Every point is critical

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Abstract. We show that, for any compact Alexandrov surface S and any point y in S, there exists a point x in S for which y is a critical point. Moreover, we prove that uniqueness characterizes the surfaces homeomorphic to the sphere among smooth orientable surfaces.

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Introduction. In this paper, by *surface* we always mean a compact Alexandrov surface with curvature bounded below and without boundary, as defined for example in [1]. It is known that our surfaces are topological manifolds. Let $\mathcal{A}$ be the space of all surfaces.

For any surface S, denote by ρ its intrinsic metric, and by ρ_x the distance function from x, given by $\rho_x(y) = \rho(x, y)$. A point $y \in S$ is called *critical* with respect to ρ_x (or to x), if for any direction v of S at y there exists a *segment* (i.e., a shortest path) from y to x whose direction at y makes an angle $\alpha \leq \pi/2$ with v. For the definition of a direction in Alexandrov surfaces, see again [1].

The survey [2] by K. Grove presents the principles, as well as applications, of the critical point theory for distance functions.

For any point x in S, denote by Q_x the set of all critical points with respect to x, and by Q_x^{-1} the set of all points $y \in S$ with $x \in Q_y$. Let M_x, F_x be the sets of all relative, respectively absolute, maxima of ρ_x. For properties of Q_x and its subsets M_x and F_x in Alexandrov spaces, see [3], [7], and the survey [5].

Our Theorem 1 establishes that $\text{card}Q_y^{-1} \geq 1$ for any point y on any surface S. This lower bound is sharp, as Theorem 2 shows. We apply it to prove a Corollary, which characterizes the smooth orientable surfaces homeomorphic to the sphere.

In a forthcoming paper [4], we provide for orientable surfaces an upper bound for $\text{card}Q_y^{-1}$ depending on the genus, and use it to estimate the cardinality of diametrally opposite sets on S. The case of points y in orientable Alexandrov surfaces, which are common maxima of several distance functions, is treated in [6].

We denote by T_x the space of directions at $x \in S$; the length λT_x of T_x satisfies $\lambda T_x \leq 2\pi$. If $\lambda T_x = 2\pi$ then x is called *smooth*, otherwise a *conical point* of S.

There might exist a direction $\tau \in T_x$ such that no segment starts at x in direction τ. On most convex surfaces, the set of such directions τ, called *singular*, is even residual in T_x, for each x (see Theorem 2 in [8]). However, the set of non-singular directions is always dense in T_x. For those τ, for which there is a geodesic Γ with direction τ at x, a so-called *cut point* $c(\tau)$ is associated, defined by the requirement that the arc $xc(\tau) \subset \Gamma$ is a segment which cannot be extended further beyond $c(\tau)$. The set of all these cut points is the *cut locus* $C(x)$ of the point x.

It is known that $C(x)$, if it is not a single point, is a local tree (i.e., each of its points z has a neighbourhood V in S such that the component $K_z(V)$ of z in $C(x) \cap V$ is a tree), even a tree if S is homeomorphic to the sphere. Theorem 4 in [9] and Theorem 1 in [8] yield the existence of surfaces S on which the set of all extremities of any cut locus is residual in S. It is, however, known that $C(x)$ has an at most countable set $C_3(x)$ of ramification points.

If S is not a topological sphere, the *cyclic part of* $C(x)$ is the minimal (with respect to inclusion) subset $C^{cp}(x)$ of $C(x)$, whose removal from S produces a topological (open) disk. It is easily seen that $C^{cp}(x)$ is a local tree with finitely many ramification points and no extremities. Let $C_3^{cp}(x)$ be the set of points of degree at least 3 in $C^{cp}(x)$. We stress that the degree is not taken in $C(x)$, but in $C^{cp}(x)$.

Recall that a *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T. The *degree* of a point y of a local tree is the number of components of $K_y(V) \setminus \{y\}$ if the neighbourhood V of y is chosen such that $K_y(V)$ is a tree. A point y of the local tree T is called an *extremity* of T if it has degree 1, and a *ramification point* of T if it has degree at least 3.

Results. Every point on a surface admits a critical point. It suffices, indeed, to take a point farthest from it. Conversely, is it true that every point is a critical point of some other point? Certainly, not every point on every surface is a farthest point from some other point!

Theorem 1 *Every point on every surface is critical with respect to some point of the surface.*

Proof. Let $S \in \mathcal{A}$ and $y \in S$. We will identify here T_y with a Euclidean circle of centre $\mathbf{0}$ and length $\lambda T_y \leq 2\pi$.

Case 1. S is homeomorphic to S^2.

If $C(y)$ is a single point, the conclusion is true. Suppose $C(y)$ is not a point, but remember it is a tree.

Let $x \in C(y)$. If all components of $T_y \setminus c^{-1}(x)$ have length at most π, then $y \in Q_x$. Suppose one component, A, has length $\lambda A > \pi$. For any non-singular $\tau \in A$, $c^{-1}(c(\tau)) \subset A$; let B_τ be the shortest subarc of A including $c^{-1}(c(\tau))$ (possibly reduced to $\{\tau\}$). Take the midpoint τ_0 of A. Then $\mathbf{0} \in \mathrm{conv} c^{-1}(c(\tau_0))$, or $B_{\tau_0} = \{\tau_0\}$ (and $c(\tau_0)$ is an extremity of $C(y)$), or $0 < \lambda B_{\tau_0} < \pi$, or else $c(\tau_0)$ is not defined.

In the first three cases, let $x' = c(\tau_0)$. In the fourth case, there is a point $x' \in C(y)$ close to y with the whole set $c^{-1}(x')$ close to τ_0 and containing points on both sides of τ_0.

In the first case, $y \in Q_{x'}$. In the last three cases, there is a single Jordan arc $J \subset C(y)$ from x to x'. The multivalued mapping $z \mapsto c^{-1}(z)$ defined on J is upper semicontinuous. Since, for $z \in J \setminus C_3(y)$ close to x and $\tau \in c^{-1}(z)$, $\lambda B_\tau > \pi$, and, for $z \in J \setminus C_3(y)$ close to x', and $\tau \in c^{-1}(z)$, $\lambda B_\tau < \pi$, there is a point $z_0 \in J$ for which $\mathbf{0} \in \mathrm{conv} c^{-1}(z_0)$. Hence $y \in Q_{z_0}$.

Case 2. S is not homeomorphic to S^2.

Consider a point $x \in C_3^{cp}(y)$, and a direction $\alpha \in c^{-1}(x)$.

Let $\alpha_-\alpha_+ \subset T_y$ be the maximal arc containing α such that, for each non-singular $\tau \in \alpha_-\alpha_+$, either $c(\tau) \notin C^{cp}(y)$ or $c(\tau) = x$. (The indices $-, +$ are taken according to a certain orientation of T_y.) Of course, $\alpha_-\alpha_+$ may be reduced to the singleton $\{\alpha\}$. For each x we have finitely many arcs of type $\alpha_-\alpha_+$.

Let $\tau \in c^{-1}(C^{cp}(y)) \setminus c^{-1}(C_3^{cp}(y))$ converge to $\alpha_- -$ (resp. $\alpha_+ +$). Then the point $g(\tau)$ of $c^{-1}(c(\tau))$ different from τ converges to some point α^- (resp. α^+), both in $c^{-1}(x)$.

Join by line-segments α^- to α_-, α_- to α_+, and α_+ to α^+. Repeating this for all directions in $c^{-1}(x)$, we obtain a cycle whose edges are the line-segments $\overline{\alpha^-\alpha_-}$, $\overline{\alpha_-\alpha_+}$, $\overline{\alpha_+\alpha^+}$ and all their analogs. And repeating the procedure for all $x \in C_3^{cp}(y)$, we obtain a graph, which is finite because S, being compact, has finite genus.

Let $\alpha_-\alpha_+ \subset T_y$ be as defined above, and $\beta_-\beta_+$ an analogous arc, the two graph vertices α_+, β_- being consecutive on T_y. Then α^+ and β^- are consecutive too, and we consider the cycle $\alpha_+\beta_-\beta^-\alpha^+$, with edges $\alpha_+\beta_-$, $\overline{\beta_-\beta^-}$, $\beta^-\alpha^+$, $\overline{\alpha^+\alpha_+}$, and all analogous cycles, in addition to the previous ones.

Moreover, consider the cycle formed by the arc $\alpha_-\alpha_+$ and the line-segment $\overline{\alpha_+\alpha_-}$, plus all analogous cycles.

Let $C_1, ..., C_n$ be all these cycles.

If $\mathbf{0} \in \cup_{j=1}^n C_j$, then $\mathbf{0}$ belongs to one of the line-segments, whence $c^{-1}(x)$ contains, for some $x \in C_3^{cp}(y)$, two diametrally opposite points of T_y, and we are done.

If not, consider the winding number $w(C_j) = w(\mathbf{0}, C_j)$ of every cycle C_j with respect to $\mathbf{0}$. We have

$$\sum_{i=1}^{n} w(C_i) = w\left(\sum_{i=1}^{n} C_i\right) = w(T_y) = 1 \pmod 2,$$

irrespective of the orientations, because each edge not in T_y is used exactly twice. This shows that $w(C_i) \neq 0$ for some cycle C_i.

If this cycle C_i is a cycle $\alpha_+\beta_-\beta^-\alpha^+$ with $\alpha_+\beta_-$ and $\beta^-\alpha^+$ of the same orientation on T_y, then the proof parallels that of Case 1 ($\tau \in \alpha_+\beta_-$ and $g(\tau) \in \alpha^+\beta^-$ move in contrary directions).

If C_i is a cycle $\alpha_+\beta_-\beta^-\alpha^+$ with $\alpha_+\beta_-$ and $\beta^-\alpha^+$ of contrary orientations on T_y, then τ and $g(\tau)$ move in the same direction, but $\mathbf{0}$ lies on different sides of $\tau g(\tau)$ for $\tau = \alpha_+$ and $\tau = \beta_-$; this and the argument of Case 1 yield the conclusion.

If C_i is a cycle $\alpha_-\alpha_+ \cup \overline{\alpha_+\alpha_-}$, then the proof again parallels that of Case 1.

Finally, if C_i is ones of the other cycles (with all edges line-segments), $w(C_i) \neq 0$ means that $\mathbf{0}$ is surrounded by C_i, which is impossible if $\mathbf{0} \notin \mathrm{conv} C_i$. By construction, $\mathrm{conv} C_i = \mathrm{conv} c^{-1}(x)$ for some $x \in C_3^{cp}(y)$. The proof is complete.

The following result shows that in general one cannot hope for a better lower bound. It extends Theorem 3 in [7] and admits a similar proof, which will therefore be omitted.

Theorem 2 *Assume $S \in \mathcal{A}$, $y \in S$ is smooth, and $x \in Q_y^{-1}$ is such that the union U of two segments from x to y separates S. If a component S' of $S \setminus U$ contains no segment from x to y then $Q_y^{-1} \cap S' = \emptyset$. In particular, if the union of any two segments from x to y separates S then $Q_y^{-1} = \{x\}$.*

Corollary *A smooth orientable surface S is homeomorphic to the sphere S^2 if and only if each point in S is critical with respect to precisely one other point of S.*

Proof. If S is homeomorphic to the sphere S^2 then $\mathrm{card} Q_y^{-1} = 1$ for any point y in S, by Theorems 1 and 2.

Next we show that every orientable surface non-homeomorphic to S^2 contains a point y with $\mathrm{card} Q_y^{-1} > 1$.

To see this, let Ω denote a shortest simple closed curve which does not separate S. Then Ω is a closed geodesic. Moreover, for any of its points z, Ω is the union of two segments of length $\lambda\Omega/2$ starting at z and ending at z_Ω. Consider the family $\mathcal{C}$ of all simple closed not contractible curves C which cut Ω at precisely one point, such that Ω separates C locally at $\Omega \cap C$. Then clearly $\mathcal{C} \neq \emptyset$, by the choice of Ω. Let Ω' be a shortest curve in $\mathcal{C}$; it is a closed geodesic too. Moreover, by the definition of $\mathcal{C}$ and by the choice of Ω', the latter is the union of two segments starting at $\{y\} = \Omega \cap \Omega'$ and ending at $y_{\Omega'}$. It follows that Q_y^{-1} contains at least two points, y_Ω and $y_{\Omega'}$.

Open question. Every orientable surface of genus $g > 0$ possesses points x, y such that y is critical with respect to x and two segments from y to x have opposite directions at y (see the proof of the Corollary). Is the same true for all surfaces homeomorphic to the sphere? Or, at least, if $\mathcal{A}_0$ denotes the space of all Alexandrov surfaces homeomorphic to the sphere, endowed with the Hausdorff-Gromov metric, is there a dense set in $\mathcal{A}_0$ with the above property? For a similar – still open – problem concerning convex surfaces, see [10].

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