ARBITRARY POLE-PLACEMENT IN THE *LQ* CONTROL PARADIGM

L. Keviczky and J. Bokor

Széchenyi University, Győr
Computer and Automation Research Institute and Control Engineering Research Group
Hungarian Academy of Sciences
H-1111 Budapest, Kende u 13-17, HUNGARY
Phone: +361-466-5435; Fax: +361-466-7503
e-mail: keviczky@sztaki.hu; bokor@sztaki.hu

Abstract: The specific relationships between the classical pole-placement state feedback, the RICCATI equation based *LQ* paradigm and the KALMAN frequency domain approach are discussed. It is shown that arbitrary pole placement is not possible by standard *LQ* optimality. A possible solution of this anomaly is to use more general *LQ* criterion with specific weights on the state, input and crossterm.

Keywords: *LQ* problem, KALMAN equation, optimality

1. Introduction

In the early time of control theory the optimization of transient processes in dynamic systems used a quadratic criterion, i.e., the integral square of error

$$I_2 = \int_0^\infty e^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} E(-s)E(s)\,\mathrm{d}s = \frac{1}{\pi}\int_0^\infty \left|E(j\omega)\right|^2 \mathrm{d}\omega \tag{1}$$

Here $e(t)$ is the error signal of a closed-loop control system. The second half of (1) is the so-called PARSEVAL theorem [3], [6], using the strictly proper $E(s)$, the LAPLACE transform of $e(t)$.

This integral criterion was very popular, because the evaluation of (1) could be performed analytically and easily computed even by the early slow computers (by preprogrammed formulas). The general theory was called WIENER approach [3] and thousands of papers were published for the different optimal designs. The first critics came from the industry: the optimal regulators minimizing (1) were not acceptable in the practice, because they resulted a very large (20~25 %) overshoot in the step response transients.

One way to overcome this problem was first to introduce a more general quadratic integral criterion, penalizing the different state variables as

$$I_{2(n)} = \int_0^\infty \left[x^2 + \tau_1^2 \dot{x}^2 + \ldots + \tau_n^2 \left(\overset{(n)}{x}\right)^2 \right] \mathrm{d}t \tag{2}$$

which is called generalized quadratic criterion. It is not difficult to show that (2) has an equivalent form

$$I_{2(n)} = \int_0^\infty \left[x + \alpha \dot{x} + \cdots + \alpha_n \overset{(n)}{x} \right]^2 \mathrm{d}t + c_\mathrm{o} \tag{3}$$

where $x(\infty) = \dot{x}(\infty) = \ldots = \overset{(n-1)}{x}(\infty) = 0$ and $c_\mathrm{o} = \alpha_1 x_\mathrm{o}^2, x_\mathrm{o} = x(0)$. The coefficients of the two forms depend on each other by the REKASIUS-FELDBAUM equations [1], [2]

$$
\begin{aligned}
\tau_1^2 &= \alpha_1^2 - 2\alpha_o\alpha_2 \\
\tau_2^2 &= \alpha_2^2 - 2\alpha_1\alpha_3 + 2\alpha_o\alpha_4 \\
\tau_3^2 &= \alpha_3^2 - 2\alpha_2\alpha_4 + 2\alpha_1\alpha_3 - 2\alpha_o\alpha_4 \\
&\vdots \\
\tau_n^2 &= \alpha_n^2
\end{aligned}
\tag{4}
$$

From (3) the minimum can be easily seen, if $x(t)$ fulfils the differential equation

$$
\alpha_n \overset{(n)}{x} + a_{n-1} \overset{(n-1)}{x} + \dots + \alpha_1 x + \alpha_o = 0 \quad ; \quad \alpha_o = 1 \tag{5}
$$

Here the signal $x(t)$ is more general than $e(t)$, because it can be one of the state variables of a linear system.

2. State feedback (*SFB*)

Consider a *SISO* continuous time linear time invariant (*LTI*) dynamic plant described by the state variable representation (*SVR*)

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\
y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}
\end{aligned}
\tag{6}
$$

Here u, y and $\boldsymbol{x}$ are the input, output and state variables of the controlled process and $^{\mathrm{T}}$ stands for transposition. The transfer function representation (*TFR*) of the open-loop system can be calculated by

$$
P(s) = \frac{\mathcal{B}(s)}{\mathcal{A}(s)} = \boldsymbol{c}^{\mathrm{T}}\left(s\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{b} = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b} \tag{7}
$$

where $\boldsymbol{I}$ is the unit matrix,

$$
\begin{aligned}
\boldsymbol{\Phi}(s) &= (s\boldsymbol{I} - \boldsymbol{A})^{-1} = \mathcal{L}\left\{e^{\boldsymbol{A}t}\right\} = \frac{\boldsymbol{\Psi}(s)}{\mathcal{A}(s)} \\
\boldsymbol{\Psi}(s) &= \mathbf{adj}(s\boldsymbol{I} - \boldsymbol{A})
\end{aligned}
\tag{8}
$$

and

$$
\mathcal{B}(s) = s^n + b_1 s^{n-1} + \ldots + b_{n-1}s + b_n = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b} \tag{9}
$$

$$
\mathcal{A}(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = \det(s\boldsymbol{I} - \boldsymbol{A}) \tag{10}
$$

are the numerator and denominator polynomials, respectively. If the feedback is restricted to a linear *SFB*, then the classical solution can be written as

$$u = k_\mathrm{r}\, r - \boldsymbol{k}^\mathrm{T} \boldsymbol{x} \tag{11}$$

where r is the reference signal, k_r is a calibrating constant and $\boldsymbol{k}^\mathrm{T}$ is the linear *SFB* vector. It is easy to check that the transfer function from the reference signal r to the output y is [4]

$$T_\mathrm{ry}(s) = \boldsymbol{c}^\mathrm{T}\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^\mathrm{T}\right)^{-1} \boldsymbol{b}\, k_\mathrm{r} = \frac{k_\mathrm{r}\, \mathcal{B}(s)}{\mathcal{A}(s) + \boldsymbol{k}^T \boldsymbol{\Psi}(s)\boldsymbol{b}} \tag{12}$$

where k_r is obtained by requiring that the static gain of T_ry should be equal to one

$$k_\mathrm{r} = \frac{\boldsymbol{k}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b} - 1}{\boldsymbol{c}^\mathrm{T}\boldsymbol{A}^{-1}\boldsymbol{b}} \tag{13}$$

The usual classical design goal is to determine the feedback gain $\boldsymbol{k}^\mathrm{T}$ so that the closed-loop system has the characteristic polynomial

$$\mathcal{R}(s) = s^n + r_1 s^{n-1} + \ldots + r_{n-1}s + r_n \tag{14}$$

The solution formally means equating the characteristic polynomial of the closed-loop with the desired polynomial ("pole placement method")

$$\mathcal{R}(s) = \det\left(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}^T\right) = \mathcal{A}(s) + \boldsymbol{k}^T\boldsymbol{\Psi}(s)\boldsymbol{b} = \mathcal{A}(s) + \mathcal{K}(s) \tag{15}$$

to compute $\boldsymbol{k}^\mathrm{T}$. The solution always exists if $P(s)$ is controllable.

If the *TFR* of the process is known then one can easily form a controllable canonical form $\left\{\boldsymbol{A}_\mathrm{c}, \boldsymbol{b}_\mathrm{c}, \boldsymbol{c}_\mathrm{c}^\mathrm{T}\right\}$ with

$$\boldsymbol{A}_\mathrm{c} = \begin{bmatrix} -\boldsymbol{a}_\mathrm{c}^\mathrm{T} \\ \mathbf{I}\ ,\ \mathbf{0} \end{bmatrix} \quad ; \quad \boldsymbol{a}_\mathrm{c} = \left[a_1, a_2, \ldots, a_n\right]^\mathrm{T} \quad ; \quad \boldsymbol{b}_\mathrm{c} = \left[1, 0, \ldots, 0\right]^\mathrm{T} \quad ; \quad \boldsymbol{c}_\mathrm{c} = \left[b_1, b_2, \ldots, b_n\right]^\mathrm{T} \tag{16}$$

and now the feedback gain is obtained from (15) as

$$\boldsymbol{k}_\mathrm{c}^\mathrm{T} = \left[r_1 - a_1, r_2 - a_2, \ldots, r_n - a_n\right] = \left[k_1, k_2, \ldots, k_n\right] \tag{17}$$

because

$$\boldsymbol{\Psi}_\mathrm{c}(s)\boldsymbol{b}_\mathrm{c} = \begin{bmatrix} s^{n-1} & \ldots & s & 1 \end{bmatrix}^\mathrm{T} \tag{18}$$

and

$$\boldsymbol{k}_\mathrm{c}^\mathrm{T}\boldsymbol{\Psi}_\mathrm{c}(s)\boldsymbol{b}_\mathrm{c} = k_1 s^{n-1} + \ldots + k_{n-1}s + k_n = \mathcal{K}(s) \tag{19}$$

The calibration factor is calculated by

$$k_{\mathrm{r}} = \frac{a_n + (r_n - a_n)}{b_n} = \frac{a_n}{b_n} \tag{20}$$

The *SVR* of the closed-loop system is described by

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)\boldsymbol{x} + k_{\mathrm{r}}\,\boldsymbol{b}\,r = \overline{\boldsymbol{A}} + k_{\mathrm{r}}\,\boldsymbol{b}\,r \\ y &= \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \end{aligned} \tag{21}$$

It is easy to see from equation (12) that $T_{\mathrm{ry}}(s)$ is now

$$T_{\mathrm{ry}}(s) = \frac{k_{\mathrm{r}}\,\mathcal{B}(s)}{\mathcal{R}(s)} \tag{22}$$

i.e., besides reaching the desired pole-placement the *SFB* leaves the open-loop zeros untouched.

3. The *LQR* (*L*inear system - *Q*uadratic criterion - *R*egulator) problem

Not only the bad transient of the error signal obtained from the optimal quadratic criterion was the problem, but also the big amplitude jumps necessary to the control action. An other way suggested to overcome the combined problem was the introduction of a penalty for the energy of the control signal. This optimization was formulated by the more general [3], [4] quadratic criterion

$$I = \frac{1}{2}\int_0^{\infty}\left[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}(t) + w_{\mathrm{u}}u^2(t)\right]\mathrm{d}t \tag{23}$$

where $\boldsymbol{x}(t)$ is the state vector, $u(t)$ is the input of the process, respectively. The positive definite $\boldsymbol{W}_{\mathrm{x}}$ stands for penalizing the variations in the state space, w_{u} is for penalizing the energy of the control action, which is more general than (2). The solution, minimizing (23) is again a negative *SFB* [7]

$$u(t) = -\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}\,\boldsymbol{x}(t) \tag{24}$$

where $\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}}$ is given by

$$\boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}} = \frac{1}{w_{\mathrm{u}}}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P} \tag{25}$$

where the symmetric positive semi definite matrix $\boldsymbol{P}$ can be obtained from the solution of the algebraic RICCATI equation [4]

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} - \frac{1}{w_{\mathrm{u}}}\boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}\boldsymbol{P} = -\boldsymbol{W}_{\mathrm{x}} \tag{26}$$

Analytic solution is not possible, because this equation is nonlinear in $\boldsymbol{P}$, therefore only numeric solution can be obtained by MATLAB© and other *CACSD* programs.

Introducing the orthogonal factorization

$$\boldsymbol{W}_{\mathrm{x}} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} \tag{27}$$

the closed-loop system is stable if the auxiliary process

$$\boldsymbol{v} = \boldsymbol{G}\,\boldsymbol{x} \tag{28}$$

is observable.

The characteristic polynomial coefficients are computed now from

$$\left[r_1, r_2, \ldots, r_n\right]^{\mathrm{T}} = \boldsymbol{k}_{\mathrm{LQ}}^{\mathrm{T}} + \left[a_1, a_2, \ldots, a_n\right]^{\mathrm{T}} \tag{29}$$

Note that this *SFB* also provides the same $T_{\mathrm{ry}}(s)$ as (22) before.

A joint use of time domain optimality criteria, prescribed constraints and pole locations are often required in practice. Optimal and partially optimal pole placement based on optimality criteria (58) was studied in [19], [20], [21] and [22]. It can be shown, however, that [20] does not provide solution for the general problem (illustrated by the examples later), mainly due to the fact that it uses only the weights $\boldsymbol{W}_{\mathrm{x}}$, $\boldsymbol{W}_{\mathrm{u}}$ but not the cross term $\boldsymbol{W}_{\mathrm{ux}}$.

4. The frequency domain solution of the *LQR* problem

The *LQR* approach is widely used for control problems in all over the world, however, in a practical problem it is not an easy task to find the best $\boldsymbol{W}_{\mathrm{x}}$ and w_{u} weights, which are usually obtained by trial and error iterative methods. The *LQR* problem has an almost forgotten frequency domain solution, too, which will give us a deterministic design process to find useful relationships between the classical pole placement *SFB* solution and the *LQR* paradigm. It can be shown that the simpler dyadic factorization [3]

$$\boldsymbol{W}_{\mathrm{x}} = \boldsymbol{g}\,\boldsymbol{g}^{\mathrm{T}} \quad ; \quad \boldsymbol{g} = \left[g_1, \ldots, g_{n-1}, g_n\right]^{\mathrm{T}} \tag{30}$$

can also be used. The frequency domain condition of the minimum of (23) is called the KALMAN equation [3] or sometimes it is named frequency domain identity (FDI)

$$w_{\mathrm{u}}\left|1 + \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right|^2 = w_{\mathrm{u}} + \left|\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right|^2 \tag{31}$$

Assuming unity weight $w_{\mathrm{u}} \equiv 1$ the equation becomes even simpler

$$\left|1 + \boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right|^2 = 1 + \left\|\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right\|_2^2 = 1 + \left|\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}\right|^2 \tag{32}$$

Using the well known relationship of complex functions

$$\left\{Z(s)\right\}^2 = \left|Z(s)\right|^2 = Z(s)Z(-s) \tag{33}$$

and introducing the $(n-1)$-th order polynomial $\mathcal{G}(s)$ as the numerator of

$$H(s)=\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}=\frac{\mathcal{G}(s)}{\mathcal{A}(s)}=\frac{g_1 s^{n-1}+\ldots+g_{n-1}s+g_n}{\mathcal{A}(s)} \tag{34}$$

the equation (32) can be rearranged into a new form

$$\underbrace{\left[\mathcal{A}(s)+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}\right]}_{\mathcal{R}(s)}\underbrace{\left[\mathcal{A}(-s)+\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(-s)\boldsymbol{b}\right]}_{\mathcal{R}(-s)}=\mathcal{A}(s)\mathcal{A}(-s)+\underbrace{\left[\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}\right]}_{\mathcal{G}(s)}\underbrace{\left[\boldsymbol{g}^{\mathrm{T}}\boldsymbol{\Psi}(-s)\boldsymbol{b}\right]}_{\mathcal{G}(-s)} \tag{35}$$

which provides the quadratic polynomial solution of the KALMAN equation. Thus the final quadratic equation, ensuring relationship between the process $\mathcal{A}(s)$, design $\mathcal{R}(s)$ and weighting $\mathcal{G}(s)$ polynomials, is

$$\mathcal{R}(s)\mathcal{R}(-s)=\mathcal{A}(s)\mathcal{A}(-s)+\mathcal{G}(s)\mathcal{G}(-s) \quad \text{or} \quad |\mathcal{R}(s)|^2=|\mathcal{A}(s)|^2+|\mathcal{G}(s)|^2 \tag{36}$$

or in the general form

$$w_{\mathrm{u}}|\mathcal{R}(s)|^2=w_{\mathrm{u}}|\mathcal{A}(s)|^2+|\mathcal{G}(s)|^2 \tag{37}$$

Observe that the solution tends to $\mathcal{R}(s)=\mathcal{A}(s)$ if $W_{\mathrm{u}}\to\infty$ and $\boldsymbol{g}^{\mathrm{T}}\boldsymbol{x}=0$ if $w_{\mathrm{u}}\to 0$. Do not forget that $\mathcal{K}(s)$ and $\mathcal{G}(s)$ are of $(n-1)$-th order [8].

5. Some anomalies in the *LQR* problem

The solution of the polynomial equation can be a direct coefficient comparison or a spectral factorization approach [5]. Consider some examples in the sequel.

Example 1
Consider a first order example with

$$\mathcal{A}(s)=s+a_1 \quad ; \quad \mathcal{R}(s)=s+r_1 \quad \text{thus} \quad \mathcal{G}(s)=g_1 \tag{38}$$

The two sides of (35) are

$$-s^2-r_1 s+r_1 s+r_1^2=-s^2-a_1 s+a_1 s+a_1^2+g_1^2 \tag{39}$$

and the solution is

$$r_1^2=a_1^2+g_1^2>0 \tag{40}$$

and

$$k_1=r_1-a_1=\sqrt{a_1^2+g_1^2}-a_1>0 \tag{41}$$

If we want to ensure (place) a required pole then the necessary weight in the *LQR* problem is

$$g_1=\sqrt{r_1^2-a_1^2} \tag{42}$$

It is easy to see that only such r_1 can be placed, which fulfills the condition

$$r_1^2 > a_1^2 \Rightarrow |r_1| > |a_1| \Rightarrow r_1 > |a_1| \tag{43}$$

for stable design polynomial $\mathcal{R}(s)$. So this example shows that only a faster pole can be placed by the *LQR* optimization comparing to the original process pole.

Example 2
Consider a second order example with

$$\mathcal{A}(s) = s^2 + a_1 s + a_2 \quad ; \quad \mathcal{R}(s) = s^2 + r_1 s + r_2 \quad \text{thus} \quad \mathcal{G}(s) = g_1 s + g_2 \tag{44}$$

The two sides of (35) are now

$$\left(s^2 + r_1 s + r_2\right)\left(s^2 - r_1 s + r_2\right) = \left(s^2 + a_1 s + a_2\right)\left(s^2 - a_1 s + a_2\right) + \left(g_1 s + g_2\right)\left(-g_1 s + g_2\right) \tag{45}$$

and the solutions are

$$r_2 = \sqrt{a_2^2 + g_2^2} > a_2 \tag{46}$$

$$r_1 = \sqrt{2\left(\sqrt{a_2^2 + g_2^2} - a_2\right) + \left(a_1^2 + g_1^2\right)} = \sqrt{2\left(r_2 - a_2\right) + \left(a_1^2 + g_1^2\right)} > a_1 > 0 \tag{47}$$

The *SFB* to be applied is given by

$$k_1 = r_1 - a_1 > 0\,; k_2 = r_2 - a_2 > 0 \tag{48}$$

For pole placement the necessary *LQR* weights are

$$g_2 = \sqrt{r_2^2 - a_2^2} \tag{49}$$

and

$$g_1 = \sqrt{r_2^2 - a_1^2 - 2\left(r_2 - a_2\right)} = \sqrt{2\left(\mu_{\mathrm{r}}^2 - \mu_{\mathrm{a}}^2\right)} \tag{50}$$

where

$$\mu_{\mathrm{r}}^2 = \frac{r_1^2 - 2r_2}{2} = \frac{\left(s_1^{\mathrm{r}}\right)^2 + \left(s_2^{\mathrm{r}}\right)^2}{2} \tag{51}$$

and

$$\mu_{\mathrm{a}}^2 = \frac{a_1^2 - 2a_2}{2} = \frac{\left(s_1^{\mathrm{a}}\right)^2 + \left(s_2^{\mathrm{a}}\right)^2}{2} \tag{52}$$

It is easy to see that there are such $\{r_1, r_2\}$ domains, which can not be reached by any $\{g_1, g_2\}$

selection !!! These conditions are

$$r_2^2 > a_2^2 \Rightarrow |r_2| > |a_2| \Rightarrow r_2 > |a_2| \tag{53}$$

and

$$r_2 \geq \frac{r_1^2}{2} - 2\mu_a^2 = \frac{r_1^2}{2} - \left(a_1^2 - 2a_2\right) \tag{54}$$

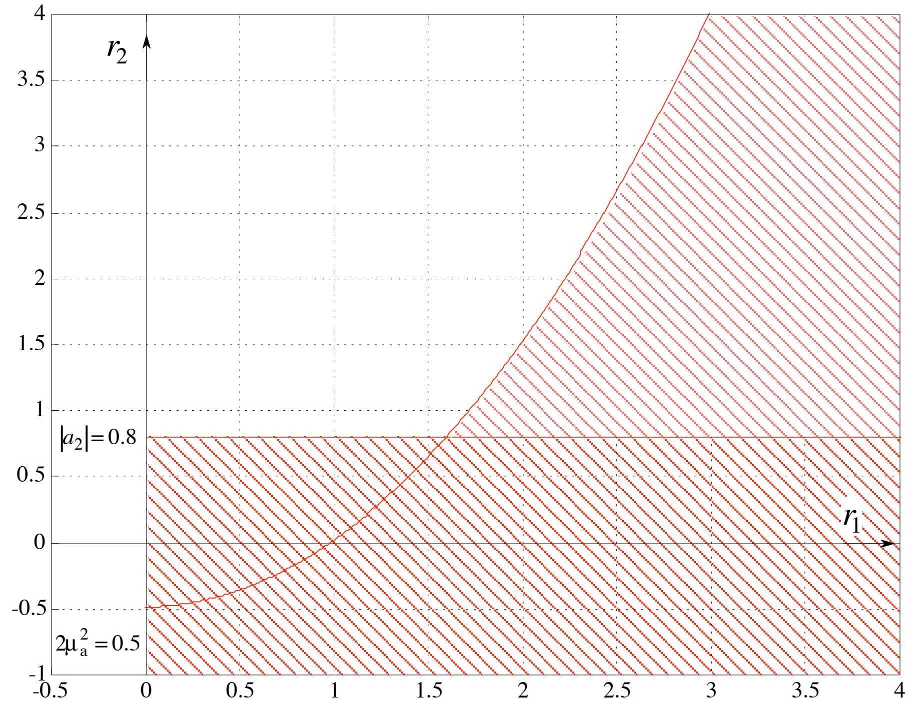

Figure 1. Unreachable design parameter domains

These conditions are graphically demonstrated on Fig.1, where the shaded area shows the unreachable design parameters for the case of open-loop process parameters $|a_2| = 0.8$ and $2\mu_a^2 = 0.5$.

One can check these results either via the solution of the RICCATI equation (very time consuming method) or by the spectral factorization approach

$$\mathcal{R}(s)\mathcal{R}(-s) = \left[\mathcal{A}(s)\mathcal{A}(-s) + \mathcal{G}(-s)\mathcal{G}(s)\right]^{+}\left[\mathcal{A}(s)\mathcal{A}(-s) + \mathcal{G}(-s)\mathcal{G}(s)\right]^{-} \tag{55}$$

as the solution of (36), i.e., by

$$\mathcal{R}(s) = \left[\mathcal{A}(s)\mathcal{A}(-s) + \mathcal{G}(s)\mathcal{G}(-s)\right]^{+} \tag{56}$$

6. Solutions for *LQ*-pole placement

We can not explain the above anomalies physically and provide unique solutions, however, offer some applicable solutions. Therefore it is necessary to discuss first the original *MIMO LQR* problem.

Infinite-horizon, continuous-time *LQ* Regulator (*LQR*)

For a continuous-time *MIMO* linear system described by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \quad ; \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{57}$$

with a *LQ* cost functional (performance index) defined as

$$J(\boldsymbol{x}_0, \boldsymbol{u}) = \frac{1}{2}\int_0^\infty \left(\boldsymbol{x}^\mathrm{T}\boldsymbol{W}_\mathrm{x}\boldsymbol{x} + \boldsymbol{u}^\mathrm{T}\boldsymbol{W}_\mathrm{u}\boldsymbol{u}\right)\mathrm{d}t \tag{58}$$

with $\boldsymbol{W}_\mathrm{x} \geq 0$ and $\boldsymbol{W}_\mathrm{u} > 0$, the stabilizing feedback control law that minimizes the value of the cost is

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x} \tag{59}$$

where $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = -\boldsymbol{W}_\mathrm{u}^{-1}\boldsymbol{B}^\mathrm{T}\boldsymbol{P} \tag{60}$$

and $\boldsymbol{P} = \boldsymbol{P}^\mathrm{T} > 0$ is the solution of the continuous time algebraic RICCATI equation

$$\boldsymbol{A}^\mathrm{T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{W}_\mathrm{u}^{-1}\boldsymbol{B}^\mathrm{T}\boldsymbol{P} + \boldsymbol{W}_\mathrm{x} = \boldsymbol{0} \tag{61}$$

It is possible to construct an even more general *LQR* performance index, which penalizes the interaction of the state and control variables, too:

$$J(\boldsymbol{x}_0, \boldsymbol{u}) = \frac{1}{2}\int_0^\infty \left(\boldsymbol{x}^\mathrm{T}\boldsymbol{W}_\mathrm{x}\boldsymbol{x} + 2\boldsymbol{u}^\mathrm{T}\boldsymbol{W}_\mathrm{ux}\boldsymbol{x} + \boldsymbol{u}^\mathrm{T}\boldsymbol{W}_\mathrm{u}\boldsymbol{u}\right)\mathrm{d}t \quad ; \quad \boldsymbol{W}_\mathrm{x} \geq 0 \quad ; \quad \boldsymbol{W}_\mathrm{u} > 0 \tag{62}$$

where the stabilizing feedback control law that minimizes the value of the cost is again

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$$

but here $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = -\boldsymbol{W}_\mathrm{u}^{-1}\left(\boldsymbol{W}_\mathrm{ux} + \boldsymbol{B}^\mathrm{T}\boldsymbol{P}\right) \tag{63}$$

and $\boldsymbol{P} = \boldsymbol{P}^\mathrm{T} > 0$ is now the solution of a more complex algebraic RICCATI equation

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\mathrm{T}\boldsymbol{P} - \left(\boldsymbol{P}\boldsymbol{B} + \boldsymbol{W}_\mathrm{ux}^\mathrm{T}\right)\boldsymbol{W}_\mathrm{u}^{-1}\left(\boldsymbol{W}_\mathrm{ux} + \boldsymbol{B}^\mathrm{T}\boldsymbol{P}\right) + \boldsymbol{W}_\mathrm{x} = \boldsymbol{0} \tag{64}$$

These results are standard facts of the *LQR* theory. For the sake of completeness a sketch of the proof for the sufficiency is given as follows: assume that $\boldsymbol{W}_\mathrm{ux} > 0$, $\boldsymbol{W}_\mathrm{x} \geq 0$, $\boldsymbol{W}_\mathrm{ux}$ are given. Then it will be shown that (62) is minimized by $\boldsymbol{K}$ in (60). The RICCATI equation can be rewritten as

$$\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{K}\boldsymbol{W}_{\mathrm{u}}^{-1}\boldsymbol{K}+\boldsymbol{W}_{\mathrm{x}}=\mathbf{0} \tag{65}$$

Pre- and post-multiplying by $\boldsymbol{x}^{\mathrm{T}}$ and $\boldsymbol{x}$, respectively and substituting $\boldsymbol{Ax}=\dot{\boldsymbol{x}}-\boldsymbol{Bu}$, it follows:

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}(\dot{\boldsymbol{x}}-\boldsymbol{Bu})+(\dot{\boldsymbol{x}}-\boldsymbol{Bu})^{\mathrm{T}}\boldsymbol{Px}-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}\boldsymbol{Kx}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}=\mathbf{0}$$

Using $-\boldsymbol{W}_{\mathrm{u}}\boldsymbol{K}=\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{W}_{\mathrm{ux}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Px}\right)=\dot{\boldsymbol{x}}\boldsymbol{Px}+\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}}$, one arrives at

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{x}}\boldsymbol{x}+2\boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{ux}}\boldsymbol{x}+\boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}\boldsymbol{u}=-\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Px}\right)+(\boldsymbol{u}-\boldsymbol{Kx})^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}(\boldsymbol{u}-\boldsymbol{Kx})$$

and by integration

$$J(\boldsymbol{x}_0,\boldsymbol{u})=\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(t_0)\boldsymbol{P}_{t_0}\boldsymbol{x}(t_0)+\frac{1}{2}\int_{t_0}^{\infty}(\boldsymbol{u}-\boldsymbol{Kx})^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}(\boldsymbol{u}-\boldsymbol{Kx})\,\mathrm{d}t$$

is obtained. Obviously $J_{\min}(\boldsymbol{x}_0)=\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(t_0)\boldsymbol{P}_{t_0}\boldsymbol{x}(t_0)$ if $\boldsymbol{u}=\boldsymbol{Kx}$.

Inverse optimality for *LQR* performance

Given a stabilizing feedback $\boldsymbol{u}=\boldsymbol{Kx}$ for (57) one can formulate the problem whether there exists an *LQR* problem of the form (58) or (62) that has the given feedback as a solution, i.e., the feedback is optimal. If the pair $(\boldsymbol{A},\boldsymbol{B})$ is controllable, then for any given spectrum Λ there is a feedback gain $\boldsymbol{K}_\Lambda$ such that $\lambda(\boldsymbol{A}+\boldsymbol{BK}_\Lambda)=\Lambda$. Concerning the pole-placement problem one can state that a spectrum Λ is *LQ* optimal if there is an associated $\boldsymbol{K}_\Lambda$ such that it is a solution of the RICCATI equation with a $\boldsymbol{W}_{\mathrm{x}}\geq 0$.

It turns out that the problem associated to the performance index (58) is nontrivial while the general case, corresponding to (62) can be always solved.

The MIMO KALMAN-FDI

In frequency domain the solution of the problem leads to the so called return difference condition. Its single input formulation is due to Kalman and was later extended by Anderson and Moore [9].

Specifically, $\boldsymbol{K}$ is optimal for $\boldsymbol{W}_{\mathrm{x}}=\boldsymbol{W}_{\mathrm{x}}^{\mathrm{T}}\geq 0$ and $\boldsymbol{W}_{\mathrm{u}}=\boldsymbol{W}_{\mathrm{u}}^{\mathrm{T}}>0$ if and only if $\boldsymbol{A}+\boldsymbol{BK}$ is stable and there exists an $\boldsymbol{W}_{\mathrm{u}}=\boldsymbol{W}_{\mathrm{u}}^{\mathrm{T}}>0$ that satisfies the return difference inequality:

$$\left[\boldsymbol{I}+\boldsymbol{H}_{\mathrm{LQ}}(-s)\right]^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}\left[\boldsymbol{I}+\boldsymbol{H}_{\mathrm{LQ}}(s)\right]\geq\boldsymbol{W}_{\mathrm{u}} \tag{66}$$

for all $s=j\omega$, $\omega\in\mathbb{R}$ or equivalently the KALMAN-FDI is also satisfied:

$$\left[\boldsymbol{I}+\boldsymbol{H}_{\mathrm{LQ}}(-s)\right]^{\mathrm{T}}\boldsymbol{W}_{\mathrm{u}}\left[\boldsymbol{I}+\boldsymbol{H}_{\mathrm{LQ}}(s)\right]=\boldsymbol{W}_{\mathrm{u}}+\boldsymbol{H}(-s)\boldsymbol{H}(s) \tag{67}$$

where

$$\boldsymbol{H}_{\mathrm{LQ}}(s)=\boldsymbol{K}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} \quad ; \quad \boldsymbol{H}(s)=\boldsymbol{G}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B} \quad ; \quad \boldsymbol{W}_{\mathrm{x}}=\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} \tag{68}$$

Choosing $\boldsymbol{W}_{\mathrm{u}}=w_{\mathrm{u}}\boldsymbol{I}$, $w_{\mathrm{u}}>0$ one has:

Proposition 1 *Consider* (58)*, then the static state feedback gain* $\boldsymbol{K}$ *is optimal for some* $\boldsymbol{W}_{\mathrm{x}}>0$*,* $\boldsymbol{W}_{\mathrm{u}}>0$ *if and only if*

- $\operatorname{Re}\lambda_i\{\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}\}<0 \quad ; \quad \forall\, i$
- $\sigma_i\left\{\mathbf{I}+\boldsymbol{K}(i\omega-\boldsymbol{A})^{-1}\boldsymbol{B}\right\}>1 \quad ; \quad \forall\, i \quad$ and $\quad \forall\, \omega$

where σ_i denotes the singular values. For *SISO* systems the KALMAN-FDI becomes:

$$w_{\mathrm{u}}\left[1+H_{\mathrm{LQ}}(-s)\right]\left[1+H_{\mathrm{LQ}}(s)\right]=w_{\mathrm{u}}+H(-s)H(s) \tag{69}$$

where $H(s)=\dfrac{\mathcal{G}(s)}{\mathcal{A}(s)}$ and $H_{\mathrm{LQ}}(s)=\boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}=\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Phi}(s)\boldsymbol{b}=\dfrac{\boldsymbol{k}^{\mathrm{T}}\boldsymbol{\Psi}(s)\boldsymbol{b}}{\mathcal{A}(s)}$.

Denoting the closed loop characteristic align by $\mathcal{R}(s)$,

$$\mathcal{R}(s)=\det\left(s\boldsymbol{I}-\boldsymbol{A}-\boldsymbol{b}\boldsymbol{k}^{\mathrm{T}}\right)=\det(s\boldsymbol{I}-\boldsymbol{A})\det\left[1+\boldsymbol{k}^{\mathrm{T}}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}\right]$$

leading to

$$\mathcal{R}(s)=\mathcal{A}(s)\left[1+H_{\mathrm{LQ}}(s)\right]$$

if $w_{\mathrm{u}}\equiv 1$ is chosen. From the KALMAN-FDI one obtains:

$$w_{\mathrm{u}}\frac{\mathcal{R}(s)\mathcal{R}(-s)}{\mathcal{A}(s)\mathcal{A}(-s)}=w_{\mathrm{u}}+\frac{\mathcal{G}(s)\mathcal{G}(-s)}{\mathcal{A}(s)\mathcal{A}(-s)} \quad \text{or} \quad \frac{\mathcal{R}(s)\mathcal{R}(-s)}{\mathcal{A}(s)\mathcal{A}(-s)}=1+w_{\mathrm{u}}^{-1}\frac{\mathcal{G}(s)\mathcal{G}(-s)}{\mathcal{A}(s)\mathcal{A}(-s)} \tag{70}$$

which corrresponds to (36). We now give a simple test for a given state feedback gain $\boldsymbol{k}$ to decide if it can be an *LQ* optimal gain.

Proposition 2 *Assume that with* $u=\boldsymbol{k}^{\mathrm{T}}\boldsymbol{x}$ *the closed loop is stable. Then* $\boldsymbol{k}$ *is optimal for some* $\boldsymbol{W}_{\mathrm{x}}\geq 0$*, and* $w_{\mathrm{u}}>0$ *if and only if*

$$\left|\frac{\mathcal{R}(i\omega)}{\mathcal{A}(i\omega)}\right|\geq 1 \quad ; \quad \forall\omega \tag{71}$$

Proof. If $\boldsymbol{k}$ is *LQ* optimal, then the closed loop is stable and from the KALMAN-FDI follows that $\left[1+H_{\mathrm{LQ}}\right]\geq 1$ and (71) is satisfied. On the contrary, if $\boldsymbol{k}$ is stabilizing and (71) is satisfied, one can find a $\boldsymbol{W}_{\mathrm{x}}\geq 0$ and $w_{\mathrm{u}}>0$ such that the KALMAN-FDI is satisfied, too, i.e., $\boldsymbol{k}$ is *LQ* optimal with this $\boldsymbol{W}_{\mathrm{x}}$ and w_{u}.

Example 3
Let the system be given as

$$\dot{x} = -2x + u$$

i.e., $\boldsymbol{A} = A = a = -2$, $\boldsymbol{B} = B = b = 1$. The open loop (plant) transfer function is

$$P(s) = \frac{1}{s+2} = \frac{b_1}{s+a_1} \quad ; \quad \mathcal{A}(s) = s + a_1$$

Applying state feedback $u = k\,x$, allocate the pole to $p_1 = -r_1 = -1$, i.e., $\mathcal{R}(s) = s + r_1$. This will be performed by $k = 1$ and the closed loop system will be

$$\dot{\overline{x}} = -1\overline{x} + u$$

i.e.,

$$\mathcal{R}(s) = s + r_1 = s + 1$$

Plotting the BODE diagram for

$$\left|\frac{\mathcal{R}(i\omega)}{\mathcal{A}(i\omega)}\right| = \frac{1}{2}\frac{|1+i\omega|}{|1+i\omega/2|}$$

one can deduce that this is below the 0 dB for small frequencies and asymptotically approaches 0 dB if $\omega \to \infty$. This shows that this k cannot be optimal for the *LQR* performance index (58).

It is seen that using static state feedback, it is not possible to "slow down" the system since $r_1 > a_1$ has to be satisfied for *LQ* optimality.

Time domain conditions

In time domain inverse optimality of the feedback gain can be described through the concept of passivity.

For a *LTI* system passivity, equivalent in this case to the positive realness, is assured in accordance with the following lemma, often termed as the KALMAN-YACUBOVICH-POPOV lemma:

Lemma 1 *A stable system* (57) *is passive, if and only if, there exists a matrix* $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0$ *such that*

$$\begin{aligned} \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} &= -\boldsymbol{W}_{\mathrm{x}} \le 0 \\ \boldsymbol{PB} &= \boldsymbol{C}^{\mathrm{T}} \end{aligned} \tag{72}$$

with $\boldsymbol{C} \in \mathrm{R}^{m\times n}$ a suitable output matrix for system (61). Then, inverse optimality is given by the following result:

Proposition 3 *A stable feedback gain-matrix* $\boldsymbol{K}$ *is optimal for a given input weighting matrix* $\boldsymbol{W}_{\mathrm{u}} > 0$ *and some state weighting matrix* $\boldsymbol{W}_{\mathrm{x}} \ge 0$*, i.e., it minimizes a performance index of the form of* (58)*, if and only if, the closed-loop system with gain-matrix*

$$\overline{\boldsymbol{K}} = \frac{1}{2}\boldsymbol{K} \tag{73}$$

is passive for an output matrix $C = -W_u K$.

Inverse optimality for *LQR* performance (67)

Including the cross term W_{ux} in the *LQR* performance index makes the problem trivial. For a stabilizing state feedback K one can find the extended matrix W (see (75)) such, that K is *LQ* optimal according to the performance (62). The procedure of deriving such weighting matrices, however, is neither trivial, nor unique. We show one possible solution that follows the procedure in [12].

It is obvious, that for any $W_u > 0$ the stabilizing feedback $u = Kx$ is optimal for the performance index:

$$J(x_0, u) = \frac{1}{2}\int_0^\infty (u - Kx)^T W_u (u - Kx)\, dt \tag{74}$$

i.e., $W_x = K^T W_u K \geq 0$ and $W_{ux} = -W_u K$ in (58). Observe that this corresponds to the solution $P = 0$ of the RICCATI equation.

A more standard solution is given by the following result:

Proposition 4 *For a given stabilizing feedback K there exists a feedback law $u = Kx$ and an extended matrix*

$$W = \begin{bmatrix} W_x & W_{ux} \\ W_{ux}^T & W_u \end{bmatrix} > 0 \tag{75}$$

such that

$$\int_0^\infty [x^T u^T] \begin{bmatrix} W_x & W_{ux} \\ W_{ux}^T & W_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \to \min_{K \in K_s} \text{tab} \tag{76}$$

if

$$\|W_u\| > \frac{\|B^T P\|}{2\|P(A + BK)\|} \tag{77}$$

where $P = P^T > 0$ satisfies the LYAPUNOV equation

$$P(A + BK) + (A + BK)^T P < 0 \tag{78}$$

Then

$$W_x = -P(A + BK) - (A + BK)^T P + K^T W_u K + K^T B^T P + PBK \tag{79}$$

$$W_{ux} = -(B^T P + W_u K) \tag{80}$$

Example 4

Consider the *Example 3* again. Let

$$\dot{x} = -2x + u$$

and apply the state feedback $k = 1$. The closed loop system $\dot{\bar{x}} = -1\bar{x} + u$ becomes stable and "slower". It can be shown that this $k = 1$ is optimal for the *LQR* performance index

$$\int_0^\infty \left(5x^2 - 4xu + u^2\right) \mathrm{d}t \tag{81}$$

Indeed, using the RICCATI-equation with

$$\boldsymbol{A} = A = a = -2 \quad ; \quad \boldsymbol{B} = B = b = 1 \quad ; \quad \boldsymbol{W}_\mathrm{x} = W_\mathrm{x} = w_\mathrm{x} = 5$$
$$\boldsymbol{W}_\mathrm{ux} = W_\mathrm{ux} = w_\mathrm{ux} = -2 \quad ; \quad \boldsymbol{W}_\mathrm{u} = W_\mathrm{u} = w_\mathrm{u} = 1$$

and

$$-4p^2 - (p-2)^2 + 5 = 0$$

and choosing the positive solution $p = 1$, the state feedback is given by

$$k = -w_\mathrm{u}^{-1}\left(bp + w_\mathrm{ux}\right) = -(1-2) = 1$$

and the closed loop matrix $\bar{\boldsymbol{A}} = \bar{A} = \bar{a} = a + bk = -2 + 1 = -1$ as required, i.e., $p_1 = -r_1 = -1$. So the closed-loop is slower !!!

This result was obtained by using the method in Proposition 4. Pick any $p > 0$ such that it is a solution of the LYAPUNOV equation $2p(a + bk) < 0$. Since $\bar{a} = a + bk = -1$ and $2p(-1) < 0$ for all $p > 0$, one can choose $p = 1$ and compute

$$w_\mathrm{u,min} = \frac{(bp)^2}{2p|\bar{a}|} = \frac{1}{2}$$

Choose any $w_\mathrm{u} > w_\mathrm{u,min}$, e.g., let $w_\mathrm{u} = 1$, then $w_\mathrm{x} = 2 + 1 + 2 = 5$ and $w_\mathrm{ux} = -(1+1) = -2$. Notice that this solution is not unique, any $W_\mathrm{u} = w_\mathrm{u} \geq 1$ would do, e.g., $w_\mathrm{u} = 2$ results in $w_\mathrm{x} = 10$, $w_\mathrm{ux} = -4$.

7. Conclusions

The paper presents the specific historical comparison of the relationships between the classical quadratic integral criterion, the pole-placement state feedback, the algebraic RICCATI equation based *LQR* paradigm and KALMAN's frequency domain approach.

Then two low order examples are shown how the obtained quadratic polynomial equation can be used. It is shown that arbitrary pole placement is not possible by standard classical *LQ* optimality by choosing only $\boldsymbol{W}_\mathrm{x}$, $\boldsymbol{W}_\mathrm{u}$ weights. For a second order case the unreachable domains are graphically demonstrated.

The *MIMO LTI* case is discussed next with more general *LQR* criterion which penalizes the interaction between the state and input variables. In this framework it is possible to obtain *LQR* solutions for the whole parameter space, but the design of the crossterm weight $\boldsymbol{W}_\mathrm{ux}$ is necessary, too. The uniqueness of the proposed solution is not guaranteed.

References

[1] Z.V. Rekasius, A general performance index for analytica of control systems, *IRE Transactions on Automatic Control*, 1960, 217.

[2] A.A. Feldbaum, *Theoretical foundations of automatic systems* (in Russian) (State Publishing House in Physics and Mathematics, Moscow, 1960).

[3] F. Csáki, *State-space methods for control systems* (Akadémiai Kiadó, Budapest, 1978).

[4] T. Kailath, *Linear systems* (Prentice Hall, 1980).

[5] M. J. Grimble and V. Kucera, *Polynomial methods for control systems design* (Springer, 1996).

[6] K.J. Åström, *Control system design Lecture Notes* (U of California, Santa Barbara, 2002).

[7] Cs. Bányász and L. Keviczky, State-feedback solutions via transfer function representations, *J. Systems Science*, 30, 2, 2004, 21-34.

[8] L. Keviczky and Cs. Bányász, Model error properties of observer-based state-feedback controller, *6. Int. Conf. System identification and control problems SICPRO'07*, Moscow, RUS, 2007, 879-888.

[9] B. D. O. Anderson and J. B. Moore, *Optimal control, Linear Quadratic Methods* (Prentice-Hall, 1989).

[10] J. Casti, The linear-quadratic control problem: some recent results and outstanding problems, *SIAM Review*, 22 (4), 1980, 459-485.

[11] J. Cigler and V. Kucera, Pole-by-Pole shifting via a linear-quadratic regulation, *17th Int. Conf. on Process Control*, XXXX, SK, 2009.

[12] A. Gattami and A. Rantzer, Linear quadratic performance criteria for cascade control. *Conf. on Decision and Control and European Control Conference* CDC-ECC'05, 2005, 3632-3637.

[13] A. Jameson, Inverse problem of linear optimal control, *SIAM Journal on Control*. 11 (1), 1973, 1-19.

[14] J. Medanic, H. S. Tharp and W. R. Perkins, Pole placement by performance criterion modification,. *IEEE Trans. on Aut. Control*, 33 (5), 1988, 469-472.

[15] D. Mehdi, Al M. Hamid and F. Perrin, Robustness and optimality of linear quadratic controller for uncertain systems, *Automatica*, 32 (7), 1996, 1081-1083.

[16] B. Molinari, The stable regulator problem and its inverse, *IEEE Trans. on Aut. Control*, 18 (5), 1973, 454-459.

[17] J. Willems, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. on Aut. Control*, 16 (6), 2003. 621-634.

[18] D. Yao, S, Zhang and X. Zhou, LQ control via semi-definite programming, *38th IEEE Conference on Decision and Control CDC99*, 1999, 1027-1032.

[19] K. Sugimoto, Partial pole placement by LQ regulators: an inverse problem approach", *IEEE Trans. on Aut. Control*, (43) 5, 1998, 706-708.

[20] A.T. Alexandridis, and G.D. Galanos, Optimal pole-placement for linear multi-input controllable systems, *IEEE Trans. Circuits Syst.*, Vol. CAS-34, 1987, 1602-1604.

[21] A.T. Alexandridis, Algorithm for the design of state feedback controllers by optimal eigenstructure assignment, Proc. ECC European Control Conference, Vol. 4b, 1995, 3365-3369.

[22] D.P. Iracleous and A.T. Alexandridis, A simple solution to the optimal eigenvalue assignment problem, *IEEE Trans. on Aut. Control* (44) 10, 1999, 1746–1749.

This work was supported in part by the Control Engineering Research Group of the HAS, at the Budapest University of Technology and Economics and by the project TAMOP 4.2.2.A-11/1/KONV-2012-2012, at the Széchenyi University of Győr.