Representation of functionals of Ito processes and their first exit times *†

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Abstract

The representation theorem is obtained for functionals of non-Markov processes and their first exit times from bounded domains. These functionals are represented via solutions of backward parabolic Ito equations. As an example of applications, analogs of forward Kolmogorov equations are derived for conditional probability density functions of Ito processes killed on the boundary. In addition, a maximum principle and a contraction property are established for SPDEs in bounded domains.

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1 Introduction

In the present paper, we study representation of integrals of stochastic non-Markov processes and their first exit times via stochastic partial differential equations. It is a generalization of the classical Kolmogorov representation for Markov diffusion processes.

Let a region $D \subset \mathbf{R}^n$ be given, let $T > 0$ be a terminal time, let $\mathcal{F}_t$ be a filtration, and let $y^{x,s}(t)$ be an Ito process adapted to $\mathcal{F}_t$ and such that $y^{x,s}(s) = x$, $x \in D$, $s < T$. Further, let $\tau^{x,s}$ be the first exit time from $D \times [0, T)$ for the vector $(y^{x,s}(t), t)$, and let Ψ and ξ be some functions.

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Our goal is to represent conditional expectations

$$\widetilde{p}(x,s,\omega) \triangleq \mathbf{E}\Big\{\Psi(y^{x,s}(T))\mathbb{I}_{\{T\le\tau^{x,s}\}}\,|\,\mathcal{F}_s\Big\} + \mathbf{E}\bigg\{\int_s^{\tau^{x,s}} \xi(y^{x,s}(t),t,\omega)\,dt\,|\,\mathcal{F}_s\bigg\} \qquad (1.1)$$

as the solutions of boundary value problems for stochastic partial differential equations. This representation has many important applications. In particular, the representation via solution of a SPDE helps to establish some regularity properties for $\widetilde{p}$ and $\tau^{x,s}$, since there is certain regularity for the solutions of SPDEs.

For the representation, we will use backward parabolic Ito equations, i.e., the equations with Cauchy condition at terminal time $t = T$. These equations are analogs of Kolmogorov backward equations for non-Markov processes. We will also consider forward parabolic Ito equations, i.e., the equations with Cauchy condition at initial time; they can be regarded as analogs of forward Kolmogorov equations.

Boundary value problems for forward parabolic Ito equations were intensively studied; see, e.g., Alós et al (1999), Bally *et al* (1994), Chojnowska-Michalik and Goldys (1995), Da Prato and Tubaro (1996), Gyöngy (1998), Kim (2004), Krylov (1999), Maslowski (1995), Pardoux (1993), Rozovskii (1990), Walsh (1986), Zhou (1992), the author's papers (1995), (2005), and the bibliography there. Note that the difference between backward and forward equations is not that important for the deterministic equations because one can always make a change of time variable and convert a backward equation to a forward one and opposite. But it cannot be done so easily for stochastic equations, because the solution needs to be adapted to the driving Brownian motion. Therefore, backward stochastic partial differential equations with boundary conditions at final time require special consideration. A possible approach is to consider so-called Ito-Bismut backward equations when the diffusion term is not given a priori but has to be found. These backward SPDEs were also widely studied; see, e.g., Pardoux and Peng (1990), Hu and Peng (1991), Dokuchaev (1992), (2003),(2010), Yong and Zhou (1999), Pardoux and Rascanu (1998), Ma and Yong (1999), Hu *et al* (2002), Confortola (2007), and references here. The duality between linear forward and backward equations was studied by Zhou (1992) for a domain without boundary, and by the author (1992) for the domains with boundaries. A different type of backward equations was described in Chapter 5 of Rozovskii (1990).

The representation of expectations (1.1) via SPDEs was established before for the following cases:

- For the classical Markovian setting then $y^{x,s}(t)$ is a diffusion Markov processes;
- For the case of non-Markov $y^{x,s}(t)$ in the entire space, i.e., when $D = \mathbf{R}^n$, i.e., for the problem without random first exit times.

The known representation theorems for non-Markov processes in $D = \mathbf{R}^n$ was never extended on the case of domains with boundary. Let us explain why it is non-trivial.

The main difficulty in the implementation of this approach to the non-Markov Ito processes and the related SPDEs is the following. One needs again a priori certain smoothness for the solution $p(\cdot)$ of a backward SPDE, to apply Ito-Ventsell formula for the process $p(y^{x,s}(t,\omega), t, \omega)$. However, the previously known results about regularity of the solution of the backward SPDE for p were insufficient for the case of domains with boundary. Therefore, the representation result was never obtained for this case. Correspondingly, it was unknown if the forward parabolic Ito equation for the conditional density of a non-Markov process in the entire space can be used for the process killed on the boundary, given additional Dirichlet boundary value condition on this boundary. As far as we know, the first attempt to solve it was made in the author's paper (1992) for a very special case. In the present paper, we have proved this fact together with representation (1.1) for some p derived from a backward parabolic Ito equation (Theorems 4.1 and Theorem 6.1).

The present paper uses the additional regularity in the form of the so-called second fundamental inequality (Theorem 3.4): the solution (p, χ) of the backward equation has L_2-integrable second derivatives for p and the first derivatives for χ. This additional regularity of the solutions of the backward equations appears to be sufficient to obtain the representation theorem. To ensure this regularity, we required additional Condition 3.5 which is a strengthened version of the standard coercivity condition (Condition 3.1). We emphasize that, without this new condition, representation theorem for (1.1) is still not established, and an equation for the probability density function of the Ito process bein killed on the boundary is still unknown (even if it easy to believe that one can use the SPDE for the density from the case of entire domain with additional the Dirichlet condition imposed on the boundary).

As a corollary, we obtained the equation for the conditional probability density function of an Ito process killed on the boundary of a domain (Theorem 6.1). This is a new result even given that the corresponding result for entire domain was known for a long time (see, e.g., Theorem 5.3.1 from Rozovskii (1990)). As an additional corollary, we obtained the "maximum principle": the solution of the forward or backward equation in the cylinder $D \times [0, T]$ is nonnegative if the free terms are nonnegative. Further, we proved that the dynamic of the homogeneous equations is of the contraction type: $\mathbf{E} \int_D |u(x, T, \omega)| dx \le \mathbf{E} \int_D |u(x, 0, \omega)| dx$ for the solutions of the forward equations, and $\operatorname{ess\,sup}_{x,\omega} |p(x, t, \omega)| \le \operatorname{ess\,sup}_{x,\omega} |p(x, T, \omega)|$ for the solutions of the backward equations. (Theorems 7.1-7.4).

The paper is organized as follows. In Section two we collect notation and definitions. Sections three contains some facts about the regularity of SPDEs, including the second fundamental inequality for backward equations. In Section four, the main result is presented. The proof of this result is given in Section five. Sections six and seven contain applications.

2 Definitions

2.1 Spaces and classes of functions.

We a given an open domain $D \subseteq \mathbf{R}^n$ such that either $D = \mathbf{R}^n$ or D is bounded with $C^{2+\alpha}$-smooth boundary ∂D for some $\alpha > 0$; if $n = 1$, then the condition of smoothness is not required. Let $T > 0$ be given, and let $Q \triangleq D \times (0, T)$.

We are given a standard complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a right-continuous filtration $\mathcal{F}_t$ of complete σ-algebras of events, $t \geq 0$; we denote by ω the elements of the set $\Omega = \{\omega\}$. We are also given a N-dimensional process $w(t) = (w_1(t), ..., w_N(t))$ with independent components such that it is a Wiener process with respect to $\mathcal{F}_t$.

We denote by $\|\cdot\|_X$ the norm in a linear normed space X, and $(\cdot, \cdot)_X$ denotes the scalar product in a Hilbert space X.

We denote Euclidean norm in $\mathbf{R}^k$ as $|\cdot|$, and $\bar{G}$ denotes the closure of a region $G \subset \mathbf{R}^k$.

We introduce some spaces of real valued functions.

We denote by $W_q^m(D)$ the Sobolev space of functions that belong to $L_q(D)$ together with first m derivatives, $q \geq 1$. In particular,

$$\|u\|_{W_2^1(D)} \triangleq \left(\|u\|_{L_2(D)}^2 + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L_2(D)}^2 \right)^{1/2}.$$

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq \overset{0}{W_2^1}(D)$ be the closure in the $W_2^1(D)$-norm of the set of all smooth functions $u : D \to \mathbf{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W_2^2(D) \cap H^1$ be the space equipped with the norm of $W_2^2(D)$. The spaces H^k and $W_2^k(D)$ are called Sobolev spaces; they are Hilbert spaces, and H^k is a closed subspace of $W_2^k(D)$, $k = 0, 1, 2$.

Let H^{-1} be the dual space to H^1, with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-k}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^0$ such that $\|v\|_{H^1} \leq 1$. H^{-k} is a Hilbert space.

We denote by ℓ_k and $\bar{\ell}_k$ the Borel measure and the Lebesgue measure in $\mathbf{R}^k$ respectively, and we denote by $\mathcal{B}_k$ the σ-algebra of Borel sets in $\mathbf{R}^k$. We denote by $\bar{\mathcal{B}}_k$ the completion of $\mathcal{B}_k$ with respect to the measure ℓ_k, or the σ-algebra of Lebesgue sets in $\mathbf{R}^k$.

We denote by $\bar{\mathcal{P}}$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathbf{P}$) of the σ-algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to $\mathcal{F}_t$.

Let $Q_s \triangleq D \times [s,T]$. For $k=-1,0,1,2$, we introduce spaces

$$X^k(s,T) \triangleq L^2([s,T]\times\Omega,\bar{\mathcal{P}},\bar{\ell}_1\times\mathbf{P};H^k),\quad Z_t^k \triangleq L^2(\Omega,\mathcal{F}_t,\mathbf{P};H^k),\quad \mathcal{C}^k(s,T)\triangleq C([s,T];Z_T^k).$$

The spaces X^k and Z_t^k are Hilbert spaces.

Further, we introduce spaces

$$Y^k(s,T) \triangleq X^k(s,T)\cap \mathcal{C}^{k-1}(s,T),\quad k\ge 0,$$

with the norm $\|u\|_{Y^k(s,T)} \triangleq \|u\|_{X^k(s,T)} + \|u\|_{\mathcal{C}^{k-1}(s,T)}$.

For brevity, we will use the notations $X^k \triangleq X^k(0,T)$, $\mathcal{C}^k \triangleq \mathcal{C}^k(0,T)$, and $Y^k \triangleq Y^k(0,T)$.

In addition, we will be using spaces

$$\mathcal{Z}_c^k \triangleq L_2(\Omega,\mathcal{F}_T,\mathbf{P};C^k(D)),\quad \mathcal{X}_c^k = L^2([0,T]\times\Omega,\,\bar{\mathcal{P}},\bar{\ell}_1\times\mathbf{P};\;C^k(\bar{D})),\quad k\ge 0,$$
$$\mathcal{W}_p^k \triangleq L^\infty([0,T]\times\Omega,\overline{\mathcal{P}},\bar{\ell}_1\times\mathbf{P};\;W_p^k(D)),\quad k=0,1,\ldots,\quad 1\le p\le+\infty.$$

The same notations will be used for the spaces of vector and matrix functions, meaning that all components belong to the corresponding spaces. In particular, $\|\cdot\|_{\mathcal{W}_p^k}$ means the sum of all this norms for all components.

We will write $(u,v)_{H^0}$ for $u\in H^{-1}$ and $v\in H^1$, meaning the obvious extension of the bilinear form from $u\in H^0$ and $v\in H^1$. Similarly, we will write $(\xi,\eta)_{X^0}$ for $\xi\in X^{-1}$ and $\eta\in X^1$.

Proposition 2.1 *Let $\xi\in X^0$, let a sequence $\{\xi_k\}_{k=1}^{+\infty}\subset L^\infty([0,T]\times\Omega,\ell_1\times\mathbf{P};\,C(\bar{D}))$ be such that all $\xi_k(\cdot,t,\omega)$ are progressively measurable with respect to $\mathcal{F}_t$, and let $\|\xi-\xi_k\|_{X^0}\to 0$. Let $t\in[0,T]$ and $j\in\{1,\ldots,N\}$ be given. Then the sequence of integrals $\int_0^t \xi_k(x,s,\omega)\,dw_j(s)$ converges in Z_t^0 as $k\to\infty$, and its limit depends on ξ, but does not depend on $\{\xi_k\}$.*

Proof follows from completeness of X^0 and from the equality

$$\mathbf{E}\int_0^t \|\xi_k(\cdot,s,\omega)-\xi_m(\cdot,s,\omega)\|_{H^0}^2\,ds = \int_D dx\,\mathbf{E}\left(\int_0^t (\xi_k(x,s,\omega)-\xi_m(x,s,\omega))\,dw_j(s)\right)^2.$$

Definition 2.1 For $\xi\in X^0$, $t\in[0,T]$, and $j\in\{1,\ldots,N\}$, we define $\int_0^t \xi(x,s,\omega)\,dw_j(s)$ as the limit in Z_t^0 as $k\to\infty$ of a sequence $\int_0^t \xi_k(x,s,\omega)\,dw_j(s)$, where the sequence $\{\xi_k\}$ is such as in Proposition 2.1.

Sometimes we will omit ω.

3 Forward and backward SPDEs

In this section, we collect some known fact for SPDEs.

3.1 Forward SPDEs

Let $s \in [0,T)$, $\varphi \in X^{-1}$, $h_i \in X^0$, and $\Phi \in Z_s^0$. Consider the boundary value problem

$$\begin{aligned} & d_t u = (\mathcal{A}u + \varphi)\, dt + \textstyle\sum_{i=1}^N [B_i u + h_i] dw_i(t), \quad t \geq s, \\ & u|_{t=s} = \Phi, \qquad u(x,t,\omega)|_{x \in \partial D} = 0. \end{aligned} \tag{3.1}$$

Here $u = u(x,t,\omega)$, $(x,t) \in Q$, $\omega \in \Omega$, and

$$\mathcal{A}v \triangleq \sum_{i,j=1}^n b_{ij}(x,t,\omega) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n f_i(x,t,\omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x,t,\omega) v(x), \tag{3.2}$$

where b_{ij}, f_i, x_i are the components of b,f, and x. Further,

$$B_i v \triangleq \frac{dv}{dx}(x)\, \beta_i(x,t,\omega) + \bar\beta_i(x,t,\omega)\, v(x), \quad i = 1, \ldots, N. \tag{3.3}$$

We assume that the functions $b(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}^{n \times n}$, $\beta_j(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}^n$, $\bar\beta_i(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}$, $f(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}^n$, $\lambda(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}$ and $\varphi(x,t,\omega) : \mathbf{R}^n \times [0,T] \times \Omega \to \mathbf{R}$ are progressively measurable for any $x \in \mathbf{R}^n$ with respect to $\mathcal{F}_t$.

To proceed further, we assume that Conditions 3.1-3.3 remain in force throughout this paper.

Condition 3.1 *The matrix $b = b^\top$ is symmetric, bounded, and progressively measurable with respect to $\mathcal{F}_t$ for all x, and there exists a constant $\delta > 0$ such that*

$$y^\top b(x,t,\omega)\, y - \frac{1}{2} \sum_{i=1}^N |y^\top \beta_i(x,t,\omega)|^2 \geq \delta |y|^2 \quad \forall\, y \in \mathbf{R}^n,\ (x,t) \in D \times [0,T],\ \omega \in \Omega. \tag{3.4}$$

Inequality (3.4) is called sometimes a coercivity condition; it means that equation (3.1) is *superparabolic*, in terminology of Rozovskii (1990).

Condition 3.2 *The functions $b(x,t,\omega) : \mathbf{R}^n \times \mathbf{R} \times \Omega \to \mathbf{R}^{n \times n}$, $f(x,t,\omega) : \mathbf{R}^n \times \mathbf{R} \times \Omega \to \mathbf{R}^n$, $\lambda(x,t,\omega) : \mathbf{R}^n \times \mathbf{R} \times \Omega \to \mathbf{R}$, are bounded and differentiable in x, and*

$$\operatorname*{ess\,sup}_{(x,t,\omega) \in Q} \left[\left| \frac{\partial b}{\partial x}(x,t,\omega) \right| + \left| \frac{\partial f}{\partial x}(x,t,\omega) \right| + \left| \frac{\partial \lambda}{\partial x}(x,t,\omega) \right| \right] < +\infty.$$

Condition 3.3 *The functions $\beta_i(x,t,\omega)$ and $\bar\beta_i(x,t,\omega)$ are bounded and differentiable in x, and* $\operatorname{ess\,sup}_{x,t,\omega} |\frac{\partial \beta_i}{\partial x}(x,t,\omega)| < +\infty$, $\operatorname{ess\,sup}_{x,t,\omega} |\frac{\partial \bar\beta_i}{\partial x}(x,t,\omega)| < +\infty$, $i = 1, \ldots, N$.

We introduce the set of parameters

$$\begin{aligned} \mathcal{P}_1 \triangleq \Bigg(& n,\ D,\ T,\ \ \delta,\ \ \operatorname{ess\,sup}_{x,t,\omega} \left[|b(x,t,\omega)| + |f(x,t,\omega)| + \left| \tfrac{\partial b}{\partial x}(x,t,\omega) \right| + \left| \tfrac{\partial f}{\partial x}(x,t,\omega) \right| \right], \\ & \operatorname{ess\,sup}_{x,t,\omega,i} \left[|\beta_i(x,t,\omega)| + |\bar\beta_i(x,t,\omega)| + \left| \tfrac{\partial \beta_i}{\partial x}(x,t,\omega) \right| + \left| \tfrac{\partial \bar\beta_i}{\partial x}(x,t,\omega) \right| \right] \Bigg). \end{aligned}$$

The definition of solution

Definition 3.1 Let $h_i \in X^0$ and $\varphi \in X^{-1}$. We say that equations (3.1) are satisfied for $u \in Y^1$ if

$$\begin{aligned} &u(\cdot,t,\omega) - u(\cdot,r,\omega) \\ &\quad = \int_r^t (\mathcal{A}u(\cdot,s,\omega) + \varphi(\cdot,s,\omega))\,ds + \sum_{i=1}^N \int_r^t [B_i u(\cdot,s,\omega) + h_i(\cdot,s,\omega)]\,dw_i(s) \end{aligned} \tag{3.5}$$

for all r, t such that $0 \le r < t \le T$, and this equality is satisfied as an equality in Z_T^{-1}.

Note that the condition on ∂D is satisfied in the following sense: $u(\cdot,t,\omega) \in H^1$ for a.e. t, ω. Further, the value of $u(\cdot,t,\omega)$ is continuous in t in Z_T^0 and uniquely defined in Z_T^0 given t, by the definitions of the space Y^1. The stochastic integrals with dw_i in (3.5) are defined as elements of Z_T^0. For an arbitrary process $u \in Y^1$, the integral with ds is defined as an element of Z_T^{-1}. However, $u \in Y^1$ presented in Definition 3.1 is such that this integral is equal to an element of Z_T^0 in the sense of equality in Z_T^{-1}.

Existence and regularity for forward SPDEs

Typically, existence and uniqueness results at different spaces for linear PDEs are based on so-called prior estimates, when a norm of the solution is estimated via a norm of the free term. For the second order equations, there are two important estimates based on L_2-norm:
so-called "the first energy inequality" or "the first fundamental inequality", and "the second energy inequality", or "the second fundamental inequality" (Ladyzhenskaya (1985)). For instance, consider a boundary value problem for the heat equation

$$\begin{aligned} &u'_t = u''_{xx} + \varphi, \quad \varphi = f'_x + g, \\ &u|_{t=0} = 0, u|_{\partial D} = 0, \quad (x,t) \in Q = D \times [0,T], \quad D \subset \mathbf{R}. \end{aligned}$$

Then the first fundamental inequality is the estimate

$$\|u\|^2_{L_2(Q)} + \|u'_x\|^2_{L_2(Q)} \le \mathrm{const}\,(\|f\|^2_{L_2(Q)} + \|g\|^2_{L_2(Q)}).$$

Respectively, the second fundamental inequality is the estimate

$$\|u'_t\|^2_{L_2(Q)} + \|u\|^2_{L_2(Q)} + \|u'_x\|^2_{L_2(Q)} + \|u''_{xx}\|^2_{L_2(Q)} \le \mathrm{const}\,\|\varphi\|^2_{L_2(Q)}.$$

The second fundamental inequality leads to existence theorem in the class of functions u such that $u''_{xx} \in L_2(Q)$. The first fundamental inequality allows more general free terms but leads to existence theorem in the class of functions u with generalized derivatives $u''_{xx} \in H^{-1}$ only.

An analog of the first and the second fundamental inequality for the forward SPDEs is given by the following two theorems.

Theorem 3.1 *[Rozovskii (1990), Ch. 3.4.1] Assume that Conditions 3.1, 3.2, and 3.3, are satisfied. Let* $\varphi \in X^{-1}(s,T)$*,* $h_i \in X^0(s,T)$*, and* $\Phi \in Z_s^0$*. Then problem (3.1) has an unique solution* u *in the class* $Y^1(s,T)$ *and the following analog of the first fundamental inequality is satisfied:*

$$\|u\|_{Y^1(s,T)} \le c\left(\|\varphi\|_{X^{-1}(s,T)} + \|\Phi\|_{Z_s^0} + \sum_{i=1}^N \|h_i\|_{X^0(s,T)}\right), \tag{3.6}$$

where $c = c(\mathcal{P}_1)$ *is a constant that depends on* $\mathcal{P}_1$ *only.*

Theorem 3.2 *[Dokuchaev (2005)] Assume that Conditions 3.1, 3.2, and 3.3, are satisfied. In addition, assume that* $\beta_i(x,t,\omega) = 0$ *for* $x \in \partial D$*,* $i = 1, ..., N$*.*
Let $\varphi \in X^0$*,* $h_i \in X^1$*, and* $\Phi \in Z_0^1$*. Then problem (3.1) has an unique solution* $u \in Y^2$ *and the following analog of the second fundamental inequality is satisfied:*

$$\|u\|_{Y^2} \le c\left(\|\varphi\|_{X^0} + \|\Phi\|_{Z_0^1} + \sum_{i=1}^N \|h_i\|_{X^1}\right), \tag{3.7}$$

where $c = c(\mathcal{P}_1)$ *is a constant that depends on* $\mathcal{P}_1$ *only.*

Introduce operators $L(s,T) : X^{-1}(s,T) \to Y^1(s,T)$, $\mathcal{M}_i(s,T) : X^0(s,T) \to Y^1(s,T)$, and $\mathcal{L}(s,T) : Z_s^0 \to Y^1(s,T)$, such that

$$u = L(s,T)\varphi + \mathcal{L}(s,T)\Phi + \sum_{i=1}^N \mathcal{M}_i(s,T)h_i,$$

where u is the solution in $Y^1(s,T)$ of problem (3.1). These operators are linear and continuous; it follows immediately from Theorem 3.1. We will denote by L, $\mathcal{M}_i$, and $\mathcal{L}$, the operators $L(0,T)$, $\mathcal{M}_i(0,T)$, and $\mathcal{L}(0,T)$, correspondingly.

3.2 Backward SPDEs

Introduce the operators being formally adjoint to the operators $\mathcal{A}$ and B_i:

$$\begin{aligned}
\mathcal{A}^* v &= \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}\Big(b_{ij}(x,t,\omega)\, v(x)\Big) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(f_i(x,t,\omega)\, v(x)\right) + \lambda(x,u,t,\omega)\, v(x), \\
B_i^* v &= -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\beta_i(x,t,\omega)\, v(x)\right) + \bar\beta_i(x,t,\omega)\, v(x).
\end{aligned}$$

Consider the boundary value problem in Q

$$\begin{aligned}
&d_t p + \Big(\mathcal{A}^* p + \sum_{i=1}^N B_i^* \chi_i + \xi\Big)\, dt = \sum_{i=1}^N \chi_i\, dw_i(t), \\
&p|_{t=T} = \Psi, \quad p(x,t,\omega)\,|_{x\in\partial D} = 0.
\end{aligned} \tag{3.8}$$

The definition of solution

Definition 3.2 We say that equation (3.8) is satisfied for $p \in Y^1$, $\xi \in X^{-1}$, $\Psi \in Z_T^0$, $\chi_i \in X^0$ if

$$p(\cdot,t) = \Psi + \int_t^T \left(\mathcal{A}^* p(\cdot,s) + \sum_{i=1}^N B_i^* \chi_i(\cdot,s) + \xi(\cdot,s) \right) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot,s)\, dw_i(s) \tag{3.9}$$

for any $t \in [0,T]$. The equality here is assumed to be an equality in the space Z_T^{-1}.

Existence and regularity for backward SPDEs

For $t \in [0,T]$, define operators $\delta_t : C([0,T]; Z_T^k) \to Z_t^k$ such that $\delta_t u = u(\cdot,t)$.

The following theorem gives an analog of the first fundamental inequality for backward SPDEs. In addition, this theorem establishes duality between forward and backward equations.

Theorem 3.3 *[Dokuchaev (1992,2010)] For any $\xi \in X^{-1}$ and $\Psi \in Z_T^0$, there exists a pair (p,χ), such that $p \in Y^1$, $\chi = (\chi_1,\dots,\chi_N)$, $\chi_i \in X^0$ and (3.8) is satisfied. This pair is uniquely defined, and the following analog of the first fundamental inequality is satisfied:*

$$\|p\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \le c(\|\xi\|_{X^{-1}} + \|\Psi\|_{Z_T^0}), \tag{3.10}$$

where $c = c(\mathcal{P}_1) > 0$ is a constant that depends on $\mathcal{P}_1$ only. Furthermore, the following duality holds between problems (3.8) and (3.1):

$$p = L^*\xi + (\delta_T L)^*\Psi, \quad \chi_i = \mathcal{M}_i^*\xi + (\delta_T \mathcal{M}_i)^*\Psi, \quad p(\cdot,0) = \mathcal{L}^*\xi + (\delta_T \mathcal{L})^*\Psi,$$

where $L^* : X^{-1} \to X^1$, $\mathcal{M}_i^* : X^0 \to X^0$, $(\delta_T L)^* : Z_0^0 \to X^1$, $(\delta_T \mathcal{M}_i)^* : Z_0^0 \to X^0$, and $(\delta_T \mathcal{L})^* : Z_T^0 \to Z_0^0$, are the operators that are adjoint to the operators $L : X^{-1} \to X^1$, $\mathcal{M}_i : X^0 \to X^1$, $\delta_T \mathcal{M}_i : X^{-1} \to Z_T^0$, $\delta_T \mathcal{M}_i : X^0 \to Z_T^0$, and $\delta_T \mathcal{L} : Z_0^0 \to Z_T^0$, respectively.

We will need an analog of the second fundamental inequality as well.

Starting from now, we assume that the following addition conditions are satisfied.

Condition 3.4 *There exist functions $\widehat{f}(x,t,\omega) : \mathbf{R}^n \times \mathbf{R}_+ \times \Omega \to \mathbf{R}^n$, $\widehat{\lambda}(x,t,\omega) : \mathbf{R}^n \times \mathbf{R}_+ \times \Omega \to \mathbf{R}$, and $\widehat{\beta}_i(x,t,\omega) : \mathbf{R}^n \times \mathbf{R}_+ \times \Omega \to \mathbf{R}$, such that*

$$\operatorname*{ess\,sup}_{x,t,\omega} \Big(|\widehat{f}(x,t,\omega)| + |\widehat{\lambda}(x,t,\omega)| + |\widehat{\beta}_i(x,t,\omega)| \Big) < +\infty,$$

and

$$\begin{aligned} \mathcal{A}^* p &= \sum_{i,j=1}^n b_{ij}(x,t,\omega) \frac{\partial^2 p}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n \hat{f}_i(x,t,\omega) \frac{\partial p}{\partial x_i}(x) - \widehat{\lambda}(x,t,\omega)\, p(x), \\ B_i^* p &= \frac{dp}{dx}(x)\, \beta_i(x,t,\omega) + \widehat{\beta}_i(x,t,\omega)\, p(x). \end{aligned}$$

Clearly, this condition is satisfied if the function $b(x,t,\omega):\mathbf{R}^n\times\mathbf{R}\times\Omega\to\mathbf{R}^{n\times n}$ is twice differentiable in x, and

$$\operatorname*{ess\,sup}_{\omega}\sup_{(x,t)\in Q}\left|\frac{\partial^2 b}{\partial x_k\partial x_m}(x,t,\omega)\right|<+\infty.$$

For an integer $M>0$, let $\Theta_b(M)$ denotes the class of all matrix functions b such that all conditions imposed in Section 3.1 are satisfied, and there exists a set $\{T_i\}_{i=0}^M$ such that $0=T_0<T_1<\cdots<T_M=T$ and that the function $b(x,t,\omega)=b(x,\omega)$ does not depend on t for $t\in[T_i,T_{i+1})$. (it follows from the assumptions that $b(x,t,\cdot)$ is $\mathcal{F}_{T_i}$-measurable for all $x\in D$, $t\in[T_i,T_{i+1})$).

Let $\Theta_b\triangleq\cup_{M>0}\Theta_b(M)$.

Let $\bar\Theta_b$ denotes the class of function b from such that all conditions imposed in Section 3.1 are satisfied, and there exists and a sequence $\{b^{(i)}\}_{i=1}^{+\infty}\subset\Theta_b$ such that $\|b-b^{(i)}\|_{\mathcal{W}^1_\infty}\to 0$ as $i\to+\infty$. (Remind that the assumptions on b are such that $b\in\mathcal{W}^1_\infty$).

Condition 3.5 *The matrix b belongs to $\bar\Theta_b$, and there exists a constant $\delta_1>0$ such that*

$$\sum_{i=1}^N y_i^\top b(x,t,\omega)\,y_i-\frac{1}{2}\left(\sum_{i=1}^N y_i^\top\beta_i(x,t,\omega)\right)^2\geq\delta_1\sum_{i=1}^N|y_i|^2$$
$$\forall\,\{y_i\}_{i=1}^N\subset\mathbf{R}^n,\ \ (x,t)\in D\times[0,T],\ \ \omega\in\Omega.\qquad(3.11)$$

Remark 3.1 *If Condition 3.5 holds, then Condition 3.1 holds. If $n=1$ and Condition 3.1 holds, then the estimate in Condition 3.5 also holds. If $n>1$, then it can happen that Condition 3.1 holds, but the estimate in Condition 3.5 does not hold. For instance, assume that $n=2$, $N=2$, $\beta_1\equiv(1,0)^\top$, $\beta_2=(0,1)^\top$, $b\equiv\frac{1}{2}(\beta_1\beta_1^\top+\beta_2\beta_2^\top)+0.01I_2=0.51I_2$, where I_2 is the unit matrix in $\mathbf{R}^{2\times 2}$. Obviously, Condition 3.1 holds and $b\in\bar\Theta_b$,. On the other hand, Condition 3.5 does not hold for this b; to see this, it suffices to take $y_1=\beta_1$ and $y_2=\beta_2$.*

Remark 3.2 *Condition 3.5 is satisfied for matrices $b\in\bar\Theta_b$ if either $n=1$ or there exists $N_0\in\{1,...,N\}$ such that $\beta_i\equiv 0$ for $i>N_0$, and there exists a constant $\delta_2>0$ such that*

$$y^\top b(x,t,\omega)\,y-\frac{N_0}{2}|y^\top\beta_i(x,t,\omega)|^2\geq\delta_2|y|^2\quad\forall\,y\in\mathbf{R}^n,\ \ (x,t)\in D\times[0,T],\ \ \omega\in\Omega,\ \ i=1,...,N_0.\qquad(3.12)$$

In particular, it is satisfied if Condition 3.1 holds and $N_0=1$.

To proceed further, we assume that Conditions 3.4- 3.5 remain in force starting from here and up to the end of this paper, as well as the previously formulated conditions.

Let $\mathcal{P}\triangleq(\mathcal{P}_1,\delta_1)$.

We will be using the following analog of the second fundamental inequality for backward SPDEs.

Theorem 3.4 *[Dokuchaev (2006)] For any $\xi \in X^0$ and $\Psi \in Z_T^1$, there exists a pair (p, χ), such that $p \in Y^2$, $\chi = (\chi_1, \ldots, \chi_N)$, $\chi_i \in X^1$ and (3.8) is satisfied. This pair is uniquely defined, and*

$$p = L^*\xi + (\delta_T L)^*\Psi, \quad \chi_i = \mathcal{M}_i^*\xi + (\delta_T \mathcal{M}_i)^*\Psi.$$

The operators $L^* : X^0 \to Y^2$, $(\delta_T L)^* : Z_T^1 \to Y^2$, and $\mathcal{M}_i^* : X^0 \to X^1$, $(\delta_T \mathcal{M}_i)^* : Z_T^1 \to X^1$, are continuous. More precisely, the following analog of the second fundamental inequality holds:

$$\|p\|_{Y^2} + \sum_{i=1}^N \|\chi_i\|_{X^1} \le c(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^1}), \tag{3.13}$$

where $c > 0$ is a constant that depends only on $\mathcal{P}$.

Semi-group property for backward equations

It is known that the dynamic of forward parabolic Ito equation has semi-group property (or causality property): if $u = L\varphi + \mathcal{L}_0\Phi$, where $\varphi \in X^{-1}$, $\Phi \in Z_0^0$, then

$$u|_{t\in[\theta,s]} = (L\varphi + \mathcal{L}_0\Phi)|_{t\in[\theta,s]} = L(\theta, s)\varphi + \mathcal{L}_\theta(\theta, s)u(\cdot, \theta). \tag{3.14}$$

We will need a similar property for the backward equations.

Theorem 3.5 *(Semi-group property for backward equations) [Dokuchaev (2010)]. Let $0 \le \theta < s < T$, and let $p = L^*\xi$, $\chi_i = \mathcal{M}_i\xi$ where $\xi \in X^{-1}$ and $\Psi \in Z_T^0$. Then*

$$p|_{t\in[\theta,s]} = L(\theta, s)^*\xi|_{t\in[\theta,s]} + (\delta_s L(\theta, s))^* p(\cdot, s), \tag{3.15}$$

$$p(\cdot, \theta) = (\delta_s \mathcal{L}_\theta(\theta, s))^* p(\cdot, s) + \mathcal{L}_\theta(\theta, s)^*\xi, \tag{3.16}$$

$$\chi_i|_{t\in[\theta,s]} = \mathcal{M}_i(\theta, s)^*\xi|_{t\in[\theta,s]} + (\delta_s \mathcal{M}_i(\theta, s))^* p(\cdot, s), \quad k = 1, ..., N. \tag{3.17}$$

Some additional regularity

Theorem 3.4 requires that $\Psi \in Z_2^1$. We will need a modification of this theorem that allows $\Psi \in Z_2^0$.

Theorem 3.6 *Let the assumptions of Theorem 3.4 be satisfied. Let $\xi \in X^0$ and $\Psi \in Z_T^0$. Let*

$$p = L^*\xi + (\delta_T L)^*\Psi, \quad \chi_i = \mathcal{M}_i^*\xi + (\delta_T \mathcal{M}_i)^*\Psi.$$

Let $\varepsilon \in (0, T)$ be given. Then

$$\|p\|_{Y^2(0,T-\varepsilon)} + \sum_{i=1}^N \|\chi_i\|_{X^1(0,T-\varepsilon)} \le \frac{c}{\sqrt{\varepsilon}}(\|\xi\|_{X^0} + \|\Psi\|_{Z_T^0}), \tag{3.18}$$

where $c = c(\mathcal{P}) > 0$ is a constant that depends only on and $\mathcal{P}$.

Proof. By Theorem 3.3 and Theorem 3.5, it follows that

$$\|p\|_{Y^2(0,T-\varepsilon)} + \sum_{i=1}^{N} \|\chi_i\|_{X^1(0,T-\varepsilon)} \le c_1(\|\xi\|_{X^0(0,T-\varepsilon)} + \|p(\cdot, T-\varepsilon)\|_{Z_T^1}), \tag{3.19}$$

where $c_1 = c_1(\mathcal{P}) > 0$ is a constant that depends only on $\mathcal{P}$. (Note that the same constant c can be used for all ε, since Theorem 3.6 holds for T replaced by $T-\varepsilon$ with any $\varepsilon \in [0,T)$). In addition, it follows from Theorem 3.4 that

$$\inf_{s\in[T-\varepsilon,T]} \|p(\cdot,s)\|_{Z_T^1}^2 \le \frac{1}{\varepsilon}\int_{T-\varepsilon}^{T} \|p(\cdot,t)\|_{Z_T^1}^2\, dt \le \frac{c_2}{\varepsilon}(\|\xi\|_{X^0}^2 + \|\Psi\|_{Z_T^0}^2),$$

where $c_2 = c_2(\mathcal{P}) > 0$ is a constant that depends only on $\mathcal{P}$. This completes the proof. □

4 The main result: the representation theorem

Let functions $\widetilde{\beta}_i : Q\times\Omega \to \mathbf{R}^n$, $i = 1,\dots,M$, be such that

$$2b(x,t,\omega) = \sum_{i=1}^{N} \beta_i(x,t,\omega)\,\beta_i(x,t,\omega)^\top + \sum_{j=1}^{M} \widetilde{\beta}_j(x,t,\omega)\,\widetilde{\beta}_j(x,t,\omega)^\top,$$

and $\widetilde{\beta}_i$ has the similar properties as β_i. (Note that, by Condition 3.1, $2b > \sum_{i=1}^N \beta_i\beta_i^\top$).

Let $\widetilde{w}(t) = (\widetilde{w}_1(t),\dots,\widetilde{w}_M(t))$ be a new Wiener process independent on $w(t)$.

Let $(x,s) \in \bar{D} \in [0,T]$ be given. Consider the following Ito equation

$$\begin{aligned} &dy(t) = \widetilde{f}(y(t),t)\,dt + \sum_{i=1}^{N} \beta_i(y(t),t)\,dw_i(t) + \sum_{j=1}^{M} \widetilde{\beta}_j(y(t),t)\,d\widetilde{w}_j(t),\\ &y(s) = x, \end{aligned} \tag{4.1}$$

where $\widetilde{f} \triangleq \hat{f} - \sum_{i=1}^N \widehat{\beta}_i\beta_i$.

Let $y(t) = y^{x,s}(t)$ be the solution of (4.1).

Set $\tau^{x,s} \triangleq \min\{t \le T : y^{x,s}(t) \notin D\}$. For $t \ge s$, set

$$\gamma^{x,s}(t) \triangleq \exp\Big[-\int_s^t \widehat{\lambda}(y^{x,s}(t),t)\,dt + \sum_{i=1}^{N}\int_s^t \widehat{\beta}_i(y^{x,s}(s),s)\,dw_i(s) - \sum_{i=1}^{N}\frac{1}{2}\int_s^t \widehat{\beta}_i(y^{x,s}(s),s)^2\,ds\Big].$$

Theorem 4.1 *Let $b \in \mathcal{X}_c^3$, $\widehat{f} \in \mathcal{X}_c^2$, $\widehat{\lambda} \in \mathcal{X}_c^1$, $\beta_i \in \mathcal{X}_c^3$ and $\widehat{\beta}_i \in \mathcal{X}_c^2$. Let $(p,\chi_1,...,\chi_N)$ be the solution of (3.8), where functions $\xi : Q\times\Omega \to \mathbf{R}$ and $\Psi : D\times\Omega$ are such that ξ is $(\mathcal{B}_{n+1}\otimes\mathcal{F},\mathcal{B}_1)$-measurable, Ψ is $(\mathcal{B}_n\otimes\mathcal{F},\mathcal{B}_1)$-measurable, $\xi \in X^0$ and $\Psi \in Z_T^0$. Then for any $s\in[0,T)$,*

$$p(x,s,\omega) = \mathbf{E}\Big\{\gamma^{x,s}(T)\Psi(y^{x,s}(T))\mathbb{I}_{\{T\ge\tau^{x,s}\}}\,|\,\mathcal{F}_s\Big\} + \mathbf{E}\Big\{\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\,\xi(y^{x,s}(t),t,\omega)\,dt\,|\,\mathcal{F}_s\Big\} \tag{4.2}$$

for a.e. x,ω.

Remind that the solution $(p,\chi_1,...,\chi_N)$ of (3.8) can be represented as

$$p = L^*\xi + (\delta_T L)^*\Psi, \qquad \chi_i = \mathcal{M}_i^*\xi + (\delta_T\mathcal{M}_i)^*\Psi, \quad i=1,...,N. \tag{4.3}$$

5 Proof of Theorem 4.1

Let us proof first the following lemma.

Lemma 5.1 *Theorem 4.1 holds even without Condition 3.1 for the case when* $\xi \in \mathcal{X}_c^0$, $\Psi \in \mathcal{Z}_c^0 \cap Z_0^1$, $p \in \mathcal{X}_c^2$, $p(\cdot, T) \in Z_c^0$, $\chi_i \in \mathcal{X}_c^1$, *where* $(p, \chi_1, ..., \chi_N)$ *is the solution of (4.8).*

Proof of Lemma 5.1. Let (x, s) be given, and let $y(t) = y^{x,s}(t)$ and $\gamma(t) = \gamma^{x,s}(t)$. We have that

$$d_t p = \mathcal{J}(p)\, dt + \sum_{i=1}^N \chi_i\, dw_i(t),$$

where

$$\mathcal{J}(p) \triangleq -\mathcal{A}^* p - \sum_{i=1}^N B_i^* \chi_i - \xi.$$

Let $\psi(t) \triangleq p(y(t, \omega), t, \omega)$.

By the Ito-Ventssel formula (see, e.g., Rozovskii (1990), Chapter 1),

$$d\psi(t) = h(y(t), t)dt + \sum_{i=1}^N \chi_i(y(t), t)\, dw_i(t) + \sum_{i=1}^N \left(\frac{\partial p}{\partial x}\beta_i\right)(y(t), t)\, dw_i(t) + \sum_{i=1}^M \left(\frac{\partial p}{\partial x}\widetilde{\beta}_i\right)(y(t), t)\, d\widetilde{w}_i(t),$$

where

$$h = h(y(t), t) = \mathcal{J}(p) + \mathcal{A}^* p + \widehat{\lambda} p - \frac{\partial p}{\partial x} \sum_{i=1}^N \widehat{\beta}_i \beta_i + \sum_{i=1}^N \frac{\partial \chi_i}{\partial x} \beta_i.$$

By (3.11), it can be rewritten as

$$h = -\xi + \widehat{\lambda} p - \frac{\partial p}{\partial x} \sum_{i=1}^N \widehat{\beta}_i \beta_i - \sum_{i=1}^N \widehat{\beta}_i \chi_i.$$

Let $\widehat{\psi}(t) \triangleq \psi(t)\, \gamma(t)$, $t \geq s$. We have that

$$d\gamma(t) = \gamma(t)\Big(-\widehat{\lambda} dt + \sum_{i=1}^N \widehat{\beta}_i(t)\, dw_i(t)\Big).$$

Using Ito formula, we derive that

$$d\widehat{\psi}(t) = -\gamma(t)\, \xi(y(t), t, \omega) + \sum_{i=1}^N \mu_i(t)\, dw_i(t) + \sum_{i=1}^M \widetilde{\mu}_i(t)\, d\widetilde{w}_i(t),$$

where $\mu_i(\cdot)$ and $\widetilde{\mu}_i(\cdot)$ are some L_2–integrable processes such that $\mu_i(t)$ and $\widetilde{\mu}_i(t)$ are independent from $w_j(r) - w_j(t)$ and $\widetilde{w}_k(r) - \widetilde{w}_k(t)$ for all $r > t$, j, k. It follows that

$$\begin{aligned}\mathbf{E}\Big\{\gamma(T)\Psi(y(T))\mathbb{I}_{\{T\le\tau^{x,s}\}}\,|\,\mathcal{F}_s\Big\} - p(x,s,\omega) &= \mathbf{E}\Big\{\,(p(y(\tau^{x,s}),\tau^{x,s},\omega) - p(x,s,\omega))\,|\,\mathcal{F}_s\Big\} \\ &= -\mathbf{E}\Big\{\int_s^{\tau^{x,s}}\gamma(t)\,\xi(y(t),t,\omega)\,dt\,|\,\mathcal{F}_s\Big\}.\end{aligned}$$

Then (4.2) follows. This completes the proof of Lemma 5.1. □

Let us continue the proof of Theorem 4.1, and let us assume first that the functions ξ and Ψ are bounded. In addition, we assume for the case when $D = \mathbf{R}^n$ that there exists a bounded domain $\widehat{D} \subset \mathbf{R}^n$ such that $\xi(x,t,\omega) = 0$ and $\Psi(x,\omega) = 0$ for all $x \notin \widehat{D}$ for all t, ω.

For functions $h \in X^0$, we introduce some transforms h_m, $m = 1, 2, ...$

(a) Let $D \neq \mathbf{R}^n$. In this case, we introduce an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $L_2(D)$ consisting of the eigenfunctions for the eigenvalue problem

$$\Delta v - v = -\lambda v, \quad v|_{\partial D} = 0. \tag{5.1}$$

Here Δ is the Laplacian. It is known that $v \in C^2(\bar{D}) \cap H^2$ (see, e.g., Theorem III.3.2 from Ladyzhenskaya and Ural'tseva (1968)). For a function $h \in X^0$, we denote by h_m the function $h_m \in X^0$ such that $h_m(\cdot,t,\omega)$ is the projection of $h(\cdot,t,\omega)$ on the subspace of $L_2(D)$ generated as the span of the functions $\{v_k\}_{k=1}^{m}$.

(b) Let $D = \mathbf{R}^n$. In this case, for a function $h \in X^0$, we denote by h_m the function $(h)_m(y,t,\omega) \triangleq \int_{\mathbf{R}^n} h(x,t,\omega)J^{(m)}(y-x)dx$ that is the corresponding Sobolev transform. Here $J(x) : \mathbf{R}^n \to \mathbf{R}$ is the Sobolev kernel: $J(x) = 0$ if $|x| \ge 1$, and $J(x) = \exp\{-|x|/(1-|x|)\}$ if $|x| < 1$, and $J^{(m)}(x) = \kappa_n m^n J(mx)$, where $\kappa_n > 0$ is such that $\int_{\mathbf{R}^n} J^{(m)}(x)dx = 1$.

The transform h_m has the following properties:

$$\begin{aligned}&h_m \in \mathcal{X}_c^2 \quad \forall h \in X,\\ &(h_m, g)_{X^0} = (h, g_m)_{X^0}, \quad \forall h, g \in X^0,\\ &\|h_m\|_{X^1} \le c\|h\|_{X^1} \quad \forall h \in X^1,\end{aligned} \tag{5.2}$$

for a constant $c > 0$ that does not depend on h. The first two properties are obvious. For the case when $D = \mathbf{R}^n$, the last property follows from the known properties of the Sobolev transform. It suffices to prove the last property for the case when $D \neq \mathbf{R}^n$. Let $D \neq \mathbf{R}^n$. For any $V \in H^0$, we have that $V = \sum_{k=1}^{\infty} c_k v_k$, where $c_k = (V, v_k)_{H^0}$, meaning the convergence of the series in H^0.

Hence

$$\|V_m\|_{H^1}^2 = (V_m, V_m - \Delta V_m)_{H^0} = \Big(\sum_{k=1}^m c_k v_k, \sum_{k=1}^m c_k v_k - \Delta \sum_{k=1}^m c_k v_k\Big)_{H^0}$$
$$= \Big(\sum_{k=1}^m c_k v_k, \sum_{k=1}^m c_k v_k + \sum_{k=1}^m \lambda_k c_k v_k\Big)_{H^0} = \sum_{k=1}^m |c_k|^2(1+\lambda_k) \le \|V\|_{H^1}^2.$$

Here λ_k are the eigenvalues of problem (5.1) that correspond to the eigenfunctions v_k. It follows that (5.2) holds for $D \neq \mathbf{R}^n$. Therefore, (5.2) holds.

Let $(p, \chi_1, ..., \chi_N) \in Y^1 \times (X^0)^N$ be such that $p = L^*\xi + (\delta_T L_i)^*\Psi$ and $\chi_i = \mathcal{M}_i^*\xi + (\delta_T \mathcal{M}_i)^*\Psi$. By Theorem 3.6, it follows that

$$(p, \chi_1, ..., \chi_N)|_{t\in[0,T-\varepsilon)} \in Y^2(0, T-\varepsilon) \times (X^1(T-\varepsilon))^N \quad \forall \varepsilon > 0. \tag{5.3}$$

In particular, it follows that $\frac{\partial^k p}{\partial x_k^2}(\cdot, t)_{t\in[0,T-\varepsilon)}$, $k = 0, 1, 2$, and $\frac{\partial \chi_i}{\partial x_i}{}_{t\in[0,T-\varepsilon)}$ belong to $X^0(0, T-\varepsilon)$.

We have that

$$d_t p_m + [(\mathcal{A}^* p)_m + \xi_m + \sum_{i=1}^N (B_i^* \chi_i)_m] dt = \sum_{i=1}^N \chi_{im} dw_i(t),$$
$$p_m(x, T, \omega) = \Psi_m(x, \omega), \quad p_m|_{x\in D} = 0.$$

It can be rewritten as

$$d_t p_m + [\mathcal{A}^* p_m + \widehat{\xi}^{(m)} + \sum_{i=1}^N B_i^* \chi_{im}] dt = \sum_{i=1}^N \chi_{im} dw_i(t),$$
$$p_m(x, T, \omega) = \Psi_m(x, \omega), \quad p_m|_{x\in\partial D} = 0.$$

Here

$$\widehat{\xi}^{(m)} \triangleq \xi_m + \eta^{(m)}, \quad \eta^{(m)} \triangleq (\mathcal{A}^* p)_m - \mathcal{A}^* p_m + \sum_{i=1}^N (B_i^* \chi_i)_m - \sum_{i=1}^N B_i^* \chi_{im}.$$

Let us show that

$$\Psi_m \to \Psi \quad \text{in} \quad Z_T^0 \quad \text{as} \quad m \to +\infty. \tag{5.4}$$

Clearly, $\Psi_m(\cdot, \omega) \to \Psi(\cdot, \omega)$ in $L_2(D)$ a.s. In addition, we have that $\|\Psi_m(\cdot, \omega)\|_{L_2(D)} \le \|\Psi(\cdot, \omega)\|_{L_2(D)}$. Hence $\|\Psi_m(, \omega) - \Psi(\cdot, \omega)\|_{L_2(D)} \le 2\|\Psi(, \omega)\|_{L_2(D)}$. By the Lebesgue's Dominated Convergence Theorem, it follows that (5.4) holds. Similarly, we obtain that

$$\xi_m \to \xi \quad \text{in} \quad X^0 \quad \text{as} \quad m \to +\infty. \tag{5.5}$$

Again, we have $\xi_m(\cdot, t, \omega) \to \xi(\cdot, t, \omega)$ in $L_2(D)$ for a.e. (t, ω). In addition, we have that $\|\xi_m(\cdot, t, \omega)\|_{L_2(D)} \le \|\xi(\cdot, t, \omega)\|_{L_2(D)}$ and $\|\xi_m(, t, \omega) - \xi(\cdot, t, \omega)\|_{L_2(D)} \le 2\|\xi(, t, \omega)\|_{L_2(D)}$. By the Lebesgue's Dominated Convergence Theorem again, it follows that (5.5) holds.

Let us show that

$$\widehat{\xi}^{(m)} \triangleq \xi_m + \eta^{(m)} \to \xi \quad \text{weakly in} \quad X^{-1} \quad \text{as } m \to +\infty. \tag{5.6}$$

By (5.5), it suffices to show that

$$\eta^{(m)} \to 0 \quad \text{weakly in} \quad X^{-1} \quad \text{as } m \to 0. \tag{5.7}$$

First, let us show that there exists a constant $c > 0$ such that

$$\|\eta^{(m)}\|_{X^{-1}} \le c \quad \forall m > 0. \tag{5.8}$$

By Theorem 3.3, it follows that $\|p\|_{X^1} \le \mathrm{const}$. Hence $\|p_m\|_{X^1} \le \mathrm{const}$. Hence

$$\|\mathcal{A}^* p_m\|_{X^{-1}} \le \mathrm{const}. \tag{5.9}$$

Further, let $B(X)$ denote the unit ball in a linear normed space X, i.e., $B(X) \triangleq \{x \in X : \|x\|_X \le 1\}$. We have that

$$\begin{aligned}\|(\mathcal{A}^*p)_m\|_{X^{-1}} &= \sup_{y\in B(X^1)} (y, (\mathcal{A}^*p)_m)_{X^0} = \sup_{y\in B(X^1)} (y_m, \mathcal{A}^*p)_{X^0} \le \sup_{y\in B(X^1)} \|\mathcal{A}y_m\|_{X^{-1}}\|p\|_{X^1} \\ &\le c_1 \sup_{y\in B(X^1)} \|y_m\|_{X^1}\|p\|_{X^1} \le c_2 \sup_{y\in B(X^1)} \|y\|_{X^1}\|p\|_{X^1} \le c_3. \end{aligned} \tag{5.10}$$

Here c_k, $k = 1, 2, 3$, are some constants that are independent from m.

Similarly, we have that, by Theorem 3.3, $\|\chi_i\|_{X^0} \le \mathrm{const}$. Hence $\|\chi_{im}\|_{X^0} \le \mathrm{const}$. Hence

$$\|B_i^*\chi_{im}\|_{X^{-1}} \le \mathrm{const}. \tag{5.11}$$

Further, we have that

$$\begin{aligned}\|(B_i^*\chi_i)_m\|_{X^{-1}} &= \sup_{y\in B(X^1)} (y, (B_i^*\chi_i)_m)_{X^0} = \sup_{y\in B(X^1)} (y_m, B_i^*\chi_i)_{X^0} \le \sup_{y\in B(X^1)} \|B_i y_m\|_{X^0}\|\chi_i\|_{X^0} \\ &\le c_1 \sup_{y\in B(X^1)} \|y_m\|_{X^1}\|\chi_i\|_{X^0} \le c_2 \sup_{y\in B(X^1)} \|y\|_{X^1}\|\chi_i\|_{X^0} \le c_3. \end{aligned} \tag{5.12}$$

Here c_k, $k = 1, 2, 3$, are some constant that are independent from m. Combining (5.9)-(5.12), we obtain (5.8).

Let $q = q(x, t, \omega)$ denote any one of the functions p, χ_i, $\partial p/\partial x_k$, $\partial^2 p/\partial x_k \partial x_m$, $\partial \chi_i/\partial x_k$, $k, m = 1, ..., n$, $i = 1, ..., N$, $t < T$. Let α denote the coefficient such that αq is presented in the expressions $\mathcal{A}^*p$ or $B_i^*\chi_i$.

For $\theta \in [0, T)$, let $X^1(\theta) \triangleq \{h \in X^1 : h(\cdot, t) \equiv 0, \quad t \in [\theta, T]\}$.

Let $\theta \in [0, T]$ and let $h \in X^1(\theta)$. It can be shown similarly to (5.5) that

$$\alpha h_m - (\alpha h)_m \to 0 \quad \text{in} \quad X^0 \quad \text{as} \quad m \to +\infty.$$

It follows that

$$((\alpha q)_m - \alpha q_m, h)_{X^0} = ((\alpha q)_m - \alpha q_m, h)_{X^0(0,\theta)} = (q, \alpha h_m - (\alpha h)_m)_{X^0(0,\theta)} \to 0 \quad \text{as } m \to \infty.$$

We have that $\eta^{(m)}$ is a sum of different terms expressed as $(\alpha q)_m - \alpha q_m$. Hence

$$(\eta^{(m)}, h)_{X^0(0,\theta)} = (\eta^{(m)}, h)_{X^0} \to 0 \quad \text{as } m \to +\infty \qquad \forall h \in X^1(\theta).$$

Clearly, the set $\cup_{\theta\in[0,T)} X^1(\theta)$ is dense in X^1. By (5.8), it follows that (5.7) holds. This completes the proof of (5.6).

Let $s \in [0, T)$ be given.

By (5.5), (5.6), and Theorem 3.3, it follows that

$$\widetilde{p}_m(\cdot, s) \to p(\cdot, s) \quad \text{weakly in} \quad Z_T^0 \quad \text{as} \quad m \to 0. \tag{5.13}$$

By Mazur's Theorem (Theorem 5.1.2 from Yosida (1995)), there exists a sequence of integer numbers $k = k_i \to +\infty$ such that there exist sets of real numbers $\{a_{mk}\}_{m=1}^k \subset [0, 1]$ such that $\sum_{m=1}^k a_{mk} = 1$ and that

$$\begin{aligned}
\widetilde{\xi}^{(k)} &\triangleq \sum_{m=1}^{k} a_{mk} \widehat{\xi}^{(m)} \to \xi \quad \text{in} \quad X^{-1} \quad \text{as} \quad k = k_i \to +\infty, \\
\widetilde{\Psi}^{(k)} &\triangleq \sum_{m=1}^{k} a_{mk} \Psi_m \to \Psi \quad \text{in} \quad Z_T^0 \quad \text{as} \quad k = k_i \to +\infty, \\
\widetilde{p}^{(k)}(\cdot, s) &\triangleq \sum_{m=1}^{k} a_{mk} p_m(\cdot, s) \to p(\cdot, s) \quad \text{in} \quad Z_T^0 \quad \text{as} \quad k = k_i \to +\infty.
\end{aligned} \tag{5.14}$$

Here $\widetilde{p}^{(k)} \triangleq \sum_{m=1}^k a_{mk} p_m$.

By Lemma 5.1, it follows that, for all s and for a.e. x, ω,

$$p_m(x, s, \omega) = \mathbf{E}\Big\{\gamma^{x,s}(T)\Psi_m(y^{x,s}(T))\mathbb{I}_{\{T\ge\tau^{x,s}\}} \,|\, \mathcal{F}_s\Big\} + \mathbf{E}\Big\{\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\, \widehat{\xi}^{(m)}(y^{x,s}(t), t, \omega)\, dt \,\Big|\, \mathcal{F}_s\Big\},$$

and

$$\widetilde{p}^{(k)}(x, s, \omega) = \mathbf{E}\Big\{\gamma^{x,s}(T)\Psi^{(k)}(y^{x,s}(T))\mathbb{I}_{\{T\ge\tau^{x,s}\}} \,|\, \mathcal{F}_s\Big\} + \mathbf{E}\Big\{\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\, \widetilde{\xi}^{(k)}(y^{x,s}(t), t, \omega)\, dt \,\Big|\, \mathcal{F}_s\Big\},$$

By the assumptions about the boundedness and the type of measurability of the functions $\xi : Q \times \Omega \to \mathbf{R}$ and $\Psi : D \times \Omega \to \mathbf{R}$, it follows that $\gamma^{x,s}(T)\Psi(y(T))\mathbb{I}_{\{T\le\tau^{x,s}\}}$ and $\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\, \xi(y(t), t, \omega)\, dt$ are bounded random variables. Let

$$\widetilde{p}(x, s, \omega) \triangleq \mathbf{E}\Big\{\gamma^{x,s}(T)\Psi(y^{x,s}(T))\mathbb{I}_{\{T\ge\tau^{x,s}\}} \,|\, \mathcal{F}_s\Big\} + \mathbf{E}\Big\{\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\, \xi(y^{x,s}(t), t, \omega)\, dt \,\Big|\, \mathcal{F}_s\Big\}. \tag{5.15}$$

Clearly, $\widetilde{p}(\cdot,s)\in Z_T^0$.

Let us show that, for a given s,

$$\widetilde{p}^{(k)}(\cdot,s)\to\widetilde{p}(\cdot,s)\quad\text{weakly in}\quad Z_T^0\quad\text{as}\quad m\to\infty. \tag{5.16}$$

By (5.14), property (5.16) implies that $p=\widehat{p}$. Therefore, if we prove (5.16) then Theorem 4.1 will be proved for the case when the functions Ψ and ξ are bounded and finitely (in x) supported.

Let us prove (5.16).

Without a loss of generality, we assume that $\Psi(x,\omega)=0$, $\Psi_m(x,\omega)=0$, $\xi(x,t,\omega)=0$, $\widehat{\xi}^{(m)}(x,t,\omega)=0$ for all $x\notin\bar{D}$. It follows that $\Psi^{(k)}(x,\omega)=0$ and $\widetilde{\xi}^{(k)}(x,t,\omega)=0$ for all $x\notin\bar{D}$. Let $\rho\in Z_s^0$. We have that

$$\begin{aligned}
|(\widetilde{p}^{(k)}(\cdot,s)-\widetilde{p}(\cdot,s),\rho)_{Z_T^0}| &\le \mathbf{E}\int_D\rho(x)\mathbf{E}\Big\{\gamma^{x,s}(T)|\Psi^{(k)}(y^{x,s}(T))-\Psi(y^{x,s}(T))|\mathbb{I}_{\{T\ge\tau^{x,s}\}}\,|\,\mathcal{F}_s\Big\}dx\\
&+\mathbf{E}\int_D\rho(x)\mathbf{E}\Big\{\int_s^{\tau^{x,s}}\gamma^{x,s}(t)\,|\widetilde{\xi}^{(k)}(y^{x,s}(t),t,\omega)-\xi(y^{x,s}(t),t,\omega)|\,dt\,\Big|\,\mathcal{F}_s\Big\}dx\\
&\le \mathbf{E}\int_D\rho(x)\mathbf{E}\Big\{\gamma^{x,s}(T)|\Psi^{(k)}(y^{x,s}(T))-\Psi(y^{x,s}(T))|\,|\,\mathcal{F}_s\Big\}dx\\
&+\mathbf{E}\int_D\rho(x)\mathbf{E}\Big\{\int_s^{T}\gamma(t)\,|\widetilde{\xi}^{(k)}(y^{x,s}(t),t,\omega)-\xi(y^{x,s}(t),t,\omega)|\,dt\,\Big|\,\mathcal{F}_s\Big\}dx.
\end{aligned}$$

Let $\rho\in Z_s^0$ be such that

$$\rho\ge 0,\quad \int_D\rho(x,\omega)dx=1,\quad \rho(x,\omega)=0 \tag{5.17}$$

for all ω. Let $a\in L_2(\Omega,\mathcal{F},\mathbf{P};\mathbf{R}^n)$ be such that $a\in D$ a.s., a has the conditional given $\mathcal{F}_s$ probability density function ρ on D, and a is independent from $(w(t)-w(t_1),\widehat{w}(t)-w(t_1))$ for all $t>t_1>s$. Let $y(t)$ be the solution of Ito equation (4.1) with initial condition $y(s)=a$, i.e., $y(t)=y^{a,s}(t)$. In addition, let $\gamma(t)=\gamma^{a,s}(t)$. Then

$$\begin{aligned}
|(\widetilde{p}^{(k)}(\cdot,s)-\widetilde{p}(\cdot,s),\rho)_{Z_T^0}| &\le \mathbf{E}\gamma(T)|\Psi^{(k)}(y(T))-\Psi(y(T))|\\
&+\mathbf{E}\int_s^T\gamma(t)\,|\widetilde{\xi}^{(k)}(y(t),t,\omega)-\xi(y(t),t,\omega)|\,dt.
\end{aligned}$$

Let $\bar{Z}_s^0=Z_s^0$ be the space defined similarly to Z_s^0 but with D replaced by $\mathbf{R}^n$. Let $u\triangleq\bar{\mathcal{L}}(s,T)\rho$, where the operator $\bar{\mathcal{L}}(s,T)$ is defined similarly to $\mathcal{L}(s,T)$ but such that D is replaced by $\mathbf{R}^n$. If $D=\mathbf{R}^n$, then $\bar{Z}_s^0=Z_s^0$ and $\bar{\mathcal{L}}(s,T)=\mathcal{L}(s,T)$. The conditions of Theorem 5.3.1 from Rozovskii (1990) are satisfied. By this theorem, it follows that

$$\int_{\mathbf{R}^n}u(x,t,\omega)\phi(x,\omega)dx=\mathbf{E}\Big\{\gamma(t)\phi(y(t),\omega)\,|\,\mathcal{F}_t\Big\}\quad\text{a.s.}$$

for all $t \in [s,T]$ for any bounded function $\phi \in \bar{Z}_t^0$. In fact, the cited theorem from Rozovskii (1990) states it for non-random ϕ, but clearly it is also correct for the case of $\phi \in \bar{Z}_t^0$ since ϕ is non-random conditionally given $\mathcal{F}_t$. (We can use also Theorem 2.2 from Dokuchaev (1995)). It follows that

$$\begin{aligned}
&|(\widetilde{p}^{(k)}(\cdot,s) - \widetilde{p}(\cdot,s),\rho)_{Z_T^0}| \\
&\le \mathbf{E}\int_{\mathbf{R}^n} u(x,T,\omega)|\Psi^{(k)}(x,\omega) - \Psi(x,\omega)|dx + \mathbf{E}\int_s^T dt \int_{\mathbf{R}^n} u(x,t,\omega)|\widetilde{\xi}^{(k)}(x,t,\omega) - \xi(x,t,\omega)|\,dx \\
&\le \|u\|_{Y^1(s,T)}\left(\|\Psi^{(k)} - \Psi\|_{Z_T^0} + \|\widetilde{\xi}^{(k)} - \xi\|_{X^{-1}}\right).
\end{aligned}$$

By (5.5), it follows that (5.16) holds for all $\rho \in Z_s^0$ such that (5.17) holds. It follows that (5.16) holds for any $\rho \in Z_s^0$, since it can be presented as $\rho = c_-\rho_+ - c_+\rho_-$, where $\rho_\pm$ are elements of $\mathcal{Z}_s^0$ such that (5.17) holds for a.e. ω, and $c_\pm \in \mathbf{R}$ are some constants.

This completes the proof of Theorem 4.1 for the case when ξ and Ψ are bounded (and finitely supported in x if $D = \mathbf{R}^n$).

For case of ξ and Ψ of the general type, it suffices to prove theorem only when $\xi \ge 0$ and $\Psi \ge 0$. The proof for ξ and Ψ with variable signs follows immediately, if we use the linearity of (4.3) and (4.2) with respect to (ξ,Ψ) and observe that $\xi = (\xi)^+ - (-\xi)^+$ and $\Psi = (\Psi)^+ - (-\Psi)^+$, where $(x)^+ \triangleq \max(0,x)$.

Let us consider ξ and Ψ such that $\xi \ge 0$ and $\Psi \ge 0$. For $M > 0$, set

$$\xi_M(x,t,\omega) \triangleq \max(\xi(x,t,\omega),M)\mathbb{I}_{\{|x|\le M\}}, \quad \Psi_M(x,\omega) \triangleq \max(\Psi(x,\omega),M)\mathbb{I}_{\{|x|\le M\}}.$$

Let $p_M \triangleq L^*\xi_M + (\delta_T L)^*\Psi_M$. We have proved that

$$p_M(x,s,\omega) = \mathbf{E}\Big\{\gamma^{x,s}(T)\Psi_M(y^{x,s}(T))\mathbb{I}_{\{T\ge\tau^{x,s}\}}\,|\,\mathcal{F}_s\Big\} + \mathbf{E}\Big\{\int_s^{\tau^{x,s}} \gamma^{x,s}(t)\,\xi_M(y^{x,s}(t),t,\omega)\,dt\,|\,\mathcal{F}_s\Big\} \quad \text{for a.e. } x,\omega.$$

By the Lebesgue's Dominated Convergence Theorem, it follows that

$$\|\xi_M - \xi\|_{X^0} + \|\Psi_M - \Psi\|_{Z_T^0} \to 0 \quad \text{as} \quad m \to +\infty.$$

By Theorem 3.3, it follows that $\|p_M - p\|_{Y^1} \to 0$. On the other hand, $\xi_M(x,t,\omega) \to \xi(x,t,\omega)$ and $\Psi_M(x,\omega) \to \Psi(x,\omega)$ from below for all x,t,ω (and these sequence are non-decreasing in m). Hence p_M converges to the right hand part of (4.2). This completes the proof of Theorem 4.1. □

Remark 5.1 *We used Theorem 3.4 to obtain (5.3) via Theorem 3.4*

6 Applications: probability density for the process killed on the boundary

Let $s \in [0,T)$. Let $\rho \in Z_s^0$ be such that $\rho \geq 0$ and $\int_D \rho(x,\omega)dx = 1$ for all ω. Let $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$ be a vector such that $a \in D$ and it has the conditional (relative to $\mathcal{F}_s$) probability density function ρ. We assume also that a is independent from $(w(t)-w(t_1), \widehat{w}(t)-\widehat{w}(t_1))$ for all $t > t_1 > s$.

Let $u = \mathcal{L}(s,T)\rho$, i.e., $u = u(x,t,\omega)$ is the solution of the problem

$$\begin{aligned} &d_t u = \mathcal{A}u\,dt + \textstyle\sum_{i=1}^N B_i u\,dw_i(t), \quad t \geq s, \\ &u|_{t=s} = \rho, \qquad u(x,t,\omega)|_{x\in\partial D} = 0. \end{aligned} \tag{6.1}$$

We assume below that the assumptions of Theorem 4.1 for $(b, \widehat{f}, \widehat{\lambda}, \beta_i, \widehat{\beta}_i)$ are satisfied.

Theorem 6.1 *Let $s \in [0,T)$. Let $y(t) = y^{a,s}(t)$ be the solution of Ito equation (4.1) with the initial condition $y(s) = a$. Then*

$$\int_D u(x,T,\omega)\Psi(x,\omega)dx = \mathbf{E}\Big\{\gamma^{a,s}(T)\Psi(y^{a,s}(T))\mathbb{I}_{\{T\leq\tau^{a,s}\}}\,|\,\mathcal{F}_T\Big\} \quad a.s. \tag{6.2}$$

for all bounded functions $\Psi \in Z_T^0$.

Note that if $D = \mathbf{R}^n$ then this theorem repeats Theorem 5.3.1 from Rozovskii (1990). However, this result is new for the case when $D \neq \mathbf{R}^n$.

Corollary 6.1 *If $\widehat{\beta}_i \equiv 0$ for all i then (6.2) means that $u(x,T,\omega)$ is the conditional (relative to $\mathcal{F}_T$) probability density function of the process $y(T) = y^{a,s}(T)$ if this process is killed at ∂D and if it is killed inside D with the rate of killing $\widehat{\lambda}$.*

Proof of Theorem 6.1. It suffices to consider $s = 0$ only. Let $\Psi \in Z_T^0$ and $\widehat{\Psi}(x,\omega) = \eta(\omega)\widehat{\Psi}(x,\omega)$, where $\eta \in L_\infty(\Omega, \mathcal{F}_T, \mathbf{P})$. Let $p \triangleq (\delta_T L)^*\widehat{\Psi}$. By Theorem 3.3, it follows that

$$(u(\cdot,T), \widehat{\Psi})_{Z_T^0} = (\delta_T \mathcal{L}\rho, \widehat{\Psi})_{Z_T^0} = (\rho, (\delta_T\mathcal{L})^*\widehat{\Psi})_{Z_T^0} = (\rho, p(\cdot,0))_{Z_T^0}.$$

By Theorem 4.1,

$$\begin{aligned} (\rho, p(\cdot,0))_{Z_T^0} = \mathbf{E}\int_D \rho(x)\gamma^{x,0}(T)\widehat{\Psi}(y^{x,0}(T)\mathbb{I}_{\{T\geq\tau^{x,0}\}}dx &= \mathbf{E}\eta\gamma^{a,0}(T)\Psi(y^{a,0}(T)\mathbb{I}_{\{T\geq\tau^{a,0}\}} \\ &= \mathbf{E}\eta\mathbf{E}\{\gamma^{a,0}(T)\Psi(y^{a,0}(T)\mathbb{I}_{\{T\geq\tau^{a,0}\}}|\mathcal{F}_T\}. \end{aligned}$$

Then

$$\mathbf{E}\eta\int_D u(x,T,\omega)\Psi(x,\omega)dx = \mathbf{E}\eta\mathbf{E}\{\gamma^{a,0}(T)\Psi(y^{a,0}(T)\mathbb{I}_{\{T\geq\tau^{a,0}\}}|\mathcal{F}_t\}.$$

Remind that $\eta \in L_\infty(\Omega, \mathcal{F}_T, \mathbf{P})$ is arbitrary. Then the proof follows. □

7 Applications: maximum principle and contraction property

Remind that the assumptions of Theorem 4.1 for $(b, \widehat{f}, \widehat{\lambda}, \beta_i, \widehat{\beta}_i)$ are satisfied.

Theorem 7.1 *(Maximum principle) Let $\xi \in X^0$ and $\Psi \in Z_T^0$ be such that $\xi(x,t,\omega) \geq 0$ and $\Psi(x,\omega) \geq 0$ for a.e. x,t,ω. Then the solution p of (3.8) is such that $p(x,t,\omega) \geq 0$ for all t for a.e. t,ω.*

Proof. Assume that $\xi(x,t,\omega) \geq 0$ and $\Psi(x,\omega) \geq 0$ for all x,t,ω and that these functions have the same measurability as described in Theorem 4.1. In this case, the proof follows immediately from Theorem 4.1. Further, let $\xi(x,t,\omega) \geq 0$ and $\Psi(x,\omega) \geq 0$ for a.e. x,t,ω. Replace these function by some equivalent non-negative functions ξ' and Ψ'. Since $p = L^*\xi + (\delta_T L)^*\Psi$, it follows that $p = L^*\xi' + (\delta_T L)^*\Psi'$ as an element of Y^2. By Theorem 4.1 again, p is nonnegative up to equivalency. Then the proof follows. □

Theorem 7.2 *(Maximum principle) Let $\varphi \in X^0$ and $\Phi \in Z_0^0$ be given such that $\varphi(x,t,\omega) \geq 0$ and $\Phi(x,\omega) \geq 0$ for a.e. x,t,ω. Then the solution u of problem (3.1) is such that $u(x,t,\omega) \geq 0$ for all t for a.e. x,ω.*

Proof. It suffices to consider $t = T$ only. Let $\Psi \in Z_T^0$ be an arbitrary function such that $\Psi \geq 0$ a.e. We have

$$(u(\cdot,T),\Psi)_{Z_T^0} = (\delta_T L\varphi + \delta_T \mathcal{L}\Phi, \Psi)_{Z_T^0} = (\varphi, p)_{Z_T^0} + (\Phi, p(\cdot,0))_{Z_T^0},$$

where $p \triangleq (\delta_T L)^*\Psi$. Then $p(x,s,t) \geq 0$ for all s for a.e. x,ω, and $(u(\cdot,T),\Psi)_{Z_T^0} \geq 0$. Then the proof follows. □

Theorem 7.3 *(Contraction property) Under the assumptions of Theorem 4.1, let $\widehat{\lambda}(x,t,\omega) \geq 0$ and $\widehat{\beta}_i \equiv 0$ for all i. Then*

$$\operatorname*{ess\,sup}_{x,\omega} |p(x,t,\omega) \leq \operatorname*{ess\,sup}_{x,\omega} |\Psi(x,\omega)| + (T-t) \operatorname*{ess\,sup}_{x,t,\omega} |\xi(x,t,\omega)| \quad \forall t \in [0,T].$$

Proof. Note that there are bounded functions ξ' and Ψ' that are equivalent to ξ and Ψ and such that

$$\operatorname*{ess\,sup}_{x,\omega} |\Psi(x,\omega)| + (T-t) \operatorname*{ess\,sup}_{x,t,\omega} |\xi(x,t,\omega)| = \sup_{x,\omega} |\Psi'(x,\omega)| + (T-t) \sup_{x,t,\omega} |\xi'(x,t,\omega)|.$$

Since $p = L^*\xi + (\delta_T L)^*\Psi$, it follows that $p = L^*\xi' + (\delta_T L)^*\Psi'$ as an element of Y^2. It follows immediately from Theorem 4.1 that

$$\operatorname*{ess\,sup}_{x,\omega} |p(x,t,\omega) \leq \sup_{x,\omega} |\Psi'(x,\omega)| + (T-t) \sup_{x,t,\omega} |\xi'(x,t,\omega)| \quad \forall t \in [0,T].$$

Then the proof follows. □

Theorem 7.4 *(Contraction property) Let $\widehat{\lambda}(x,t,\omega) \geq 0$ and $\widehat{\beta}_i \equiv 0$ for all i, and let $\varphi \in X^0$ and $\Phi \in Z_0^0$ be given. Then the following holds for the solution u of problem (3.1):*

(a) If $\varphi \equiv 0$, then

$$\mathbf{E}\int_D |u(x,T,\omega)|dx \leq \mathbf{E}\int_D |\Phi(x,\omega)|dx.$$

(b) If $\Phi = 0$, then

$$\mathbf{E}\int_D |u(x,T,\omega)|dx \leq \frac{1}{T}\mathbf{E}\int_Q |\varphi(x,t,\omega)|dxdt.$$

Proof. Let $\Psi \in Z_T^0$ be an arbitrary function. By Theorem 3.3, it follows that

$$(u(\cdot,T),\Psi)_{Z_T^0} = (\delta_T L\varphi + \delta_T \mathcal{L}\Phi, \Psi)_{Z_T^0} = (\varphi, p)_{Z_T^0} + (\Phi, p(\cdot,0))_{Z_T^0},$$

where $p \triangleq (\delta_T L)^*\Psi$. Then the proof follows from Theorem 7.3. □

Conclusions

We obtained the representation theorem for non-Markov Ito processes in bounded domains when the first exit times are involved. This result is not particularly surprising; the similar result without first exit times for the processes in the entire space was obtained long time ago. However, the setting with first exit times required to overcome one crucial obstacle: insufficiency of the known regularity for backward SPDEs in domains with boundaries. Consequently, there is a little known about first exit times of non-Markov processes. The representation theorem opens some further opportunities for studying first exit times for non-Markov processes. It is unclear yet if it is possible to relax the strengthened coercivity required by Condition 3.5. Probably, is some cases, this condition may be lifted via the estimates from Dokuchaev (2008). To cover more general models, we suggest to include the case of infinite number of driving Wiener processes and more general boundary conditions. We leave it for future research.

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