with $\inf_\Omega f > 0$, problem (12) has at least one solution $u \in BV(\Omega)$ satisfying $0 < \inf_\Omega f \leq u \leq \sup_\Omega f$.

We find that, under appropriate assumptions, we can prove the existence of a Lipschitz solution for problem (12) as detailed below.

Theorem 4.1 (Existence of a Lipschitz solution) *Suppose that Ω is a bounded open domain in R^N and $f \in L^1(\Omega)$ with $0 < \inf_\Omega f \leq f \leq \sup_\Omega f < \infty$. Then problem (12) has at least one Lipschitz solution if for any $x \in \partial\Omega$, there exist two planes in R^{N+1}, $z = \pi^+(x)$ and $z = \pi^-(x)$ such that*

(i) $\pi^-(x) \leq f(x) \leq \pi^+(x)$, $\forall x \in \partial\Omega$;

(ii) *the slopes of these planes are uniformly bounded, independent of x, by a constant K. That is, $|D\pi^\pm| \leq K$ for all $x \in \Omega$.*

Proof Consider the approximation of problem (12), that is, replacing $\phi(s)$ by $\phi^\epsilon(s) + |s| \log(1 + |s|)$. By the general elliptic theory (see [8]), there exists a minimizer u_ϵ for each $\epsilon > 0$. The proof in Section 10 of [11] shows that u_ϵ has a uniform gradient bound which is not greater than K, then, by applying to a subsequence of u_ϵ as $\epsilon \to 0$, we get a Lipschitz function u, and the rest of the proof is straightforward. $\qquad \square$

Theorem 4.2 (Uniqueness) *Let $f > 0$ be in $L^\infty(\Omega)$, then problem (12) has at most one solution $\tilde{u}$ such that $0 < \tilde{u} < 2f$.*

Proof Letting $\tau(u) = \log u + f/u$, we have $\tau'(u) = 1/u + (u - f)/u^2$ and $\tau''(u) = (2f - u)/u^3$. We thus deduce that if $0 < u < 2f$, then τ is strictly convex. Using (7) and noting that $\phi : R \to R^+$

is convex, increasing on R^+ with linear growth at infinity, we obtain the uniqueness of a minimizer. □

Theorem 4.3 (Comparison principle) *Let f_1 and f_2 be in $L^\infty(\Omega)$ with $\inf_\Omega f_1 > 0$ and $\inf_\Omega f_2 > 0$. Assume that $f_1 < f_2$ in Ω. We denote by u_1 (resp., u_2) a solution of* (12) *for $f = f_1$ (resp., $f = f_2$). Then we have $u_1 \leq u_2$.*

Proof Let $J(u) = \int_\Omega \phi(Du)$. From Theorem 4 in [5], we know that u_1 and u_2 exist. Since u_i is a minimizer with data f_i ($i = 1, 2$), we have

$$\int_\Omega \left(\log(\inf(u_1, u_2)) + \frac{f_1}{\inf(u_1, u_2)} \right) + J(\inf(u_1, u_2))$$
$$\geq \int_\Omega \left(\log u_1 + \frac{f_1}{u_1} \right) + J(u_1) \tag{13}$$

and

$$\int_\Omega \left(\log(\sup(u_1, u_2)) + \frac{f_2}{\sup(u_1, u_2)} \right) + J(\sup(u_1, u_2))$$
$$\geq \int_\Omega \left(\log u_2 + \frac{f_2}{u_2} \right) + J(u_2). \tag{14}$$

Using the fact that $J(\inf(u_1, u_2)) + J(\sup(u_1, u_2)) \leq J(u_1) + J(u_2)$ (see [12] or [13] for details), and adding these two inequalities (13) and (14), we obtain

$$\int_\Omega \left(\log(\inf(u_1, u_2)) + \log(\sup(u_1, u_2)) + \frac{f_1}{\inf(u_1, u_2)} + \frac{f_2}{\sup(u_1, u_2)} \right)$$
$$\geq \int_\Omega \left(\log u_1 + \log u_2 + \frac{f_1}{u_1} + \frac{f_2}{u_2} \right). \tag{15}$$

Writing $\Omega = \{u_1 > u_2\} \cup \{u_1 \leq u_2\}$, we easily get that

$$\int_{\{u_1 > u_2\}} \left(\log u_2 + \log u_1 + \frac{f_1}{u_2} + \frac{f_2}{u_1} \right) \geq \int_{\{u_1 > u_2\}} \left(\log u_1 + \log u_2 + \frac{f_1}{u_1} + \frac{f_2}{u_2} \right),$$

this is,

$$\int_{\{u_1 > u_2\}} (f_1 - f_2) \frac{u_1 - u_2}{u_1 u_2} \geq 0.$$

Since $f_1 < f_2$, we thus deduce that $\{u_1 > u_2\}$ has a zero Lebesgue measure; *i.e.*, $u_1 \leq u_2$ a.e. in Ω. □

Remark The above result agrees with observation in image processing. Suppose that f_1 and f_2 are two images with noise. u_1 and u_2 are from f_1 and f_2 after denoising by some method, respectively. If the image f_1 is dimer than f_2 at almost every pixel, then u_1 is not naturally lighter than u_2.

5 Conclusion

In this paper, we study the mathematical properties of an important variational model proposed in our recent work [5] for removing multiplicative noise. For the first time, many important properties of the model, including the lower semicontinuity, the differential property, the convergence and regularization property are established and proved. The well-posedness of the underlying mathematical problem for the model has also been established. The comparison principle of solutions for the model has also been obtained.

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