Families of completely positive maps associated with monotone metrics

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Abstract

An operator convex function on $(0, \infty)$ which satisfies the symmetry condition $k(x^{-1}) = xk(x)$ can be used to define a type of non-commutative multiplication by a positive definite matrix (or its inverse) using the primitive concepts of left and right multiplication and the functional calculus. The operators for the inverse can be used to define quadratic forms associated with Riemannian metrics which contract under the action of completely positive trace-preserving maps.

We study the question of when these operators define maps which are also completely positive (CP). Although $A \mapsto D^{-1/2}AD^{-1/2}$ is the only case for which both the map and its inverse are CP, there are several well-known one parameter families for which either the map or its inverse is CP. We present a complete analysis of the behavior of these families, as well as the behavior of lines connecting an extreme point with the smallest one and some results for geometric bridges between these points.

Our primary tool is an order relation based on the concept of positive definite functions. Although some results can be obtained from known properties, we also prove new results based on the positivity of the Fourier transforms of certain functions. Concrete computations of certain Fourier transforms not only yield new examples of positive

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definite functions, but also examples in the much stronger class of infinitely divisible functions.

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Contents

1 Introduction

On a commutative algebra the operations of multiplication and "division" by elements of the positive cone take the positive cone into itself. However, this is not the case for non-commutative algebras, on which these operations are not even uniquely defined.

Various non-commutative versions of multiplication and division (i.e., multiplication by the inverse) by elements of the cone of positive definite matrices or operators correspond to maps on matrix algebras, and some of these maps play an important role in many contexts. For $D > 0$, some naive definitions of multiplication by the inverse are given by the maps $X \mapsto D^{-1/2} X D^{-1/2}$, $X \mapsto D^{-1}X$ and $X \mapsto XD^{-1}$. The first maps the cone of positive operators to itself, while the other two do not even preserve self-adjointness. There are many other possible generalizations of multiplication by D^{-1}. In Section 4 we consider several different one parameter families of such maps.

Perhaps, the best known example is

$$\Omega_D(X) = \int_0^\infty (D + tI)^{-1} X (D + tI)^{-1}\, dt\,, \tag{1.1}$$

which gives a well-defined (and highly symmetric) notion of non-commutative multiplication by D^{-1}. Its inverse is well-known to be

$$\Omega_D^{-1}(Y) = \int_0^1 D^t\, Y D^{1-t}\, dt \tag{1.2}$$

which is a form of non-commutative multiplication by D. The quadratic form $\operatorname{Tr} A^* \Omega_D^{-1}(B)$ is known as the Bogoliubov or Kubo-Mori inner product. Although the inverse relationship between (1.2) and (1.1) is well-known, and follows from more general results given later, we include an explicit proof in Appendix C.2.

We consider here a class of such maps which arise in quantum information theory, in the context of what are known as monotone Riemannian metrics [45, 33].

We study the question of which maps within this class have the property known as completely positivity defined in Section 2.3. The map in (1.1) has this property, but its inverse (1.2) does not.

This question was motivated by an observation in [33, 54] about bounds on the contraction of monotone metrics under the action of completely positive trace-preserving (CPT) maps which are also known as *quantum channels* in view of their important role as noise models in quantum information theory. These bounds are discussed briefly in Section 2.4 and Appendix B.

This paper is organized as follows. Section 2 describes the various concepts we need and introduces the notations we will use. Section 2.4 contains a brief summary of the background and motivation behind this work. In Section 3 we present some powerful tools we will use formulated in terms of positive kernels, and a closely related partial order. In Section 4 we present a large number of examples of one parameter families of functions which provide a number of inequivalent classes of maps used to define non-commutative multiplication by the inverse of a positive matrix. In most cases, we can also provide precise ranges for which these maps or their inverses are CP. Section 5 proves new results about positive kernels based on Fourier transforms, which are needed to prove some of our results. These results are of interest in their own right. In Section 6 we complete the proofs of those results stated in Section 4 which require the results of Section 5.

There are also three appendices. The first contains the detailed proof of the integral representation needed in Section 2.1. The second gives more details about the motivation in terms of contraction under CPT maps of the Riemannian metrics associated with our maps. Finally, for the benefit of non-experts, we present some very pedestrian arguments which clarify some well-known results that are often glossed over.

2 Preliminaries

2.1 Basics

For each $d \in \mathbf{N}$ we write $\mathbb{M}_d$, $\mathbb{H}_d$, $\mathbb{P}_d$ and $\overline{\mathbb{P}}_d$ for the sets of $d \times d$ complex, Hermitian, positive definite and positive semi-definite matrices, respectively. Functions of matrices in $\mathbb{H}_d$ can be defined by using the spectral theorem; this is sometimes called "functional calculus" (see, e.g., [49, Section VII.1]).

A real function f on $(0, \infty)$ is said to be *operator monotone* (or operator monotone increasing) if $A \geq B$ implies $f(A) \geq f(B)$ for every $A, B \in \mathbb{P}_d$ with any $d \in \mathbf{N}$, *operator monotone decreasing* if $-f$ is operator monotone. A real function k on $(0, \infty)$ is said to be *operator convex* if

$$k(\lambda A + (1-\lambda)B) \leq \lambda k(A) + (1-\lambda)k(B)$$

for all $A, B \in \mathbb{P}_d$ with any $d \in \mathbf{N}$ and all $\lambda \in (0,1)$, and *operator concave* if $-k$ is operator convex. The theory of operator monotone and operator convex functions was initiated by Löwner [36] and Kraus [29], respectively. It is well-known [7, Section V.4] (also [2, 12, 18]) that operator monotone (also operator convex) functions on $(0, \infty)$ have an analytic continuation into the upper half-plane of $\mathbf{C}$. Moreover, a necessary and sufficient condition for a function on $(0, \infty)$ to be operator monotone increasing (resp., decreasing) is that it has the *"Pick" mapping property* that the analytic continuation maps the upper half-plane *into* the upper (resp., lower) half-plane. The integral representation theory for Pick functions and operator monotone functions are also well-known, see, e.g., [1, Section 59] or [7, Section V.4] and [2, 12, 18].

Definition 2.1. Let $\mathcal{K}$ denote the class of functions $k : (0, \infty) \to (0, \infty)$ which are operator convex and satisfy the symmetry condition $xk(x) = k\big(x^{-1}\big)$ and the normalization condition $k(1) = 1$.

There are a number of equivalent characterizations of the class $\mathcal{K}$ which are given in Theorem 2.4 below. Its proof uses an integral representation, which is important in its own right and presented in Theorem 2.3. We also observe that although $k(x) \in \mathcal{K}$ may diverge as $x \searrow 0$, it cannot diverge more rapidly than x^{-1}. This was proved in [22]. For completeness, we include its proof as well as the proof of Theorem 2.3 in Appendix A.

Proposition 2.2. *Let $k : (0, \infty) \to (0, \infty)$ be operator convex. Then $\lim_{x \to 0} xk(x)$ exists and is finite. When $xk(x) = k\big(x^{-1}\big)$, $\lim_{x \to \infty} k(x)$ also exists and is finite.*

Theorem 2.3. *For every $k \in \mathcal{K}$, there exists a unique probability measure m on $[0, 1]$ such that for $x \in (0, \infty)$*

$$\begin{aligned} k(x) &= \int_{[0,1]} \frac{1 + x}{(x + \nu)(1 + \nu x)} \cdot \frac{(1 + \nu)^2}{2} \, dm(\nu) \\ &= \int_{[0,1]} \left(\frac{1}{x + \nu} + \frac{1}{1 + x\nu} \right) \frac{(1 + \nu)}{2} \, dm(\nu). \end{aligned}$$

Theorem 2.4. *For each function $k : (0, \infty) \to (0, \infty)$ the following are equivalent*:

(a) *k is operator convex and $xk(x) = k\big(x^{-1}\big)$;*

(b) *k is operator monotone decreasing and $xk(x) = k\big(x^{-1}\big)$;*

(c) *$f(x) \equiv 1/k(x)$ is operator concave and $f(x) = xf\big(x^{-1}\big)$;*

(d) *$f(x) \equiv 1/k(x)$ is operator monotone and $f(x) = xf\big(x^{-1}\big)$.*

Proof. The equivalence of $xk(x) = k\big(x^{-1}\big)$ and $f(x) = xf\big(x^{-1}\big)$ is easily checked. The equivalence (c) $\Leftrightarrow$ (d) for positive functions on $(0, \infty)$ is well-known (see, e.g., [7, Theorem V.2.5]), and (b) $\Leftrightarrow$ (d) follows from the fact that $x \mapsto 1/x$ is operator monotone decreasing on $(0, \infty)$. The implication (a) $\Rightarrow$ (b) follows immediately from Theorem 2.3 and the well-known fact that the map $x \mapsto 1/(\alpha x + \beta)$ is operator monotone decreasing on $(0, \infty)$ for any fixed $\alpha, \beta \geq 0$. We finally show that (b) $\Rightarrow$ (a). Assume (b); then $1/k(x)$ is operator monotone and so operator concave on $(0, \infty)$. This implies (a) since x^{-1} is operator monotone decreasing and operator convex. **QED**

As shown in the above proof, the implication (b) $\Rightarrow$ (a) and the equivalence of (b)–(d) hold true without the symmetry assumption $xk(x) = k\big(x^{-1}\big)$. However, the reverse implication (a) $\Rightarrow$ (b) only holds under this additional assumption and appears to be new.

The next result is easy to verify, but stated explicitly for completeness.

Proposition 2.5. *The map $k(x) \mapsto \widehat{k}(x) \equiv 1/k\big(x^{-1}\big)$ is a bijection on $\mathcal{K}$ and the map $f(x) \mapsto \widehat{f}(x) \equiv 1/f\big(x^{-1}\big)$ induces the same bijection with $f(x) = 1/k(x)$ and $\widehat{f}(x) = 1/\widehat{k}(x)$.*

2.2 The multiplication map and its inverse Ω_D^k

For each $D \in \mathbb{P}_d$ we write L_D and R_D for the left and the right multiplication operators, respectively, i.e., $L_D X \equiv DX$ and $R_D X \equiv XD$ for $X \in \mathbb{M}_d$. Note that L_D and R_D are commuting positive invertible operators on $\mathbb{M}_d$ considered as a Hilbert space equipped with the Hilbert-Schmidt inner product $\langle X, Y \rangle_{\mathrm{HS}} \equiv \operatorname{Tr} X^* Y$, where Tr denotes the usual trace functional on $\mathbb{M}_d$. The operator $L_A R_B^{-1}$ was used by Araki [5] to define the relative entropy of positive operators A, B in far more general situations than matrix algebras, and is often called the *relative modular operator*.

For a fixed function $k \in \mathcal{K}$ we define, for any $D \in \mathbb{P}_d$, the linear map $\Omega_D^k : \mathbb{M}_d \to \mathbb{M}_d$ by

$$\Omega_D^k(X) \equiv R_D^{-1} k\left(L_D R_D^{-1}\right) X = L_D^{-1} k\left(R_D L_D^{-1}\right) X, \qquad X \in \mathbb{M}_d. \tag{2.1}$$

Since both of the commuting operators L_D and R_D are positive with respect to the Hilbert-Schmidt inner product, it is clear that Ω_D^k is also positive (in the same sense). Each map Ω_D^k can be considered as a non-commutative generalization of multiplication by D^{-1}; indeed, if $DX = XD$ then $\Omega_D^k(X) = D^{-1}X$ independently of $k \in \mathcal{K}$. The equivalence of the two expressions in (2.1) follows from the symmetry condition $xk(x) = k\big(x^{-1}\big)$ since

$$\begin{aligned} R_D^{-1} k\left(L_D R_D^{-1}\right) &= L_D^{-1} L_D R_D^{-1} k\left(L_D R_D^{-1}\right) \\ &= L_D^{-1} k\left(\left(L_D R_D^{-1}\right)^{-1}\right) = L_D^{-1} k\left(R_D L_D^{-1}\right). \end{aligned}$$

To better understand the action of Ω_D^k, we consider the two-variable function $\phi^k(x,y) \equiv (1/y)\, k(x/y)$ for $x, y > 0$ and observe that $\Omega_D^k = \phi^k(L_D, R_D)$. When D is a diagonal matrix with eigenvalues λ_j, it is an easy consequence of the functional calculus that the action of Ω_D^k on the matrix with entries x_{ij} is

$$x_{ij} \longmapsto \phi^k(\lambda_j, \lambda_k) x_{ij} = \frac{1}{\lambda_j} k\left(\frac{\lambda_i}{\lambda_j}\right) x_{ij}$$

which is the Schur (or Hadamard or pointwise) product $A \circ X$ with the matrix A with entries $a_{ij} = \phi^k(\lambda_i, \lambda_j)$. More generally, let U be a unitary which diagonalizes D so that

$$D = U \mathrm{diag}(\lambda_1, \ldots, \lambda_d) U^*.$$

Then

$$\Omega_D^k(X) = U\Big([\phi^k(\lambda_i, \lambda_j)] \circ [U^* X U]\Big) U^*. \tag{2.2}$$

Since $L_D^{-1} = L_{D^{-1}}$ and $R_D^{-1} = R_{D^{-1}}$, it might be tempting to think that $\Omega_D^{-1}(X) = \Omega_{D^{-1}}(X)$. However, this is easily seen to be false by considering specific examples (including (1.1) and (1.2)). Instead we have for any $D \in \mathbb{P}_d$,

$$J_D^f \equiv (\Omega_D^k)^{-1} = R_D f\big(L_D R_D^{-1}\big) = R_D \widehat{k}(L_D^{-1} R_D) = \Omega_{D^{-1}}^{\widehat{k}} \tag{2.3}$$

with $f(x) = 1/k(x) = \widehat{k}\big(x^{-1}\big)$. To see this, observe that it follows from (2.2) that $(\Omega_D^k)^{-1} = (1/\phi^k)(L_D, R_D)$. From the relation

$$\frac{1}{\phi^k}(x,y) = \frac{y}{k(x/y)} = y f(x/y) = x f(y/x),$$

the functional calculus implies (2.3).

2.3 Complete positivity of Ω_D^k

A linear map $\Phi : \mathbb{M}_d \mapsto \mathbb{M}_d$ is called *positive* if it is positivity-preserving in the sense that $A > 0$ implies $\Phi(A) \geq 0$, i.e., $\Phi(\mathbb{P}_d) \subseteq \overline{\mathbb{P}}_d$. A linear map $\Phi : \mathbb{M}_d \mapsto \mathbb{M}_d$ is called *completely positive* (CP) if $\Phi \otimes \mathcal{I}_n$ is positive on $\mathbb{M}_d \otimes \mathbb{M}_n$ for all $n \in \mathbf{N}$ with $\mathcal{I}_n$ the identity map on $\mathbb{M}_n$. The notion of complete positivity, introduced by Stinespring [53] and discussed in, e.g., [42, Chapter 6] plays an important role in quantum information theory. (See, e.g, [40, 47].)

The recognition in (2.2) that Ω_D^k can be implemented as a Schur product yields a simply stated condition to test that it is CP. In general, complete positivity of a map on $\mathbb{M}_d$ is a much stronger condition than positivity. However, for Schur products, it is well-known (see, e.g., [42, Theorem 3.7]) that both positivity conditions for the map $\Phi_A(X) = A \circ X$ $(A, X \in \mathbb{M}_d)$ are equivalent to positivity of A. Indeed, the map $\Phi_A \otimes \mathcal{I}_n$ on $\mathbb{M}_n(\mathbb{M}_d) \cong \mathbb{M}_d \otimes \mathbb{M}_n$ can be realized as Schur multiplication with $A \otimes J_n$, where J_n is the $n \times n$ matrix with all entries 1. Therefore, in our setting, the requirement that the map Ω_D^k is CP is equivalent to the weaker positivity requirement, as we explicitly state for completeness in the following:

Theorem 2.6. *The following conditions for $k \in \mathcal{K}$ are equivalent*:

(a) $\Omega_D^k : \mathbb{M}_d \to \mathbb{M}_d$ *is CP for every $D \in \mathbb{P}_d$ with any $d \in \mathbf{N}$;*

(b) $\Omega_D^k : \mathbb{M}_d \to \mathbb{M}_d$ *is positive for every $D \in \mathbb{P}_d$ with any $d \in \mathbf{N}$;*

(c) *the $d \times d$ matrix*
$$A = \left[\frac{1}{w_j} k\left(\frac{w_i}{w_j}\right)\right]_{1 \leq i,j \leq d} \tag{2.4}$$
is positive semi-definite for every $w_1, \ldots, w_d > 0$ with any $d \in \mathbf{N}$.

The next result allows us to replace the matrix A in part (c) of Theorem 2.6 by some closely related matrices for which the positivity condition may be more easily checked in some situations.

Proposition 2.7. *The matrix A in* (2.4) *is positive if and only if one* (*and hence both*) *of the following matrices are positive*:
$$\left[w_i k\left(\frac{w_i}{w_j}\right)\right]_{1 \leq i,j \leq d}, \qquad \left[\sqrt{\frac{w_i}{w_j}} k\left(\frac{w_i}{w_j}\right)\right]_{1 \leq i,j \leq d}. \tag{2.5}$$

Proof. Let W be the diagonal matrix with entries $w_i \delta_{ij}$. Then the matrices above correspond to $W^* A W$ and $(W^*)^{1/2} A W^{1/2}$, respectively. **QED**

2.4 Background and motivation

For each $k \in \mathcal{K}$, the map Ω_D^k can be used to define a quadratic form
$$\Gamma_D^k(X, Y) \equiv \langle X, \Omega_D^k(Y) \rangle_{\text{HS}} = \operatorname{Tr} X^* \Omega_D^k(Y) \tag{2.6}$$
which can be interpreted as a metric on the Riemannian manifold $\mathcal{D}_d \equiv \{D \in \mathbb{P}_d : \operatorname{Tr} D = 1\}$ of invertible density matrices in $\mathbb{M}_d$. Here, the matrices in $\mathbb{H}_d$ with trace zero form the

tangent space, denoted by $\mathbb{H}_d^0$, of $\mathcal{D}_d$ at each foot point D. This metric is *monotone* in the sense that for any completely positive and trace-preserving map $\Phi : \mathbb{M}_d \to \mathbb{M}_m$ $(d, m \in \mathbf{N})$,

$$\Gamma^k_{\Phi(D)}\big(\Phi(X), \Phi(Y)\big) \le \Gamma^k_D(X, Y), \qquad D \in \mathcal{D}_d,\ X, Y \in \mathbb{H}_d^0. \tag{2.7}$$

The theory of monotone Riemannian metrics was largely developed by Petz [45] after Morozova and Chentsov [38] introduced the concept. It was shown in [45] that each $k \in \mathcal{K}$ defines a family of monotone metrics of the form (2.6) with $D \in \mathcal{D}_d$ for all $d \in \mathbf{N}$, and that any Riemannian metric on $\mathcal{D}_d$, $d \in \mathbf{N}$, which satisfies the contraction condition (2.7) must be of the form (2.6) for some $k \in \mathcal{K}$. (See also [32].)

In [45], the operator J_D^f defined in (2.3) was used to define monotone metrics in the equivalent form as

$$\Gamma^k_D(X, Y) = \langle X, (J_D^f)^{-1}(Y)\rangle_{\mathrm{HS}} = \operatorname{Tr} X^*(J_D^f)^{-1}(Y).$$

It might seem more natural to work with J_D^f which is a non-commutative version of multiplication by D rather than using its inverse Ω_D^k (introduced in [33]). However, in this paper we use Ω_D^k instead of J_D^f since it avoids the need to take inverses to define our target maps.

In [54], monotone metrics of the form $\Gamma^k_Q(P-Q, P-Q)$ with $P, Q \in \mathcal{D}_d$ played an important role in the study of mixing times of Markov processes. It was observed in [54, Section III.B] and [33, Section IV.C] that when both Ω_D^k and its inverse $(\Omega_D^k)^{-1}$ are positivity-preserving, one can obtain a useful upper bound on the contraction of Riemannian metrics, which is described in more detail in Appendix B. In [54] this bound was used in the case $k(x) = x^{-1/2}$ for which both $\Omega_D^k(A) = D^{-1/2}AD^{-1/2}$ and the inverse $D^{1/2}AD^{1/2}$ clearly map $\mathbb{P}_d$ into itself. (In fact, for every $D \in \mathbb{P}_d$ they are bijections on $\mathbb{P}_d$.) Theorem 3.5 below implies that $k(x) = x^{-1/2}$ is the only function in $\mathcal{K}$ with this property.

The study of *quasi-entropies* was also initiated by Petz in [43, 44, 41], which can be defined from any operator convex function g on $(0, \infty)$ with $g(1) = 0$ as

$$H_g(A, B, K) \equiv \langle K, g\big(L_A R_B^{-1}\big) R_B K\rangle_{\mathrm{HS}} = \operatorname{Tr} \sqrt{B} K^* g\big(L_A R_B^{-1}\big)(K\sqrt{B}) \tag{2.8}$$

for $A, B \in \mathbb{P}_d$, $K \in \mathbb{M}_d$. It was later observed in [46, 48] that for any $D \in \mathcal{D}_d$ the Hessian

$$-\frac{\partial^2}{\partial a \partial b} H_g(D + aX, D + bY, I)\Big|_{a=b=0}, \qquad X, Y \in \mathbb{H}_d^0,$$

can be associated with a monotone Riemannian metric in some important examples. This was proved for more general g in [33, Theorem II.8], where it was also noticed that $g(x) = (1-x)^2 k(x)$ with $k \in \mathcal{K}$ is an operator convex function on $(0, \infty)$ with $g(1) = 0$. Moreover, the symmetry condition $xk(x) = k\big(x^{-1}\big)$ implies that $xg\big(x^{-1}\big) = g(x)$ and the quasi-entropy with $K = I$ has the symmetry property

$$H_g(A, B, I) = H_g(B, A, I). \tag{2.9}$$

and that every quasi-entropy with this symmetry property comes from a $k \in \mathcal{K}$. The quantity $H_g(A, B, I)$ is often called an *f-divergence*. See [21] for a thorough discussion of f-divergences (without the additional symmetry condition).

When $k(x) = 4/(1+\sqrt{x})^2$ so that $g(x) = 4(1-\sqrt{x})^2$, the function $H_g(A, A, K)$ is the *Wigner-Yanase skew information* [56], which Dyson suggested extending to the case including the parameter $p \in (0, 1)$, that is equivalent to using $g(x) = 4(1-x^p)(1-x^{1-p})$. This led to Lieb's seminal work on concave trace functions [34], in which he showed that $(A, B) \mapsto \mathrm{Tr}\, K^* A^p K B^{1-p}$ is jointly concave in $A, B \in \mathbb{P}_d$ when $p \in (0, 1)$. It is implicit[1] in Ando's paper [3] that the quasi-entropy $H_g(A, B, K)$ can be extended to $g(x) = (1-x^p)(1-x^{1-p})/p(1-p)$ with $p \in [-1, 2]$. Hasegawa [17] seems to have been the first to use well-known properties of monotone and convex operator functions to explicitly recognize that replacing 4 by $1/p(1-p)$ allows one to extend the quasi-entropy[2] for the WYD skew information and the associated Riemannian metric to the maximal range $p \in [-1, 2]$ (with $p = 0, 1$ defined as limits[3]). See also [23] where equality conditions were given for the convexity of $H_g(A, B, K)$ and some other inequalities for the extended WYD family.

In this paper, we make use of tools developed by Hiai and Kosaki [19, 20] in study of means of operators.[4] Motivated by this work, whenever $k \in \mathcal{K}$, we define

$$M^k(x, y) \equiv \frac{y}{k(x/y)}, \qquad x, y > 0, \tag{2.10}$$

$$M^k(A, B) \equiv R_B\Big(k\big(L_A R_B^{-1}\big)\Big)^{-1}, \qquad A, B \in \mathbb{P}_d. \tag{2.11}$$

From (2.1) and (2.3) we have in particular

$$M^k(D, D) = \big(\Omega_D^k\big)^{-1} = \Omega_{D^{-1}}^{\widehat{k}} = J_D^f. \tag{2.12}$$

The function $M^k(x, y)$ is called a *symmetric homogeneous mean* for positive scalars, i.e., $M = M^k : (0, \infty) \times (0, \infty) \to (0, \infty)$ is a continuous function such that

(1) $M(x, y) = M(y, x)$,

(2) $M(tx, ty) = tM(x, y)$ for $t > 0$,

(3) $M(x, y)$ is non-decreasing in x, y,

(4) $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

With $f = 1/\widehat{k}$, definition (2.10) is equivalent to $M^k(x, y) = y\, f(x/y)$ which was used in [19, 20] under the weaker condition that f is non-decreasing in the numerical sense. It follows from Proposition 2.5 that as k runs through $\mathcal{K}$ both conventions generate the same set of operators of the form (2.11).

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[1] Ando found an alternate proof of Lieb's concavity results and also showed convexity for $p \in (1, 2)$. Both Lieb and Ando ignored the linear term $\mathrm{Tr}\, K^* A K$ in the skew information, since it was irrelevant to convexity.

[2] Hasegawa actually used the asymmetric $g(x) = (1 - x^p)/p(1 - p)$. However, it follows from Eq. (37) in [33] that this yields the same $k(x)$ given by (4.6) as the symmetric version above.

[3] Lindblad [35] was the first to observe that one could recover joint convexity of the usual relative entropy by taking $\lim_{p\to 1}$ in Lieb's result.

[4] This work was motivated by inequalities for unitarily invariant norms. The term *mean* used there does not, in general, yield the mean of a pair of operators in the sense of Kubo and Ando [31].

2.5 The convex sets $\mathcal{K}$ and $\mathcal{K}^+$

Recall that $\mathcal{K}$ denotes the set of functions $k : (0,\infty) \to (0,\infty)$ satisfying any of the equivalent conditions of Theorem 2.4 and $k(1) = 1$. With $\widehat{k}(x) = 1/k\big(x^{-1}\big)$ given in Proposition 2.5, $k \mapsto \widehat{k}$ is a bijective transformation on $\mathcal{K}$.

For $\nu \in [0,1]$ let us set

$$k_\nu^{\mathrm{ext}} \equiv \frac{(1+\nu)^2}{2} \cdot \frac{1+x}{(x+\nu)(1+\nu x)} = \frac{(1+\nu)}{2}\left(\frac{1}{x+\nu} + \frac{1}{1+x\nu}\right). \tag{2.13}$$

For any fixed $x \in (0,\infty)$, by computing the derivative of $k_\nu(x)$ in ν one can easily verify that $k_\nu(x)$ is non-increasing in $\nu \in [0,1]$ so that

$$k_1^{\mathrm{ext}}(x) = \frac{2}{1+x} \le k_\nu^{\mathrm{ext}}(x) \le \frac{1+x}{2x} = k_0^{\mathrm{ext}}(x), \qquad \nu \in (0,1). \tag{2.14}$$

Since Theorem 2.3 can be rewritten as

$$k(x) = \int_{[0,1]} k_\nu^{\mathrm{ext}}(x)\; dm(\nu) \tag{2.15}$$

with m a probability measure, one moreover has

$$\frac{2}{1+x} \le k(x) \le \frac{1+x}{2x}, \qquad k \in \mathcal{K}. \tag{2.16}$$

Thus, $\mathcal{K}$ has the smallest element $k_1^{\mathrm{ext}}(x) = 2/(1+x)$ and the largest element $k_0^{\mathrm{ext}}(x) = (1+x)/2x$ in the pointwise order.

Now we may consider $\mathcal{K}$ as a subset of the locally convex topological vector space consisting of real functions on $(0,\infty)$ with the pointwise convergence topology. Then it is obvious from (2.16) that $\mathcal{K}$ is a convex and compact subset. The uniqueness of the representing measure m in Theorem 2.3 implies that $\mathcal{K}$ is a Choquet simplex with the extreme points k_ν^{ext} for $\nu \in [0,1]$ (that is the reason for the notation k_ν^{ext}). Furthermore, since $\nu \mapsto k_\nu^{\mathrm{ext}}$ is a homeomorphism from the interval $[0,1]$ into $\mathcal{K}$, one sees that $\mathcal{K}$ is a so-called Bauer simplex (as in [16]).

Motivated by the work on contraction bounds in [33, 54] which is described in Appendix B, we define two subsets $\mathcal{K}^+$ and $\mathcal{K}^-$ of $\mathcal{K}$ as

$$\mathcal{K}^+ \equiv \{k \in \mathcal{K} : \Omega_D^k \text{ is CP for every } D \in \mathbb{P}_d,\ d \in \mathbf{N}\},$$
$$\mathcal{K}^- \equiv \{k \in \mathcal{K} : (\Omega_D^k)^{-1} \text{ is CP for every } D \in \mathbb{P}_d,\ d \in \mathbf{N}\}.$$

It follows from (2.3) that

$$k \in \mathcal{K}^+ \Longleftrightarrow \widehat{k} \in \mathcal{K}^- \quad \text{where} \quad \widehat{k}(x) = 1/k\big(x^{-1}\big). \tag{2.17}$$

It follows from from Theorem 2.6 that $\mathcal{K}^+$ and $\mathcal{K}^-$ are closed under pointwise convergence. Although $\mathcal{K}^+$ is convex, $\mathcal{K}^-$ is not convex (as shown in Example 4.4 below). Since $\mathcal{K}^+$ is a compact convex subset of $\mathcal{K}$, it is the closed convex hull of its extreme points by the Krein-Milman theorem. However, determining all the extreme points of $\mathcal{K}^+$ seems quite challenging. Some non-trivial ones are described in Example 4.3.

In this paper, we have chosen to formulate most of our results in terms of functions $k \in \mathcal{K}$. As is clear from Theorem 2.4 we can also define the convex set of functions $\mathcal{F}$ with $f = 1/k$ which satisfy property (c) or (d). Although our choice is partly a matter of taste, in some situations, one may be more convenient than the other. We find it useful here to let

$$\mathcal{F}^{\pm} \equiv \{f \in \mathcal{F} : (\Omega_D^{1/f})^{\pm 1} \text{ is CP for every } D \in \mathbb{P}_d,\ d \in \mathbf{N}\},$$

so that $\mathcal{K}^{\pm}$ corresponds to $\mathcal{F}^{\pm}$ by $k \leftrightarrow f = 1/k$. Since $1/k(x) = \widehat{k}(x^{-1}) = x\,\widehat{k}(x)$, it is obvious by (2.17) that

$$\mathcal{F} = \{xk(x) : k \in \mathcal{K}\}, \qquad \mathcal{F}^{+} = \{xk(x) : k \in \mathcal{K}^{-}\}, \qquad \mathcal{F}^{-} = \{xk(x) : k \in \mathcal{K}^{+}\}.$$

Hence $k \leftrightarrow xk(x)$ gives an affine correspondence between $\mathcal{K}$ and $\mathcal{F}$, by which $\mathcal{K}^{+}$ is isomorphic to $\mathcal{F}^{-}$. Therefore, $\mathcal{F}^{-}$ is also convex and the extreme points of $\mathcal{F}$ are

$$f_\nu^{\text{ext}}(x) = x\,k_\nu^{\text{ext}}(x) = \frac{(1+\nu)^2}{2} \cdot \frac{x\,(1+x)}{(x+\nu)(1+\nu x)}. \tag{2.18}$$

3 Positive kernels and induced order

3.1 Basic definitions

In principle, the condition of Theorem 2.6 (c) gives a simple criterion for complete positivity. But in practice, it is not easy to verify that either the matrix A in (2.4) or one of those in (2.5) is positive semi-definite. Only a few examples can be resolved using this criterion. However, there is another equivalent condition based on the theory of functions which define positive kernels.

Definition 3.1. A continuous function $h : \mathbf{R} \mapsto \mathbf{C}$ is called *positive definite* if $h(x-y)$ is a positive semi-definite kernel, i.e., $\big[h(t_i - t_j)\big]_{1 \le i,j \le d}$ is positive semi-definite for any $t_1, \ldots, t_d \in \mathbf{R}$ with any $d \in \mathbb{M}$, or equivalently,

$$\iint \overline{\varphi(s)} h(s-t) \varphi(t)\, ds\, dt \ge 0, \qquad \varphi \in C_0^\infty(\mathbf{R}),$$

where $C_0^\infty(\mathbf{R})$ denotes the smooth compactly supported functions on $\mathbf{R}$. Functions satisfying this condition are sometimes called "functions of positive type" [50, Section IX.2] or "positive in the sense of Bochner".

Moreover, h is called *infinitely divisible* if $h(t)^r$ is positive definite for every $r > 0$, or equivalently, $h(t)^{1/n}$ is positive definite for every $n \in \mathbf{N}$.

For convenience, some basic properties of positive definite functions stated here:

(a) A positive definite function h is uniformly bounded on $\mathbf{R}$ as $|f(t)| \le f(0)$.

(b) Bochner's theorem (see [50, Theorem IX.9], [1, Section 60]) says that h is positive definite if and only if it is the Fourier transform of a finite positive measure on $\mathbf{R}$. Thus, positive definiteness of h can be checked, in principle, by testing positivity of its Fourier transform.

(c) The product of positive definite functions is positive definite. This immediately follows from the well-known fact that the Fourier transform of the convolution of two finite measures is the product of their Fourier transforms, or from the Schur product theorem for positive semi-definite matrices.

In this paper we only consider positive definite functions on $\mathbf{R}$ so that we shall omit "on $\mathbf{R}$" in the rest. Positive definite functions played an important role in the work [19, 20] on means of operators, where a partial order was introduced. The following definition is its adaptation to functions in $\mathcal{K}$:

Definition 3.2. For $k_1, k_2 \in \mathcal{K}$ we write $k_1 \preccurlyeq k_2$ if either of the following equivalent conditions holds:

(a) the function $k_1(e^t)/k_2(e^t)$ is positive definite on $\mathbf{R}$;

(b) the matrix

$$\left[\frac{k_1(w_i/w_j)}{k_2(w_i/w_j)}\right]_{1\le i,j\le d}$$

is positive semi-definite for every $w_1, \ldots, w_d > 0$ with any $d \in \mathbf{N}$.

It is easily verified as in [19, 20] that $\preccurlyeq$ is really a partial order in $\mathcal{K}$, and $k_1 \preccurlyeq k_2$ implies $k_1(x) \le k_2(x)$ on $(0, \infty)$, i.e., $k_1 \le k_2$ pointwise.

The stronger condition that $k_1(e^t)/k_2(e^t)$ is infinitely divisible (following Definition 3.1), was studied in [8]. Results given there sometimes play a role in showing that the one-parameter families studied in Section 4.2 are monotonic in the $\preccurlyeq$ order. Moreover, infinite divisibility is important in the discussion of geometric bridges in Sections 4.3 and 6.3. Some examples considered here require new results for specific functions which are obtained in Sections 5.3 and 5.4.

The next useful lemma on positive definite functions will often be used in this paper. See [25, Appendix B] and [9, Theorem 3.2] for the proof of (1). On the other hand, (2) was first proved in [9, Theorem 5.1] while the "if part" was pointed out earlier in [57].

Lemma 3.3.

(1) *The function* $\sinh \alpha t/\sinh t$ *is positive definite for* $\alpha \in (0, 1)$.

(2) *For* $\beta > -1$, *the function* $(\cosh t + \beta)^{-1}$ *is positive definite if and only if* $\beta \le 1$.

3.2 Basic applications

The next theorem gives a basic characterization of the class $\mathcal{K}^+$. The equivalence of (a)–(c) follows immediately from Theorem 2.6 with A replaced by the second matrix in (2.5). The equivalence of (b) and (d) is an adaptation of [19, Theorem 1.1] via (2.12) in the present situation.

Theorem 3.4. *The following conditions for* $k \in \mathcal{K}$ *are equivalent*:

(a) $k \in \mathcal{K}^+$, *i.e.*, Ω_D^k *is CP for every* $D \in \mathbb{P}_d$ *with any* $d \in \mathbf{N}$;

(b) $k \preccurlyeq x^{-1/2}$;

(c) $e^{t/2}k(e^t)$ *is positive definite*;

(d) *there exists a symmetric probability measure* ν *on* $\mathbf{R}$ *such that*

$$\Omega_D^k(X) = \int_{-\infty}^{\infty} D^{-\frac{1}{2}+it} X D^{-\frac{1}{2}-it}\, d\nu(t) \tag{3.1}$$

for all $D \in \mathbb{P}_d$ *and* $X \in \mathbb{M}_d$ *with any* $d \in \mathbf{N}$.

It is a well-known consequence of the Stinespring representation theorem that a CP map Φ on the matrix algebra $\mathbb{M}_d$ can be represented in the form $\Phi(A) = \sum_j F_j A F_j^*$ with at most d^2 matrices $F_j \in \mathbb{M}_d$ (see, e.g., [30, 10], [42, Proposition 4.7] or [24, Appendix A]). Thus, for any fixed $D \in \mathbb{P}_d$, when Ω_D^k is CP, one can find matrices F_j in $\mathbb{M}_d$ such that $\Omega_D^k(X) = \sum_{j=1}^m F_j X F_j^*$ with $m \le d^2$. But, for fixed $k \in \mathcal{K}$, the representation will change with D (hence with d). (Even for fixed D the F_j in the representation are only determined up to a unitary transformation $F_j \mapsto \sum_i u_{ij} F_i$ with u_{ij} entries of a unitary matrix.) However, when we are allowed to use integral representation, Theorem 3.4 says that we have the standard representation given in (3.1), from which the CP of the map Ω_D^k is directly seen. Moreover, one sometimes has different integral expressions of Ω_D^k or $(\Omega_D^k)^{-1}$; a typical example is (1.1) for Ω_D^k in case of $k(x) = \log x/(x-1)$ (see Appendix C.2).

It follows immediately from (2.17) and Theorem 3.4 that $k \in \mathcal{K}^-$ if and only if $k \succcurlyeq x^{-1/2}$. Consequently, $x^{-1/2}$ is the largest element of $\mathcal{K}^+$ and the smallest of $\mathcal{K}^-$. Moreover, since $\preccurlyeq$ is a partial order on $\mathcal{K}$, we conclude

Theorem 3.5. *The only function in* $\mathcal{K}$ *for which both* Ω_D *and* Ω_D^{-1} *are CP for every* $D \in \mathbb{P}_d$, $d \in \mathbf{N}$, *is* $x^{-1/2}$.

It follows from Theorem 3.4 that the problem of determining whether or not $k \in \mathcal{K}$ belongs to $\mathcal{K}^+$ can be reduced to the computation of the Fourier transform of the function $e^{t/2}k(e^t)$. However, this is often a hard task as will be seen in Section 5.

In contrast to $\mathcal{K}$, it does not seem easy to find extreme points of $\mathcal{K}^+$ other than $x^{-1/2}$ and $2/(1+x)$ which are the largest and the smallest elements of $\mathcal{K}^+$, respectively, in the order $\preccurlyeq$ as well as the pointwise order. However, some new extreme points will be described in Example 4.3 and Theorem 6.2. In addition, a natural boundary point will be found in Example 4.8 which is conjectured to be an extreme point.

By comparing part (b) of the next result to (2.14), one immediately sees that $\preccurlyeq$ is stronger than the pointwise order.

Proposition 3.6. *The following relations hold.*

(a) $k_1^{\rm ext}(x) = \dfrac{2}{1+x} \preccurlyeq k_\nu^{\rm ext}$ *and* $\widehat{k}_\nu^{\rm ext} \preccurlyeq \dfrac{1+x}{2x} = k_0^{\rm ext}(x)$ *for all* $\nu \in [0,1]$.

(b) $k_\nu^{\rm ext} \not\preccurlyeq \dfrac{1+x}{2x} = k_0^{\rm ext}(x)$ *and* $k_1^{\rm ext}(x) = \dfrac{2}{1+x} \not\preccurlyeq \widehat{k}_\nu^{\rm ext}$ *for all* $\nu \in (0,1)$.

(c) $2/(1+x) \preccurlyeq x^{-1/2} \preccurlyeq (1+x)/2x$.

Proof. A straightforward computation gives

$$\frac{k_1^{\mathrm{ext}}(e^t)}{k_\nu^{\mathrm{ext}}(e^t)} = \frac{\widehat{k}_\nu^{\mathrm{ext}}(e^{-t})}{k_0^{\mathrm{ext}}(e^{-t})} = \frac{4}{(1+\nu)^2} \cdot \frac{(e^t+\nu)(1+\nu e^t)}{(1+e^t)^2}$$
$$= \frac{4}{(1+\nu)^2} \cdot \frac{\nu(e^t+e^{-t}+2)+(1-\nu)^2}{e^t+e^{-t}+2}$$
$$= \frac{4\nu}{(1+\nu)^2} + \frac{2(1-\nu)^2}{(1+\nu)^2} \cdot \frac{1}{\cosh t + 1}$$

from which (a) follows by using $\beta = 1$ in Lemma 3.3 (2). Similarly

$$\frac{k_\nu^{\mathrm{ext}}(e^t)}{k_0^{\mathrm{ext}}(e^t)} = \frac{k_1^{\mathrm{ext}}(e^{-t})}{\widehat{k}_\nu^{\mathrm{ext}}(e^{-t})} = (1+\nu)^2 \, \frac{e^t}{(e^t+\nu)(1+\nu e^t)}$$
$$= (1+\nu)^2 \, \frac{1}{\nu(e^t+e^{-t})+1+\nu^2}$$
$$= \frac{(1+\nu)^2}{2\nu} \cdot \frac{1}{\cosh t + \frac{1+\nu^2}{2\nu}}.$$

Since $(1+\nu^2)/2\nu > 1$ for $\nu \in (0,1)$, this proves (b) by Lemma 3.3 (2) again. Finally (c) follows easily from

$$e^{t/2} k_1^{\mathrm{ext}}(e^t) = \frac{e^{-t/2}}{k_0^{\mathrm{ext}}(e^t)} = \frac{2e^{t/2}}{e^t+1} = \frac{1}{\cosh(t/2)}.$$

QED

Proposition 2.2 implies that for every $k \in \mathcal{K}$, $xk(x)$ is bounded on $(0,b)$ and $k(x)$ is bounded on (a,∞) for any $a,b > 0$. Theorem 3.4 implies that a necessary condition for $k \in \mathcal{K}^+$ is the stronger property that $x^{1/2}k(x)$ is bounded on $(0,\infty)$. However, this is not a sufficient condition. Indeed, it holds for all $k_\nu^{\mathrm{ext}}(x)$ with $\nu \in (0,1]$. Yet, as will be seen in Example 4.1 $k_\nu^{\mathrm{ext}}(x) \in \mathcal{K}^+$ only for $\nu = 1$. The following result will be used in Example 4.4 to analyze convex combinations of $x^{-1/2}$ and k_ν^{ext}.

Proposition 3.7. *Assume that* $k \in \mathcal{K} \setminus \mathcal{K}^+$ *and* $\lim_{x\to\infty} x^{1/2}k(x) = 0$. *Then for every* $\lambda \in (0,1]$,

$$\lambda k(x) + (1-\lambda)x^{-1/2} \notin \mathcal{K}^+.$$

Proof. Assume that $k \in \mathcal{K}$ satisfies $\lim_{x\to\infty} x^{1/2}k(x) = 0$ and $\lambda k(x) + (1-\lambda)x^{-1/2} \in \mathcal{K}^+$ with some $\lambda \in (0,1]$. Then, thanks to Theorem 3.4 and Bochner's theorem there is a probability measure μ on $\mathbf{R}$ satisfying

$$\lambda e^{t/2}k(e^t) + (1-\lambda) = \hat{\mu}(t) \equiv \int_{-\infty}^{\infty} e^{its}\, d\mu(s), \qquad t \in \mathbf{R}.$$

However, the symmetry condition $xk(x) = k\big(x^{-1}\big)$ implies $e^{t/2}k(e^t) = e^{-t/2}k(e^{-t})$, $t \in \mathbf{R}$, and hence $\lim_{|t|\to\infty} e^{t/2}k(e^t) = 0$. Therefore, we have

$$\mu(\{0\}) = \lim_{|t|\to\infty} \hat{\mu}(t) = 1 - \lambda$$

(see [20, Corollary A.8]). This means $e^{t/2}k(e^t) = \hat{\mu}_0(t)$, $t \in \mathbf{R}$, with the probability measure $\mu_0 = \lambda^{-1}(\mu - \mu(\{0\})\delta_0)$, implying the contradiction $k \in \mathcal{K}^+$. **QED**

4 Examples

In this section we list known families of functions in $\mathcal{K}$ and investigate which functions in those families belong to $\mathcal{K}^+$ (or $\mathcal{K}^-$). In this way we will see that $\mathcal{K}^+$ indeed contains a variety of functions even though it occupies only a small part of $\mathcal{K}$.

4.1 Extreme points and simple averages

Example 4.1. (Extreme points of $\mathcal{K}$) The extreme points of $\mathcal{K}$ are $k_\nu^{\rm ext}$, $\nu \in [0,1]$, given in (2.13). These are not in $\mathcal{K}^+$ unless $\nu = 1$ for which we have $k_1^{\rm ext}(x) = 2/(1+x)$. Indeed, for $\nu \in (0,1]$ we find

$$e^{t/2}k_\nu^{\rm ext}(e^t) = \frac{(1+\nu)^2}{2\nu} \cdot \frac{\cosh(t/2)}{\cosh t + \frac{1+\nu^2}{2\nu}}.$$

If $e^{t/2}k_\nu^{\rm ext}(e^t)$ is positive definite, then so is its product with the positive definite $1/\cosh(t/2)$. But this yields (up to a constant) a function of the form in Lemma 3.3 (2), which is not positive definite for $\beta = (1+\nu^2)/2\nu > 1$ when $\nu \in (0,1)$.

It was shown in [6, Example 9] that $k_\nu^{\rm ext}(x) \le x^{-1/2}$ (in the pointwise order) for all $x > 0$ if and only if $3 - 2\sqrt{2} \le \nu \le 1$. This example provides another demonstration that the $\preccurlyeq$ order is stronger and $\preccurlyeq x^{-1/2}$ is the key to determining whether or not a function $k \in \mathcal{K}^+$.

Example 4.2. (Convex combinations involving $k_0^{\rm ext}$) Consider the convex combinations

$$a_{1,0,\lambda}(x) \equiv \lambda k_0^{\rm ext}(x) + (1-\lambda)k_1^{\rm ext}(x) = \lambda\frac{1+x}{2x} + (1-\lambda)\frac{2}{1+x}, \qquad \lambda \in [0,1],$$

of the smallest element of $\mathcal{K}^+$ and the largest element of $\mathcal{K}$. Since

$$e^{t/2}a_{1,0,\lambda}(e^t) = \frac{1-\lambda}{\cosh(t/2)} + \lambda\cosh(t/2)$$

is unbounded for any $\lambda \in (0,1]$, it cannot be positive definite and hence combining an arbitrarily small amount of $k_0^{\rm ext}$ (the largest element of $\mathcal{K}^-$) with the smallest element of $\mathcal{K}^+$ moves out of $\mathcal{K}^+$.

A similar argument can be used to show that any $k \in \mathcal{K}$ for which the measure m in (2.15) has the property that $m(\{0\}) > 0$ cannot be in $\mathcal{K}^+$.

Example 4.3. (Convex combinations of k_1^{ext} and k_ν^{ext}) Replacing k_0^{ext} in the previous example with another k_ν^{ext} does sometimes yield convex combination in $\mathcal{K}^+$. To be precise, let

$$a_{1,\nu,\lambda}(x) \equiv \lambda k_\nu^{\text{ext}}(x) + (1-\lambda)\,\frac{2}{1+x}, \qquad \lambda \in [0,1], \tag{4.1}$$

be a convex combination of the smallest $k_1^{\text{ext}}(x) = 2/(1+x)$ of $\mathcal{K}^+$ and other extreme points k_ν^{ext} of $\mathcal{K}$, $\nu \in (0,1)$. We are interested in the problem to determine ν, λ for which $a_{1,\nu,\lambda}$ belongs to $\mathcal{K}^+$. Our result is that for every $\nu \in [0,1)$, $a_{1,\nu,\lambda}$ is in $\mathcal{K}^+$ if and only if

$$0 \le \lambda \le \frac{2\sqrt{\nu}}{(1+\sqrt{\nu})^2} = \frac{2}{\left(\nu^{1/4}+\nu^{-1/4}\right)^2}. \tag{4.2}$$

Moreover, $a_{1,\nu,\lambda}$ is an extreme point of $\mathcal{K}^+$ if and only if equality holds in (4.2).

Since the proofs require some technical results from Section 5, they are postponed to Section 6.2. Note that the right-hand side of (4.2) is $< \frac{1}{2}$ but $a_{1,1,\lambda}(x) = 2/(1+x) \in \mathcal{K}^+$ for $\lambda \in [0,1]$. Thus, this example exhibits some discontinuous behavior at $\nu = 1$.

It is straightforward (see the last paragraph of Section 2) to extend these results to show that the function

$$a_{1,\nu,\lambda}(x^{-1}) = \lambda f_\nu^{\text{ext}}(x) + (1-\lambda)\,\frac{2x}{1+x}, \qquad \lambda \in [0,1),$$

is in $\mathcal{F}^-$ if and only if the inequality holds in (4.2) and that it is an extreme point of $\mathcal{F}^-$ if and only if equality holds.

Example 4.4. (Extended Heron means) Consider the convex combinations of $x^{-1/2}$ and extreme points of $\mathcal{K}$, i.e.,

$$\lambda k_\nu^{\text{ext}}(x) + (1-\lambda)x^{-1/2} \tag{4.3}$$

which are sometimes known as Heron means when $\nu = 1$, in which case (4.3) is obviously in $\mathcal{K}^+$ for all $\lambda \in [0,1]$. However, for $\nu = 0$ the function (4.3) is in $\mathcal{K}^+$ only for $\lambda = 0$ since $x^{1/2}k_0^{\text{ext}}(x)$ is unbounded. Furthermore, it follows from Proposition 3.7 that for $\nu \in (0,1)$ and $\lambda \ne 0$ the function in (4.3) is never in $\mathcal{K}^+$ because $k_\nu^{\text{ext}} \notin \mathcal{K}^+$ and $x^{1/2}k_\nu^{\text{ext}}(x) \to 0$ as $x \to \infty$.

Next, consider (4.3) with $\nu = 0$ as the convex combination of the largest and the smallest elements of $\mathcal{K}^-$ for $\lambda \in \left(0, \frac{1}{2}\right)$. Since

$$e^{-t}\left(\lambda\,\frac{1+e^{2t}}{2e^{2t}} + (1-\lambda)e^{-t}\right)^{-1} = \frac{1}{\lambda}\cdot\frac{1}{\cosh t + \frac{1-\lambda}{\lambda}}$$

with $(1-\lambda)/\lambda > 1$ is not positive definite by Lemma 3.3 (2), we have $\lambda \widehat{k}_1^{\text{ext}}(x) + (1-\lambda)x^{-1/2} \notin \mathcal{K}^-$, showing that $\mathcal{K}^-$ is not convex. However, the dual set $\mathcal{F}^-$ is convex and $f \mapsto k = 1/f$ transforms $\mathcal{F}^-$ to $\mathcal{K}^-$. Thus, although $\mathcal{K}^-$ is not convex, harmonic means of functions in $\mathcal{K}^-$ are in $\mathcal{K}^-$.

4.2 Families of classic functions in $\mathcal{K}$

Examples given so far suggest us that $\mathcal{K}^+$ is a rather thin subset of $\mathcal{K}$. Therefore, it is a bit surprising that we find a number of one-parameter families in $\mathcal{K}^+$ in the examples below. Each of these families shows some type of symmetry and monotonicity in the $\preccurlyeq$ order on maximally suitable intervals. In all these cases, the symmetry condition $xk(x) = k\big(x^{-1}\big)$ can be easily checked and it is rather straightforward to use the Pick mapping property to verify that they are in $\mathcal{K}$. Although the most intriguing family is associated with the WYD skew information, it is also rather complex.

Example 4.5. (Heinz type means) The family of functions

$$k^{\mathrm{H}}_\alpha(x) \equiv \frac{2}{x^\alpha + x^{1-\alpha}}, \qquad \alpha \in [0,1], \tag{4.4}$$

has the dual family

$$\widehat{k}^{\mathrm{H}}_\alpha(x) \equiv \frac{1}{k^{\mathrm{H}}_\alpha\big(x^{-1}\big)} = \frac{x^{-\alpha} + x^{-1+\alpha}}{2}, \qquad 0 \le \alpha \le 1.$$

which were used in [54]. One easily recovers the Heinz type means via (2.10) since

$$\frac{y}{k^{\mathrm{H}}_\alpha(x/y)} = \frac{x^\alpha y^{1-\alpha} + x^{1-\alpha} y^\alpha}{2}, \qquad \alpha \in [0,1].$$

In addition to $k^{\mathrm{H}}_{1/2}(x) = x^{-1/2} = \widehat{k}^{\mathrm{H}}_{1/2}(x)$, important special cases are

$$k^{\mathrm{H}}_0(x) = k^{\mathrm{H}}_1(x) = \frac{2}{1+x} = k^{\mathrm{ext}}_1(x), \qquad \widehat{k}^{\mathrm{H}}_0(x) = \widehat{k}^{\mathrm{H}}_1(x) = \frac{1+x}{2x} = k^{\mathrm{ext}}_0(x)$$

reflecting the obvious symmetry around $x = \frac{1}{2}$.

Since $e^{t/2} k^{\mathrm{H}}_\alpha(e^t) = 1/\cosh\big((\alpha - \frac{1}{2})t\big)$ is positive definite, $k^{\mathrm{H}}_\alpha \in \mathcal{K}^+$ for any $\alpha \in [0,1]$ and $\widehat{k}^{\mathrm{H}}_\alpha \in \mathcal{K}^-$ for any $\alpha \in [0,1]$. A different proof of the former was in [6, Example 3].

If $0 \le \alpha \le \beta \le \frac{1}{2}$, then $k^{\mathrm{H}}_\alpha \preccurlyeq k^{\mathrm{H}}_\beta$ (see [19, Section 2]) so that the pair of functions k^{H}_α for $\alpha \in \big[0, \frac{1}{2}\big]$ and $\widehat{k}^{\mathrm{H}}_\alpha$ for $\alpha \in \big[\frac{1}{2}, 1\big]$ can be regarded as a single family which increases in the $\preccurlyeq$ order from the smallest to the largest element of $\mathcal{K}$.

Moreover, whenever $0 \le \alpha \le \beta \le \frac{1}{2}$,

$$\frac{k^{\mathrm{H}}_\alpha(e^t)}{k^{\mathrm{H}}_\beta(e^t)} = \frac{\cosh\big(\big(\frac{1}{2} - \beta\big)\, t\big)}{\cosh\big(\big(\frac{1}{2} - \alpha\big)\, t\big)}$$

is infinitely divisible by [8, Theorem 1].

Example 4.6. (Binomial means or power means) The functions

$$k^{\mathrm{B}}_\alpha(x) \equiv \left(\frac{2}{x^\alpha + 1}\right)^{1/\alpha}, \qquad \alpha \in [-1,1],$$

are easily verified to be in $\mathcal{K}$ as observed in [39, Theorem 3 (i)] and correspond to the binomial (or power) means

$$\frac{y}{k_\alpha^{\mathrm{B}}(x/y)} = \left(\frac{x^\alpha + y^\alpha}{2}\right)^{1/\alpha}, \qquad \alpha \in [-1,1].$$

Interesting special cases are

$$\begin{aligned}
k_{-1}^{\mathrm{B}}(x) &= \frac{1+x}{2x} = k_0^{\mathrm{ext}}(x), & k_{-1/2}^{\mathrm{B}}(x) &= \frac{(1+\sqrt{x})^2}{4x} = \widehat{k}_{1/2}^{\mathrm{WYD}}(x),\\
k_0^{\mathrm{B}}(x) &= \lim_{\alpha\to 0} k_\alpha^{\mathrm{B}}(x) = x^{-1/2}, & k_{1/2}^{\mathrm{B}}(x) &= \frac{4}{(1+\sqrt{x})^2} = k_{1/2}^{\mathrm{WYD}}(x),\\
k_1^{\mathrm{B}}(x) &= \frac{2}{1+x} = k_1^{\mathrm{ext}}(x),
\end{aligned}$$

where k_p^{WYD} is given in Example 4.8. Moreover, $k_\alpha^{\mathrm{B}}(x) = \widehat{k}_{-\alpha}^{\mathrm{B}}(x)$ which implies that for this family

$$(\Omega_D^\alpha)^{-1} = \Omega_{D^{-1}}^{-\alpha}, \qquad \alpha \in [-1,1],$$

with the obvious abuse of notation. It follows from [27, Theorem 9] that if $-1 \le \beta \le \alpha \le 1$ then $k_\alpha^{\mathrm{B}} \preccurlyeq k_\beta^{\mathrm{B}}$, so that we have a decreasing family in the $\preccurlyeq$ order. Since $k_0^{\mathrm{B}}(x) = x^{-1/2}$, we conclude

- $k_\alpha^{\mathrm{B}} \in \mathcal{K}^+$ if and only if $\alpha \in [0,1]$,
- $k_\alpha^{\mathrm{B}} \in \mathcal{K}^-$ if and only if $\alpha \in [-1,0]$.

Moreover, $k_\alpha^{\mathrm{B}}(e^t)/k_\beta^{\mathrm{B}}(e^t)$ is infinitely divisible whenever $\beta \le \alpha$ [27, Theorem 9].

Example 4.7. (Power difference means) The family of functions

$$k_\alpha^{\mathrm{PD}}(x) \equiv \frac{\alpha}{\alpha-1}\cdot\frac{x^{\alpha-1}-1}{x^\alpha-1}, \qquad \alpha \in [-1,2],$$

gives the family of power difference means considered in [19, 20]. In fact,

$$\frac{y}{k_\alpha^{\mathrm{PD}}(x/y)} = M_\alpha(x,y) \equiv \frac{\alpha-1}{\alpha}\cdot\frac{x^\alpha - y^\alpha}{x^{\alpha-1}-y^{\alpha-1}}, \tag{4.5}$$

whose family is also called the A-L-G interpolation means since it interpolates the arithmetic, the logarithmic and the geometric means by allowing us to recover all of these as special cases

$$\begin{aligned}
k_{-1}^{\mathrm{PD}}(x) &= \frac{1+x}{2x}, & k_0^{\mathrm{PD}}(x) &= \lim_{\alpha\to 0} k_\alpha^{\mathrm{PD}}(x) = \frac{x-1}{x\log x},\\
k_{1/2}^{\mathrm{PD}}(x) &= x^{-1/2}, & k_1^{\mathrm{PD}}(x) &= \lim_{\alpha\to 1} k_\alpha^{\mathrm{PD}}(x) = \frac{\log x}{x-1},\\
k_2^{\mathrm{PD}}(x) &= \frac{2}{1+x}.
\end{aligned}$$

It is known [19, Proposition 4.2] that $k_\alpha^{\mathrm{PD}} \in \mathcal{K}$ for all $\alpha \in [-1,2]$. Moreover, we have $k_\alpha^{\mathrm{PD}} = \widehat{k}_{1-\alpha}^{\mathrm{PD}}$, which implies that for this family

$$(\Omega_D^\alpha)^{-1} = \Omega_{D^{-1}}^{1-\alpha}, \qquad \alpha \in [-1,2],$$

with the obvious abuse of notation. If $-1 \le \beta \le \alpha \le 2$ then $k_\alpha^{\mathrm{PD}} \preccurlyeq k_\beta^{\mathrm{PD}}$ (see [19, Theorem 2.1]), so that we have another increasing family. Thus, since $k_{1/2}^{\mathrm{PD}}(x) = x^{-1/2}$, we can conclude

- $k_\alpha^{\rm PD} \in \mathcal{K}^+$ if and only if $\alpha \in \left[\frac{1}{2}, 2\right]$,
- $k_\alpha^{\rm PD} \in \mathcal{K}^-$ if and only if $\alpha \in \left[-1, \frac{1}{2}\right]$.

Moreover, the monotonicity can be strengthened to the infinite divisibility of $k_\alpha^{\rm PD}(e^t)/k_\beta^{\rm PD}(e^t)$ for $\beta \le \alpha$ by [27, Theorem 5].

Example 4.8. (WYD family) One of the best known families in $\mathcal{K}$ is an outgrowth of the Wigner-Yanase-Dyson skew information discussed in Section 2.4 which leads to the functions

$$k_p^{\rm WYD}(x) \equiv \frac{1}{p(1-p)} \cdot \frac{(1-x^p)(1-x^{1-p})}{(1-x)^2}, \qquad p \in [-1, 2]. \tag{4.6}$$

This family is symmetric around $p = \frac{1}{2}$, and the special cases $p = 0, 1$ should be understood by continuity, i.e.,

$$k_1^{\rm WYD}(x) = k_0^{\rm WYD}(x) = \lim_{p \to 1} k_p^{\rm WYD}(x) = \frac{\log x}{x-1}.$$

Other important special cases are

$$k_{1/2}^{\rm WYD}(x) = \frac{4}{(1+\sqrt{x})^2}, \qquad k_{-1/2}^{\rm WYD}(x) = k_{3/2}^{\rm WYD}(x) = \frac{4}{3} \cdot \frac{1+\sqrt{x}+x}{\sqrt{x}(1+\sqrt{x})^2},$$
$$k_{-1}^{\rm WYD}(x) = k_2^{\rm WYD}(x) = \frac{1+x}{2x} = k_0^{\rm ext}(x).$$

We can summarize the CP situation for this family as follows:

(a) $k_p^{\rm WYD} \in \mathcal{K}^+$ if and only if $p \in [0, 1]$,

(b) $k_p^{\rm WYD} \in \mathcal{K}^-$ if and only if $p \in \left[-1, -\frac{1}{2}\right] \cup \left[\frac{3}{2}, 2\right]$.

For $p \in \left[\frac{1}{2}, 2\right]$ the functions $k_p^{\rm WYD}$ increase monotonically with respect to the $\preccurlyeq$ order. Set $r \equiv p + q - 1$, $\alpha \equiv p/r$, $\beta \equiv q/r$ so that $r > 0$ and $0 < \alpha < \beta$. We note

$$\begin{aligned}
\frac{k_p^{\rm WYD}(x)}{k_q^{\rm WYD}(x)} &= \frac{q(1-q)}{p(1-p)} \cdot \frac{(1-x^p)(1-x^{1-p})}{(1-x^q)(1-x^{1-q})} \\
&= \frac{\beta(\alpha-1)}{\alpha(\beta-1)} \cdot \frac{(1-x^{r\alpha})(1-x^{r(\beta-1)})}{(1-x^{r\beta})(1-x^{r(\alpha-1)})} = \frac{M_\alpha(x^r, 1)}{M_\beta(x^r, 1)},
\end{aligned}$$

where $M_\alpha(x, y)$ is the power difference mean defined by (4.5) (for any real parameter α). Therefore, when $\frac{1}{2} \le p \le q \le 2$, $k_p^{\rm WYD}(e^t)/k_q^{\rm WYD}(e^t)$ is infinitely divisible by [27, Theorem 5] and in particular $k_p^{\rm WYD} \preccurlyeq k_q^{\rm WYD}$.

Thus, the functions $k_p^{\rm WYD}$ form a smooth family which are in $\mathcal{K}^+$ up to $p = 1$ when p increases from $\frac{1}{2}$. Therefore, $k_1^{\rm WYD}$ lies on the boundary of $\mathcal{K}^+$, and we conjecture that it is an extreme point of $\mathcal{K}^+$.

The operator Ω_D^k for $k = k_1^{\rm WYD}$ is given by (1.1), which implies that $k_p^{\rm WYD} \in \mathcal{K}^+$ for $p = 0, 1$. In the proof of [6, Theorem 2], explicit (double) integral representations were obtained for Ω_D^k when $k = k_p^{\rm WYD}$ for $p \in (0, 1)$ in such a way that the CP of Ω_D^k immediately follows and hence $k_p^{\mathrm{WYD}} \in \mathcal{K}^+$ for $p \in (0,1)$. This gives the "if" part of (a). An alternate proof of this, as well as details for the remaining claim above are given in Section 6.1. This requires results from Section 5.1 which are of independent interest.

Unlike other families we consider, the functions $\widehat{k}_p^{\mathrm{WYD}}(x) = 1/k_p^{\mathrm{WYD}}\big(x^{-1}\big)$ do not belong to the WYD family. Despite the extensive study of WYD metrics, there seems to have been little attention given to this dual family

$$\widehat{k}_p^{\mathrm{WYD}}(x) \equiv p(1-p)\frac{(1-x)^2}{(x-x^{1-p})(x-x^p)}, \qquad p \in [-1,2].$$

This is symmetric around $p = \frac{1}{2}$ and special cases are

$$\widehat{k}_{1/2}^{\mathrm{WYD}}(x) = \frac{(1+\sqrt{x})^2}{4x}, \qquad \widehat{k}_1^{\mathrm{WYD}}(x) = \frac{x-1}{x\log x}, \qquad \widehat{k}_2^{\mathrm{WYD}}(x) = \frac{2}{1+x} = k_1^{\mathrm{ext}}(x).$$

By (2.17) and (b) above the functions $\widehat{k}_p^{\mathrm{WYD}}$ are in $\mathcal{K}^+$ for $p \in \big[\frac{3}{2}, 2\big]$ and in $\mathcal{K}^-$ for $p \in [0,1]$.

Example 4.9. (Stolarsky means) As in the WYD example above, the dual of the Stolarsky family gives a different family. In this case, we introduce both

$$k_\alpha^{\mathrm{St}}(x) \equiv \left(\frac{x^\alpha - 1}{\alpha(x-1)}\right)^{\frac{1}{1-\alpha}}, \qquad \widehat{k}_\alpha^{\mathrm{St}}(x) \equiv \left(\frac{x^{1-\alpha} - x}{\alpha(1-x)}\right)^{\frac{1}{\alpha-1}}, \qquad \alpha \in [-2,2].$$

It is known [39, Theorem 3 (iii)] (also [6, Theorem 3]) that $k_\alpha^{\mathrm{St}} \in \mathcal{K}$ for $\alpha \in [-2,2]$ and this range for α such that $k_\alpha^{\mathrm{St}} \in \mathcal{K}$ is optimal. The functions $k_\alpha^{\mathrm{St}}(x)$ correspond to the familiar family of Stolarsky means as

$$\frac{y}{k_\alpha^{\mathrm{St}}(x/y)} = y\,\widehat{k}_\alpha^{\mathrm{St}}(y/x) = S_\alpha(x,y) \equiv \left(\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right)^{\frac{1}{\alpha-1}}. \tag{4.7}$$

The mean $S_1(x,y) = e^{-1}(x^x/y^y)^{1/(x-y)}$ for $\alpha = 1$ is called the identric mean.

The functions k_α^{St} include more familiar special cases than $\widehat{k}_\alpha^{\mathrm{St}}$ as follows:

$$k_2^{\mathrm{St}}(x) = \frac{2}{1+x}, \qquad k_1^{\mathrm{St}}(x) = \lim_{\alpha\to 1} k_\alpha^{\mathrm{St}}(x) = e\,x^{\frac{x}{1-x}}, \qquad k_{1/2}^{\mathrm{St}}(x) = \frac{4}{(1+\sqrt{x})^2},$$
$$k_0^{\mathrm{St}}(x) = \lim_{\alpha\to 0} k_\alpha^{\mathrm{St}}(x) = \frac{\log x}{x-1}, \qquad k_{-1}^{\mathrm{St}}(x) = x^{-1/2}, \qquad k_{-2}^{\mathrm{St}}(x) = \left(\frac{1+x}{2x^2}\right)^{1/3},$$

which provide an interesting comparison with the other families, as shown in Table 1.

When

$$S_{\alpha,\beta}(x,y) \equiv \left(\frac{\beta(x^\alpha - y^\alpha)}{\alpha(x^\beta - y^\beta)}\right)^{\frac{1}{\alpha-\beta}},$$

it was proved in [27, Theorem 12] that $S_{\alpha,\beta}(e^x,1)/S_{\alpha',\beta'}(e^x,1)$ is infinitely divisible as long as $\alpha \le \alpha'$ and $\beta \le \beta'$. Since in particular $S_{\alpha,1}(x,y) = S_\alpha(x,y)$, it follows from (4.7) that this implies that when $\alpha \le \beta$ the dual family $\widehat{k}_\alpha^{\mathrm{St}} \preccurlyeq \widehat{k}_\beta^{\mathrm{St}}$ increases and $k_\beta^{\mathrm{St}} \preccurlyeq k_\alpha^{\mathrm{St}}$ decreases. We can then conclude that

- $k_\alpha^{\rm St} \in \mathcal{K}^+$ and $\widehat{k}_\alpha^{\rm St} \in \mathcal{K}^-$ if and only if $\alpha \in [-1, 2]$,
- $k_\alpha^{\rm St} \in \mathcal{K}^-$ and $\widehat{k}_\alpha^{\rm St} \in \mathcal{K}^+$ if and only if $\alpha \in [-2, -1]$.

As remarked above, the dual functions form a different family with special cases

$$\widehat{k}_{-2}^{\rm St}(x) = \left(\frac{2}{x(1+x)}\right)^{1/3}, \qquad \widehat{k}_{-1}^{\rm St}(x) = x^{-1/2}, \qquad \widehat{k}_0^{\rm St}(x) = \lim_{\alpha\to 0} \widehat{k}_\alpha^{\rm St}(x) = \frac{x-1}{x\log x},$$
$$\widehat{k}_{1/2}^{\rm St}(x) = \frac{(1+\sqrt{x})^2}{4x}, \qquad \widehat{k}_1^{\rm St}(x) = \lim_{\alpha\to 1} \widehat{k}_\alpha^{\rm St}(x) = e^{-1}\, x^{\frac{1}{x-1}}, \qquad \widehat{k}_2^{\rm St}(x) = \frac{1+x}{2x} = \widehat{k}_0^{\rm ext}(x).$$

The pair $k_{1-\alpha}^{\rm St}$ for $-1 \le \alpha \le 2$ and $\widehat{k}_\alpha^{\rm St}$ for $-1 \le \alpha \le 2$ can be regarded as a single family which increases in the $\preccurlyeq$ order from $k_0^{\rm ext}$ to $k_1^{\rm ext}$. The functions $k_1^{\rm St}$ and $\widehat{k}_{-2}^{\rm St}$ give new members of $\mathcal{K}^+$ which do not not appear in any of the other families. Moreover, $\widehat{k}_{-2}^{\rm St}$ must lie on the boundary of both $\mathcal{K}^+$ and $\mathcal{K}$, which implies that $\mathcal{K}^+$ touches the boundary of $\mathcal{K}$ at the interior of a face. It seems reasonable to conjecture that $\widehat{k}_{-2}^{\rm St}$ is an extreme point of $\mathcal{K}^+$.

It is interesting to compare the behavior of these examples as the parameters α and p change, as summarized in Table 1 and Figure 1.

	4.6 $k_\alpha^{\rm B}$	4.7 $k_\alpha^{\rm PD}$	4.8 $k_p^{\rm WYD}$	4.9 $k_\alpha^{\rm St}$
$\frac{2}{1+x}$	1	2		2
$ex^{x/(1-x)}$				1
$\frac{4}{(1+\sqrt{x})^2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\log x}{x-1}$		1	0	0
$x^{-1/2}$	0	$\frac{1}{2}$		-1

Table 1: Summary of common crossing points

The Stolarsky family is the only one which goes through all of the indicated points. The WYD family is the only one which does not begin and end at the smallest and largest elements, and moves outside of both $\mathcal{K}^+$ and $\mathcal{K}^-$ for some parameter range.

4.3 Geometric bridges

In Examples 4.2 and 4.3 of Section 4.1 we considered arithmetic weighted averages of $2/(1+x)$ (the smallest of $\mathcal{K}^+$) or $x^{-1/2}$ (the largest of $\mathcal{K}^+$) with extreme points of $\mathcal{K}$, and noticed that such averages can be in $\mathcal{K}^+$ in rather limited cases. In this section we consider a different type of averages, often called a *geometric bridge*, which is defined as weighted geometric means $[k_1(x)]^{1-\lambda}[k_2(x)]^\lambda$, $0 \le \lambda \le 1$, of $k_1, k_2 \in \mathcal{K}$. We first show that $\mathcal{K}$ and $\mathcal{K}^\pm$ are all closed under geometric bridge interpolations as far as some infinite divisibility condition is satisfied for $\mathcal{K}^\pm$. The equivalence of (ii) and (iii) in the next theorem implies that a similar result holds for $\mathcal{F}$ and $\mathcal{F}^\pm$.

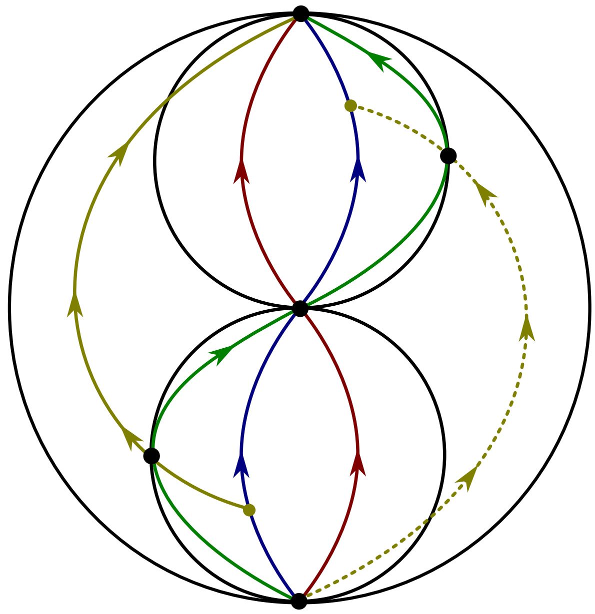

Figure 1: Schematic diagram of families in $\mathcal{K}$ parameterized so that they increase in the $\preccurlyeq$ order. The lower ball corresponds to $\mathcal{K}^+$ and the upper ball to $\mathcal{K}^-$. The three curves inside $\mathcal{K}^+ \cup \mathcal{K}^-$ beginning at the smallest member $2/(1+x)$ are described from right to left. The rightmost curve (red) describes the Heinz family k_α^{H} $(0 \le \alpha \le 1/2)$ and $\widehat{k}_\alpha^{\mathrm{H}}$ $(1/2 \le \alpha \le 1)$; the next (blue) curve the binomial family $k_{-\alpha}^{\mathrm{B}}$ $(-1 \le \alpha \le 1)$; the next (green) curve the power difference family $k_{-\alpha}^{\mathrm{PD}}$ $(-2 \le \alpha \le 1)$. The brown curve on the left the WYD family k_p^{WYD} in the range $p \in [\frac{1}{2}, 2]$ and the dotted brown curve on the right the dual WYD family. The crossings at $4/(1+\sqrt{x})^2$ and $\log x/(x-1)$ can easily be seen. The complex Stolarsky family, which is the only one which starts at the smallest $2/(1+x)$ and goes through both of these crossings while remaining in $\mathcal{K}^+$ before reaching $x^{-1/2}$, is not shown.

Proposition 4.10. *If $k_1, k_2 \in \mathcal{K}$, then for every $\lambda \in [0,1]$ the function $[k_1(x)]^{1-\lambda}[k_2(x)]^\lambda$ is also in $\mathcal{K}$. Moreover, if $k_1, k_2 \in \mathcal{K}^+$ (resp., $\mathcal{K}^-$) and one of the following conditions is satisfied, then for every $\lambda \in [0,1]$ the function $[k_1(x)]^{1-\lambda}[k_2(x)]^\lambda$ is also in $\mathcal{K}^+$ (resp., $\mathcal{K}^-$):*

(i) *both $e^{t/2}k_1(e^t)$ and $e^{t/2}k_2(e^t)$ are infinitely divisible,*

(ii) *$k_2(e^t)/k_1(e^t)$ is infinitely divisible,*

(iii) *$k_1(e^t)/k_2(e^t)$ is infinitely divisible.*

Proof. To prove the first assertion, let $k_1, k_2 \in \mathcal{K}$; then by Theorem 2.4 they have the Pick mapping property, from which it follows that $[k_1(x)]^{1-\lambda}[k_2(x)]^\lambda$ also has this property and hence is operator monotone decreasing. Since the symmetry condition in Theorem 2.4 is obvious, we conclude that $k_1^{1-\lambda}k_2^\lambda \in \mathcal{K}$.

To prove the second assertion, let $k_1, k_2 \in \mathcal{K}^+$ and $0 < \lambda < 1$. When (i) is satisfied, $[e^{t/2}k_1(e^t)]^{1-\lambda}$ and $[e^{t/2}k_2(e^t)]^\lambda$ are positive definite and hence so is the product

$e^{t/2}[k_1(e^t)]^{1-\lambda}[k_2(e^t)]^{\lambda}$. We note

$$e^{t/2}[k_1(e^t)]^{1-\lambda}[k_2(e^t)]^{\lambda} = e^{t/2}k_1(e^t)\left(\frac{k_2(e^t)}{k_1(e^t)}\right)^{\lambda} = e^{t/2}k_2(e^t)\left(\frac{k_1(e^t)}{k_2(e^t)}\right)^{1-\lambda}$$

so that we get the desired positive definiteness from either (ii) or (iii).

Finally, the assertion for $\mathcal{K}^-$ is easily verified by taking $\widehat{k}_j(x) \equiv 1/k_j\big(x^{-1}\big)$ and using (2.17). **QED**

Recall that all the one-parameter families in $\mathcal{K}^+$ given in Examples 4.5–4.9 satisfy the property of infinite divisibility (an order stronger than $\preccurlyeq$). Therefore, the above proposition implies that geometric bridges joining k_1, k_2 in each of these family sits inside $\mathcal{K}^+$.

Example 4.11. Consider the bridge

$$k_\alpha(x) \equiv [k_\alpha^{\mathrm{St}}(x)]^{1-\alpha}[x^{-1/2}]^{\alpha} = \frac{x^{\alpha/2} - x^{-\alpha/2}}{\alpha(x-1)}.$$

By Proposition 4.10 this is in $\mathcal{K}^+$ for $\alpha \in [0,1]$. In fact, $k_\alpha \in \mathcal{K}$ in the larger range $\alpha \in [0,2]$. One way to see this is to observe that $g_\beta(x) = x^{-\beta}(1-x) = x^{-\beta} - x^{1-\beta}$ is operator convex for $\beta \in [0,1]$. Then it follows from [33, Theorem II.13] that

$$\frac{g_\beta(x) + x g_\beta(x^{-1})}{(x-1)^2} = \frac{x^\beta - x^{-\beta}}{x-1} = 2\beta\, k_{2\beta}(x)$$

is a multiple of a function in $\mathcal{K}$ for $\beta \in [0,1]$.

Since

$$e^t k_\alpha(e^{2t}) = \frac{1}{\alpha}\cdot\frac{\sinh(\alpha t)}{\sinh t},$$

it is easy to see by Lemma 3.3 (1) that $k_\alpha \in \mathcal{K}^+$ if and only if $\alpha \in [0,1]$ while $k_\alpha \in \mathcal{K}^-$ if and only if $\alpha \in [1,2]$. It is also known [8, Theorem 2] that $k_\alpha(e^t)/k_\beta(e^t)$ is infinitely divisible whenever $\alpha \leq \beta$. Given the special cases

$$k_0(x) = \lim_{\alpha\to 0} k_\alpha(x) = \frac{\log x}{x-1}, \qquad k_{1/2}(x) = \frac{2}{x^{1/4} + x^{3/4}} = k_{1/4}^{\mathrm{H}}(x),$$
$$k_1(x) = x^{-1/2}, \qquad k_2(x) = \frac{1+x}{2x} = k_0^{\mathrm{ext}}(x),$$

it follows that $k_\alpha(x)$ is a family which increases on $[0,2]$ in the $\preccurlyeq$ order from $\log x/(x-1)$ to $(1+x)/2x$.

The connection between g_β and $k_{2\beta}$ is interesting because, as mentioned in Section 2.4, $g(x) = (x-1)^2 k(x)$ is always an operator convex function with the properties needed to define a symmetric quasi-entropy. Although one can begin with a function $g(x)$ which does *not* satisfy $g(x) = xg(x^{-1})$ and generate a function $k \in \mathcal{K}$, it is not at all obvious how to reverse the process without obtaining a symmetric g. In this case, we have found an asymmetric g, in particular $g_{1/2}(x) = x^{-1/2} - x^{1/2}$, which generates the key function $k(x) = x^{-1/2} \in \mathcal{K}$. The associated quasi-entropies do not seem to have been studied previously, but appeared recently in [52].

The remaining examples are concerned with geometric bridges joining k_1^{ext} and other extreme points of $\mathcal{K}$ which require a more difficult analysis.

Example 4.12. For $\mu, \nu, \lambda \in [0,1]$ we define

$$g_{\mu,\nu,\lambda}(x) \equiv k_\mu^{\text{ext}}(x)^{1-\lambda} k_\nu^{\text{ext}}(x)^\lambda = k_\mu^{\text{ext}}(x) \left(\frac{k_\nu^{\text{ext}}(x)}{k_\mu^{\text{ext}}(x)} \right)^\lambda$$

with k_ν^{ext} given by (2.13). This is in $\mathcal{K}$ by Proposition 4.10. A special case

$$g_{1,0,\lambda}(x) = k_1^{\text{ext}}(x) \left(\frac{k_0^{\text{ext}}(x)}{k_1^{\text{ext}}(x)} \right)^\lambda = x^{-\lambda} \left(\frac{2}{1+x} \right)^{1-2\lambda}, \qquad 0 \le \lambda \le 1,$$

was treated in [6, Example 5]. We have

$$e^{t/2} g_{1,0,\lambda}(e^t) = \left(\frac{1}{\cosh(t/2)} \right)^{1-2\lambda},$$

which is positive definite exactly when $0 \le \lambda \le \frac{1}{2}$ since $1/\cosh t$ is infinitely divisible (see [8, Theorem 1] for instance). Therefore, $g_{1,0,\lambda}$ is in $\mathcal{K}^+$ if and only if $0 \le \lambda \le \frac{1}{2}$.

Example 4.13. For the more general case

$$g_{1,\nu,\lambda}(x) = k_1^{\text{ext}}(x) \left(\frac{k_\nu^{\text{ext}}(x)}{k_1^{\text{ext}}(x)} \right)^\lambda = \left(\frac{2}{1+x} \right)^{1-2\lambda} \left(\frac{(1+\nu)^2}{(x+\nu)(1+\nu x)} \right)^\lambda, \tag{4.8}$$

which increase pointwise with $\lambda \in [0,1]$ from k_1^{ext} to k_ν^{ext}. Its behavior (in the present context) when $\nu \in (0,1)$ seems much more mysterious. Our results here are:

(i) the pointwise order of $g_{1,\nu,\lambda}$ in λ can be also strengthened to the $\preccurlyeq$ order, and consequently the set $\{\lambda \in [0,1] : g_{1,\nu,\lambda} \in \mathcal{K}^+\}$ is a subinterval $[0, \lambda_c(\nu)]$,

(ii) for each $\nu \in (0,1)$ the critical value $\lambda_c(\nu)$ satisfies

$$\frac{1}{4} \le \lambda_c(\nu) \le \frac{1}{3}. \tag{4.9}$$

The proof requires some lengthy computations of Fourier transforms, which will be presented in Section 6.3. Unfortunately, we do not have any information about the form of $\lambda_c(\nu)$.

Example 4.14. It is worth noting that a family of modified bridges

$$g_{1,1-\lambda,\lambda}(x) = \left(\frac{2}{1+x} \right)^{1-2\lambda} \left(\frac{(2-\lambda)^2}{(1+x-\lambda)(1+(1-\lambda)x)} \right)^\lambda, \qquad 0 \le \lambda \le 1, \tag{4.10}$$

joining k_1^{ext} and k_0^{ext} was constructed in [15] for the explicit purpose of finding a one-parameter family which increases from k_1^{ext} and k_0^{ext} in the pointwise order and all of whose elements except for $\lambda = 1$ are regular (here $k \in \mathcal{K}$ is regular if $\lim_{x \searrow 0} k(x) < +\infty$). Without the regularity requirement, the families in Examples 4.5–4.7 and 4.9 have this property in the stronger $\preccurlyeq$ order.

From the same computation as in Lemma 6.4 (Section 6.3) we observe that the set $\Lambda \equiv \{\lambda \in [0,1] : g_{1,1-\lambda,\lambda} \in \mathcal{K}^+\}$ includes $\left[0, \frac{1}{4}\right]$. Koenraad Audenaert did some numerical work suggesting that $g_{1,1-\lambda,\lambda}$ is not in $\mathcal{K}^+$ for $\lambda \geq 0.3$ giving a CP crossing at a point slightly smaller than 0.3 which would be consistent with (4.9). However, we do not know strong monotonicity in the $\preccurlyeq$ order for the family (4.10). To conclude that Λ is of the form $[0, a]$ we would need a stronger result, e.g., that $\lambda_c(\nu)$ is monotone in ν.

5 Positive definite functions

In this section, we present results on positive definiteness and infinite divisibility of certain functions involving hyperbolic functions, which are needed in our proofs. The study here is considered as a continuation of [26, 28, 27], which are of independent interest.

5.1 Positive definiteness of sinh ratios

We investigate positive definiteness of the function

$$f(t) \equiv \frac{\sinh(at)\sinh(bt)}{\sinh^2 t}$$

with $a, b > 0$. If $a, b \leq 1$, then $f(t)$ is a positive definite function as the product of two such functions (see Lemma 3.3 (1)). It is actually infinitely divisible as is explained in [8, 26] for instance. We will show that the converse also holds true.

Theorem 5.1. *The function $f(t)$ is positive definite if and only if $a, b \leq 1$.*

When $a + b > 2$, we have $\lim_{t\to\pm\infty} f(t) = \infty$ so that $f(t)$ cannot be positive definite. When $a + b = 2$ and $a \neq b$, the obvious estimate

$$f(0) = ab < \left(\frac{a+b}{2}\right)^2 = 1 = \lim_{t\to\pm\infty} f(t)$$

also shows failure of positive definiteness.

We will assume $a + b < 2$, and we must show that $f(t)$ is not positive definite as long as $a > 1$ (and hence $0 < b < 1$). For this purpose it suffices to deal with a, b rational. Indeed, if $f(t)$ were positive definite for such a, b (and the result is known for such rational parameters), then with a', b' rational satisfying $1 < a' \leq a$ and $0 < b' \leq b$ the product

$$f(t)\,\frac{\sinh(a't)\sinh(b't)}{\sinh(at)\sinh(bt)} = \frac{\sinh(a't)\sinh(b't)}{\sinh^2 t}$$

would be positive definite, a contradiction.

Hence, we will assume that a, b are rational in the rest. Obviously we can further assume

$$a = \tfrac{m}{n} > 1,\ \ b = \tfrac{k}{n} > 0,\ \ a + b < 2 \text{ with } n, m, k \in \mathbf{N} \text{ even.} \tag{5.1}$$

The most delicate part in our proof for Theorem 5.1 is covered in the next lemma, and the rest of the subsection will be devoted to its proof.

Lemma 5.2. *The function $f(t)$ cannot be positive definite for a, b rational described by* (5.1).

For a fixed $s \in \mathbf{R}$ we set

$$F(z) \equiv f(z)\, e^{isz} \quad \left(= \frac{\sinh(az)\sinh(bz)}{\sinh^2 z}\, e^{isz} = \frac{\sinh\left(\frac{m}{n}\, z\right)\sinh\left(\frac{k}{n}\, z\right)}{\sinh^2 z}\, e^{isz} \right)$$

with a, b given by (5.1), and compute its integral along the following rectangle Γ:

$$\begin{array}{lll} \Gamma_1 & z = t, & t : -R \to R, \\ \Gamma_2 & z = R + is, & s : 0 \to n\pi, \\ \Gamma_3 & z = t + in\pi, & t : R \to -R, \\ \Gamma_4 & z = -R + is, & s : n\pi \to 0. \end{array}$$

We observe

$$\begin{aligned} &\sinh(t + in\pi) = \sinh t, \\ &\sinh\left(\tfrac{m}{n}\,(t + in\pi)\right) = \sinh\left(\tfrac{m}{n}\, t + im\pi\right) = \sinh\left(\tfrac{m}{n}\, t\right), \\ &\sinh\left(\tfrac{k}{n}\,(t + in\pi)\right) = \sinh\left(\tfrac{k}{n}\, t + ik\pi\right) = \sinh\left(\tfrac{k}{n}\, t\right) \end{aligned}$$

(since n, m, k are even) so that we have $F(t + in\pi) = f(t)\, e^{is(t+in\pi)}$ and

$$\begin{aligned} \int_{\Gamma_1 \cup \Gamma_3} F(z)\, dz &= \int_{-R}^{R} f(t)\, e^{ist} dt + \int_{R}^{-R} f(t)\, e^{ist} e^{-n\pi s} dt \\ &= \left(1 - e^{-n\pi s}\right) \int_{-R}^{R} f(t)\, e^{ist} dt \\ &= 2e^{-n\pi s/2} \sinh(n\pi s/2) \int_{-R}^{R} f(t)\, e^{ist} dt. \end{aligned} \tag{5.2}$$

Since $a + b < 2$, we have $f(z) \to 0$ uniformly on the strip $\{z \in \mathbf{C};\, 0 \le \operatorname{Im} z \le n\pi\}$ as $\operatorname{Re} z \to \pm\infty$ and hence

$$\lim_{R \to \infty} \int_{\Gamma_2 \cup \Gamma_4} F(z)\, dz = 0. \tag{5.3}$$

Therefore, the Fourier transform of $f(t)$ can be computed from $\int_\Gamma F(z)\, dz$. If the Fourier transform fails to be positive, then Lemma 5.2 follows from Bochner's theorem.

Note that $z = 0, in\pi$ are zeros of $\sinh^2 z$ of order 2. However, these two points are also zeros for $\sinh\left(\frac{m}{n}\, z\right)$, $\sinh\left(\frac{k}{n}\, z\right)$ so that $z = 0, in\pi$ are removable singularities for $F(z)$. The poles (inside of Γ) closest to Γ are

$$z_1 = i\pi \quad \text{and} \quad z_{n-1} = i(n-1)\pi.$$

Note that z_1, z_{n-1} are zeros for $\sinh^2 z$ (appearing in the denominator) of order 2 and that they are not zeros for $\sinh\left(\frac{m}{n}\, z\right)$ and $\sinh\left(\frac{k}{n}\, z\right)$ (due to $1 < a = \frac{m}{n} < 2$ and $0 < b = \frac{k}{n} < 1$). Thus, we conclude that $z = z_1, z_{n-1}$ are double poles for $F(z)$.

We begin with computation of the residue $\operatorname{Res}(F(z); z_1)$ at $z = z_1$. Thanks to $\sinh z = -\sinh(z - i\pi)$ (or by direct computation) the power series expansion of $\sinh z$ around $z_1 = i\pi$ is given by

$$\sinh z = -\left((z - z_1) + (z - z_1)^3/3! + (z - z_1)^5/5! + \cdots\right)$$

$$= -(z-z_1)\big(1+(z-z_1)^2/3!+(z-z_1)^4/5!+\cdots\big).$$

We thus get the following Laurent series expansion:

$$\begin{aligned}
\frac{1}{\sinh^2 z} &= \frac{1}{(z-z_1)^2}\cdot\frac{1}{\big(1+(z-z_1)^2/3!+(z-z_1)^4/5!+\cdots\big)^2}\\
&= \frac{1}{(z-z_1)^2}\cdot\frac{1}{1+(z-z_1)^2/3+\text{higher even powers}}\\
&= \frac{1}{(z-z_1)^2}\left(1-(z-z_1)^2/3+\text{higher even powers}\right). \qquad (5.4)
\end{aligned}$$

Since

$$\frac{d^\ell}{dz^\ell}\,e^{isz}\Big|_{z=z_1} = (is)^\ell e^{isz}\big|_{z=z_1} = (is)^\ell e^{-\pi s},$$

we have

$$e^{isz} = e^{-\pi s}\left(1+is(z-z_1)-s^2(z-z_1)^2/2+\cdots\right). \qquad (5.5)$$

Computations

$$\begin{aligned}
&\sinh\left(\tfrac{m}{n}\,z_1\right) = i\sin\left(\tfrac{m}{n}\,\pi\right)\ \ \big(= i\sin(at)\big),\\
&\frac{d}{dz}\sinh\left(\tfrac{m}{n}\,z\right)\Big|_{z=z_1} = \tfrac{m}{n}\cosh\left(\tfrac{m}{n}\,z\right)\big|_{z=z_1} = \tfrac{m}{n}\cos\left(\tfrac{m}{n}\,\pi\right)\big(= a\cos(at)\big)
\end{aligned}$$

give rise to

$$\sinh\left(\tfrac{m}{n}\,z\right) = i\sin(a\pi)+a\cos(a\pi)(z-z_1)+\cdots, \qquad (5.6)$$

and similarly

$$\sinh\left(\tfrac{k}{n}\,z\right) = i\sin(b\pi)+b\cos(b\pi)(z-z_1)+\cdots. \qquad (5.7)$$

From (5.4)–(5.7) the Laurent series expansion of $F(z)$ around $z=z_1$ is given by

$$\begin{aligned}
&\frac{e^{-\pi s}}{(z-z_1)^2}\left(1-(z-z_1)^2/3+\text{higher even powers}\right)\left(1+is(z-z_1)+\cdots\right)\\
&\quad\times\big(i\sin(a\pi)+a\cos(a\pi)(z-z_1)+\cdots\big)\big(i\sin(b\pi)+b\cos(b\pi)(z-z_1)+\cdots\big). \qquad (5.8)
\end{aligned}$$

The residue $\mathrm{Res}(F(z);z_1)$ is nothing but the coefficient of $(z-z_1)^{-1}$ here, i.e., that of $(z-z_1)$ in the product of the above four brackets (multiplied by $e^{-\pi s}$). Since the starting term is 1 and a $(z-z_1)$-term is absent in the first bracket, what we have to compute is the coefficient of $(z-z_1)$ in the product of the last three brackets. In this way we arrive at

$$\begin{aligned}
\mathrm{Res}(F(z);z_1) &= e^{-\pi s}\big(i\sin(a\pi)\cdot b\cos(b\pi)+a\cos(a\pi)\cdot i\sin(b\pi)\\
&\qquad\qquad + is\cdot i\sin(a\pi)\cdot i\sin(b\pi))\\
&= ie^{-\pi s}\big(a\cos(a\pi)\sin(b\pi)+b\cos(b\pi)\sin(a\pi)-s\sin(a\pi)\sin(b\pi)\big).
\end{aligned}$$

We next move to computation of the residue $\mathrm{Res}(F(z);z_{n-1})$ at $z=z_{n-1}$ $\big(=i(n-1)\pi\big)$. Because of $\sinh z = -\sinh(z-(n-1)\pi i)$ $\big(=-\sinh(z+\pi i)\big)$ with n even we have

$$\begin{aligned}
\sinh z &= -\big((z-z_{n-1})+(z-z_{n-1})^3/3!+(z-z_{n-1})^5/5!+\cdots\big)\\
&= -(z-z_{n-1})\big(1+(z-z_{n-1})^2/3!+(z-z_{n-1})^4/5!+\cdots\big)
\end{aligned}$$

(with the identical coefficients as in the power expansion around $z = z_1$), and hence we have

$$\frac{1}{\sinh^2 z} = \frac{1}{(z - z_{n-1})^2} \left(1 - (z - z_{n-1})^2/3 + \text{higher even powers}\right)$$

again (see (5.4)). Also, since $\frac{d^\ell}{dz^\ell} e^{isz}\big|_{z=z_{n-1}} = (is)^\ell e^{isz}\big|_{z=z_{n-1}} = (is)^\ell e^{-(n-1)\pi s}$, (5.5) has to be replaced by

$$e^{isz} = e^{-(n-1)\pi s} \left(1 + is(z - z_{n-1}) - s^2(z - z_{n-1})^2/2 + \cdots\right).$$

So far we have not seen changes of coefficients except the obvious modification that the factor $e^{-\pi s}$ in (5.5) was replaced by $e^{-(n-1)\pi s}$. On the other hand, since

$$\begin{aligned}
\sinh\left(\tfrac{m}{n}\, z_{n-1}\right) &= i \sin\left(\tfrac{m}{n}\,(n-1)\pi\right) = i \sin\left(m\pi - \tfrac{m}{n}\,\pi\right) \\
&= -i \sin\left(\tfrac{m}{n}\,\pi\right) \ \ \big(= -i\sin(at)\big), \\
\frac{d}{dz} \sinh\left(\tfrac{m}{n}\, z\right)\big|_{z=z_{n-1}} &= \tfrac{m}{n} \cosh\left(\tfrac{m}{n}\, z\right)\big|_{z=z_{n-1}} = \tfrac{m}{n} \cos\left(\tfrac{m}{n}\,(n-1)\pi\right) \\
&= \tfrac{m}{n} \cos\left(m\pi - \tfrac{m}{n}\,\pi\right) = \tfrac{m}{n} \cos\left(\tfrac{m}{n}\,\pi\right) \ \ \big(= a\cos(at)\big),
\end{aligned}$$

the power series expansions (5.6) and (5.7) are replaced by

$$\begin{aligned}
\sinh\left(\tfrac{m}{n}\, z\right) &= -i\sin(a\pi) + a\cos(a\pi)(z - z_{n-1}) + \cdots, \\
\sinh\left(\tfrac{k}{n}\, z\right) &= -i\sin(b\pi) + b\cos(b\pi)(z - z_{n-1}) + \cdots
\end{aligned}$$

with constant terms of the opposite sign. The four relevant expansions are now at our disposal, and the same reasoning as before (see the product (5.8)) gives us the following conclusion:

$$\begin{aligned}
\mathrm{Res}(F(z); z_{n-1}) &= e^{-(n-1)\pi s}\big(-i\sin(a\pi)\cdot b\cos(b\pi) - a\cos(a\pi)\cdot i\sin(b\pi) \\
&\qquad\qquad + is\,(-i\sin(a\pi))(-i\sin(b\pi))\big) \\
&= ie^{-(n-1)\pi s}\big(-a\cos(a\pi)\sin(b\pi) - b\cos(b\pi)\sin(a\pi) \\
&\qquad\qquad - s\sin(a\pi)\sin(b\pi)\big).
\end{aligned}$$

The sum (multiplied by $2\pi i$) of the two residues we have computed so far can be rearranged in the following way:

$$\begin{aligned}
&2\pi i\big(\mathrm{Res}(F(z); z_1) + \mathrm{Res}(F(z); z_{n-1})\big) \\
&\ = 2\pi\big[e^{-\pi s}\big(-a\cos(a\pi)\sin(b\pi) - b\cos(b\pi)\sin(a\pi) + s\sin(a\pi)\sin(b\pi)\big) \\
&\qquad + e^{-(n-1)\pi s}\big(a\cos(a\pi)\sin(b\pi) + b\cos(b\pi)\sin(a\pi) + s\sin(a\pi)\sin(b\pi)\big)\big] \\
&\ = 2\pi e^{-n\pi s/2}\big[e^{(n/2-1)\pi s}\big(-a\cos(a\pi)\sin(b\pi) - b\cos(b\pi)\sin(a\pi) + s\sin(a\pi)\sin(b\pi)\big) \\
&\qquad + e^{-(n/2-1)\pi s}\big(a\cos(a\pi)\sin(b\pi) + b\cos(b\pi)\sin(a\pi) + s\sin(a\pi)\sin(b\pi)\big)\big] \\
&\ = 4\pi e^{-n\pi s/2}\big[-\big(a\cos(a\pi)\sin(b\pi) + b\cos(b\pi)\sin(a\pi)\big)\sinh\left((n/2-1)\pi s\right) \\
&\qquad\qquad + s\sin(a\pi)\sin(b\pi)\cosh\left((n/2-1)\pi s\right)\big].
\end{aligned}$$

Therefore, by recalling (5.2) and (5.3) we conclude

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= \frac{1}{\sinh(n\pi s/2)} \Big[\sin(a\pi)\sin(b\pi) \cdot s\cosh\left((n/2-1)\pi s\right) \\ - \big(a\cos(a\pi)\sin(b\pi) + b\cos(b\pi)\sin(a\pi)\big)\sinh\left((n/2-1)\pi s\right) \\ + \text{lower order terms}\Big]. \tag{5.9}$$

A few remarks concerning "lower order terms" are in order. Other candidates for poles (inside of Γ) of $F(z)$ are

$$z_\ell = i\ell\pi \quad (\text{for } \ell = 2, 3, \dots, n-2),$$

where nature of singularities at these points (i.e., removable singularities or poles of order at most 2) is determined according to values of $\sinh\left(\frac{m}{n} z_\ell\right)$, $\sinh\left(\frac{k}{n} z_\ell\right)$ appearing in the numerator. Anyway, residues arising from them give us linear combinations of factors of the forms

$$\sinh\left(\left(n/2 - \ell'\right)\pi s\right), \quad \cosh\left(\left(n/2 - \ell'\right)\pi s\right) \quad \text{with } \ell' = 2, 3, \dots, n/2$$

in the above big bracket (5.9) (possibly with the linear factor s for double poles). Indeed, the only source for exponential factors is the power series expansions of e^{isz} around $z = z_\ell$ (see (5.5)), which actually gives rise to

$$e^{isz_\ell} = e^{-\ell\pi s} = e^{-n\pi s/2} e^{(n/2-\ell)\pi s} \quad (\ell = 2, 3, \dots, n-2).$$

Thus, by recalling the factor $e^{-n\pi s/2}$ appearing in (5.2), we get the assertion.

The dominant term (as $s \to \pm\infty$) in the numerator of the Fourier transform is

$$\sin(a\pi)\sin(b\pi) \cdot s\cosh\left((n/2-1)\pi s\right),$$

and we observe

$$\sin(a\pi)\sin(b\pi) < 0$$

thanks to $1 < a < 2$ and $0 < b < 1$ (see (5.1)). Consequently, the Fourier transform takes negative values for $|s|$ large (i.e., failure of positive definiteness for $f(s)$), and Lemma 5.2 has been proved.

5.2 Fourier transform of $((\cosh(t/2) + \alpha)(\cosh t + \beta))^{-1}$

Detailed information on positive definiteness for $\left(\cosh^k(t/2)(\cosh t + \beta)^m\right)^{-1}$ will be needed to prove results on geometric bridges in Example 4.13. However, a direct computation for its Fourier transform based on residue calculus seems hopeless due to the fact that poles of higher orders have to be considered. Instead, in this subsection we compute the Fourier transform in the special case $k = m = 1$ with the additional parameter α as in the theorem below (and then in Section 5.4 we will check higher order partial derivatives relative to α and β to achieve our goal).

Theorem 5.3. *For $\alpha \in (-1, 1)$ and $\beta > 1$ we have*

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{ist}\, dt}{\big(\cosh(t/2) + \alpha\big)\big(\cosh t + \beta\big)}$$

$$= \frac{1}{\sinh(2\pi s)} \left[\frac{\sinh(2\theta s)}{\sqrt{1-\alpha^2}\bigl(2\alpha^2 - 1 + \beta\bigr)} - \frac{\sqrt{\frac{\beta-1}{2}} \cos(\lambda s) \sinh(\pi s) - \alpha \sin(\lambda s) \cosh(\pi s)}{\sqrt{\beta^2-1}\bigl(\frac{\beta-1}{2} + \alpha^2\bigr)} \right], \tag{5.10}$$

where $\theta = \cos^{-1}\alpha \in (0, \pi)$ *and* $\lambda = \log\left(\beta + \sqrt{\beta^2 - 1}\right)$*, i.e.,* $\lambda > 0$ *is a solution of* $\cosh\lambda = \beta$.

Proof. For a fixed $s \in \mathbf{R}$ we set

$$F(z) \equiv \frac{e^{isz}}{\bigl(\cosh(z/2) + \alpha\bigr)\bigl(\cosh z + \beta\bigr)}$$

and compute its integral along the following rectangle Γ:

$$\begin{array}{lll} \Gamma_1 & z = t, & t : -R \to R, \\ \Gamma_2 & z = R + is, & s : 0 \to 4\pi, \\ \Gamma_3 & z = t + 4\pi i, & t : R \to -R, \\ \Gamma_4 & z = -R + is, & s : 4\pi \to 0. \end{array}$$

Due to $\cosh((t + 4\pi i)/2) = \cosh(t/2)$, $\cosh(t + 4\pi i) = \cosh t$ and

$$\lim_{R\to\infty} \int_{\Gamma_2 \cup \Gamma_4} f(z)\, dz = 0$$

we have

$$\begin{aligned} \lim_{R\to\infty} \int_{\Gamma} f(z)\, dz &= \lim_{R\to\infty} \int_{\Gamma_1 \cup \Gamma_3} f(z)\, dz \\ &= \lim_{R\to\infty} (1 - e^{-4\pi s}) \int_{-R}^{R} \frac{e^{ist} ds}{\bigl(\cosh(t/2) + \alpha\bigr)\bigl(\cosh t + \beta\bigr)} \\ &= 2e^{-2\pi s} \sinh(2\pi s) \int_{-\infty}^{\infty} \frac{e^{ist} ds}{\bigl(\cosh(t/2) + \alpha\bigr)\bigl(\cosh t + \beta\bigr)}, \end{aligned}$$

and we will compute $\int_\Gamma f(z)\, dz$ by residue calculus.

It is easy to see that we have the following six simple poles inside of Γ:

$$z_0 = 2i(\pi - \theta), \quad z_1 = 2i(\pi + \theta), \quad \xi_0^{\pm} = i\pi \pm \lambda, \quad \xi_1^{\pm} = 3i\pi \pm \lambda.$$

When $\alpha = 1$ (i.e., $\theta = 0$), $2\pi i$ is a double pole. However, we assumed $\alpha \in (-1, 1)$ to avoid this complication. Note that the Fourier transform formula (5.10) itself remains valid for $\alpha = 1$ by the obvious limiting argument with the understanding

$$\left.\frac{\sinh(2\theta s)}{\sqrt{1-\alpha^2}}\right|_{\alpha=1} = \lim_{\alpha \nearrow 1} \frac{\sinh(2\theta s)}{\sqrt{1-\alpha^2}} = 2s$$

(see (5.17) below). We note

$$\mathrm{Res}(z_j, F(z)) = \frac{e^{isz_j}}{\frac{1}{2}\sinh(z_j/2)\bigl(\cosh z_j + \beta\bigr)}, \qquad j = 0, 1,$$

and observe

$$
\begin{aligned}
&e^{isz_0} = e^{-2(\pi-\theta)s}, \quad e^{isz_1} = e^{-2(\pi+\theta)s},\\
&\sinh(z_0/2) = i\sin(\pi-\theta) = i\sin\theta = i\sqrt{1-\alpha^2},\\
&\sinh(z_1/2) = i\sin(\pi+\theta) = -i\sin\theta = -i\sqrt{1-\alpha^2},\\
&\cosh(z_0) = \cos(2(\pi-\theta)) = \cos(2\theta) = 2\cos^2\theta - 1 = 2\alpha^2-1,\\
&\cosh(z_1) = \cos(2(\pi+\theta)) = \cos(2\theta) = 2\alpha^2-1.
\end{aligned}
$$

Thus, we compute

$$
\begin{aligned}
\mathrm{Res}(z_0;F(z)) &= \frac{e^{-2(\pi-\theta)s}}{\frac{i}{2}\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)} = -\frac{2ie^{-2(\pi-\theta)s}}{\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)},\\
\mathrm{Res}(z_1;F(z)) &= \frac{e^{-2(\pi+\theta)s}}{-\frac{i}{2}\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)} = \frac{2ie^{-2(\pi+\theta)s}}{\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)},
\end{aligned}
$$

and consequently we have

$$
\begin{aligned}
\mathrm{Res}(z_0;F(z)) + \mathrm{Res}(z_1;F(z)) &= -\frac{2ie^{-2\pi s}\left(e^{2\theta s}-e^{-2\theta s}\right)}{\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)}\\
&= -\frac{4ie^{-2\pi s}\sinh(2\theta s)}{\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)}.
\end{aligned} \tag{5.11}
$$

We note

$$
\mathrm{Res}(\xi_j^{\pm};F(z)) = \frac{e^{is\xi_j^{\pm}}}{\big(\cosh(\xi_j^{\pm}/2)+\alpha\big)\sinh\xi_j^{\pm}}, \qquad j=0,1.
$$

We observe

$$
\begin{aligned}
&e^{is\xi_0^{\pm}} = e^{-\pi s \pm i\lambda s}, \quad e^{is\xi_1^{\pm}} = e^{-3\pi s \pm i\lambda s},\\
&\cosh(\xi_0^{\pm}/2) = \cosh((i\pi\pm\lambda)/2) = \pm i\sinh(\lambda/2) = \pm i\sqrt{\frac{\cosh\lambda-1}{2}} = \pm i\sqrt{\frac{\beta-1}{2}},\\
&\cosh(\xi_1^{\pm}/2) = \cosh((3i\pi\pm\lambda)/2) = \mp i\sinh(\lambda/2) = \mp i\sqrt{\frac{\beta-1}{2}},\\
&\sinh(\xi_0^{\pm}) = \sinh(i\pi\pm\lambda) = \mp\sinh\lambda = \mp\sqrt{\beta^2-1},\\
&\sinh(\xi_1^{\pm}) = \sinh(3i\pi\pm\lambda) = \mp\sqrt{\beta^2-1},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{Res}(\xi_0^{\pm};F(z)) &= \frac{e^{-\pi s\pm i\lambda s}}{\left(\pm i\sqrt{\frac{\beta-1}{2}}+\alpha\right)\left(\mp\sqrt{\beta^2-1}\right)} = \frac{ie^{-\pi s\pm i\lambda s}}{\sqrt{\beta^2-1}\left(\sqrt{\frac{\beta-1}{2}}\mp i\alpha\right)}\\
&= \frac{ie^{-\pi s\pm i\lambda s}\left(\sqrt{\frac{\beta-1}{2}}\pm i\alpha\right)}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2}+\alpha^2\right)},
\end{aligned}
$$

$$\operatorname{Res}(\xi_1^{\pm}; F(z)) = \frac{e^{-3\pi s \pm i\lambda s}}{\left(\mp i\sqrt{\frac{\beta-1}{2}} + \alpha\right)\left(\mp\sqrt{\beta^2-1}\right)} = -\frac{i e^{-3\pi s \pm i\lambda s}}{\sqrt{\beta^2-1}\left(\sqrt{\frac{\beta-1}{2}} \pm i\alpha\right)}$$
$$= -\frac{i e^{-3\pi s \pm i\lambda s}\left(\sqrt{\frac{\beta-1}{2}} \mp i\alpha\right)}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}.$$

We compute

$$\operatorname{Res}(\xi_0^+; F(z)) + \operatorname{Res}(\xi_0^-; F(z)) = \frac{i e^{-\pi s}\left[e^{i\lambda s}\left(\sqrt{\frac{\beta-1}{2}} + i\alpha\right) + e^{-i\lambda s}\left(\sqrt{\frac{\beta-1}{2}} - i\alpha\right)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}$$
$$= \frac{2i e^{-\pi s}\left[\sqrt{\frac{\beta-1}{2}}\cos(\lambda s) - \alpha\sin(\lambda s)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)},$$
$$\operatorname{Res}(\xi_1^+; F(z)) + \operatorname{Res}(\xi_1^-; F(z)) = -\frac{i e^{-3\pi s}\left[e^{i\lambda s}\left(\sqrt{\frac{\beta-1}{2}} - i\alpha\right) + e^{-i\lambda s}\left(\sqrt{\frac{\beta-1}{2}} + i\alpha\right)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}$$
$$= -\frac{2i e^{-3\pi s}\left[\sqrt{\frac{\beta-1}{2}}\cos(\lambda s) + \alpha\sin(\lambda s)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}.$$

Both the quantities have sin, cos, and we conclude

$$\operatorname{Res}(\xi_0^+; F(z)) + \operatorname{Res}(\xi_0^-; F(z)) + \operatorname{Res}(\xi_1^+; F(z)) + \operatorname{Res}(\xi_1^-; F(z))$$
$$= \frac{2i\left[\sqrt{\frac{\beta-1}{2}}\cos(\lambda s)\left(e^{-\pi s} - e^{-3\pi s}\right) - \alpha\sin(\lambda s)\left(e^{-\pi s} + e^{-3\pi s}\right)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}$$
$$= \frac{4i e^{-2\pi s}\left[\sqrt{\frac{\beta-1}{2}}\cos(\lambda s)\sinh(\pi s) - \alpha\sin(\lambda s)\cosh(\pi s)\right]}{\sqrt{\beta^2-1}\left(\frac{\beta-1}{2} + \alpha^2\right)}. \tag{5.12}$$

The desired Fourier transform formula (5.10) is obtained as the sum of (5.11) and (5.12) (multiplied by $2\pi i$). **QED**

5.3 Analysis of $((\cosh(t/2) + \alpha)(\cosh t + \beta))^{-1}$

Here, we recall the Kolmogorov theorem (a version of Lévy-Khintchine formula): A function $f(t)$ on $\mathbf{R}$ is the characteristic function of an infinitely divisible probability measure with finite second moment if and only if there exist a finite positive measure ν and a $\gamma \in \mathbf{R}$ such that

$$\log f(t) = i\gamma t + \int_{-\infty}^{\infty} \left(\frac{e^{its} - 1 - its}{s^2}\right) d\nu(s).$$

Detailed accounts can be found in [37, 14] for instance. We note that functions $f(t)$ we are dealing with here are all smooth and hence the "finite second moment" condition is automatic (see [37, Section 2.3] for instance).

In the following lemma we state two explicit examples of the Kolmogorov theorem obtained in [27, Lemma 2 (ii) and Lemma 16] for later use and for the convenience of the reader:

Lemma 5.4.

(i) *For $a > 0$ and $\theta \in [0, \pi)$,*

$$\log\left(\frac{1+\cos\theta}{\cosh(at)+\cos\theta}\right) = \int_{-\infty}^{\infty} \left(e^{its} - 1 - ist\right) \frac{\cosh(\theta s/a)}{s\sinh(\pi s/a)}\, ds.$$

(ii) *For $a > 0$ and $\lambda \geq 0$,*

$$\log\left(\frac{1+\cosh\lambda}{\cosh(at)+\cosh\lambda}\right) = \int_{-\infty}^{\infty} \left(e^{its} - 1 - ist\right) \frac{\cos(\lambda s/a)}{s\sinh(\pi s/a)}\, ds.$$

When $\alpha = 0$ (i.e., $\theta = \pi/2$), (5.10) reduces to

$$\int_{-\infty}^{\infty} \frac{e^{ist}\, dt}{\cosh(t/2)\,(\cosh t + \beta)} = 2\pi\, \frac{1 - \sqrt{\frac{2}{\beta+1}}\cos(\lambda s)}{(\beta-1)\cosh(\pi s)} \quad (\geq 0), \tag{5.13}$$

which corresponds to the special case $\alpha = 0$ in the next result ([11, Theorem 4.13] and see also [27, Section 7]).

Corollary 5.5. *We set*

$$G(t) \equiv \frac{1}{\big(\cosh(t/2)+\alpha\big)\big(\cosh t + \beta\big)}$$

with $\alpha, \beta > -1$.

(i) *When $\beta > 1$, $G(t)$ is positive definite if and only if $\alpha \in (-1, 0]$.*

(ii) *When $-1 < \beta \leq 1$, $G(t)$ is infinitely divisible for each $\alpha \in (-1, \infty)$.*

Proof. Assume $\beta > 1$. Due to (5.13) $G(t)$ is positive definite for $\alpha = 0$ and remains so for $\alpha \in (-1, 0]$ as well thanks to positive definiteness of

$$\frac{\cosh(t/2)}{\cosh(t/2)+\alpha} = 1 + \frac{-\alpha}{\cosh(t/2)+\alpha}$$

(see Lemma 3.3 (2)). When $\alpha \in (0, 1)$, we have $\theta = \cos^{-1}\alpha \in (0, \pi/2)$ in (5.10). Thus, the dominant terms in the big bracket in the right side of (5.10) are $\cos(\lambda s)\sinh(\pi s)$ and $\sin(\lambda s)\cosh(\pi s)$ so that the quantity in the big bracket takes both positive and negative values for $|s|$ large. To prove (i), it remains to show failure of positive definiteness for $\alpha \geq 1$. However this follows from positive definiteness of

$$\frac{\cosh(t/2)+\alpha}{\cosh(t/2)+\frac{1}{2}} = 1 + \frac{\alpha - \frac{1}{2}}{\cosh(t/2)+\frac{1}{2}}$$

for instance (and the already known failure of positive definiteness for $\alpha = \frac{1}{2}$).

Next, assume $-1 < \beta \le 1$. The statement (ii) is obvious for $\alpha \in (-1, 1]$, $G(t)$ being the product of two infinitely divisible functions under these circumstances. On the other hand, when $\alpha > 1$, we have

$$\log\bigl((1+\alpha)(1+\beta)G(t)\bigr) = \int_{-\infty}^{\infty} \left(e^{ist} - 1 - ist\right) \left(\frac{\cos(2\theta s)}{s\sinh(2\pi s)} + \frac{\cosh(\lambda s)}{s\sinh(\pi s)}\right) ds$$

with $\theta = \log\left(\alpha + \sqrt{\alpha^2 - 1}\right)$ and $\lambda = \cos^{-1}\beta$ (by Lemma 5.4). The density here can be written as

$$\frac{\cos(2\theta s) + 2\cosh(\pi s)\cosh(\lambda s)}{s\sinh(2\pi s)}, \tag{5.14}$$

which is certainly positive. **QED**

5.4 Analysis of $\left(\cosh^k(t/2)(\cosh t + \beta)^m\right)^{-1}$

In this subsection we obtain a result which will be used in Theorem 6.6 of Section 6.3 to obtain a bound for the interval in which $g_{1,\nu,\lambda}$ is in $\mathcal{K}^+$.

We assume $\beta > 1$ and $\alpha \in (-1, 0]$ as in Corollary 5.5 (i). Under these circumstances the density (5.14) is switched to

$$\log\bigl((1+\alpha)(1+\beta)G(t)\bigr) = \int_{-\infty}^{\infty} \left(e^{its} - 1 - ist\right) \frac{\cosh(2\theta s) + 2\cos(\lambda s)\cosh(\pi s)}{s\sinh(2\pi s)}\, ds$$

with $\theta = \cos^{-1}\alpha \in [\pi/2, \pi)$ and $\lambda = \log\left(\beta + \sqrt{\beta^2 - 1}\right)$. Thus, the positive definite function $G(t)$ (Corollary 5.5 (i)) is infinitely divisible if and only if

$$\cosh(2\theta s) + 2\cos(\lambda s)\cosh(\pi s) \ge 0, \qquad s \in \mathbf{R}.$$

This is quite a delicate condition, but for the extreme value $\alpha = 0$ (i.e., $\theta = \pi/2$) the condition simply means

$$(1 + 2\cos(\lambda s))\cosh(\pi s) \ge 0,$$

and it is never fulfilled for any $\beta > 1$ (which is exactly [27, Theorem 15]).

We take higher order partial derivatives $\partial_\beta^{m-1}\partial_\alpha^{k-1}$ from the Fourier transform formula (5.10) (with the variable s fixed). It is obvious that from the left side we get a scalar multiple of

$$\int_{-\infty}^{\infty} \frac{e^{ist}\, ds}{\bigl(\cosh(t/2) + \alpha\bigr)^k \bigl(\cosh t + \beta\bigr)^m},$$

and for the special value $\alpha = 0$ the above integral reduces to

$$\int_{-\infty}^{\infty} \frac{e^{ist}\, ds}{\cosh^k(t/2)\bigl(\cosh t + \beta\bigr)^m}. \tag{5.15}$$

Therefore, behavior on the Fourier transform (5.15) can be seen by computing $\partial_\beta^{m-1}\partial_\alpha^{k-1}$ of the right side of (5.10) at first and then by substituting $\alpha = 0$.

The right side $R(\alpha,\beta)$ of the formula (5.10) consists of the three terms:

$$R(\alpha,\beta) = F_0(\alpha,\beta)\,\frac{\sinh(2\theta s)}{\sinh(2\pi s)} - F_c(\alpha,\beta)\cos(\lambda s)\,\frac{\sinh(\pi s)}{\sinh(2\pi s)} + F_s(\alpha,\beta)\sin(\lambda s)\,\frac{\cosh(\pi s)}{\sinh(2\pi s)}$$

$$= F_0(\alpha,\beta)\,\frac{\sinh(2\theta s)}{\sinh(2\pi s)} - F_c(\alpha,\beta)\,\frac{\cos(\lambda s)}{2\cosh(\pi s)} + F_s(\alpha,\beta)\,\frac{\sin(\lambda s)}{2\sinh(\pi s)} \tag{5.16}$$

with

$$F_0(\alpha,\beta) = \frac{1}{\sqrt{1-\alpha^2}\big(2\alpha^2-1+\beta\big)},$$
$$F_c(\alpha,\beta) = \frac{\sqrt{\frac{\beta-1}{2}}}{\sqrt{\beta^2-1}\big(\frac{\beta-1}{2}+\alpha^2\big)},$$
$$F_s(\alpha,\beta) = \frac{\alpha}{\sqrt{\beta^2-1}\big(\frac{\beta-1}{2}+\alpha^2\big)}.$$

We note

$$\frac{d\theta}{d\alpha} = -\frac{1}{\sin\theta} = -\frac{1}{\sqrt{1-\alpha^2}}, \qquad \frac{d\lambda}{d\beta} = \frac{1}{\sqrt{\beta^2-1}}, \tag{5.17}$$

and consequently

$$\partial_\alpha \sinh(2\theta s) = -\frac{2s\cosh(2\theta s)}{\sqrt{1-\alpha^2}}, \qquad \partial_\alpha \cosh(2\theta s) = -\frac{2s\sinh(2\theta s)}{\sqrt{1-\alpha^2}},$$
$$\partial_\beta \sin(\lambda s) = \frac{s\cos(\lambda s)}{\sqrt{\beta^2-1}}, \qquad \partial_\beta \cos(\lambda s) = -\frac{s\sin(\lambda s)}{\sqrt{\beta^2-1}}.$$

We begin with the first term in (5.16). Since $\sinh(2\theta s)$ and $\cosh(2\theta s)$ behave like constants against ∂_β, $\partial_\beta^{m-1}\partial_\alpha^{k-1}$ of the first term is a polynomial of s of degree at most $k-1$ with coefficients $\sinh(2\theta s)$, $\cosh(2\theta s)$, $1/\sinh(2\pi s)$ and so on. Therefore, the substitution $\alpha = 0$ (i.e., $\theta = \pi/2$) gives rise to a polynomial of s of degree at most $k-1$ with coefficients containing

$$\left.\frac{\sinh(2\theta s)}{\sinh(2\pi s)}\right|_{\theta=\pi/2} = \frac{1}{2\cosh(\pi s)}, \qquad \left.\frac{\cosh(2\theta s)}{\sinh(2\pi s)}\right|_{\theta=\pi/2} = \frac{1}{2\sinh(\pi s)}.$$

The same procedure for the second and third terms in (5.16) obviously gives rise to a polynomial of s of degree at most $m-1$. It is important to make sure that the order is exactly $m-1$, and we will closely check the coefficient of s^{m-1}. For this purpose we begin with the third term in (5.16) and we note

$$F_s(\alpha,\beta) = \frac{1}{2\sqrt{\beta^2-1}}\left(\frac{1}{\alpha+i\sqrt{\frac{\beta-1}{2}}} + \frac{1}{\alpha-i\sqrt{\frac{\beta-1}{2}}}\right),$$

$$\partial_\alpha^{k-1}F_s(\alpha,\beta) = \frac{(-1)^{k-1}(k-1)!}{2\sqrt{\beta^2-1}}\left(\frac{1}{\Big(\alpha+i\sqrt{\frac{\beta-1}{2}}\Big)^k} + \frac{1}{\Big(\alpha-i\sqrt{\frac{\beta-1}{2}}\Big)^k}\right),$$

$$\partial_\alpha^{k-1}\left(F_s(\alpha,\beta)\cdot\frac{\sin(\lambda s)}{2\sinh(\pi s)}\right)=\partial_\alpha^{k-1}F_s(\alpha,\beta)\,\frac{\sin(\lambda s)}{2\sinh(\pi s)}.$$

So far no s-terms show up because λ just depends on β. We then take derivatives relative to β. It is plain to see that the highest s^{m-1}-term arises from

$$\partial_\alpha^{k-1}F_s(\alpha,\beta)\,\partial_\beta^{m-1}\left(\frac{\sin(\lambda s)}{2\sinh(\pi s)}\right)=\partial_\alpha^{k-1}F_s(\alpha,\beta)\,\frac{\partial_\beta^{m-1}\sin(\lambda s)}{2\sinh(\pi s)}.$$

From (5.17) we also easily observe

$$\partial_\beta^{m-1}\sin(\lambda s)=\begin{cases}\pm\dfrac{s^{m-1}}{(\beta^2-1)^{(m-1)/2}}\ \sin(\lambda s)+\text{lower } s\text{-terms} & (\text{for } m \text{ odd}),\\ \pm\dfrac{s^{m-1}}{(\beta^2-1)^{(m-1)/2}}\ \cos(\lambda s)+\text{lower } s\text{-terms} & (\text{for } m \text{ even}).\end{cases}\tag{5.18}$$

By substituting $\alpha=0$, we observe

$$\partial_\alpha^{k-1}F_s(\alpha,\beta)\Big|_{\alpha=0}=\frac{(-1)^{k-1}(k-1)!}{2\sqrt{\beta^2-1}}\left(\frac{1}{\left(i\sqrt{\frac{\beta-1}{2}}\right)^k}+\frac{1}{\left(-i\sqrt{\frac{\beta-1}{2}}\right)^k}\right)\neq 0$$

as long as k is even. From the discussion so far, for k even the highest s^{m-1}-term arising from

$$\partial_\beta^{m-1}\partial_\alpha^{k-1}\left(F_s(\alpha,\beta)\cdot\frac{\sin(\lambda s)}{2\sinh(\pi s)}\right)\Bigg|_{\alpha=0}$$

is a non-zero scalar (of course depending upon β) multiple of

$$\frac{s^{m-1}\sin(\lambda s)}{\sinh(\pi s)}\quad(\text{for } m \text{ odd})\quad\text{or}\quad\frac{s^{m-1}\cos(\lambda s)}{\sinh(\pi s)}\quad(\text{for } m \text{ even})$$

depending upon the parity of m.

We next move to the second term in (5.16). We note

$$F_c(\alpha,\beta)=\frac{1}{2i\sqrt{\beta^2-1}}\left(\frac{1}{\alpha+i\sqrt{\frac{\beta-1}{2}}}-\frac{1}{\alpha-i\sqrt{\frac{\beta-1}{2}}}\right),$$

$$\partial_\alpha^{k-1}F_c(\alpha,\beta)=\frac{(-1)^{k-1}(k-1)!}{2i\sqrt{\beta^2-1}}\left(\frac{1}{\left(\alpha+i\sqrt{\frac{\beta-1}{2}}\right)^k}-\frac{1}{\left(\alpha-i\sqrt{\frac{\beta-1}{2}}\right)^k}\right).$$

The presence of the minus sign this time in the big bracket enables us to conclude

$$\partial_\alpha^{k-1}F_c(\alpha,\beta)\Big|_{\alpha=0}\neq 0$$

for k odd. Since the formula akin to (5.18) is available to $\cos(\lambda s)$, for k odd the highest s^{m-1}-term arising from

$$\partial_\beta^{m-1}\partial_\alpha^{k-1}\left(F_c(\alpha,\beta)\,\frac{\cos(\lambda s)}{2\cosh(\pi s)}\right)\Big|_{\alpha=0}$$

is a non-zero scalar multiple of

$$\frac{s^{m-1}\cos(\lambda s)}{\cosh(\pi s)}\quad\text{(for } m \text{ odd)}\quad\text{or}\quad\frac{s^{m-1}\sin(\lambda s)}{\cosh(\pi s)}\quad\text{(for } m \text{ even)}$$

this time.

Summing up the discussions so far, we conclude: For s large the leading terms of $\partial_\beta^{m-1}\partial_\alpha^{k-1}R(\alpha,\beta)$ (which arise form the first term and the last two terms in (5.16) respectively) are non-zero scalar multiples of

$$\frac{s^{k-1}}{e^{\pi s}}\quad\text{and}\quad\frac{s^{m-1}\sin(\lambda s+\delta)}{e^{\pi s}}$$

regardless of the parity of k.

We are now ready to prove the following result:

Theorem 5.6. *We assume $\beta > 1$ and set*

$$H(t) \equiv \frac{1}{\cosh^k(t/2)\big(\cosh t+\beta\big)^m}$$

for positive integers k, m.

(i) *$H(t)$ is positive definite if and only if $k \geq m$.*

(ii) *$H(t)$ is infinitely divisible if and only if $k \geq 2m$.*

Proof. We set $\lambda = \log\left(\beta+\sqrt{\beta^2-1}\right) > 0$ as in Theorem 5.3.

(i) Firstly we assume $k \geq m$. Since

$$\frac{1}{\cosh^k(t/2)\big(\cosh t+\beta\big)^m} = \frac{1}{\cosh^{k-m}(t/2)}\left(\frac{1}{\cosh(t/2)\big(\cosh t+\beta\big)}\right)^m,$$

positive definiteness of $H(t)$ follows from Corollary 5.5 (i) (or rather the paragraph before the corollary). On the other hand, when $k < m$, for s large the dominant term in (5.15) (i.e., the Fourier transform of $H(t)$) is

$$\frac{s^{m-1}\sin(\lambda s+\delta)}{e^{\pi s}}$$

as was mentioned in the paragraph right before the theorem. Thus, the Fourier transform admits both positive and negative values and hence $H(t)$ cannot be positive definite.

(ii) By Lemma 5.4 we have

$$\log\big((1+\beta)^m H(t)\big) = \int_{-\infty}^{\infty}\left(e^{ist}-1-ist\right)F(s)\,ds$$

with

$$F(s)=\frac{k}{2s\sinh(\pi s)}+\frac{m\cos(\lambda s)}{s\sinh(\pi s)}=\frac{k+2m\cos(\lambda s)}{2s\sinh(\pi s)}.$$

Thus, $H(t)$ is infinitely divisible if and only if $k+2m\cos(\lambda s)\ge 0$, i.e., $k\ge 2m$. **QED**

Note that the optimal case in (ii) (i.e., case $k=2m$) corresponds to infinite divisibility of $\big((\cosh t+1)(\cosh t+\beta)\big)^{-1}$ (see [27, §7.1]) because of $\cosh^2(t/2)=(\cosh t+1)/2$. The function $1/\cosh(t/2)$ is positive definite (and indeed infinitely divisible) while $1/(\cosh t+\beta)$ (with $\beta>1$) is not (see Lemma 3.3 (2)). Thus, intuition might suggest that as far as the function $H(t)$ is concerned one has higher (resp., lower) chance for positive definiteness and/or infinite divisibility as k (resp., m) increases. The theorem completely clarifies where proper balance is taken.

6 Proofs of results in Section 4

6.1 Results on WYD family

For the functions k_p^{WYD}, $p\in[-1,2]$, defined by (4.6), we will prove the next results stated in Example 4.8.

Theorem 6.1.

(a) *The function k_p^{WYD} belongs to $\mathcal{K}^+$ if and only if $p\in[0,1]$,*

(b) *The function k_p^{WYD} belongs to $\mathcal{K}^-$ if and only if $p\in\big[-1,-\frac{1}{2}\big]\cup\big[\frac{3}{2},2\big]$.*

Proof. We may and do assume $p\in\big[\frac{1}{2},2\big]$ in view of the symmetry $k_p^{\mathrm{WYD}}=k_{1-p}^{\mathrm{WYD}}$, and set

$$f_p(t)\equiv e^t k_p^{\mathrm{WYD}}(e^{2t})=\frac{1}{p(1-p)}\cdot\frac{\sinh(pt)\sinh((1-p)t)}{\sinh^2 t},$$

$$g_p(t)\equiv e^t/k_p^{\mathrm{WYD}}(e^{-2t})=p(1-p)\cdot\frac{\sinh^2 t}{\sinh(pt)\sinh((1-p)t)}$$

for convenience. Theorem 3.4 says that we have to determine when $f_p(t)$ and $g_p(t)$ are positive definite.

Positive definiteness for $f_p(t)$ with $p\in\big[\frac{1}{2},1\big]$ is well-known (where $f_1(t)$ is understood as $t/\sinh t$). When $p\in(1,2]$, we have

$$f_p(t)=\frac{1}{p(p-1)}\cdot\frac{\sinh(pt)\sinh((p-1)t)}{\sinh^2 t},$$

which fails to be positive definite thanks to Theorem 5.1 (due to the presence of $\sinh(pt)$ with $p>1$ in the numerator). Thus, we have shown that $f_p(t)$ is positive definite if and only if $p\in\big[\frac{1}{2},1\big]$, that is, (a) is proved.

To prove (b), we next check positive definiteness for $g_p(t)$. When $p\in\big[\frac{1}{2},1\big]$, we have

$$\lim_{t\to\pm\infty}g_p(t)=+\infty$$

(where $g_1(t)$ is understood as $\sinh t/t$) so that $g_p(t)$ cannot be positive definite. We now move to the case $p \in (1, 2]$ so that we use the expression

$$g_p(t) = p(p-1)\,\frac{\sinh^2 t}{\sinh(pt)\sinh((p-1)t)}.$$

When $2 > p + (p-1)$ (i.e., $p < \frac{3}{2}$), $g_p(t)$ once again diverges as $t \to \pm\infty$ and fails to be positive definite. Thus, it remains to show positive definiteness for $p \geq \frac{3}{2}$. For the extreme value $p = \frac{3}{2}$ we compute

$$\begin{aligned} g_{3/2}(t) &= \frac{3}{4}\cdot\frac{\sinh^2 t}{\sinh(3t/2)\sinh(t/2)} = \frac{3}{4}\cdot\frac{\sinh^2 t}{\sinh^2(t/2)\left(4\cosh^2(t/2)-1\right)} \\ &= \frac{3\cosh^2(t/2)}{4\cosh^2(t/2)-1} = \frac{3\left(\cosh t+1\right)/2}{2\left(\cosh t+1\right)-1} = \frac{3}{4}\left(1+\frac{\frac{1}{2}}{\cosh t+\frac{1}{2}}\right). \end{aligned}$$

Since $1/\left(\cosh t + \frac{1}{2}\right)$ is positive definite (see Lemma 3.3 (2)), so is $g_{3/2}(t)$. Finally, the obvious identity

$$\frac{\sinh^2 t}{\sinh(pt)\sinh((p-1)t)} = \frac{\sinh^2 t}{\sinh(3t/2)\sin(t/2)}\cdot\frac{\sinh(3t/2)\sin(t/2)}{\sinh(pt)\sin((p-1)t)}$$

takes care of the remaining case (i.e., $p \in \left(\frac{3}{2}, 2\right]$). **QED**

6.2 Proofs for Example 4.3

We now prove the claims in Example 4.3.

Theorem 6.2. *We assume $\nu \in (0,1)$ and $\lambda \in [0,1]$. Then, the function $a_{1,\nu,\lambda}$ defined in* (4.1) *belongs to $\mathcal{K}^+$ if and only if*

$$\lambda \leq \frac{2\sqrt{\nu}}{(1+\sqrt{\nu})^2} = \frac{2}{\left(\nu^{1/4}+\nu^{-1/4}\right)^2}. \tag{6.1}$$

Moreover, $a_{1,\nu,\lambda}$ is an extreme point in $\mathcal{K}^+$ if and only if equality holds in (6.1).

Proof. With $\beta = (1+\nu^2)/2\nu$ (> 1) we compute

$$\begin{aligned} e^{t/2}a_{1,\nu,\lambda}(e^t) &= \lambda(\beta+1)\,\frac{\cosh(t/2)}{\cosh t+\beta} + (1-\lambda)\,\frac{1}{\cosh(t/2)} \\ &= \frac{\lambda(\beta+1)\left(\cosh t+1\right)/2 + (1-\lambda)\left(\cosh t+\beta\right)}{\cosh(t/2)\left(\cosh t+\beta\right)} \\ &= \frac{\left(\lambda(\beta-1)/2+1\right)\left(\cosh t+\beta\right) - \lambda(\beta^2-1)/2}{\cosh(t/2)\left(\cosh t+\beta\right)} \\ &= \frac{\lambda(\beta-1)/2+1}{\cosh(t/2)} - \frac{\lambda(\beta^2-1)/2}{\cosh(t/2)\left(\cosh t+\beta\right)}. \end{aligned}$$

Let us recall (5.13) (where the symbol λ for $\cosh^{-1}\beta$ there is changed to α to avoid the obvious confusion) and

$$\int_{-\infty}^{\infty}\frac{e^{ist}\,dt}{\cosh(t/2)}=\frac{2\pi}{\cosh(\pi s)}.$$

The Fourier transform is thus given by

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}e^{t/2}a_{1,\nu,\lambda}(e^t)\,e^{ist}\,dt&=\frac{\lambda(\beta-1)+2}{\cosh(\pi s)}-\frac{\lambda(\beta+1)\left(1-\sqrt{\frac{2}{\beta+1}}\cos(\alpha s)\right)}{\cosh(\pi s)}\\&=\frac{N(s)}{\cosh(\pi s)}\end{aligned}$$

with the numerator

$$N(s)\equiv 2(1-\lambda)+\lambda\sqrt{2\,(\beta+1)}\cos(\alpha s).$$

This computation says that $e^{t/2}a_{1,\nu,\lambda}(e^t)$ is positive definite (i.e., $a_{1,\nu,\lambda}\in\mathcal{K}^+$) if and only if

$$\lambda\sqrt{2\,(\beta+1)}\le 2(1-\lambda).$$

It is obviously satisfied for $\lambda=0$ (which corresponds to the obvious positive definiteness of $e^{t/2}k_1^{\mathrm{ext}}(e^t)$) while for $\lambda>0$ the requirement is the same as

$$\sqrt{\frac{1+\beta}{2}}\le\frac{1-\lambda}{\lambda}\iff\lambda\le\left(\sqrt{\frac{1+\beta}{2}}+1\right)^{-1}.$$

Finally, we compute

$$\sqrt{\frac{1+\beta}{2}}+1=\sqrt{\frac{(1+\nu)^2}{4\nu}}+1=\frac{(1+\sqrt{\nu})^2}{2\sqrt{\nu}},$$

which proves the first part.

Now, we set $\lambda(\nu)\equiv 2\sqrt{\nu}/(1+\sqrt{\nu})^2$ and prove the second part. It is obvious that $a_{1,\nu,\lambda}$ is not an extreme point of $\mathcal{K}^+$ if $\lambda<\lambda(\nu)$. To prove the converse we assume that $a_{1,\nu,\lambda(\nu)}(x)=\lambda k_1(x)+(1-\lambda)k_2(x)$ with some $\lambda\in(0,1)$, $k_1,k_2\in\mathcal{K}^+$ and hence

$$k_i(x)=\int_{[0,1]}k_\nu^{ext}(x)\,dm_i(\nu),\qquad i=1,2,$$

with representing probability measures m_i on $[0,1]$. From the uniqueness of a representing measure we have

$$\lambda m_1+(1-\lambda)m_2=(1-\lambda(\nu))\delta_1+\lambda(\nu)\delta_\nu.$$

In particular, we have

$$\begin{cases}\lambda m_1(\{1\})+(1-\lambda)m_2(\{1\})=1-\lambda(\nu),\\ \lambda m_1(\{\nu\})+(1-\lambda)m_2(\{\nu\})=\lambda(\nu),\end{cases}\tag{6.2}$$

and they sum up to

$$\lambda\big(m_1(\{1\})+m_1(\{\nu\})\big)+(1-\lambda)\big(m_2(\{1\})+m_2(\{\nu\})\big)=1.\tag{6.3}$$

On the other hand, we have $m_i(\{1\}) + m_i(\{\nu\}) \le 1$ $(i = 1, 2)$ because m_i's are probability measures. Therefore, (6.3) guarantees $m_i(\{1\}) + m_i(\{\nu\}) = 1$ $(i = 1, 2)$. Hence, both of m_1, m_2 are supported on the two-point set $\{\nu, 1\}$, that is, of the form

$$m_i = (1 - a_i)\delta_1 + a_i\delta_\nu \quad (i = 1, 2)$$

with $a_i = m_i(\{\nu\}) \in [0, 1]$. Since $k_i \in \mathcal{K}^+$ $(i = 1, 2)$, the first part of the theorem implies $a_i \le \lambda(\nu)$ and hence the second equation of (6.2) forces $a_1 = a_2 = \lambda(\nu)$, i.e., $k_1 = k_2 = a_{1,\nu,\lambda(\nu)}$. **QED**

Since $2\sqrt{\nu}/\left(1 + \sqrt{\nu}\right)^2 < \frac{1}{2}$ (with $\nu \ne 1$), we have $a_{1,\nu,\lambda} \notin \mathcal{K}^+$ as long as $\lambda \ge \frac{1}{2}$ (and $\nu \in [0, 1)$). Choose the extreme value $\lambda = \lambda(\nu) = \left(\sqrt{(\beta+1)/2} + 1\right)^{-1}$ with $\beta = (1+\nu^2)/2\nu$. Then, it is straightforward to compute

$$e^{t/2} a_{1,\nu,\lambda_0}(e^t) = \sqrt{\frac{\beta+1}{2}} \cdot \frac{\cosh t + \sqrt{2(\beta+1)} - 1}{\cosh(t/2)\,(\cosh t + \beta)}.$$

It is not clear if this positive definite function is infinitely divisible.

6.3 Results on geometric bridges

We consider the geometric bridges $g_{1,\nu,\lambda}(x)$, $0 \le \lambda \le 1$, between $k_1^{\rm ext}$ and $k_\nu^{\rm ext}$ with $\nu \in [0, 1)$ given by (4.8) and described in Example 4.13. Since the case $\nu = 0$ was settled in Example 4.12:, we assume $\nu \in (0, 1)$. Our main result is

Theorem 6.3. *For each fixed $\nu \in (0, 1)$, there is a critical λ_c (dependent on ν) such that the function $g_{1,\nu,\lambda}$ is in $\mathcal{K}^+$ for $\lambda \in \left[0, \lambda_c\right]$. Moreover,*

$$\frac{1}{4} \le \lambda_c(\nu) \le \frac{1}{3} \qquad \textit{for each} \quad \nu \in (0, 1). \tag{6.4}$$

This will follow from a series of lemmas and theorems below, which are of independent interest. First, observe that one can rewrite (4.8) as

$$g_{1,\nu,\lambda}(x) = x^{-1/2} \left(\frac{2}{x^{1/2} + x^{-1/2}}\right)^{1-2\lambda} \left(\frac{(1+\nu)^2}{2\nu\left(\frac{x+x^{-1}}{2} + \frac{1+\nu^2}{2\nu}\right)}\right)^{\lambda}.$$

and define for $\beta = \frac{1+\nu^2}{2\nu} > 1$,

$$f_{\nu,\lambda}(t) \equiv e^{t/2} g_{1,\nu,\lambda}(e^t) = \frac{1}{\cosh^{1-2\lambda}(t/2)} \left(\frac{1+\beta}{\cosh t + \beta}\right)^{\lambda}.$$

Recall that Theorem 3.4 implies that $g_{1,\nu,\lambda} \in \mathcal{K}^+$ if and only if $f_{\nu,\lambda}(t)$ is positive definite.

Lemma 6.4. *The function $f_{\nu,\lambda}(t)$ is infinitely divisible if and only if $0 \le \lambda \le \frac{1}{4}$.*

Proof. With $\alpha = \log\left(\beta + \sqrt{\beta^2 - 1}\right) > 0$ by Lemma 5.4 we have

$$\log f_{\nu,\lambda}(t) = \lambda \log\left(\frac{1+\beta}{\cosh t + \beta}\right) + (1-2\lambda)\log\left(\frac{1}{\cosh(t/2)}\right)$$
$$= \int_{-\infty}^{\infty} \left(e^{ist} - 1 - ist\right) F(s)\, ds$$

with

$$F(s) = \lambda \frac{\cos(\alpha s)}{s \sinh(\pi s)} + (1-2\lambda)\frac{1}{2s\sinh(\pi s)} = \frac{2\lambda\big(\cos(\alpha s) - 1\big) + 1}{2s\sinh(\pi s)}.$$

The minimum of $\cos(\alpha s) - 1$ is -2 so that the above density $F(s)$ is non-negative exactly when $-4\lambda + 1 \geq 0$. **QED**

We prove that for fixed ν the functions $g_{1,\nu,\lambda}$ increase monotonically with λ in the $\preccurlyeq$ order.

Lemma 6.5. *If $\lambda' \leq \lambda$, then $g_{1,\nu,\lambda'} \preccurlyeq g_{1,\nu,\lambda}$*

Proof. It suffices to show positive definiteness of

$$\frac{f_{\nu,\lambda'}(t)}{f_{\nu,\lambda}(t)} = \frac{g_{1,\nu,\lambda'}(e^t)}{g_{1,\nu,\lambda}(e^t)} \tag{6.5}$$

for $\lambda' \leq \lambda$. However, this ratio is equal to

$$\frac{1}{\cosh^{2(\lambda-\lambda')}(t/2)}\left(\frac{\cosh t + \beta}{1+\beta}\right)^{\lambda-\lambda'} = \left(\frac{1}{\cosh^2(t/2)} \cdot \frac{\cosh t + \beta}{1+\beta}\right)^{\lambda-\lambda'}$$
$$= \left(\frac{2}{1+\beta} \cdot \frac{\cosh t + \beta}{\cosh t + 1}\right)^{\lambda-\lambda'}.$$

On the other hand, by Lemma 5.4 (ii) we have

$$\log\left(\frac{2}{1+\beta} \cdot \frac{\cosh t + \beta}{\cosh t + 1}\right) = \int_{-\infty}^{\infty} \left(e^{ist} - 1 - ist\right) \frac{1 - \cos(\alpha s)}{s\sinh(\pi s)}\, ds$$

with $\alpha = \log\left(\beta + \sqrt{\beta^2 - 1}\right) > 0$. The density $(1 - \cos(\alpha s))/s\sinh(\pi s)$ here is positive so that $(\cosh t + \beta)/(\cosh t + 1)$ is infinitely divisible and hence the ratio (6.5) (with $\lambda' \leq \lambda$) is positive definite. **QED**

The monotonicity shown above implies that for each fixed ν, the set

$$\left\{\lambda \in [0,1] : g_{1,\nu,\lambda} \in \mathcal{K}^+\right\}$$

is a subinterval $[0, \lambda_c]$, for which we prove that the critical value $\lambda_c = \lambda_c(\nu)$ satisfies (6.4) above. The lower bound $\frac{1}{4}$ follows immediately from Lemma 6.4. The upper bound follows immediately from Theorem 6.6 below.

Theorem 6.6. *When $\lambda > \frac{1}{3}$, the function $g_{1,\nu,\lambda}$ does not belong to $\mathcal{K}^+$ for any $\nu \in (0,1)$.*

Proof. By Theorem 3.4, this is equivalent to showing that $f_{\nu,\lambda}(t)$ is not positive definite when $\lambda > \frac{1}{3}$, for which we will prove by contradiction. We choose a rational $\frac{m}{n}$ with

$$\frac{1}{3} < \frac{m}{n} \le \lambda. \tag{6.6}$$

Then by Lemma 6.5, positive definiteness of $f_{\nu,\lambda}(t)$ implies that so is $f_{\nu,\frac{m}{n}}(t)$. Since $f_{\nu,\frac{m}{n}}(t)$ is equal to

$$\frac{1}{\cosh^{1-\frac{2m}{n}}(t/2)\big(\cosh t+\beta\big)^{\frac{m}{n}}}$$

up to a positive constant, its nth power

$$\frac{1}{\cosh^{k}(t/2)\big(\cosh t+\beta\big)^{m}}$$

with $k = n-2m$ would be also positive definite. However, this contradicts Theorem 5.6 (i) because of (6.6), i.e., $k < m$. **QED**

How the critical value $\lambda_c(\nu)$ depends on $\nu \in (0,1)$ seems to be an interesting problem. We note that $f_{\nu,1/3}(t)$ is equal to

$$\frac{1}{\cosh^{1/3}(t/2)\left(\cosh t+\frac{1+\nu^2}{2\nu}\right)^{1/3}}$$

up to a positive scalar. Although the function $\big(\cosh(t/2)(\cosh t+(1+\nu^2)/2\nu)\big)^{-1}$ is known not to be infinitely divisible ([27, Theorem 15] and see also Theorem 5.6 (ii)), its cubic root might be positive definite for some values of ν). However, the authors are unable to handle this delicate phenomenon.

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A Proofs from Section 2

A.1 Proof of Proposition 2.2

The following is from the proof of [22, Corollary 2.2]. Since k is an operator convex function on $(0,\infty)$, the function $g(x) = k(1+x)$ is operator convex on $(-1,1)$. By Kraus' theorem [29] (see [2, Lemma III.1] or [7, Theorem V.3.10]) the divided difference function $h(x) \equiv (g(x)-g(0))/x$

is operator monotone on $(-1,1)$. Then by Löwner's integral representation (see [2, Theorem II.1] or [7, Corollary V.4.5]) there exists a (unique) finite measure μ on $[-1,1]$ such that

$$h(x) = a + \int_{[-1,1]} \frac{x}{1-\lambda x}\, d\mu(\lambda)$$

with $a = h(0) = k'(1)$. Thus

$$\begin{aligned}(1+x)k(1+x) &= (1+x)(ax+b) + (1+x)\int_{[-1,1]} \frac{x^2}{1-\lambda x}\, d\mu(\lambda) \\ &= (1+x)(ax+b) + \mu(\{-1\}) + x^2 \int_{(-1,1]} \frac{1+x}{1-\lambda x}\, d\mu(\lambda)\end{aligned}$$

with $b = k(1)$. Since $(x+1)(1-\lambda x) \le 1$ whenever $x \in (-1,1)$ and $\lambda \in (-1,1]$, the Lebesgue dominated convergence theorem implies that

$$\lim_{x\searrow -1} \int_{(-1,1]} \frac{1+x}{1-\lambda x}\, d\mu(\lambda) = 0$$

so that

$$\lim_{x\searrow 0} x\, k(x) = \lim_{x\searrow -1} (1+x)\, k(1+x) = \mu(\{-1\}).$$

A.2 Proof of Theorem 2.3

Since the divided difference function $h(x) \equiv (k(x)-k(1))/(x-1)$ is operator monotone on $(0,\infty)$ as in Section A.1, it is known (see, e.g., [13, Theorem 1.9]) that there exist a (unique) $\gamma \ge 0$ and a (unique) positive measure μ on $[0,\infty)$ with $\int_{[0,\infty)} (1+\lambda)^{-1}\, d\mu(\lambda) < +\infty$ such that

$$h(x) = h(1) + \gamma(x-1) + \int_{[0,\infty)} \frac{x-1}{x+\lambda}\, d\mu(\lambda), \qquad x \in (0,\infty).$$

Therefore,

$$k(x) = k(1) + k'(1)(x-1) + \gamma(x-1)^2 + \int_{[0,\infty)} \frac{(x-1)^2}{x+\lambda}\, d\mu(\lambda), \quad x \in (0,\infty). \tag{A.1}$$

From the symmetry property $xk(x) = k\big(x^{-1}\big)$ we notice that

$$\lim_{x\to\infty} \frac{k(x)}{x} = \lim_{x\to\infty} x^{-2}k\big(x^{-1}\big) = \lim_{x\searrow 0} x^2 k(x) = \lim_{x\searrow 0} \int_{[0,\infty)} \frac{x^2}{x+\lambda}\, d\mu(\lambda) = 0$$

by the Lebesgue convergence theorem. On the other hand, since $(x-1)^2/x(x+\lambda) \nearrow 1$ as $x \nearrow \infty$, we have

$$\lim_{x\to\infty} \frac{1}{x} \int_{[0,\infty)} \frac{(x-1)^2}{x+\lambda}\, d\mu(\lambda) = \int_{[0,\infty)} d\mu(\lambda)$$

by the monotone convergence theorem, and hence

$$\lim_{x\to\infty} \frac{k(x)}{x} = k'(1) + \gamma\cdot(+\infty) + \int_{[0,\infty)} d\mu(\lambda).$$

Therefore, $\gamma = 0$, μ is a finite measure, and $k'(1) + \int_{[0,\infty)} d\mu(\lambda) = 0$. From

$$\frac{(x-1)^2}{x+\lambda} = x - 1 - \frac{(x-1)(1+\lambda)}{x+\lambda}$$

it follows that

$$\begin{aligned} k(x) &= k(1) - \int_{[0,\infty)} \frac{(x-1)(1+\lambda)}{x+\lambda}\, d\mu(\lambda) \\ &= k(1) + \int_{[0,\infty)} \left(\frac{1+\lambda}{x+\lambda} - 1\right)(1+\lambda)\, d\mu(\lambda), \end{aligned}$$

which is operator monotone decreasing since so is $(1+\lambda)/(x+\lambda)$. Moreover, since $(x-1)(1+\lambda)/(x+\lambda) \nearrow 1+\lambda$ as $x \nearrow \infty$, we have $0 \le k(1) - \int_{[0,\infty)}(1+\lambda)\, d\mu(\lambda)$ so that $\int_{[0,\infty)}(1+\lambda)\, d\mu(\lambda) \le k(1) < +\infty$. Now, defining a finite positive measure ν on $[0,\infty]$ by

$$d\nu(\lambda) \equiv (1+\lambda)\, d\mu(\lambda) \quad \text{on} \quad [0,\infty), \qquad \nu(\{\infty\}) \equiv k(1) - \int_{[0,\infty)} (1+\lambda)\, d\mu(\lambda),$$

we write

$$k(x) = \int_{[0,\infty]} \frac{1+\lambda}{x+\lambda}\, d\nu(\lambda), \qquad x \in (0,\infty), \tag{A.2}$$

where $(1+\lambda)/(x+\lambda) \equiv 1$ for $\lambda = \infty$. Letting $d\tilde{\nu}(\lambda) \equiv d\nu(\lambda^{-1})$ on $[0,\infty]$ we also write

$$k(x) = \int_{[0,\infty]} \frac{1+\lambda^{-1}}{x+\lambda^{-1}}\, d\tilde{\nu}(\lambda) = \int_{[0,\infty]} \frac{1+\lambda}{1+\lambda x}\, d\tilde{\nu}(\lambda)$$

so that

$$k\bigl(x^{-1}\bigr) = \int_{[0,\infty]} \frac{x(1+\lambda)}{x+\lambda}\, d\tilde{\nu}(\lambda).$$

This is the familiar integral expression of the operator monotone function $k\bigl(x^{-1}\bigr)$ with a unique representing measure $\tilde{\nu}$. Hence the measure ν satisfying (A.2) is unique (this fact itself is also well-known). Since

$$k(x) = x^{-1}k\bigl(x^{-1}\bigr) = \int_{[0,\infty]} \frac{1+\lambda}{1+\lambda x}\, d\nu(\lambda) = \int_{[0,\infty]} \frac{1+\lambda}{x+\lambda}\, \tilde{\nu}(\lambda),$$

it follows that $\nu = \tilde{\nu}$. Define

$$dm(\lambda) \equiv 2d\nu(\lambda) \quad \text{on} \quad [0,1), \qquad m(\{1\}) \equiv \nu(\{1\}),$$

to obtain

$$\begin{aligned} k(x) &= \int_{[0,1)} \left(\frac{1+\lambda}{x+\lambda} + \frac{1+\lambda^{-1}}{x+\lambda^{-1}}\right) d\nu(\lambda) + \frac{2}{1+x}\,\nu(\{1\}) \\ &= \int_{[0,1]} \frac{1+x}{(x+\lambda)(1+\lambda x)} \cdot \frac{(1+\lambda)^2}{2}\, dm(\lambda). \end{aligned}$$

Finally, note that the uniqueness of m is immediate from that of ν in (A.2).

The integral expression (A.1) was also given in [33], which was considerably extended in [13, Theorem 5.1]. There is another route to prove the two theorems in Section 2.1. It was proved in [4, Theorem 3.1] that a function $k:(0,\infty)\to(0,\infty)$ is operator monotone decreasing if and only if it is operator convex and non-increasing in the numerical sense. It is easy to see that if an operator convex function k satisfies the symmetry condition $xk(x)=k\big(x^{-1}\big)$, then it is non-increasing numerically. Hence we have the implication (a) $\Rightarrow$ (b) of Theorem 2.4. All other parts of Theorem 2.4 are plain or well-known. Then we can prove Theorem 2.3 by applying the familiar integral expression to a symmetric operator monotone function $k\big(x^{-1}\big)$ as above (indeed, this part of the proof is the same as the proof of [31, Theorem 4.4]).

B Contraction bounds

As stated in (2.7), monotone Riemannian metrics contract under the action of quantum channels, i.e., CPT (CP and trace-preserving) maps. It is well-known [21, 44, 33, 55] that the quasi-entropies $H_g(A,B)\equiv H_g(A,B,I)$ given by (2.8) with $K=I$ contract under CPT maps, i.e.,

$$H_g\big(\Phi(A),\Phi(B)\big)\le H_g(A,B),\qquad A,B\in\mathbb{P}_d,$$

whenever g is operator convex on $(0,\infty)$. In the rest of this subsection let $g(x)=(1-x)^2k(x)$ with $k\in\mathcal{K}$ as in Section 2.4. In applications, the maximal contraction rate plays an important role, which motivated in [33] the following definitions of *contraction coefficients*:

$$\eta_k^{\mathrm{RelEnt}}(\Phi)\equiv\sup_{\rho,\gamma\in\mathcal{D}_d,\,\rho\ne\gamma}\frac{H_g\big(\Phi(\rho),\Phi(\gamma)\big)}{H_g(\rho,\gamma)}$$

and

$$\eta_k^{\mathrm{Riem}}(\Phi)\equiv\sup_{\rho\in\mathcal{D}_d}\ \sup_{X\in\mathbb{H}_d^0,\,X\ne0}\frac{\Gamma_{\Phi(\rho)}^k\big((\Phi(X),\Phi(X)\big)}{\Gamma_\rho^k(X,X)}.$$

A contraction coefficient was also defined [51] for the trace norm $\|X\|_1\equiv\mathrm{Tr}\,|X|=\mathrm{Tr}\,(X^*X)^{1/2}$ distance which also contracts under CPT maps, i.e.,

$$\eta^{\mathrm{Dob}}(\Phi)\equiv\sup_{\rho,\gamma\in\mathcal{D}_d,\,\rho\ne\gamma}\frac{\|\Phi(\rho-\gamma)\|_1}{\|\rho-\gamma\|_1},$$

where the superscript reflects the fact that this is the quantum analogue of the classical Dobrushin coefficient.

For any CPT map Φ, it was shown in [33, Theorem IV.2] that

$$\eta_k^{\mathrm{Riem}}(\Phi)\le\eta_k^{\mathrm{RelEnt}}(\Phi)\le1\qquad\text{for any}\quad k\in\mathcal{K},$$

and in [54, Theorems 13, 14] that

$$\eta_{x^{-1/2}}^{\mathrm{RelEnt}}(\Phi)\le\eta^{\mathrm{Dob}}(\Phi)\le\sqrt{\eta_k^{\mathrm{Riem}}(\Phi)}\tag{B.1}$$

when $k(x)$ is given by (4.4). The upper bound in (B.1), given in [51, Theorem 3] for the particular case $k_0^{\text{ext}}(x) = (1+x)/2x$ and in [54] for $k = \widehat{k}_\alpha^{\text{H}}$ in Example 4.5, holds for any $k \in \mathcal{K}$. Our work here was motivated by the lower bound in (B.1) based on the following observations from [33, 54]. Applying the max-min principle to the eigenvalue problem

$$\big(\widehat{\Phi} \circ \Omega^k_{\Phi(\rho)} \circ \Phi\big)(X) = \lambda\, \Omega^k_\rho(X) \tag{B.2}$$

(for which $X = I$ always yields the largest eigenvalue $\lambda_1 = 1$) implies that

$$\eta_k^{\text{Riem}}(\Phi) = \sup_{\rho \in \mathcal{D}_d} \lambda_2^k(\Phi, \rho),$$

where $\widehat{\Phi}$ is the adjoint of Φ (with respect to the Hilbert-Schmidt inner product) and $\lambda_2^k(\Phi,\rho)$ denotes the second largest eigenvalue of (B.2). This is equivalent to the eigenvalue problem

$$\Upsilon^k_{\rho,\Phi}(\Phi(X)) = \big(\Omega^k_\rho\big)^{-1} \circ \widehat{\Phi} \circ \Omega^k_{\Phi(\rho)}(\Phi(X)) = \lambda X$$

for the trace-preserving map $\Upsilon^k_{\rho,\Phi} \equiv \big(\Omega^k_\rho\big)^{-1} \circ \widehat{\Phi} \circ \Omega^k_{\Phi(\rho)}$ restricted on $\mathbb{H}_d^0$. When $\Upsilon^k_{\rho,\Phi}$ is positivity-preserving,

$$\lambda_2^k(\Phi,\rho) = \sup_{X \in \mathbb{H}_d^0} \frac{\|\Upsilon^k_\rho(\Phi(X))\|_1}{\|X\|_1} \le \sup_{X \in \mathbb{H}_d^0} \frac{\|\Phi(X)\|_1}{\|X\|_1} = \eta^{\text{Dob}}(\Phi).$$

A sufficient condition for $\Upsilon^k_{\rho,\Phi}$ to be positivity-preserving is that both $\big(\Omega^k_\rho\big)^{-1}$ and Ω^k_ρ are CP,[5] which we have seen holds if and only if $k(x) = x^{-1/2}$. There may be particular maps Φ for which $\Upsilon^k_{\rho,\Phi}$ is positivity-preserving even when Ω^k_ρ and/or its inverse are not. Whether or not the bound $\eta^{\text{RelEnt}}_{x^{-1/2}}(\Phi) \le \eta^{\text{Dob}}(\Phi)$ holds for other $k \in \mathcal{K}$ even though $\Upsilon^k_{\rho,\Phi}$ is not positivity-preserving is an open question.

C Some pedestrian arguments

In this section we present, for the benefit of non-experts, some very pedestrian ways to see certain well-known results used in this paper.

C.1 Functional calculus for L_D and R_D

It is basic that when D has the spectral decomposition $D = \sum_j w_j |\xi_j\rangle\langle\xi_j|$ (where $|\xi_j\rangle\langle\xi_j|$ is the physicists's notation for the spectral projection onto the eigenspace of the eigenvector $|\xi_j\rangle$), $\varphi(D) = \sum_j \varphi(w_j)|\xi_j\rangle\langle\xi_j|$ for any function φ on $(0,\infty)$. It then follows that

$$L_{\varphi(D)}(X) = \varphi(L_D)(X) = \sum_j \varphi(w_j)|\xi_j\rangle\langle\xi_j|X \qquad \text{and}$$
$$R_{\psi(D)}(X) = \psi(R_D)(X) = \sum_j \psi(w_j)X|\xi_j\rangle\langle\xi_j|.$$

[5] Unfortunately, in [33] it was claimed that Υ^k_ρ is positivity-preserving for $k(x) = \log x/(x-1)$. Although Ω^k_ρ given by (1.1) is clearly positivity-preserving, the inverse $\big(\Omega^k_\rho\big)^{-1}$ given by (1.2) is not.

Then the product

$$(\psi(R_D)\varphi(L_D))(X) = \sum_{i,j} \varphi(w_i)\psi(w_j)\,|\xi_i\rangle\langle\xi_j|\,\langle\xi_i, X\xi_j\rangle.$$

Since L_D and R_D commute, it follows that for an arbitrary function $\phi(x, y)$

$$\phi(L_D, R_D)(X) = \sum_{i,j} \phi(w_i, w_j)\,|\xi_i\rangle\langle\xi_j|\,\langle\xi_i, A\xi_j\rangle$$

which is exactly the Hadamard product of $A \circ X$ when $a_{ij} = \phi(w_i, w_j)$ and X is represented in the basis $|\xi_j\rangle$.

C.2 Integral representation and inversion of BKM operator

Although it is well-known that Ω_D^k and its inverse for $k(x) = \log x/(x-1)$ are given by (1.1) and (1.2), most proofs rely on an explicit expansion in eigenvalues as in [34]. Using L_D and R_D allows one to see this more directly in terms of integrals and anti-derivatives starting from the elementary formula

$$\log x = \int_0^\infty \left(\frac{1}{1+u} - \frac{1}{x+u}\right) du$$

to write

$$\begin{aligned}
\big(\log L_D R_D^{-1}\big)(X) &= \big(\log L_D - \log R_D\big)(X) = \big(L_{\log D} - R_{\log D}\big)(X) \\
&= \int_0^\infty \left(X\,\frac{1}{D+tI} - \frac{1}{D+tI}X\right) dt \\
&= \int_0^\infty \frac{1}{D+tI}\,(DX - XD)\,\frac{1}{D+tI}\,dt \\
&= \int_0^\infty \frac{1}{D+tI}\,(L_D - R_D)(X)\,\frac{1}{D+tI}\,dt
\end{aligned}$$

from which it follows that

$$\Omega_D^k(X) = \frac{\log L_D - \log R_D}{L_D - R_D}(X) = \int_0^\infty \frac{1}{D+tI}\,X\,\frac{1}{D+tI}\,dt.$$

Now, observe that

$$\begin{aligned}
\int_0^1 D^t\big[(\log D)X - X\log D\big]D^{1-t}\,dt &= \int_0^1 \frac{d}{dt} D^t X D^{1-t}\,dt \\
&= DX - XD = (L_D - R_D)(X)
\end{aligned}$$

so that $\int_0^1 D^t\,\Omega_D^k(X)D^{1-t}\,dt = X$, which implies (1.2), i.e.,

$$\big(\Omega_D^k\big)^{-1}(Y) = \int_0^1 D^t\,Y D^{1-t}\,dt.$$

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