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TWO-PART AND k-SPERNER FAMILIES: NEW PROOFS USING PERMUTATIONS*

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Abstract. This is a paper about the beauty of the permutation method. New and shorter proofs are given for the theorem [P. L. Erdős and G. O. H. Katona, *J. Combin. Theory. Ser. A*, 43 (1986), pp. 58–69; S. Shahriari, *Discrete Math.*, 162 (1996), pp. 229–238] determining all extremal two-part Sperner families and for the uniqueness of k-Sperner families of maximum size [P. Erdős, *Bull. Amer. Math. Soc.*, 51 (1945), pp. 898–902].

Key words. extremal problems, Sperner families, permutation method

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1. Introduction. Let X be a finite set of n elements. A family $\mathcal{F}$ of subsets of X is called *Sperner* (or *inclusion-free*, or an *antichain*) if $E, F \in \mathcal{F}$ implies $E \not\subset F$. The classic result of Sperner [15] states that

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1}$$

with equality only when $\mathcal{F}$ consists either of all sets of size $\lfloor \frac{n}{2} \rfloor$ or of all sets of size $\lceil \frac{n}{2} \rceil$.

There are several generalizations and elegant proofs. However, frequently the case of equality is left to the reader, since it could be rather complicated. The aim for this paper is to illustrate the strength of the permutation method by presenting new shorter proofs for Sperner-type theorems. We will give two proofs, one using the permutation method and another using cyclic permutations, a method developed by the senior author [8], [9] and applied successfully to Sperner theorems by Füredi (see [10]).

1.1. Two-part families. Kleitman [11] and Katona [7] independently observed that the statement of the Sperner theorem remains unchanged if the conditions are weakened in the following way. Let $X = X_1 \cup X_2$ be a partition of the underlying set X, $|X_i| = n_i$, $n_1 + n_2 = n$. Suppose $n_1 \ge n_2$ for the entire paper. We say that $\mathcal{F}$ is a *two-part Sperner family* if and only if $E, F \in \mathcal{F}$ $(E \ne F), E \subset F$ implies $(F - E) \not\subset X_1, X_2$. Kleitman [11] and Katona [7] proved that the size of a two-part Sperner family cannot exceed the right-hand side of (1).

The family of all $\lfloor \frac{n}{2} \rfloor$-element subsets gives equality here, too. There are, however, many other optimal constructions. A family $\mathcal{F}$ is called *homogeneous* (with respect

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to the partition X_1, X_2) if $F \in \mathcal{F}$ implies $E \in \mathcal{F}$ for all sets satisfying $|E \cap X_1| = |F \cap X_1|$, $|E \cap X_2| = |F \cap X_2|$. A homogeneous family can be described with the set $I(\mathcal{F}) = \{(i_1, i_2) : |F \cap X_1| = i_1, |F \cap X_2| = i_2 \text{ for some } F \in \mathcal{F}\}$. If $\mathcal{F}$ is a homogeneous two-part Sperner family, then $I(\mathcal{F})$ cannot contain pairs with the same first or second components, respectively. Consequently we have $|I(\mathcal{F})| \le n_2 + 1$. We say that a homogeneous family $\mathcal{F}$ is *full* if $|I(\mathcal{F})| = n_2 + 1$. Then for every i_2 $(0 \le i_2 \le n_2)$ there is a unique $f(i_2)$ such that $(f(i_2), i_2) \in I(\mathcal{F})$. A homogeneous family is called *well-paired* if it is full and

$$\binom{n_2}{i} < \binom{n_2}{j} \text{ implies } \binom{n_1}{f(i)} \le \binom{n_1}{f(j)} \tag{2}$$

for every pair $1 \le i, j \le n_2$.

Here "well-paired" roughly means that every binomial coefficient of order n_2 obtains a match from the set of binomial coefficients of order n_1 and a larger value obtains a larger match. Of course this procedure is not unique. Let us illustrate the definition by an example. Let $n_1 = 8, n_2 = 5$. Since $(n_2 + 1 =)6$ largest binomial coefficients of order $n_1 = 8$ should be chosen, $\{f(0), f(1), f(2), f(3), f(4), f(5)\}$ is either $\{1, 2, 3, 4, 5, 6\}$ or $\{2, 3, 4, 5, 6, 7\}$. Choose the first case. $\binom{5}{2}$ and $\binom{5}{3}$ are the largest ones of the binomial coefficients of order 5; therefore $\binom{8}{f(2)}$ and $\binom{8}{f(3)}$ should be two largest ones from the binomial coefficients of order 8. Choose, for instance, $f(3) = 4, f(2) = 5$. Now $\binom{5}{1}$ and $\binom{5}{4}$ are larger than $\binom{5}{0}$ and $\binom{5}{5}$, so $\binom{8}{f(1)}$ and $\binom{8}{f(4)}$ should be next two largest ones after $\binom{8}{4}$ and $\binom{8}{5}$. Choose $f(4) = 3$ and $f(1) = 6$. Finally, let $f(0) = 1, f(5) = 2$. In this way we obtained a well-paired family $\mathcal{F}$ which consists of all subsets F satisfying $|F \cap X_1| = i_1$ and $|F \cap X_2| = i_2$, where $(i_1, i_2) \in \{(1, 0), (6, 1), (5, 2), (4, 3), (3, 4), (2, 5)\}$.

The following characterization (although not in this form) was proved in [5]. Later Shahriari [14] found an alternative proof.

THEOREM 1.1. *Let $\mathcal{F}$ be a two-part Sperner family with parts X_1, X_2, $|X_1| + |X_2| = n$. Then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds with equality if and only if $\mathcal{F}$ is a homogeneous well-paired family.

We give two new, probably shorter proofs in section 3 of the present paper.

Homogeneity type results are also true in a much more general setting. See the paper by Füredi et al. [6] or the joint paper of the present authors with Frankl [4]. In those papers it is shown, that there is a homogeneous optimal construction. Here we see that no other family can be optimal.

1.2. Families with no $k + 1$-chains. To prove Theorem 1.1 we need another extension of the Sperner theorem, which is due to Paul Erdős. A family $\mathcal{F}$ of sets is called *k-Sperner* if it contains no *chain* $F_0 \subset F_1 \subset \cdots \subset F_k$ of $k + 1$ different sets. It was proved in [3] that if a family $\mathcal{F}$ of subsets of an n-element set is k-Sperner, then $|\mathcal{F}|$ is at most the sum of the k largest binomial coefficients of order n. The following theorem determines the cases of equality. This result is part of the folklore, but we do not know any written reference for it. The proof is a direct generalization of the uniqueness proof of the original Sperner theorem, due to the second author.

THEOREM 1.2. *Let $\mathcal{F}$ be a k-Sperner family of subsets of an n-element set. Then*

$$|\mathcal{F}| \leq \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i} \tag{3}$$

holds with equality if and only if $\mathcal{F}$ is the family of all sets of sizes either in the interval $[\lfloor \frac{(n-k+1)}{2} \rfloor, \lfloor \frac{(n+k-1)}{2} \rfloor]$ or in the interval $[\lceil \frac{(n-k+1)}{2} \rceil, \lceil \frac{(n+k-1)}{2} \rceil]$.

This theorem will be proved in section 2. The upper bound in the following result is an immediate corollary. Denote by $\binom{X}{i}$ the family of all i-element subsets of X; it is called the ith *level* in X.

THEOREM 1.3. *Let $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ be a disjoint union of k-Sperner families of subsets of an n-element set X. Then $|\mathcal{F}|$ satisfies* (3) *with equality if and only if $\mathcal{F}_i = \binom{X}{r_i}$ holds for $1 \leq i \leq k$, where $r_1, \ldots, r_k$ is a permutation of the elements either of the interval $[\lfloor \frac{(n-k+1)}{2} \rfloor, \lfloor \frac{(n+k-1)}{2} \rfloor]$ or of the interval $[\lceil \frac{(n-k+1)}{2} \rceil, \lceil \frac{(n+k-1)}{2} \rceil]$.*

2. Uniqueness in Erdős theorem and in the generalized YBLM-inequality. First we will prove a sharper version of Paul Erdős's theorem (Theorem 1.2) and will characterize the cases of equality of this sharper one. $\mathcal{F}$ is called *homogeneous* if $F \in \mathcal{F}, E \subset X$, and $|E| = |F|$ imply $E \in \mathcal{F}$. If $\mathcal{F}$ is a family of subsets, $f_i(\mathcal{F})$ will denote the number of i-element members of $\mathcal{F}$.

THEOREM 2.1. *Let $\mathcal{F}$ be a k-Sperner family. Then*

$$\sum_{i=0}^{n} \frac{f_i(\mathcal{F})}{\binom{n}{i}} \leq k \tag{4}$$

with equality only when $\mathcal{F}$ is homogeneous and contains sets of k distinct sizes.

The inequality part of this theorem can be found in [4, Theorem 5a] and is a generalization of the well-known YBLM-inequality [16], [1], [12], [13].

Proof. The method of cyclic permutations is used. The main point of this method is to reduce the original problem into an analogous problem on a fixed cyclic permutation.

If $\emptyset \in \mathcal{F}$, then $\mathcal{F} \setminus \{\emptyset\}$ is a $(k-1)$-Sperner family, and we can use induction on k. The case $X \in \mathcal{F}$ is similar. So from now on (in this section) we suppose that $f_0 = f_n = 0$ and $n > k$.

Let C be a cyclic permutation of X and let $\mathcal{F}(C)$ denote the subfamily of $\mathcal{F}$ consisting of all sets forming an interval (i.e., an arc) in C. $\mathcal{F}(C)$ is said to be *homogeneous* if $F \in \mathcal{F}(C)$ implies that every interval E along C of the same size ($|E| = |F|$) is in $\mathcal{F}(C)$. The proof is based on the following lemma.

LEMMA 2.2.

$$|\mathcal{F}(C)| \leq nk. \tag{5}$$

Here equality holds if and only if $\mathcal{F}(C)$ is homogeneous and it contains k distinct sizes.

Proof of Lemma 2.2. Since $\emptyset, X \notin \mathcal{F}$ at most k sets may start at any fixed element of X along C in one direction. This establishes (5).

In the case of equality there must be exactly k intervals in $\mathcal{F}(C)$ starting from each point of C. Let $B_i(j)$ ($1 \leq i \leq n$, $1 \leq j \leq k$) denote the jth interval starting from the ith point where $|B_i(1)| < |B_i(2)| < \cdots < |B_i(k)|$ is supposed. We claim that $|B_i(j)| \leq |B_{i+1}(j)|$ holds. Indeed, otherwise $B_{i+1}(1) \subset B_{i+1}(2) \subset \cdots \subset B_{i+1}(j) \subset$

$B_i(j) \subset \cdots \subset B_i(k)$ would be a chain of intervals of length $k+1$, a contradiction. Hence we have $|B_1(j)| \le |B_2(j)| \le \cdots \le |B_n(j)| \le |B_1(j)|$ implying $|B_i(j)| = |B_{i+1}(j)|$ for all $1 \le i < n$ and $1 \le j \le k$. ∎

Let us return to the proof of Theorem 2.1. Lemma 2.2 yields

$$\sum_C \sum_{F\in\mathcal{F}(C)} 1 = \sum_C |\mathcal{F}(C)| \le (n-1)!nk = n!k. \tag{6}$$

The number of cyclic permutations C containing a given set F as an interval is $|F|!(n-|F|)!$ (if $|F| \ne 0, n$). Hence

$$\sum_{F\in\mathcal{F}} \sum_{C:F\in\mathcal{F}(C)} 1 = \sum_{F\in\mathcal{F}} |F|!(n-|F|)! \tag{7}$$

holds. Comparing (7) and (6) we obtain (4), the inequality part of Theorem 2.1.

Formula (4) can hold with equality only when (7) and (6) are equal, that is, when (5) holds with equality for all cyclic permutations: $\mathcal{F}(C)$ is homogeneous for each C. Consider any two subsets A and B ($\subset X$) of equal cardinality. It is obvious that there is a cyclic permutation C in which they are both intervals. Therefore either $A, B \in \mathcal{F}$ or $A, B \notin \mathcal{F}$ holds, and consequently $\mathcal{F}$ is also homogeneous. ∎

We need a simple inequality; for completeness we supply a sketch of the proof, standard in linear programming.

LEMMA 2.3. *Suppose that for integers $n \ge k \ge 1$ and nonnegative reals $f_1, f_2 \ldots, f_{n-1}$ the following inequalities hold:*

$$\sum_{1\le i\le n-1} \frac{f_i}{\binom{n}{i}} \le k,$$

$$f_i \le \binom{n}{i}.$$

Then

$$\sum_{1\le i\le n-1} f_i \le \sum_{i=\lfloor (n-k+1)/2\rfloor}^{\lfloor (n+k-1)/2\rfloor} \binom{n}{i} := f(n,k).$$

Here equality holds if and only if

(a) *in the case $n \not\equiv k \pmod 2$, $f_i = \binom{n}{i}$ for $(n-k+1)/2 \le i \le (n+k-1)/2$ and $f_i = 0$ otherwise,*

(b) *in the case $n \equiv k \pmod 2$, $f_i = \binom{n}{i}$ for $(n-k+2)/2 \le i \le (n+k-2)/2$ and $f_{(n-k)/2} + f_{(n+k)/2} = \binom{n}{(n-k)/2}$ and $f_i = 0$ otherwise.*

Proof. Consider a vector $\mathbf{f} = (f_1, f_2, \ldots, f_{n-1})$ which maximizes $\sum f_i$. (The domain is compact; maximum(s) exists.) For $\binom{n}{j} < \binom{n}{i}$ the inequalities $f_i < \binom{n}{i}, 0 < f_j$ lead to a contradiction, since replacing them by $f_i + \varepsilon\binom{n}{i}$ and $f_j - \varepsilon\binom{n}{j}$ keeps the constraint the lemma but increases the sum $\sum f_i$. ∎

Proof of Theorem 1.2. The constraint of Lemma 2.3 holds for the sequence $f_1(\mathcal{F}), \ldots, f_{n-1}(\mathcal{F})$ by (4) and since $f_i(\mathcal{F}) \le \binom{n}{i}$ is obvious. This implies the Erdős theorem.

We can have equality in this theorem only when (4) holds with equality. Then Theorem 2.1 implies that $\mathcal{F}$ is homogeneous and consists of k distinct sizes. ∎

Proof of Theorem 1.3. The inequality part is trivial, since $\mathcal{F}$ is a k-Sperner family. It is clear from the previous proof that the equality implies equality in (4). Since $\mathcal{F}_i$ $(1 \le i \le k)$ is a Sperner family, (4) holds for $\mathcal{F}_i$ with $k = 1$. Hence (4) with $k = 1$ must hold with equality for each $\mathcal{F}_i$. Therefore $\mathcal{F}_i = \binom{X}{r_i}$ for some r_i. Since $\mathcal{F}_i$ are disjoint, r_i must be different, $\mathcal{F}$ is a union of k distinct levels. The maximality of $|\mathcal{F}|$ implies that these k levels must be the k middle ones. $\square$

2.1. Uniqueness in the Erdős theorem using intervals. Here we give another proof for Theorem 1.2.

Let $\mathcal{F}$ be a k-Sperner family on the n-element underlying set $X = [n]$. We may suppose that $\emptyset, X \notin \mathcal{F}$ because these cases can easily be reduced to the general case. As in the classical proofs, consider a *permutation* π of X. The initial segments of π, i.e., the sets of the form $\{\pi(1), \pi(2), \dots, \pi(i)\}_{1 \le i < n}$ form a *chain* $\mathcal{C}(\pi)$ of length $n-1$. The k-Sperner property of $\mathcal{F}$ implies that $\mathcal{C}(\pi)$ contains at most k members of $\mathcal{F}$, so we have

$$\sum_{F: F \in \mathcal{F}, F \in \mathcal{C}(\pi)} \binom{n}{|F|} \le \sum k \,\text{largest binomial coefficients} := f(n,k). \tag{8}$$

Add this up for all the $n!$ permutations.

$$\sum_{\pi} \sum_{F \in \mathcal{F}, F \in \mathcal{C}(\pi)} \binom{n}{|F|} \le n! f(n,k).$$

Here the left-hand side can be determined exactly.

$$\sum_{F: F \in \mathcal{F}} \sum_{\pi: F \in \mathcal{C}(\pi)} \binom{n}{|F|} = \sum_{F} |F|!(n-|F|)! \binom{n}{|F|} = n! |\mathcal{F}|.$$

This gives $|\mathcal{F}| \le f(n,k)$.

If $|\mathcal{F}| = f(n,k)$, then equality holds in (8) for every π, so the sizes of the members of $\mathcal{F}$ in $\mathcal{C}(\pi)$ form a middle interval of length k. In the case $n \not\equiv k \pmod 2$ this middle interval is unique; we get that $\mathcal{F}$ is homogeneous, and it consists of all sets of sizes at least $(n-k+1)/2$ and at most $(n+k-1)/2$. In the case $n \not\equiv k \pmod 2$ there are two possibilities for a middle interval, so $f_i = \binom{n}{i}$ for $(n-k+2)/2 \le i \le (n+k-2)/2$ and $f_{(n-k)/2} + f_{(n+k)/2} = \binom{n}{(n-k)/2}$ and $f_i = 0$ otherwise. We also obtain that for $|F'| = (n-k)/2$, $|F''| = (n+k)/2$, $F' \subset F''$, one and only one of $\{F', F''\}$ belongs to $\mathcal{F}$. Suppose that there exists an $F \in \mathcal{F}$, $|F| = (n-k)/2$. We claim that $f_{(n-k)/2} = \binom{n}{(n-k)/2}$ and then $f_{(n+k)/2} = 0$, and we are done.

Consider an arbitrary pair $x \in F$ and $y \in X \setminus F$. We claim that $F \setminus \{x\} \cup \{y\} \in \mathcal{F}$. Indeed, consider a permutation π where $F \setminus \{x\}$, F and $F \cup \{y\}$ are initial segments, and let π' be a permutation obtained from π be exchanging the places of x and y. The largest member of $\mathcal{F}$ in $\mathcal{C}(\pi)$ has $(n+k-2)/2$ elements, so the same is true for $\mathcal{C}(\pi')$. Since the sizes of the members of $\mathcal{C}(\pi') \cap \mathcal{F}$ form a middle interval, the smallest member has $(n-k)/2$ elements. This smallest member is $F \setminus \{x\} \cup \{y\}$.

Call two $(n-k)/2$-element sets F_1 and F_2 neighbors if $|F_1 \cap F_2| = |F_1| - 1$. Then the above property of the extremal $\mathcal{F}$ can be formulated as it contains all neighbors of F whenever $F \in \mathcal{F}$. It follows that in that case it contains the second, third, etc. neighbors, so $\mathcal{F}$ contains the whole $((n-k)/2)$th level. $\square$

3. Two-part Sperner families. In the method of cyclic permutations a given problem on subsets is reduced to intervals in a cyclic permutation of the underlying set. In the present proof the problem will be reduced to a family of certain mixed objects, pairs (A, B), where A is a subset of X_1 and B is an interval along a fixed cyclic permutation of X_2. Therefore the method can be called the *mixcyc* method.

First proof of Theorem 1.1. Let C_2 be a cyclic permutation of X_2 and $\mathcal{F}$ a family of subsets of X. Then $\mathcal{F}(C_2)$ will denote those members of $\mathcal{F}$ for which $F \cap X_2$ is an interval along C_2.

Introduce the notation

$$t(j) = \begin{cases} n_2 & \text{if} \quad j = 0, n_2, \\ 1 & \text{if} \quad 1 \le j \le n_2 - 1. \end{cases}$$

The double sum

$$\sum_{\substack{(C_2,F)\\ F\in\mathcal{F}(C_2)}} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} \tag{9}$$

will be evaluated in two different ways. First

$$\begin{aligned} &\sum_{F\in\mathcal{F}C_2:}\ \sum_{F\in\mathcal{F}(C_2)} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} \\ &\qquad = \sum_{F\in\mathcal{F}} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} \sum_{C_2:\ F\in\mathcal{F}(C_2)} 1. \end{aligned}$$

Here

$$\sum_{C_2:\ F\in\mathcal{F}(C_2)} 1 = \begin{cases} (n_2-1)! & \text{if } F\cap X_2 = \emptyset \text{ or } X_2, \\ |F\cap X_2|!\,(n - |F\cap X_2|)! & \text{otherwise.} \end{cases}$$

Therefore

$$(9) = \sum_{F\in\mathcal{F}} n_2! = |\mathcal{F}|n_2!.$$

On the other hand, (9) is equal to

$$\sum_{C_2}\quad \sum_{F\in\mathcal{F}(C_2)} t\left(|F\cap X_2|\right)\binom{n_2}{|F\cap X_2|}. \tag{10}$$

Introduce the notation

$$w(i) = t(i)\binom{n_2}{i}, \qquad i = 0, \ldots, n_2,$$

and let $(j_0, j_1, \ldots, j_{n_2})$ be one of the permutations of $(0, 1, \ldots, n_2)$ satisfying $w(j_0) \ge w(j_1) \ge \cdots \ge w(j_{n_2}) = n_2$. There are four cases of w with value n_2. Suppose that j_{n_2-1} and j_{n_2} are chosen to be 0 and n_2, respectively. Now fix a cyclic permutation $C_2 = (c_1, \ldots, c_n)$ of X_2 and decompose its intervals into n_2 *chains* of intervals: define

$$\mathcal{L}_1 = \{\emptyset, \{c_1\}, \{c_1, c_2\}, \ldots, \{c_1, c_2, \ldots, c_{n_2-1}\}, \{c_1, \ldots, c_{n_2}\}\},$$

while for $i = 2, \ldots, n_2$ let

$$\mathcal{L}_i = \{\{c_i\}, \{c_i, c_{i+1}\}, \ldots, \{c_i, c_{i+1}, \ldots, c_{n_2}, c_1, \ldots, c_{i-3}\}, \{c_i, \ldots, c_{i-2}\}\}.$$

Consider the subsum

$$\sum_{(F\cap X_2)\in\mathcal{L}_1} t(|F \cap X_2|)\binom{n_2}{|F \cap X_2|} = \sum_{i=0}^{n_2} |\mathcal{F}(j_i)|\, w(j_i), \tag{11}$$

where $\mathcal{F}(j)$ is defined by

$$\mathcal{F}(j) = \{F \cap X_1 : F \in \mathcal{F}, |F \cap X_2| = j \text{ and } F \cap X_2 \in \mathcal{L}_1\}.$$

It is easy to see that the family $\mathcal{F}(j)$ is Sperner for every j and that $\mathcal{F}(j_k)\cap\mathcal{F}(j_l) = \emptyset$ holds when $k \neq l$. Formula (11) can be written as

$$\begin{aligned}
(11) = &\Big(|\mathcal{F}(j_0)| + \cdots + |\mathcal{F}(j_{n_2})|\Big)w(j_{n_2})\\
&+ \Big(|\mathcal{F}(j_0)| + \cdots + |\mathcal{F}(j_{n_2-1})|\Big)\Big(w(j_{n_2-1}) - w(j_{n_2})\Big)\\
&+ \cdots + \Big(|\mathcal{F}(j_0)| + |\mathcal{F}(j_1)|\Big)\Big(w(j_1) - w(j_2)\Big)\\
&+ |\mathcal{F}(j_0)|\Big(w(j_0) - w(j_1)\Big).
\end{aligned} \tag{12}$$

By the Erdős theorem the total size of k pairwise disjoint Sperner families in X_1 cannot exceed the k largest levels. Therefore if $m(i) = \binom{n_1}{i}$ and $(l_0, l_1, \ldots, l_{n_1})$ is one of the permutations of $(0, 1, \ldots, n_1)$ satisfying $m(l_0) \geq m(l_1) \geq \cdots \geq m(l_{n_1})$, then

$$\begin{aligned}
(12) \leq &\Big(m(l_0) + m(l_1) + \cdots + m(l_{n_2})\Big)w(j_{n_2})\\
&+ \Big(m(l_0) + m(l_1) + \cdots + m(l_{n_2-1})\Big)\Big(w(j_{n_2-1}) - w(j_{n_2})\Big) + \cdots\\
&+ \Big(m(l_0) + m(l_1)\Big)\Big(w(j_1) - w(j_2)\Big) + m(l_0)\Big(w(j_0) - w(j_1)\Big)\\
= &\sum_{i=0}^{n_2} m(l_i)w(j_i).
\end{aligned} \tag{13}$$

The same estimations can be applied for the other $n_2 - 1$ chains $\mathcal{L}_k$ $(k = 2, \ldots, n_2)$:

$$\sum_{F\cap X_2\in\mathcal{L}_k} t(|F \cap X_2|)\binom{n_2}{|F \cap X_2|} \leq \sum_{i=0}^{n_2-2} m(l_i)w(j_i).$$

Using the fact that the number of cyclic permutations C_2 is $(n_2 - 1)!$ and putting together the previous inequalities, we obtain

$$\begin{aligned}
(10) &\leq \sum_{C_2}\left(n_2 \sum_{i=0}^{n_2-2} m(l_i)w(j_i) + m(l_{n_2-1})w(j_{n_2-1}) + m(l_{n_2})w(j_{n_2})\right)\\
&= n_2! \sum_{i=0}^{n_2} \binom{n_1}{l_i}\binom{n_2}{j_i} = n_2! \sum_{i=0}^{n_2} \binom{n_1}{\lceil \frac{n_1+n_2}{2}\rceil + i}\binom{n_2}{i}\\
&= n_2! \sum_{i=0}^{n_2} \binom{n_1}{\lfloor \frac{n_1+n_2}{2}\rfloor - i}\binom{n_2}{i} = \binom{n}{\lfloor \frac{n}{2}\rfloor}.
\end{aligned} \tag{14}$$

(9) = (10) $\leq$ (14) finishes the proof of the two-part Sperner theorem.

To prove the equality part of Theorem 1.1 we only have to check carefully the cases of equality in the above proof of the two-part Sperner theorem.

Define

$$\mathcal{F}_1(B) = \{A: \ A \subset X_1, A \cup B \in \mathcal{F}\} \quad \text{for } B \subset X_2.$$

If $\mathcal{F}$ is a family satisfying equality in the Erdős theorem (in the form of Theorem 1.3), then there must be equality between (12) and (13), that is,

$$|\mathcal{F}(j_0)| + |\mathcal{F}(j_1)| + \cdots + |\mathcal{F}(j_r)| = m(l_0) + m(l_1) + \cdots + m(l_r) \tag{15}$$

holds whenever $w(j_r) - w(j_{r+1}) > 0$ (where $w(j_{n_2+1}) = 0$). It is obvious that every second of these differences is zero, and the other ones are positive. If n_2 is even, then $w(j_0) - w(j_1)$ is positive, $w(j_1) - w(j_2)$ is zero, $w(j_2) - w(j_3)$ is positive, and so on. On the other hand, if n_2 is odd, then this sequence starts with a zero. We should not forget, however, that there are some irregularities at the end. First, the last coefficient $w(j_{n_2})$ (first in (12)) is always positive; second, it is preceded by three zeros. This implies, by Theorem 1.3, that in the case of even $n_2, \mathcal{F}(j_0)$ must be one of the (one or two) largest levels in X_1; $\mathcal{F}(j_0), \mathcal{F}(j_1), \mathcal{F}(j_2)$ must be the three largest levels; and so on. Hence $\mathcal{F}(j_1)$ and $\mathcal{F}(j_2)$ are the two levels next or equal in size. The same holds for $\mathcal{F}(j_{2s+1})$ and $\mathcal{F}(j_{2s+2})$ for $0 \leq s \leq \frac{n_2-6}{2}$. If n_2 is odd, then $\mathcal{F}(j_0)$ and $\mathcal{F}(j_1)$ are the two largest levels, $\mathcal{F}(j_2)$ and $\mathcal{F}(j_3)$ are the next two levels, and so on. In general $\mathcal{F}(j_{2s})$ and $\mathcal{F}(j_{2s+1})$ $(0 \leq s \leq \frac{n_2-5}{2})$ are a pair of the $(2s+1)$st and $(2s+2)$th largest levels.

Since $w(j_{n_2}) > 0$ holds, $\mathcal{F}(j_0), \ldots, \mathcal{F}(j_{n_2})$ are the $n_2 + 1$ largest levels in X_1. However, we have some freedom in choosing their order, but this order must satisfy the conditions above. Until now we have proved a restricted version of the homogeneity of $\mathcal{F}$, namely, that the subfamily $\{F: \ F \in \mathcal{F}, F \cap X_2 \in \mathcal{L}_1\}$ is a homogenous full family. That is, the family $\{F \cap X_1: \ F \in \mathcal{F}, F \cap X_2 = \{c_1, \ldots, c_j\}\} = \mathcal{F}_1(\{c_1, \ldots, c_j\}) = \mathcal{F}(j)$ is equal to $\binom{X_1}{w}$ for some w. Let this w be denoted by $f^*(j)$. It remained to check that this restriction of $\mathcal{F}$ is well-paired; that is, this ordering satisfies (2).

If n_2 is even, then the left-hand side of (2),

$$\binom{n_2}{j_u} < \binom{n_2}{j_v} \ (u < n_2 - 3), \tag{16}$$

holds if and only if $v \leq u$ and u is not an even integer $= v + 1$. Then

$$\binom{n_1}{f^*(j_u)} \leq \binom{n_1}{f^*(j_v)} \tag{17}$$

is obvious. The case when n_2 is odd is analogous. That is, the order follows (2) up to $n_2 - 4$. Consider now the case when $u = n_2 - 3, n_2 - 2, n_2 - 1, n_2$ and $n_2 - 3 > v$. Since $\{j_{n_2}, j_{n_2-1}\} = \{0, n_2\}$ by definition, consequently we have $\{j_{n_2-2}, j_{n_2-3}\} = \{1, n_2 - 1\}$, and hence the last few $\binom{n_2}{j_u}$ are $n_2, n_2, 1, 1$. (16) holds in these cases; therefore (17) also must hold. It is really true since $\mathcal{F}(j_0), \ldots, \mathcal{F}(j_{n_2-4})$ are $n_2 - 3$ largest levels in X_1. We do not know the monotonicity among the last four u's. An important consequence is that $f^*(j_v)$ cannot be $\lfloor \frac{n_1-n_2}{2} \rfloor$ or $\lceil \frac{n_1+n_2}{2} \rceil$ when $n_2 - 3 > v$.

The above ideas are valid for all cyclic permutations of X_2; therefore $\mathcal{F}_1(B)$ is defined for all $B \subset X_2$ and it is a full level $\binom{X_1}{j}$ for some $j = j(B)(\lfloor \frac{n_1-n_2}{2} \rfloor \leq j \leq \lceil \frac{n_1+n_2}{2} \rceil)$.

We have to show that $\mathcal{F}_1(B)$ depends only on the size of B, that is, $|B_1| = |B_2|$ implies $\mathcal{F}_1(B_1) = \mathcal{F}_1(B_2)$. It is sufficient to verify this statement for "neighboring" sets, that is, when $|B_1 - B_2| = 1$. Let $B_1 = \{x_1, x_2, \ldots, x_l\}, B_2 = \{x_2, x_3, \ldots, x_l, x_{l+1}\}$. Consider the cyclic permutations $C = (x_2, x_3, \ldots, x_l, x_1, x_{l+1}, x_{l+2} \ldots, x_{n_2}), C' = (x_2, x_3, \ldots, x_l, x_{l+1}, x_1, x_{l+2} \ldots, x_{n_2})$. They define the chains (of length $n_2 + 1$) $\mathcal{L}_1$ and $\mathcal{L}'_1$, which differ only in one member. The function $\mathcal{F}_1$ associates a family $\binom{X_1}{j}(\lfloor \frac{n_1-n_2}{2} \rfloor \le j \le \lceil \frac{n_1+n_2}{2} \rceil)$ with each member of these chains, where the j's are different for one chain. If n_1 and n_2 have the same parities, then there are $n_2 + 1$ choices for j and therefore $\mathcal{F}_1(B_1) = \mathcal{F}_1(B_2)$. If their parities are different, then $\mathcal{F}_1(B_1)$ and $\mathcal{F}_1(B_2)$ may be different: one is $\binom{X_1}{\lfloor \frac{n_1-n_2}{2} \rfloor}$ and the other is $\binom{X_1}{\lceil \frac{n_1+n_2}{2} \rceil}$. It is clear from the monotonicity (17) that this can happen only when $|B_1| = 1$ or $n_2 - 1$. This proves the statement $\mathcal{F}_1(B_1) = \mathcal{F}_1(B_2)$ for $1 < |B_1| = |B_2| < n - 1$. Moreover,

$$\mathcal{F}_1(B) = \text{ either } \binom{X_1}{\lfloor \frac{n_1-n_2}{2} \rfloor} \text{ or } \binom{X_1}{\lceil \frac{n_1+n_2}{2} \rceil} \text{ if } |B| = 1, n - 1.$$

Since $\mathcal{F}$ is a two-part Sperner family, $B \subset C$ implies $\mathcal{F}_1(B) \ne \mathcal{F}_1(C)$ (in fact, they must be disjoint). Suppose, e.g., that $j(\{x\}) = \lfloor \frac{n_1-n_2}{2} \rfloor$ holds for some $x \in X_2$. Then $j(C)$ must be $\lceil \frac{n_1+n_2}{2} \rceil$ for all $n_2 - 1$-element C with the possible exception of $X_2 - x$. But these sets cover X_2; therefore $j(\{x\}) = \lfloor \frac{n_1-n_2}{2} \rfloor$ must hold for all $x \in X_2$, and consequently $j(C) = \lceil \frac{n_1+n_2}{2} \rceil$ for all $n_2 - 1$-element $C \in X_2$. We have proved that $\mathcal{F}$ is homogeneous and full, and the function f is defined by $f(i) = j(B)$, where $i = |B|$.

It is almost proved that $\mathcal{F}$ is well-paired, by (17). The only possible exception is that the right-hand side of (2) does not hold for one or more of the pairs $(0, 1), (0, n_2 - 1), (n_2, 1), (n_2, n_2 - 1)$. Suppose, e.g., that the pair $(0, 1)$ is such a one. Then

$$|\mathcal{F}| = \sum_{i=0}^{n_2} \binom{n_2}{i} \binom{n_1}{f(i)}$$

can be increased by interchanging the values $f(0)$ and $f(1)$. (It increases the sum only when $n_2 > 1$ but the case $n_2 = 1$ is trivial.) This contradiction shows that $\mathcal{F}$ is well-paired. $\square$

The interested reader should check [5], where the optimal constructions for all four cases (depending on the parities of n_1 and n_2, resp.,) are illustrated with figures.

3.1. Extremal two-part Sperner families and intervals. Here we give another proof for Theorem 1.1. We need two simple lemmas. Suppose that $u \ge v \ge 1$ are integers, $a_1 \ge a_2 \ge \cdots a_u \ge 0$, $b_1 \ge b_2 \ge \cdots \ge b_v$ are reals, and $g : [v] \to [u]$ is an arbitrary injection (i.e., $g(i) \ne g(j)$ for $i \ne j$). Then we say that the two sequences are *well-paired* by g if $b_i < b_j$ implies $a_{g(i)} \le a_{g(j)}$. Observe that if this definition is applied for the binomial coefficients of ranks n_1 and n_2, respectively, and for the function defined by a homogenous two-part Sperner family, then definition (2) is obtained, again.

LEMMA 3.1. *Suppose that $u \ge v \ge 1$ are integers, $a_1 \ge a_2 \ge \cdots \ge a_u \ge 0$, $b_1 \ge b_2 \ge \cdots \ge b_v$ are reals, and $g : [v] \to [u]$ is an arbitrary injection. Then*

$$\sum_i a_{g(i)} b_i \le \sum_{1 \le i \le v} a_i b_i,$$

and here equality holds if and only if the sequences are well-paired by g.

LEMMA 3.2. *Let the $a_1, a_2, \ldots, a_{n_1+1}$ be the sequence of binomial coefficients of rank n_1 in decreasing order, and let $b_1, \ldots, b_{n_2+1}$ be the binomial coefficients of rank n_2 again in decreasing order. (We have $a_i = \binom{n_1}{\lfloor (n_1+i)/2 \rfloor}$ and $b_j = \binom{n_2}{\lfloor (n_2+j)/2 \rfloor}$.) Then $\sum_i a_i b_i = \binom{n}{\lfloor n/2 \rfloor}$.*

Second proof of Theorem 1.1. Let $\mathcal{F}$ be a two-part Sperner family on the n-element underlying set $X = [n]$, with parts X_1, X_2, $|X_i| = n_i$, $n_1 \geq n_2 > 0$. Suppose that $|\mathcal{F}|$ is maximal; then we have $|\mathcal{F}| \geq \binom{n}{\lfloor n/2 \rfloor}$. Let $\pi_i \in S_{[n_i]}$ be a permutation of X_i, $i = 1, 2$. Define the $(n_1+1) \times (n_2+1)$ matrix $M = M(\pi_1, \pi_2)$ as follows. Label the rows by $0, 1, \ldots, n_1$ and the columns by $0, 1, \ldots, n_2$, and for the i, j entry, $M_{i,j}$ equals 1 if the unions of the two initial segments $\{\pi_1(1), \pi_1(2), \ldots, \pi_1(i)\} \cup \{\pi_2(1), \ldots, \pi_2(j)\}$ belong to $\mathcal{F}$, and $M_{i,j} = 0$ for the other entries. Such an M contains at most one nonzero entry in each row and column.

Suppose that M is an arbitrary $(n_1 + 1) \times (n_2 + 1)$ matrix, labeled as above, and suppose that each entry is 0 or 1 and each row and column contains at most one 1. Define a two-part Sperner family $\mathcal{H}(M)$ by taking all sets $F \subset X$ with $M_{|F \cap X_1|, |F \cap X_2|} = 1$. Then $|\mathcal{H}(M)| = \sum_{M_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j}$. By Lemmas 3.1 and 3.2 we have

$$|\mathcal{H}(M)| \leq \sum_{i,j} a_i b_j = \binom{n}{\lfloor n/2 \rfloor}$$

with equality only when M contains a 1 in each column and the mapping defined by M is well-paired with respect the binomial coefficients of ranks n_1 and n_2, respectively.

We obtain

$$\begin{aligned}
|\mathcal{F}| n_1! n_2! \geq \binom{n}{\lfloor n/2 \rfloor} n_1! n_2! &\geq \sum_{(\pi_1, \pi_2)} |\mathcal{H}(M(\pi_1, \pi_2))| \\
&= \sum_{F \in \mathcal{F}} \sum_{\substack{\pi_1, \pi_2 \\ F \cap X_i \text{ is initial in } \pi_i}} \binom{n_1}{|F \cap X_1|}\binom{n_2}{|F \cap X_2|} \\
= \sum_{F \in \mathcal{F}} |F \cap X_1|!(n_1 - |F \cap X_1|)!|F \cap X_2|!&(n_2 - |F \cap X_2|)! \binom{n_1}{|F \cap X_1|}\binom{n_2}{|F \cap X_2|} \\
&= |\mathcal{F}| n_1! n_2!.
\end{aligned}$$

Thus equality holds here, i.e., $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$, and so it does for each $|\mathcal{H}(M(\pi_1, \pi_2))|$. It also follows that for each (π_1, π_2), the matrix $M(\pi_1, \pi_2)$ has a 1 in each column and the mapping defined by $M(\pi_1, \pi_2)$ is well-paired. This can be heuristically expressed by saying that the restrictions of $\mathcal{F}$ for a fixed pair of permutations (of X_1 and X_2) is full and well-paired. We have to show that $\mathcal{F}$ is homogeneous, too. In other words, we know that the matrices $M(\pi_1, \pi_2)$ are very similar (there is a little freedom in choosing a 1 in each column), but we have to show that they are identical. Since every permutation can be obtained by interchanging neighboring elements, it is sufficient to show that $M(\pi_1', \pi_2)$ and $M(\pi_1, \pi_2')$ are the same as $M(\pi_1, \pi_2)$ if π_i' is obtained from π_i by interchanging two neighboring elements.

First check what happens if π_2' is obtained from π_2 by interchanging the elements v and $v + 1$ in X_2 ($1 \leq v < n_2$). The initial segments in X_2 are the same for the two permutations π_2 and π_2', except possibly the v-element initial segments. Therefore the new matrices $M = M(\pi_1, \pi_2)$ and $M' = M(\pi_1, \pi_2')$ have the same columns, except eventually the vth one. Since M and M' are full, there are indices u and u' such that $M_{u,v} = 1$ and $M'_{u',v} = 1$. We claim that $u = u'$; the two matrices are identical. Indeed, calculating the cardinalities $|\mathcal{H}(M(\pi_1, \pi_2))|$ and $|\mathcal{H}(M(\pi_1, \pi'_2))|$, both have maximal values. They differ only in the one term, the one containing the factor $\binom{n_2}{v}$. This is multiplied with $\binom{n_1}{u}$ and $\binom{n_1}{u'}$, respectively. Therefore $\binom{n_1}{u} = \binom{n_1}{u'}$ must hold. Hence either $u = u'$ (and we are done) or $u + u' = n_1$. In the latter case consider again the sums

$$\sum_{M_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j} = \sum_{M'_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j}.$$

In the second sum there is no $\binom{n_1}{u}$, and in the first there is no $\binom{n_1}{n_1-u}$. By symmetry, $u < n_1 - u$ can be supposed. By the lemmas, the first sum contains the largest $n_2 + 1$ values of binomial coefficients of rank n_1; this implies that none of $\binom{n_1}{i}(i < u, n_1 - u \leq i$ may occur. On the other hand, all other ones are there: $u \leq i < n_1 - u$. Since the matrix is full, it contains a 1 in each column, and we have $n_1 - 2u = n_2 + 1$ binomial coefficients of rank n_1. The smallest one of them is $\binom{n_1}{u}$. M is well-paired; therefore it must be paired (multiplied) with (one of the) smallest binomial coefficient of rank n_2, namely, $\binom{n_2}{0}$ or $\binom{n_2}{n_2}$. Hence we have $v = 0$ or n_2 in contradiction with the assumption $1 \leq v < n_2$.

Compare now the pairs of permutations (π_1, π_2) and (π'_1, π_2), where π'_1 is obtained from π_1 by interchanging the elements u and $u + 1$ in $X_1 (1 \leq u < n_1)$. The matrices $M(\pi_1, \pi_2)$ and $M(\pi'_1, \pi_2)$ are equal except possibly in the uth row. Suppose that both of them have an entry 1 in the uth row and in the vth and in the v'th columns, respectively, where $v \neq v'$. The matrix $M(\pi_1, \pi_2)$ has exactly one 1 in each column, and there is an entry $M_{i,j} = 1$ with $i \neq u, j = v'$. Then $M(\pi'_1, \pi_2)$ has two entries 1 in the v'th column. This contradiction shows $v = v'$; that is, the two matrices are identical. If neither of the two matrices has a 1 in the uth row, then they are the same, again. Finally, if one has a 1 in the uth row and the other one has none, then the sums $\mathcal{H}(M(\pi_1, \pi_2))$ and $\mathcal{H}(M(\pi'_1, \pi_2))$ differ in one positive term; they cannot be (maximally) equal. This contradiction completes the proof of the fact that one change in either permutation does not change the matrix $M(\pi_1, \pi_2)$; they are all the same, and $\mathcal{F}$ is a homogeneous family. $\square$

The interested reader can find further applications of the permutation method in the excellent monograph [2].

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