Optimal switching instants for a switched-capacitor DC/DC power converter ⋆

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Abstract

We consider a switched-capacitor DC/DC power converter with variable switching instants. The determination of optimal switching instants giving low output ripple and strong load regulation is posed as a non-smooth dynamic optimization problem. By introducing a set of auxiliary differential equations and applying a time-scaling transformation, we formulate an equivalent optimization problem with semi-infinite constraints. Existing algorithms can be applied to solve this smooth semi-infinite optimization problem. Existence of an optimal solution is also established. For illustration, the optimal switching instants for a practical switched-capacitor DC/DC power converter are determined using this approach.

Key words: Power converter, Impulsive dynamical system, Switched linear system, Semi-infinite programming, Control parametrization enhancing transform.

1 Introduction

DC/DC power converters are used in mobile electronic systems such as laptop computers and cellular phones to generate different DC voltages from a single battery source. Over the past two decades, many modern DC/DC power converters have been developed that can be realized primarily using capacitors and switches (see, for example, Chung & Mok (1999) or Chung, Chow, Hui, & Lee (2000) and the references cited therein). Such power converters are called *switched-capacitor DC/DC power converters*. Free of bulky inductive elements, they are ideal for small-size applications requiring low electromagnetic interference and high power density.

The capacitors in a switched-capacitor DC/DC power converter are used to store and supply energy. The circuit topology and, in particular, the function of each capacitor changes according to the switch configuration. For each switch configuration, some of the capacitors act as the power supply and deliver energy to the load; the remainder are charged by the input source. The converter operates by switching between the different topologies so that the role of each capacitor is changed regularly. More specifically, when a topology switch occurs, those capacitors that were previously discharging energy to the load begin to charge up, while those that were previously charging start to release energy as output voltage. For more detailed information, the reader can consult Ioinovici (2001) and the references cited therein.

Ideally, any DC/DC power converter should supply a steady voltage to the attached appliance. However, the switching mechanism inherent in the operation of a switched-mode power converter induces a ripple in the output voltage. Hence, although the ripple may be reduced by increasing the switching frequency, it is impossible to eliminate it entirely. On the other hand, topology switches are accompanied by an energy loss, and so excessive switching should be avoided (see Arntzen & Maksimović (1998)). Furthermore, the input voltage and load resistance influence the converter output through the circuit dynamics. This influence should be minor so that uncertainties in the input and changes to the load do not cause large variations in the output

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voltage.

A switched-capacitor DC/DC power converter can be controlled by varying the duty cycle — that is, the time spent in each topology — using a pulse-width modulation technique. In view of the previous discussion, an ideal control scheme would achieve the following two objectives: (i) minimize the output voltage ripple; and (ii) ensure output voltage regulation in the presence of uncertainties. Many different feedback control methodologies for achieving one, or both, of these objectives have been proposed in the literature. See, for example, Leung, Tam, & Li (1993), Khayatian & Taylor (1994), Garofalo, Marino, Scala, & Vasca (1994), or Choi, Lim, & Choi (2001), and the references cited therein. The majority of these methods are based on a linear time invariant approximate model of the switched-capacitor DC/DC power converter. However, since its governing dynamics change at the switching instants, a switched-mode power converter actually constitutes a highly non-linear and time-varying dynamical system. Hence, the performance of these existing control schemes can only be guaranteed under a small signal assumption.

In contrast, the problem of determining optimal switching instants *a priori* has received little attention in the literature. In Ho, Ling, Liu, Tam, & Teo (2008), a novel method for the offline computation of these switching instants was proposed. Specifically, the problem was formulated as a dynamic optimization problem, where the switching instants are chosen to minimize a cost function subject to a dynamic model of the power converter. This problem can be solved using existing optimization software such as MISER (see Jennings, Fisher, Teo, & Goh (2004)). The switching instants obtained can then be used to operate the power converter.

Unlike previous control schemes, this approach avoids the use of averaging and linearization; instead, a more accurate switched system dynamic model of the power converter is used. The time-varying and non-linear nature of a switched-mode power converter is therefore explicitly taken into account in the offline formulation of an optimal switching regime. Incidently, the optimization and control of switched systems has been an active research area over the past decade, and we direct the interested reader to Xu & Antsaklis (2004), Bengea & DeCarlo (2005), and Seatzu, Cornoa, Giua, & Bemporad (2006) for information on some recent developments.

Note also that the cost function used by Ho et al. (2008) contains terms to penalize *both* the output voltage ripple and the output sensitivity. Hence, objectives (i) and (ii) above are simultaneously considered in the determination of an optimal switching scheme. Calculating these output sensitivity terms, however, is a complicated task involving matrix inversion, eigenvalue computation, and a formula consisting of five nested summations. Therefore, computing the cost function and, in particular, its gradient, is highly involved. Nevertheless, this computation is necessary to solve the optimization problem effectively. Furthermore, the eigenvalues of the system coefficient matrices need to be derived analytically as functions of the load resistance. Such analytical expressions are only possible if the system coefficient matrices have dimension less than or equal to four. Thus, the method proposed in Ho et al. (2008) is only applicable to problems with small dimension. This is a serious restriction, and hence there is an urgent need to develop a more efficient method that can be applied to the large-scale problems encountered in practice.

With this motivation, in this paper we formulate the determination of optimal switching instants as a different optimization problem to that discussed in Ho et al. (2008). We penalize output voltage ripple over the entire time horizon, and not separately over each topology. Hence, the switching loss penalty terms introduced by Ho et al. (2008) become redundant. Furthermore, a novel method is developed to calculate the output sensitivity terms via an auxiliary system of differential equations. This auxiliary system can be solved simultaneously with the state system using any standard differential equation solver. Thus, the computation of the sensitivity terms is a simple and straightforward exercise, in contrast with the arduous task required in Ho et al. (2008). We also establish the existence of an optimal solution in Section 5 before applying our method to a practical example in Section 6.

2 Problem formulation

Consider a switched-capacitor DC/DC power converter containing m capacitors. Suppose that during the time horizon $[0, T]$, the converter switches topology n times. In other words, it cycles through $n+1$ different circuit topologies in each switching period. Since physical considerations limit the maximum rate of switching, there is a minimum time duration $\rho > 0$ that must be spent in each topology. On this basis, define

$$\Gamma := \{\, \boldsymbol{\tau} \in \mathbb{R}^n : \ \tau_i + \rho \leq \tau_{i+1}, \ i = 0, \ldots, n \,\},$$

where $\tau_0 = 0$ and $\tau_{n+1} = T$. The power converter can be operated using the components of a given $\boldsymbol{\tau} \in \Gamma$ as the topology switching instants. Accordingly, any $\boldsymbol{\tau} \in \Gamma$ is referred to as a feasible vector of switching instants.

Now, for each $i = 1, \ldots, m$, let $x_i(t) \in \mathbb{R}$ denote the voltage across the ith capacitor at time t. Furthermore, let $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_n]^T \in \Gamma$. Topology switches are accompanied by a voltage loss from the capacitors in the converter. We assume that this voltage leak can be expressed as a given function of the voltage across the capacitors immediately before the switch. Accordingly, the state voltage $\mathbf{x}(t) = [x_1(t), \ldots, x_m(t)]^T \in \mathbb{R}^m$ experiences a jump at each switching instant:

$$\mathbf{x}(\tau_i^+) = \mathbf{x}(\tau_i^-) + \boldsymbol{\phi}^i(\mathbf{x}(\tau_i^-)), \qquad i = 1, \dots, n, \tag{1}$$

where the negative and positive superscripts denote the limit from the left and right, respectively, and $\boldsymbol{\phi}^i : \mathbb{R}^m \to \mathbb{R}^m$, $i = 1, \dots, n$, are given continuously differentiable functions.

During the ith topology, $i = 1, \dots, n+1$, the state voltage is governed by a linear time invariant dynamical system as follows:

$$\dot{\mathbf{x}}(t) = A^i \mathbf{x}(t) + B^i \boldsymbol{\sigma}, \qquad t \in (\tau_{i-1}, \tau_i), \tag{2}$$

where $\boldsymbol{\sigma} = [\sigma_1, \dots, \sigma_r]^T \in \mathbb{R}^r$ is the DC input voltage, $R_L \in \mathbb{R}$ is the given load resistance and, for each $i = 1, \dots, n+1$, $A^i := A^i(R_L) : \mathbb{R} \to \mathbb{R}^{m \times m}$ and $B^i := B^i(R_L) : \mathbb{R} \to \mathbb{R}^{m \times r}$ are given matrix-valued functions of the load resistance. We assume that each of these functions is continuously differentiable.

The initial condition for the dynamics (2) is:

$$\mathbf{x}(0) = \mathbf{x}(0^+) = \mathbf{x^0}, \tag{3}$$

where $\mathbf{x^0} \in \mathbb{R}^m$ is the initial voltage across the capacitors.

The output voltage delivered by the converter during the ith topology, $i = 1, \dots, n+1$, is given by

$$y(t) = C^i \mathbf{x}(t) + D^i \boldsymbol{\sigma}, \qquad t \in [\tau_{i-1}, \tau_i), \tag{4}$$

where, for each $i = 1, \dots, n+1$, $C^i := C^i(R_L) : \mathbb{R} \to \mathbb{R}^{1 \times m}$ and $D^i := D^i(R_L) : \mathbb{R} \to \mathbb{R}^{1 \times r}$ are given matrix-valued functions of the load resistance. As before, each of these functions is assumed to be continuously differentiable. Also, at the terminal time, we set $y(T) := y(T^-)$.

If the switching instants are chosen *a priori* — that is, the components of a given $\boldsymbol{\tau} \in \Gamma$ are used for the topology switching instants — then the state voltage of the power converter will evolve according to the switched dynamical system (1)-(3). Let $\mathbf{x}(\cdot|\boldsymbol{\tau}) := \mathbf{x}(\cdot|\boldsymbol{\tau}, \boldsymbol{\sigma}, R_L)$ denote this state voltage. The corresponding output voltage from (4) is denoted by $y(\cdot|\boldsymbol{\tau}) := y(\cdot|\boldsymbol{\tau}, \boldsymbol{\sigma}, R_L)$. Clearly, different choices of switching instants will result in different output voltage profiles.

Recall that the DC/DC power converter should (ideally) deliver a steady voltage to the attached appliance. Hence, the switching instants should be chosen so that the resulting output voltage ripple,

$$\sup_{t \in [0,T]} y(t|\boldsymbol{\tau}) - \inf_{t \in [0,T]} y(t|\boldsymbol{\tau}),$$

is small. Moreover, for each $\boldsymbol{\tau} \in \Gamma$, the *sensitivity* of the output voltage with respect to the load resistance and input voltage is given, respectively, by

$$\sup_{t \in [0,T]} \left| \frac{\partial y(t|\boldsymbol{\tau})}{\partial R_L} \right| \qquad \text{and} \qquad \sup_{t \in [0,T]} \left\| \frac{\partial y(t|\boldsymbol{\tau})}{\partial \boldsymbol{\sigma}} \right\|_\infty,$$

where $\| \cdot \|_\infty$ denotes the maximum norm in $\mathbb{R}^r$. The switching instants should also be selected to minimize these sensitivity terms, so that changes in the input and load do not induce a large change in the output voltage.

On the basis of the above discussion, we define the following optimization problem for the computation of optimal switching instants.

Problem (P1). Given the system (1)-(4), choose $\boldsymbol{\tau} \in \Gamma$ such that the cost function

$$\begin{aligned} J(\boldsymbol{\tau}) :=& \alpha \left(\sup_{t \in [0,T]} y(t|\boldsymbol{\tau}) - \inf_{t \in [0,T]} y(t|\boldsymbol{\tau}) \right) \\ &+ \beta \sup_{t \in [0,T]} \left| \frac{\partial y(t|\boldsymbol{\tau})}{\partial R_L} \right| + \gamma \sup_{t \in [0,T]} \left\| \frac{\partial y(t|\boldsymbol{\tau})}{\partial \boldsymbol{\sigma}} \right\|_\infty \end{aligned}$$

is minimized over Γ, where α, β, and γ are non-negative weighting factors.

Remark. In Problem (P1), the switching sequence is assumed known and only the switching times are decision variables to be determined optimally. A more general problem — which, potentially, could yield an improved control scheme — would involve optimally choosing *both* the switching sequence and switching times. This generalized problem is a mixed discrete/continuous optimization problem and is therefore much harder to solve numerically than Problem (P1). Since, in practice, the switching sequence for a switched-mode power converter is normally known, we restrict our attention to Problem (P1) in this paper.

3 Problem transformation

The solution of Problem (P1) furnishes a switching regime which, when used to operate the switched-capacitor DC/DC power converter, results in a steady and robust output voltage profile. Note, however, that Problem (P1) is a non-smooth optimization problem and, as yet, we have no way of computing the output sensitivity terms appearing in the cost function $J(\cdot)$. Thus, we cannot use conventional optimization techniques to solve it. To proceed, we introduce the following auxiliary system of m jump differential equations corresponding to each $\boldsymbol{\tau} \in \Gamma$:

$$\dot{\boldsymbol{\psi}}(t) = \frac{\partial A^i}{\partial R_L}\mathbf{x}(t|\boldsymbol{\tau}) + A^i\boldsymbol{\psi}(t)+\frac{\partial B^i}{\partial R_L}\boldsymbol{\sigma}, \quad t \in (\tau_{i-1}, \tau_i), \quad i = 1, \ldots, n+1, \tag{5}$$

with

$$\boldsymbol{\psi}(0) = \boldsymbol{\psi}(0^+) = \mathbf{0} \tag{6}$$

and, for each $i = 1, \ldots, n$,

$$\boldsymbol{\psi}(\tau_i^+) = \boldsymbol{\psi}(\tau_i^-) + \frac{\partial \boldsymbol{\phi}^i(\mathbf{x}(\tau_i^-|\boldsymbol{\tau}))}{\partial \mathbf{x}}\boldsymbol{\psi}(\tau_i^-). \tag{7}$$

Similarly, for each $j = 1, \ldots, r$, consider another system of m auxiliary jump differential equations:

$$\dot{\boldsymbol{\varphi}}^j(t) = A^i\boldsymbol{\varphi}^j(t) + B^i_j, \qquad t \in (\tau_{i-1}, \tau_i), \quad i = 1, \ldots, n+1, \tag{8}$$

where, for any matrix M, M_j denotes its jth column; with

$$\boldsymbol{\varphi}^j(0) = \boldsymbol{\varphi}^j(0^+) = \mathbf{0} \tag{9}$$

and, for each $i = 1, \ldots, n$,

$$\boldsymbol{\varphi}^j(\tau_i^+) = \boldsymbol{\varphi}^j(\tau_i^-) + \frac{\partial \boldsymbol{\phi}^i(\mathbf{x}(\tau_i^-|\boldsymbol{\tau}))}{\partial \mathbf{x}}\boldsymbol{\varphi}^j(\tau_i^-). \tag{10}$$

Theorem 1. For each $\boldsymbol{\tau} \in \Gamma$, the state sensitivity functions $\frac{\partial \mathbf{x}(\cdot|\boldsymbol{\tau})}{\partial R_L}$ and $\frac{\partial \mathbf{x}(\cdot|\boldsymbol{\tau})}{\partial \sigma_j}$, $j = 1, \ldots, r$, are the unique solutions of (5)-(7) and (8)-(10), respectively.

Proof. It follows from the theory of differential equations that any solution to the system defined by (5)-(7) is unique. Now, for each $i = 1, \ldots, n+1$, the solution of the system (1)-(3) on $t \in (\tau_{i-1}, \tau_i)$ satisfies the following integral equation:

$$\mathbf{x}(t|\boldsymbol{\tau}) = \mathbf{x}(\tau_{i-1}^+|\boldsymbol{\tau}) + \int_{\tau_{i-1}}^t \left(A^i\mathbf{x}(\eta|\boldsymbol{\tau}) + B^i\boldsymbol{\sigma}\right) d\eta.$$

Differentiating this equation with respect to R_L using Leibniz's Rule yields

$$\frac{\partial \mathbf{x}(t|\boldsymbol{\tau})}{\partial R_L} = \frac{\partial \mathbf{x}(\tau_{i-1}^+|\boldsymbol{\tau})}{\partial R_L} + \int_{\tau_{i-1}}^t \frac{\partial A^i}{\partial R_L}\mathbf{x}(\eta|\boldsymbol{\tau})d\eta + \int_{\tau_{i-1}}^t \left(A^i\frac{\partial \mathbf{x}(\eta|\boldsymbol{\tau})}{\partial R_L} + \frac{\partial B^i}{\partial R_L}\boldsymbol{\sigma}\right) d\eta.$$

By differentiating the above equation with respect to t, it is evident that $\frac{\partial \mathbf{x}(\cdot|\boldsymbol{\tau})}{\partial R_L}$ satisfies (5). Moreover, it follows from (3) that

$$\frac{\partial \mathbf{x}(0|\boldsymbol{\tau})}{\partial R_L} = \frac{\partial}{\partial R_L}\left\{\mathbf{x^0}\right\} = \mathbf{0}.$$

Hence, (6) is satisfied. Finally, differentiating (1) with respect to R_L shows that the jump conditions (7) are also satisfied. The proof for the sensitivity functions $\frac{\partial \mathbf{x}(\cdot|\boldsymbol{\tau})}{\partial \sigma_j}$, $j = 1, \ldots, r$, is similar. ■

Let $\boldsymbol{\psi}(\cdot|\boldsymbol{\tau})$ and $\boldsymbol{\varphi}^j(\cdot|\boldsymbol{\tau})$, $j = 1, \ldots, r$, denote the respective solutions of (5)-(7) and (8)-(10) corresponding to $\boldsymbol{\tau} \in \Gamma$. Define the following function:

$$w(t|\boldsymbol{\tau}) := C^i\boldsymbol{\psi}(t|\boldsymbol{\tau}) + \frac{\partial C^i}{\partial R_L}\mathbf{x}(t|\boldsymbol{\tau}) + \frac{\partial D^i}{\partial R_L}\boldsymbol{\sigma}, \quad t \in [\tau_{i-1}, \tau_i), \quad i = 1, \ldots, n+1. \tag{11}$$

Similarly, for each $j = 1, \ldots, r$, let

$$z^j(t|\boldsymbol{\tau}) := C^i\boldsymbol{\varphi}^j(t|\boldsymbol{\tau}) + D^i_j, \quad t \in [\tau_{i-1}, \tau_i), \quad i = 1, \ldots, n+1. \tag{12}$$

In addition, we set $w(T|\boldsymbol{\tau}) := w(T^-|\boldsymbol{\tau})$ and $z^j(T|\boldsymbol{\tau}) := z^j(T^-|\boldsymbol{\tau})$, $j = 1, \ldots, r$. By virtue of Theorem 1, $w(\cdot)$ and $z^j(\cdot)$, $j = 1, \ldots, r$, are, respectively, $\frac{\partial y(\cdot|\boldsymbol{\tau})}{\partial R_L}$ and $\frac{\partial y(\cdot|\boldsymbol{\tau})}{\partial \sigma_j}$. Note also that (5)-(7) and (8)-(10) depend on the solution of the state system, and are in the same form as (1)-(3). It follows that we can solve the state system and the auxiliary system simultaneously as an expanded system of jump differential equations. This provides us with a much simpler and straightforward procedure for computing the sensitivity terms compared with that reported in Ho et al. (2008). On this basis, we can now introduce a new optimization problem equivalent to Problem (P1). In this new problem, the non-smoothness inherent in Problem (P1) is also removed via the addition of a set of continuous constraints. We define Problem (P2) as follows.

Problem (P2). Given the system (1)-(4), (5)-(10) and (11)-(12), choose $\boldsymbol{\tau} \in \Gamma$ and $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T \in \mathbb{R}^4$ such that the cost function

$$H(\boldsymbol{\tau}, \boldsymbol{\zeta}) := \alpha\zeta_1 + \alpha\zeta_2 + \beta\zeta_3 + \gamma\zeta_4$$

is minimized over $\Gamma \times \mathbb{R}^4$ subject to:

$$y(t|\boldsymbol{\tau}) \leq \zeta_1, \quad t \in [0, T], \tag{13}$$
$$-y(t|\boldsymbol{\tau}) \leq \zeta_2, \quad t \in [0, T], \tag{14}$$
$$-\zeta_3 \leq w(t|\boldsymbol{\tau}) \leq \zeta_3, \quad t \in [0, T], \tag{15}$$
$$-\zeta_4 \leq z^j(t|\boldsymbol{\tau}) \leq \zeta_4, \quad t \in [0, T], \quad j = 1, \ldots, r. \tag{16}$$

Note that the difficulties inherent in Problem (P1) are not present in Problem (P2). However, the constraints (13)-(16) are semi-infinite. Nevertheless, reliable methods are available for handling these types of constraints (see, for example, Teo, Rehbock, & Jennings (1993)).

We have the following result establishing the equivalence of Problems (P1) and (P2).

Theorem 2. Let $\boldsymbol{\tau}^* \in \Gamma$ and $\boldsymbol{\zeta}^* = [\zeta_1^*, \zeta_2^*, \zeta_3^*, \zeta_4^*]^T \in \mathbb{R}^4$. Then $(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*)$ is optimal for Problem (P2) if and only if $\boldsymbol{\tau}^*$ is optimal for Problem (P1) and

$$\zeta_1^* = \sup_{t\in[0,T]} y(t|\boldsymbol{\tau}^*), \tag{17}$$

$$\zeta_2^* = -\inf_{t\in[0,T]} y(t|\boldsymbol{\tau}^*), \tag{18}$$

$$\zeta_3^* = \sup_{t\in[0,T]} \left| \frac{\partial y(t|\boldsymbol{\tau}^*)}{\partial R_L} \right|, \tag{19}$$

$$\zeta_4^* = \sup_{t\in[0,T]} \left\| \frac{\partial y(t|\boldsymbol{\tau}^*)}{\partial \boldsymbol{\sigma}} \right\|_\infty . \tag{20}$$

Proof. Suppose that Problem (P1) has an optimal solution $\boldsymbol{\tau}^*$. Then $\boldsymbol{\tau}^* \in \Gamma$ and we can define $\boldsymbol{\zeta}^* = [\zeta_1^*, \zeta_2^*, \zeta_3^*, \zeta_4^*]^T$ according to (17)-(20). Note that $y(t|\boldsymbol{\tau}^*) \leq \zeta_1^*$ and $-y(t|\boldsymbol{\tau}^*) \leq \zeta_2^*$ for all $t \in [0,T]$. In addition, by Theorem 1,

$$-\zeta_3^* \leq w(t|\boldsymbol{\tau}^*) = \frac{\partial y(t|\boldsymbol{\tau}^*)}{\partial R_L} \leq \zeta_3^*$$

and

$$-\zeta_4^* \leq z^j(t|\boldsymbol{\tau}^*) = \frac{\partial y(t|\boldsymbol{\tau}^*)}{\partial \sigma_j} \leq \zeta_4^*, \quad j = 1, \ldots, r,$$

for all $t \in [0,T]$. Thus, $(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*)$ is feasible for Problem (P2). It follows from the definition of $\boldsymbol{\zeta}^*$ that

$$H(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*) = J(\boldsymbol{\tau}^*). \tag{21}$$

Suppose that $\boldsymbol{\tau} \in \Gamma$ and $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T \in \mathbb{R}^4$ also satisfy the constraints (13)-(16). Then by noting that $\boldsymbol{\tau}^*$ is optimal for Problem (P1), it follows from (21) that

$$\begin{aligned} H(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*) \leq J(\boldsymbol{\tau}) = \alpha \left(\sup_{t\in[0,T]} y(t|\boldsymbol{\tau}) - \inf_{t\in[0,T]} y(t|\boldsymbol{\tau}) \right) \\ + \beta \sup_{t\in[0,T]} \left| \frac{\partial y(t|\boldsymbol{\tau})}{\partial R_L} \right| + \gamma \sup_{t\in[0,T]} \left\| \frac{\partial y(t|\boldsymbol{\tau})}{\partial \boldsymbol{\sigma}} \right\|_\infty . \end{aligned} \tag{22}$$

It is clear from the constraints (13)-(16) that ζ_1 is an upper bound for $y(t|\boldsymbol{\tau})$, $-\zeta_2$ is a lower bound for $y(t|\boldsymbol{\tau})$, and ζ_3 and ζ_4 are upper bounds for $|w(t|\boldsymbol{\tau})| = \left| \frac{\partial y(t|\boldsymbol{\tau})}{\partial R_L} \right|$ and $|z^j(t|\boldsymbol{\tau})| = \left| \frac{\partial y(t|\boldsymbol{\tau})}{\partial \sigma_j} \right|$, $j = 1, \ldots, r$, respectively. Thus, by virtue of (22), we have

$$H(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*) \leq \alpha (\zeta_1 + \zeta_2) + \beta \zeta_3 + \gamma \zeta_4 = H(\boldsymbol{\tau}, \boldsymbol{\zeta})$$

and so $(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*)$ is optimal for Problem (P2).

Conversely, suppose that Problem (P2) has an optimal solution $(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*)$. Then $\boldsymbol{\tau}^*$ is obviously feasible for Problem (P1). Equations (17)-(20) must hold, since assuming otherwise would allow us to replace $\boldsymbol{\zeta}^*$ with the values on the right hand sides of (17)-(20), retaining feasibility whilst lowering the value of $H(\cdot, \cdot)$. Suppose now, that $\boldsymbol{\tau}^*$ is not optimal for Problem (P1). Then there exists a $\hat{\boldsymbol{\tau}} \in \Gamma$ such that $J(\hat{\boldsymbol{\tau}}) < J(\boldsymbol{\tau}^*)$. Define $\hat{\boldsymbol{\zeta}} = [\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3, \hat{\zeta}_4]^T$ according to equations (17)-(20) with the left hand sides replaced by the components of $\hat{\boldsymbol{\zeta}}$, and $\boldsymbol{\tau}^*$ on the right hand side replaced by $\hat{\boldsymbol{\tau}}$. Then the same arguments from the first part of the theorem can be used to show that $(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\zeta}})$ is feasible for Problem (P2) and $H(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\zeta}}) = J(\hat{\boldsymbol{\tau}})$. Also, we have $H(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*) = J(\boldsymbol{\tau}^*)$. Since $J(\hat{\boldsymbol{\tau}}) < J(\boldsymbol{\tau}^*)$, it is clear that the optimality of $(\boldsymbol{\tau}^*, \boldsymbol{\zeta}^*)$ for Problem (P2) has been violated. Hence, $\boldsymbol{\tau}^*$ must be optimal for Problem (P1). ■

As it stands, Problem (P2) cannot be solved directly using existing numerical techniques. The main difficulty is the dependence of the system state on the variable switching instants. We follow the advice suggested in Wu & Teo (2006) and employ a time-scaling transformation to map these switching instants into a fixed set of time points in a new time horizon.

Firstly, define

$$\Theta := \left\{ \boldsymbol{\theta} \in \mathbb{R}^{n+1} : \theta_i \geq \rho, \; i = 1, \ldots, n+1; \; \sum_{i=1}^{n+1} \theta_i = T \right\}.$$

Let $s \in [0, n+1]$ be a new time variable with switching instants occurring at the fixed locations $s = i$, $i = 1, \ldots, n$. For each $\boldsymbol{\theta} \in \Theta$, define $\mu(\cdot|\boldsymbol{\theta}) : [0, n+1] \to \mathbb{R}$ by

$$\mu(s|\boldsymbol{\theta}) := \begin{cases} \displaystyle\sum_{j=1}^{\lfloor s \rfloor} \theta_j + \theta_{\lfloor s \rfloor + 1}(s - \lfloor s \rfloor), & \text{if } s \in [0, n+1), \\ T, & \text{if } s = n+1, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. It can be readily verified that $\mu(\cdot|\boldsymbol{\theta})$ is continuous and strictly increasing on $[0, n+1]$. Consequently, $\mu(\cdot|\boldsymbol{\theta}) : [0, n+1] \to [0, T]$ is a bijection. Under this mapping, the new uniform switching instants are mapped to the following values in the original time scale:

$$\mu(i|\boldsymbol{\theta}) = \sum_{j=1}^{i} \theta_j, \qquad i = 0, \ldots, n+1. \tag{23}$$

Let $\tilde{\mathbf{x}}(s) = \mathbf{x}(\mu(s|\boldsymbol{\theta}))$, $\tilde{\boldsymbol{\psi}}(s) = \boldsymbol{\psi}(\mu(s|\boldsymbol{\theta}))$ and $\tilde{\boldsymbol{\varphi}}^j(s) = \boldsymbol{\varphi}^j(\mu(s|\boldsymbol{\theta}))$, $j = 1, \ldots, r$. Then, from (2), (5), (8) and the definition of $\mu(\cdot|\boldsymbol{\theta})$, we have

$$\dot{\tilde{\mathbf{x}}}(s) = \theta_i A^i \tilde{\mathbf{x}}(s) + \theta_i B^i \boldsymbol{\sigma}, \tag{24}$$

$$\dot{\tilde{\boldsymbol{\psi}}}(s) = \theta_i \frac{\partial A^i}{\partial R_L} \tilde{\mathbf{x}}(s) + \theta_i A^i \tilde{\boldsymbol{\psi}}(s) + \theta_i \frac{\partial B^i}{\partial R_L} \boldsymbol{\sigma}, \tag{25}$$

$$\dot{\tilde{\boldsymbol{\varphi}}}^j(s) = \theta_i A^i \tilde{\boldsymbol{\varphi}}^j(s) + \theta_i B^i_j, \quad j = 1, \ldots, r, \tag{26}$$

for $s \in (i-1, i)$, $i = 1, \ldots, n+1$. For each $i = 1, \ldots, n$, we have the jump conditions:

$$\tilde{\mathbf{x}}(i^+) = \tilde{\mathbf{x}}(i^-) + \boldsymbol{\phi}^i(\tilde{\mathbf{x}}(i^-)), \tag{27}$$

$$\tilde{\boldsymbol{\psi}}(i^+) = \tilde{\boldsymbol{\psi}}(i^-) + \frac{\partial \boldsymbol{\phi}^i(\tilde{\mathbf{x}}(i^-))}{\partial \mathbf{x}} \tilde{\boldsymbol{\psi}}(i^-), \tag{28}$$

$$\tilde{\boldsymbol{\varphi}}^j(i^+) = \tilde{\boldsymbol{\varphi}}^j(i^-) + \frac{\partial \boldsymbol{\phi}^i(\tilde{\mathbf{x}}(i^-))}{\partial \mathbf{x}} \tilde{\boldsymbol{\varphi}}^j(i^-), \quad j = 1, \ldots, r. \tag{29}$$

From (3), (6), (9), and (23), the initial conditions for the transformed expanded system are

$$\tilde{\mathbf{x}}(0) = \mathbf{x^0}, \tag{30}$$

$$\tilde{\boldsymbol{\psi}}(0) = \mathbf{0}, \tag{31}$$

$$\tilde{\boldsymbol{\varphi}}^j(0) = \mathbf{0}, \quad j = 1, \ldots, r. \tag{32}$$

Let $\tilde{\mathbf{x}}(\cdot|\boldsymbol{\theta}) := \tilde{\mathbf{x}}(\cdot|\boldsymbol{\theta}, \boldsymbol{\sigma}, R_L)$, $\tilde{\boldsymbol{\psi}}(\cdot|\boldsymbol{\theta}) := \tilde{\boldsymbol{\psi}}(\cdot|\boldsymbol{\theta}, \boldsymbol{\sigma}, R_L)$ and $\tilde{\boldsymbol{\varphi}}^j(\cdot|\boldsymbol{\theta}) := \tilde{\boldsymbol{\varphi}}^j(\cdot|\boldsymbol{\theta}, \boldsymbol{\sigma}, R_L)$, $j = 1, \ldots, r$, denote the solutions of (24)-(32) corresponding to $\boldsymbol{\theta} \in \Theta$. We define

$$\tilde{y}(s|\boldsymbol{\theta}) := C^i \tilde{\mathbf{x}}(s|\boldsymbol{\theta}) + D^i \boldsymbol{\sigma}, \tag{33}$$

$$\tilde{w}(s|\boldsymbol{\theta}) := C^i \tilde{\boldsymbol{\psi}}(s|\boldsymbol{\theta}) + \frac{\partial C^i}{\partial R_L} \tilde{\mathbf{x}}(s|\boldsymbol{\theta}) + \frac{\partial D^i}{\partial R_L} \boldsymbol{\sigma}, \tag{34}$$

$$\tilde{z}^j(s|\boldsymbol{\theta}) := C^i \tilde{\boldsymbol{\varphi}}^j(s|\boldsymbol{\theta}) + D^i_j, \tag{35}$$

for $s \in [i-1, i)$, $i = 1, \ldots, n+1$. Moreover, $\tilde{y}(n+1|\boldsymbol{\theta})$, $\tilde{w}(n+1|\boldsymbol{\theta})$ and $\tilde{z}^j(n+1|\boldsymbol{\theta})$, $j = 1, \ldots, r$, are defined in an obvious manner.

The constraints (13)-(16) become

$$\tilde{y}(s|\boldsymbol{\theta}) \leq \zeta_1, \quad s \in [0, n+1], \tag{36}$$

$$-\tilde{y}(s|\boldsymbol{\theta}) \leq \zeta_2, \quad s \in [0, n+1], \tag{37}$$

$$-\zeta_3 \leq \tilde{w}(s|\boldsymbol{\theta}) \leq \zeta_3, \quad s \in [0, n+1], \tag{38}$$

$$-\zeta_4 \leq \tilde{z}^j(s|\boldsymbol{\theta}) \leq \zeta_4, \quad s \in [0, n+1], \quad j = 1, \ldots, r. \tag{39}$$

It is clear from (23) that $[\mu(1|\boldsymbol{\theta}), \ldots, \mu(n|\boldsymbol{\theta})]^T \in \Gamma$. Moreover, for each $\boldsymbol{\tau} \in \Gamma$, we can choose a $\boldsymbol{\theta} \in \Theta$ such that $\tau_i = \mu(i|\boldsymbol{\theta})$, $i = 1, \ldots, n$. On this basis, Problem (P2) is equivalent to the following Problem (P3).

Problem (P3). Given the system (24)-(35), choose $\boldsymbol{\theta} \in \Theta$ and $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T \in \mathbb{R}^4$ such that the cost function

$$\tilde{H}(\boldsymbol{\theta}, \boldsymbol{\zeta}) := \alpha\zeta_1 + \alpha\zeta_2 + \beta\zeta_3 + \gamma\zeta_4$$

is minimized over $\Theta \times \mathbb{R}^4$ subject to (36)-(39).

Problem (P3) is a smooth dynamic optimization problem with fixed switching times and continuous inequality constraints. In the next section, we will demonstrate how it can be solved using existing optimization techniques. Note that the optimal switching instants for Problem (P1) can be easily recovered from equation (23) once Problem (P3) has been solved.

4 Solving Problem (P3)

Problem (P3) is essentially a semi-infinite programming problem that can be solved using the algorithm developed in Teo et al. (1993). To apply this algorithm, we first express the continuous constraints (36)-(39) as

$$g_i(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) \leq 0, \quad s \in [0, n+1], \quad i = 1, \ldots, 2r+4, \tag{40}$$

where

$$\begin{aligned}
g_1(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= \tilde{y}(s|\boldsymbol{\theta}) - \zeta_1, \\
g_2(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= -\tilde{y}(s|\boldsymbol{\theta}) - \zeta_2, \\
g_3(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= \tilde{w}(s|\boldsymbol{\theta}) - \zeta_3, \\
g_4(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= -\tilde{w}(s|\boldsymbol{\theta}) - \zeta_3, \\
g_{2j+3}(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= \tilde{z}^j(s|\boldsymbol{\theta}) - \zeta_4, \qquad j = 1, \ldots, r, \\
g_{2j+4}(s|\boldsymbol{\theta}, \boldsymbol{\zeta}) &= -\tilde{z}^j(s|\boldsymbol{\theta}) - \zeta_4, \qquad j = 1, \ldots, r.
\end{aligned}$$

Now, for a given $\epsilon > 0$ and $\vartheta > 0$, consider the following auxiliary optimization problem.

Problem ($\tilde{\mathbf{P}}_{\epsilon,\vartheta}$). Given the system (24)-(35), choose $\boldsymbol{\theta} \in \Theta$ and $\boldsymbol{\zeta} = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T \in \mathbb{R}^4$ such that the cost function

$$\tilde{H}_{\epsilon,\vartheta}(\boldsymbol{\theta}, \boldsymbol{\zeta}) := \tilde{H}(\boldsymbol{\theta}, \boldsymbol{\zeta}) + \vartheta \sum_{i=1}^{2r+4} \int_0^{n+1} \chi_\epsilon(g_i(s|\boldsymbol{\theta}, \boldsymbol{\zeta})) ds$$

is minimized over $\Theta \times \mathbb{R}^4$, where

$$\chi_\epsilon(\eta) = \begin{cases} 0, & \text{if } \eta < -\epsilon, \\ (\eta + \epsilon)^2 / 4\epsilon, & \text{if } -\epsilon \leq \eta \leq \epsilon, \\ \eta, & \text{if } \eta > \epsilon. \end{cases}$$

Note that Problem ($\tilde{\mathrm{P}}_{\epsilon,\vartheta}$) is a smooth optimization problem governed by an impulsive dynamical system. Hence, the gradient of the objective function $\tilde{H}_{\epsilon,\vartheta}(\cdot,\cdot)$ with respect to the decision variables can be calculated according to formulae reported in Liu, Teo, Jennings, & Wang (1998), Jennings et al. (2004), and Wu & Teo (2006). On this basis, any of the gradient-based optimization algorithms discussed in Nocedal & Wright (1999) and Luenberger (2005) can be employed to solve Problem $(\tilde{\mathrm{P}}_{\epsilon,\vartheta})$. Furthermore, in Teo et al. (1993), it is shown that for any $\epsilon > 0$, there exists a corresponding $\vartheta(\epsilon) > 0$ such that an optimal solution of Problem $(\tilde{\mathrm{P}}_{\epsilon,\vartheta})$ with $\vartheta > \vartheta(\epsilon)$ satisfies the continuous constraints of Problem (P3). By virtue of this result, a solution to the semi-infinite optimization Problem (P3) can be obtained by solving a sequence of approximate Problems $(\tilde{\mathrm{P}}_{\epsilon,\vartheta})$. A detailed algorithm for updating ϵ and ϑ during this sequence, as well as several important convergence results, are given in Teo et al. (1993).

5 Existence of an optimal solution

Note that Problem (P1) is essentially a non-linear optimization problem involving the minimization of a cost function over a compact set. However, because of the unconventional form of the cost function, it is not obvious that an optimal solution exists. On the other hand, the feasible region for the equivalent Problems (P2) and (P3) is not compact. If an optimal solution to Problem (P1) does not exist, then its suitability as a mathematical formulation of a practical electronic design scenario must be reconsidered. In this section, we will show that Problem (P1) does indeed admit an optimal solution.

Firstly, we present the following two preliminary lemmas, whose proofs are similar to those of Lemma 6.4.2 and Lemma 6.4.3 in Teo, Goh, & Wong (1991), respectively.

Lemma 1. There exists a positive real number $\Lambda > 0$ such that, for all $s \in [0, n+1]$ and $\boldsymbol{\theta} \in \Theta$,

$$\begin{aligned}
&|\tilde{y}(s|\boldsymbol{\theta})| \leq \Lambda, \\
&|\tilde{w}(s|\boldsymbol{\theta})| \leq \Lambda, \\
&|\tilde{z}^j(s|\boldsymbol{\theta})| \leq \Lambda, \qquad j = 1, \ldots, r.
\end{aligned}$$

Lemma 2. Suppose that $\{\boldsymbol{\theta}^k\}_{k=1}^{\infty} \subset \Theta$ is a sequence converging to $\boldsymbol{\theta} \in \Theta$. Then for each $s \in [0, n+1]$,

$$\begin{aligned}
&\lim_{k\to\infty} \tilde{y}(s|\boldsymbol{\theta}^k) = \tilde{y}(s|\boldsymbol{\theta}), \\
&\lim_{k\to\infty} \tilde{w}(s|\boldsymbol{\theta}^k) = \tilde{w}(s|\boldsymbol{\theta}), \\
&\lim_{k\to\infty} \tilde{z}^j(s|\boldsymbol{\theta}^k) = \tilde{z}^j(s|\boldsymbol{\theta}), \qquad j = 1, \ldots, r.
\end{aligned}$$

We now state and prove the main result of this section.

Theorem 3. Problem (P1) admits an optimal solution.

Proof. Note that Problem (P1) is equivalent to Problem (P3). Hence, it suffices to consider the existence of an optimal element for Problem (P3). For each $\boldsymbol{\theta} \in \Theta$, define $\bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}) = \left[\bar{\zeta}_1(\boldsymbol{\theta}), \bar{\zeta}_2(\boldsymbol{\theta}), \bar{\zeta}_3(\boldsymbol{\theta}), \bar{\zeta}_4(\boldsymbol{\theta})\right]^T$ by

$$\begin{aligned}
\bar{\zeta}_1(\boldsymbol{\theta}) &:= \sup_{s\in[0,n+1]} \tilde{y}(s|\boldsymbol{\theta}), \\
\bar{\zeta}_2(\boldsymbol{\theta}) &:= -\inf_{s\in[0,n+1]} \tilde{y}(s|\boldsymbol{\theta}), \\
\bar{\zeta}_3(\boldsymbol{\theta}) &:= \sup_{s\in[0,n+1]} |\tilde{w}(s|\boldsymbol{\theta})|, \\
\bar{\zeta}_4(\boldsymbol{\theta}) &:= \sup_{s\in[0,n+1]} \max_{1\leq j\leq r} |\tilde{z}^j(s|\boldsymbol{\theta})|,
\end{aligned}$$

where Lemma 1 ensures that the above expressions are well-defined. Now, select an arbitrary $\hat{\kappa} \in \{1, \ldots, r\}$ and $\hat{s} \in [0, n+1]$. For any $\boldsymbol{\theta} \in \Theta$, we have

$$\begin{aligned}
\tilde{H}(\boldsymbol{\theta}, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta})) &= \alpha\bar{\zeta}_1(\boldsymbol{\theta}) + \alpha\bar{\zeta}_2(\boldsymbol{\theta}) + \beta\bar{\zeta}_3(\boldsymbol{\theta}) + \gamma\bar{\zeta}_4(\boldsymbol{\theta}) \\
&\geq \alpha\tilde{y}(\hat{s}|\boldsymbol{\theta}) - \alpha\tilde{y}(\hat{s}|\boldsymbol{\theta}) + \beta\,|\tilde{w}(\hat{s}|\boldsymbol{\theta})| \\
&\qquad + \gamma\left|\tilde{z}^{\hat{\kappa}}(\hat{s}|\boldsymbol{\theta})\right| \\
&\geq 0.
\end{aligned}$$

Hence, we can find a non-negative $\omega \in \mathbb{R}$ such that

$$\omega = \inf\left\{\, \tilde{H}(\boldsymbol{\theta}, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta})) : \ \boldsymbol{\theta} \in \Theta \,\right\}.$$

That is, for each $k \in \mathbb{N}$, there exists some $\boldsymbol{\theta}^k \in \Theta$ such that

$$\tilde{H}(\boldsymbol{\theta}^k, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}^k)) < \omega + \frac{1}{k}.$$

Clearly, $\tilde{H}(\boldsymbol{\theta}^k, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}^k)) \to \omega$ as $k \to \infty$. In addition, it follows from Lemma 1 that for each $k \geq 1$,

$$-\Lambda \leq \bar{\zeta}_p(\boldsymbol{\theta}^k) \leq \Lambda, \qquad p = 1, 2, 3, 4.$$

Hence, since Θ is a compact set, $\left\{(\boldsymbol{\theta}^k, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}^k))\right\}_{k=1}^{\infty}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\left\{(\boldsymbol{\theta}^{k_i}, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}^{k_i}))\right\}_{i=1}^{\infty}$ converging to an element $(\boldsymbol{\theta}^*, \boldsymbol{\zeta}^*)$. Applying Lemma 2 to the constraints (36)-(39), we see that $(\boldsymbol{\theta}^*, \boldsymbol{\zeta}^*)$ is feasible for Problem (P3).

Furthermore,

$$\begin{aligned}
\omega &= \lim_{i\to\infty} \tilde{H}(\boldsymbol{\theta}^{k_i}, \bar{\boldsymbol{\zeta}}(\boldsymbol{\theta}^{k_i})) \\
&= \alpha\zeta_1^* + \alpha\zeta_2^* + \beta\zeta_3^* + \gamma\zeta_4^* = \tilde{H}(\boldsymbol{\theta}^*, \boldsymbol{\zeta}^*).
\end{aligned}$$

Thus, $(\boldsymbol{\theta}^*, \boldsymbol{\zeta}^*)$ is optimal for Problem (P3). ■

6 Numerical simulation

Based on the procedure outlined in Section 4, a Fortran 90 program was written to solve Problem (P3) corresponding to the switched-capacitor DC/DC power converter reported in Umeno, Takahashi, Oota, Ueno,

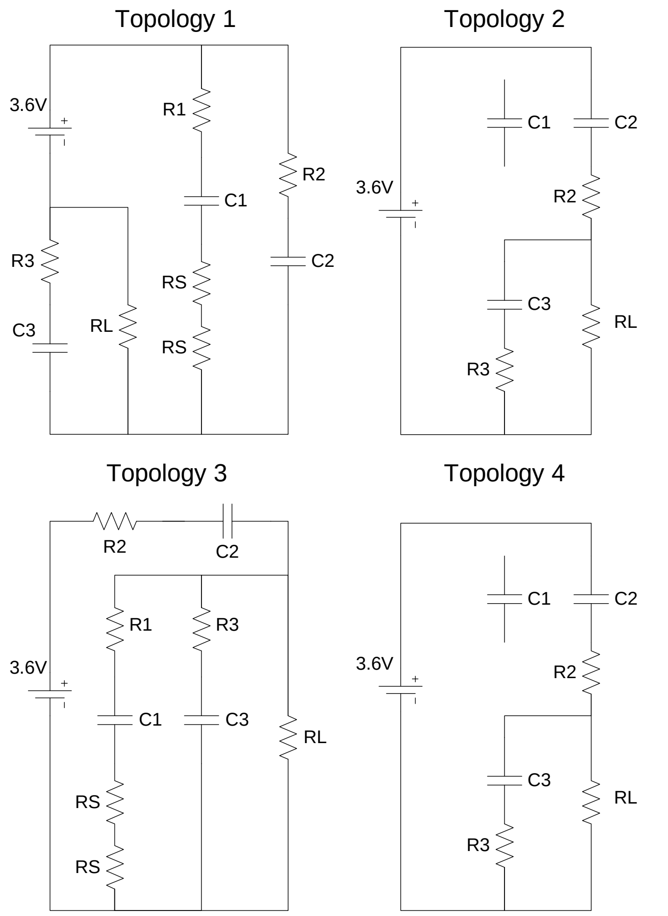

Fig. 1. Circuit topologies 1-4.

& Inoue (1990). This power converter contains three capacitors and cycles through four circuit topologies in each period. Circuit schematics for the different topologies are shown in Figure 1. The state space matrices A^i, B^i, C^i, and D^i, $i = 1, 2, 3, 4$, can be readily obtained by applying Kirchhoff's laws. To solve the state system, our Fortran program utilized the ordinary differential equation solver LSODA (see Hindmarsh (1982)). The optimization process was handled by the routine NLPQLP (see Schittkowski (2007)).

The cycle length and minimum topology duration were chosen as $T = 2.0 \times 10^{-5}$ and $\rho = 1.0 \times 10^{-6}$, respectively. The other circuit parameters were $C_1 = C_2 = C_3 = 30.0 \times 10^{-6}$F, $R_1 = R_2 = R_3 = 0.02\Omega$, $R_S = 0.01\Omega$ and $R_L = 75.0\Omega$. We also assumed that topology changes are accompanied by a 5% voltage leak from the capacitors in the circuit. Hence,

$$\boldsymbol{\phi}^i(\mathbf{x}(\tau_i^-)) = -0.05\mathbf{x}(\tau_i^-), \qquad i = 1, 2, 3.$$

We initially used our program to determine the optimal switching instants for the first period after start-up (that is, the initial voltage across the capacitors is zero). We then repeated this process for subsequent periods until it was evident that the converter had reached steady state. The output voltage profile of the power converter under the optimal switching regime is plotted in Figure 2. Figures 3-5 show the corresponding voltage across each of the capacitors. Note that, as expected, this switched-capacitor DC/DC power converter acts as a voltage halver at the steady state.

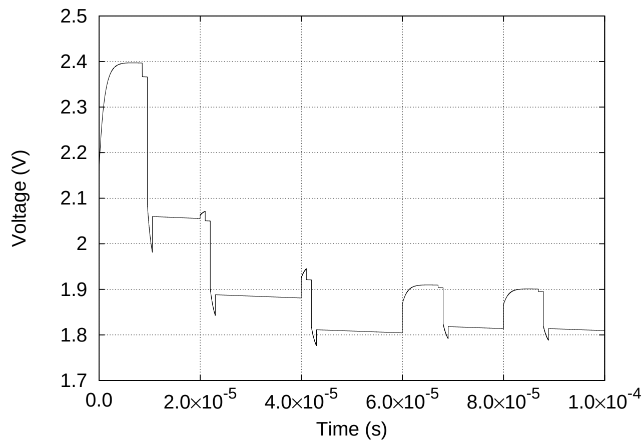

Fig. 2. Output voltage profile under the optimal switching regime.

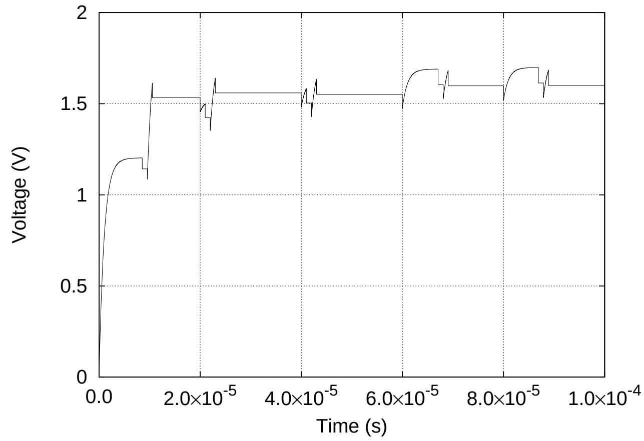

Fig. 3. Voltage across capacitor 1.

The optimal steady state switching instants are $\tau_1^* = 6.8853 \times 10^{-6}$, $\tau_2^* = 7.8853 \times 10^{-6}$ and $\tau_3^* = 8.8853 \times 10^{-6}$. Furthermore, the sensitivities of the output voltage with respect to the load resistance and input are 1.7927×10^{-4} and 6.6595×10^{-1}, respectively. Since these values — in particular the sensitivity with respect to the load resistance — are quite small, the optimal switching regime is rather insensitive to changes in the load and small perturbations in the input voltage. Note also that the steady state output voltage ripple is $1.1260 \times 10^{-1}V$.

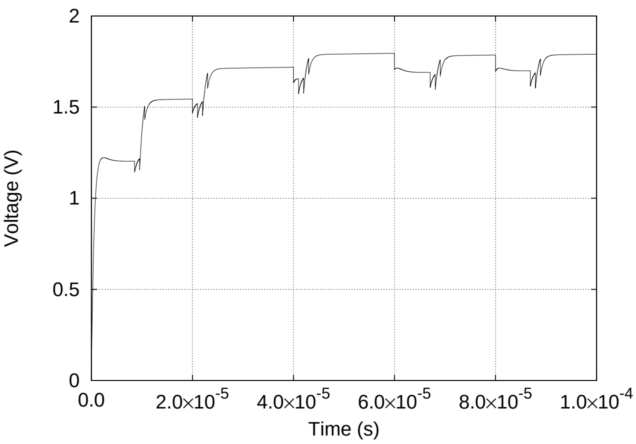

Fig. 4. Voltage across capacitor 2.

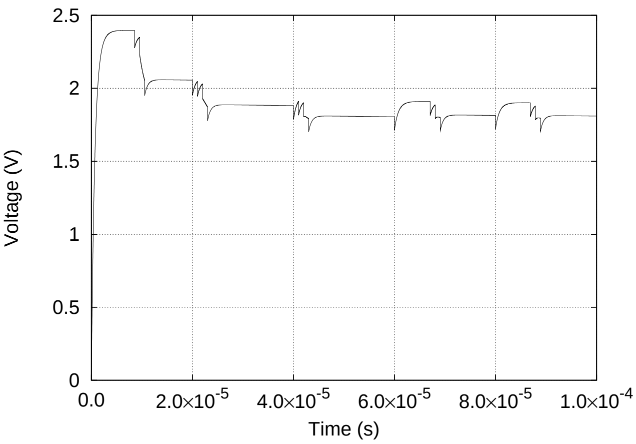

Fig. 5. Voltage across capacitor 3.

7 Conclusion

We have discussed a numerical optimization approach to determining the optimal switching instants for a switched-capacitor DC/DC power converter. Note that the analysis performed in this paper is not restricted to switched-capacitor DC/DC power converters and can be applied to any switched linear system where Problem (P1) is practically relevant. Compared to the work in Ho et al. (2008), our new formulation avoids tedious and at times difficult hand/computer algebra computations and can be readily implemented using existing optimization software such as MISER. We have also established the existence of an optimal solution and used our method to solve an example problem. Interestingly, the results from this example indicate that a uniform switching scheme is not optimal. Future research should involve formulating realistic state jump functions that can accurately reflect the energy losses involved in switching. More advanced optimal parameter selection models, where design parameters such as device capacitances and switching frequency are included in the optimal decision making process, can also be considered.

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