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Upward Morley's theorem downward

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By a celebrated theorem of Morley, a theory T is $\aleph_1$-categorical if and only if it is κ-categorical for all uncountable κ. In this paper we are taking the first steps towards extending Morley's categoricity theorem "to the finite". In more detail, we are presenting conditions, implying that certain finite subsets of certain $\aleph_1$-categorical T have at most one n-element model for each natural number $n \in \omega$ (counting up to isomorphism, of course).

1 Introduction

1.1 Motivation and results

By a celebrated theorem of Morley, a (countable, first order) theory T is $\aleph_1$-categorical if and only if it is κ-categorical for all uncountable κ, cf. [7] or [1, Theorem 7.1.14]. In this paper we are taking the first steps towards extending Morley's categoricity theorem "to the finite". The most natural generalization would be that if a first order theory T is $\aleph_1$-categorical then, up to isomorphism, T has a unique n-element model for each finite natural number n. We shall see below that this statement is obviously false. If we are dealing with finite models, then it is natural to consider finite subsets of T. More concretely, if Φ is a (finite) set of formulas then we shall say that $\mathcal{A}$ is a Φ-elementary substructure of $\mathcal{B}$ iff $A \subseteq B$ and for every $\varphi \in \Phi$ and $\bar{d} \in A$, the statements $\mathcal{A} \models \varphi(\bar{d})$ and $\mathcal{B} \models \varphi(\bar{d})$ are equivalent. We shall study Φ-elementary substructures of certain $\aleph_1$-categorical structures. If Φ is finite then such a Φ-elementary substructure may still remain finite.

We shall investigate some conditions on T, which, together with T being $\aleph_1$-categorical, imply that

For every large enough finite subset $\Phi \subseteq T$, up to isomorphism, models of T have at most one Φ-elementary substructure of cardinality n for all $n \in \omega$. $(*)$

Infinitely categorical structures are $\aleph_0$-categorical and $\aleph_0$-stable. Studying $\aleph_0$-categorical, $\aleph_0$-stable structures in their own right has a great tradition. In this direction we refer to [3, 15, 16], where, among other results, it was shown that $\aleph_0$-categorical, $\aleph_0$-stable structures are smoothly approximable, particularly, they are not finitely axiomatizable. For more recent related results we refer to Cherlin-Hrushovski [2]. By a personal communication with Zilber and Cherlin, it turned out that $(*)$ follows for $\aleph_0$-categorical, $\aleph_0$-stable theories from already known results. However, to show this, $\aleph_0$-categoricity plays a critical role. In this paper we do not assume $\aleph_0$-categoricity.

At that point one would be tempted to think that if T is $\aleph_1$-categorical then $(*)$ would follow without any additional condition. In fact, the situation is more complicated. To illustrate the nature of $(*)$, we insert here three simple examples.

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Example 1: By a result of Peretyatkin [8] there exists a finitely axiomatizable $\aleph_1$-categorical structure $\mathcal{A}$; let T be the theory of $\mathcal{A}$. Recall that a theory is pseudo-finite if each finite subset of it has a finite model, for more details, cf. [2]. Infinite structures with a finitely axiomatizable theory cannot be pseudo-finite, so large enough finite subsets of T do not have finite models. Consequently, $(*)$ holds for T, for trivial reasons.

Example 2: Let T be the theory of algebraically closed fields of a fixed positive characteristic. Then T is $\aleph_1$-categorical but not $\aleph_0$-categorical and already the field axioms are finitely categorical: two finite fields are isomorphic iff their cardinalities are the same. We also note that T is not pseudo-finite, hence—similarly to the previous example—$(*)$ holds for it.

Example 3: The theory of dense linear orders are not stable (hence are not $\aleph_1$-categorical), but each pairs of finite linear orders of the same cardinality are isomorphic.

In order to provide conditions for T which makes $(*)$ true, we shall deal with 'finitary analogues' of some classical notions such as elementary and Φ-elementary substructures. Here are some more 'finitary' notions we shall need below.

Definition 1.1 If $\mathcal{A}$ is a structure $X \subseteq A$ and Δ is a set of formulas then by $\mathsf{acl}^{\mathcal{A}}_{\Delta}(X)$ we understand the smallest (w.r.t. inclusion) set Y containing X which is closed under Δ-algebraic formulas, i.e., whenever $\varphi \in \Delta$, $\bar{y} \in Y$ and $A_0 = \{a : \mathcal{A} \models \varphi(a, \bar{y})\}$ is finite then $A_0 \subseteq Y$.

We may omit the superscript $\mathcal{A}$ when it is clear from the context.

It is worth to note here that if $v_0 = v_1 \in \Delta$ then $\mathsf{acl}^{\mathcal{A}}_{\Delta}$ is a closure operator. In addition, $\mathsf{acl}^{\mathcal{A}}_{\Delta}(X)$ is not the same as the set of those elements which are algebraic over X witnessed by a formula in Δ. In fact, if we denote this latter set by X^{Δ} then

$$\mathsf{acl}^{\mathcal{A}}_{\Delta}(X) = \bigcup_{n\in\omega} X_n,$$

where $X_0 = X$ and $X_{n+1} = X_n^{\Delta}$ for all $n \in \omega$.

Following, e.g., [5, p. 167], by an elementary mapping we understand a partial map which preserves all the formulas. If Δ is a set of formulas and $\mathcal{A}$, $\mathcal{B}$ are structures then a partial function $f : A \to B$ is said to be Δ-*elementary* if it preserves formulas in Δ, i.e., for any $\varphi \in \Delta$ and $\bar{x} \in \mathrm{dom}(f)$ we have $\mathcal{A} \models \varphi(\bar{x})$ if and only if $\mathcal{B} \models \varphi(f(\bar{x}))$. Isomorphisms and embeddings are supposed to be total functions.

For a fixed first order (relational) language L we write $\mathsf{Form}(L)$ (or simply Form if L is clear from the context) for the set of L-formulas. If X is a set then by Form_X we understand the set of formulas in the language extended with constant symbols for $x \in X$.

We write CB_X for the usual Cantor-Bendixson rank over the parameter set X (the definition will be recalled in §2). Our aim is to prove the following theorem.

Theorem 1.2 *Suppose $\mathcal{A}$ is an $\aleph_1$-categorical structure satisfying (a)-(b) below:*

(a) *For any finite set ε of formulas there exists another finite set $\Delta \supseteq \varepsilon$ of formulas such that whenever $\Delta' \supseteq \Delta$ is finite and g is a Δ'-elementary mapping then there exists a Δ-elementary mapping h extending g such that* $\mathrm{dom}(h) = \mathsf{acl}_{\Delta'}(\mathrm{dom}(g))$.

(b) *For each finite $\bar{a} \in A$ and each infinite subset E of A definable over $\bar{a}$ there exists a function ∂_E : $\mathsf{Form}_{\bar{a}} \to \mathsf{Form}_{\bar{a}}$ such that $\mathsf{CB}_{\bar{a}}(\partial_E\varphi) = 0$ for all formulas φ; and in addition $\varphi(\bar{x}, \bar{d})$ defines an atom of the Boolean–algebra of E–definable relations of $\mathcal{A}$ if and only if $\mathcal{A} \models \partial_E\varphi(\bar{d})$.*

Then, up to isomorphism, every large enough $T \subseteq \mathsf{Th}(\mathcal{A})$ has at most one n-element model for each $n \in \omega$.

We note that every elementary mapping f can be extended to an elementary mapping to $\mathsf{acl}(\mathrm{dom}(f))$; clause (a) is a finitary analogue of this well known fact. We shall informally refer to (b) as "E-atoms have a definition schema", for infinite, definable E (cf. Definition 2.5 below). We are going to discuss these two notions in detail in §2 (in fact, §2 is completely devoted to a brief motivation, explanation and analysis of these notions).

We say that a theory T has the *Finite Morley Property* iff it satisfies $(*)$ (the conclusions of Theorem 1.2). As we mentioned, here we are investigating sufficient conditions for the Finite Morley Property.

Before going further, let us list a couple of examples for which our theorem can be applied (i.e., structures satisfying clauses (a) and (b) above).

Example A1: Infinite dimensional vector spaces $\mathcal{V} = \langle V, +, \lambda\rangle_{\lambda\in\mathbb{F}}$ over a finite field $\mathbb{F}$. Here the language contains a binary function symbol for addition and a unary function symbol for each scalar in the field. Then $\mathcal{V}$, as it is $\aleph_0$-categorical, satisfies our clause (b) by Proposition 2.10. Further, it is easy to check clause (a): if a function preserves unnested atomic formulas then it is a linear map, therefore it extends to an automorphism of $\mathcal{V}$. Note that $\mathcal{V}$ is pseudo-finite and clearly any two vector spaces of the same finite dimension are isomorphic.

Example A2: Let $\mathbf{F}$ be an algebraically closed field with a given positive characteristic. Then, similarly to the case of vector spaces, $\mathbf{F}$ satisfies condition (a), and since it is strongly minimal, Proposition 4.6 below implies that it satisfies (b) and hence it has the Finite Morley Property. Note that $\mathbf{F}$ is not $\aleph_0$-categorical.

As we noted in Example A2, our main results may be applied to algebraically closed fields with a given positive characteristics, and these structures are not $\aleph_0$-categorical. For completeness, we note that one part of condition (b) of Theorem 1.2 is also satisfied by these structures (we emphasize again that, as explained above, the Finite Morley Property of these structures follows from our results without checking this property). In more detail, if E is an infinite definable subset of $\mathbf{F}$ then E-atoms have a definition schema (this is condition (b) of Theorem 1.2 without the assumption $\mathsf{CB}(\partial_E\varphi) = 0$). To check this let E' be the subfield of $\mathbf{F}$ generated by E. We claim that $E' = \mathbf{F}$. For, assume, seeking a contradiction, that $a \in \mathbf{F} - E'$. Then, for any $b \in E' - \{0\}$ we also have $a \cdot b \notin E'$, thus $\mathbf{F} - E'$ would be infinite. On the other hand, $\mathbf{F}$ is strongly minimal, hence $\mathbf{F} - E$ is finite, as well as $\mathbf{F} - E'$; this contradiction verifies our claim. It follows that $\mathsf{dcl}(E) = E' = \mathbf{F}$, hence each E-atom consists a single element of $\mathbf{F}$. In other words, for any formula φ and parameters $\bar{d} \in E$, the relation defined by $\varphi(v, \bar{d})$ is an E-atom iff $\varphi(v, \bar{d})$ can be realized by a unique element of $\mathbf{F}$; this is of course, a first order property of $\bar{d}$.

Example B: Take any finite structure $\mathcal{X}$ (in a finite language) and let $\mathcal{A} = \bigsqcup_\omega \mathcal{X}$ be the disjoint union of $\aleph_0$ many copies of $\mathcal{X}$. If a function g preserves the diagram of $\mathcal{X}$ then it extends to an automorphism h of $\mathcal{A}$, hence clause (a) holds. Since $\mathcal{A}$ is $\aleph_0$-categorical clause (b) holds, too (cf. Proposition 2.10). $\mathcal{A}$ has, for any finite set Δ of formulas, a Δ-elementary substructure, and any two of them, for large enough Δ, are isomorphic.

Example C: The structure $\mathcal{A} = \langle A, U, g\rangle$ where $g : {}^nU \to A \setminus U$ is a one-to-one mapping and U is a one-place relation symbol. $\mathcal{A}$ is $\aleph_1$-categorical [1, Chapter 7, pp. 483] and it is not hard to see that the theory of $\mathcal{A}$ admits elimination of quantifiers. Then, by Proposition 2.11, $\mathcal{A}$ also satisfies the conditions of our Theorem 1.2.

Example D: Let $n \in \omega$ be fixed and let $A_0, \dots, A_{n-1}$ be pairwise disjoint sets of the same infinite cardinality. Further, for all $i < n$ let $f_i : A_0 \to A_i$ be a bijection and set $A = \bigcup_{i<n} A_i$. It is not hard to check that the structure $\mathcal{A} = \langle A, A_0, \dots, A_{n-1}, f_0, \dots, f_{n-1}\rangle$ satisfies all of the assumptions of Theorem 1.2.

Example E: Let $q \in \omega$ be a prime power. Consider the group $\bigoplus_\omega \mathbb{Z}/q\mathbb{Z}$. It is totally categorical and has a finite base for elimination of quantifiers[1]. By Proposition 2.11 this structure satisfies the assumptions of Theorem 1.2. We also note that by total categoricity, this group has finite Δ-elementary substructures for all finite Δ (which, for large enough Δ, are unique up to isomorphism, according to our Theorem 1.2).

Example F: Any $\aleph_1$-categorical structure having a finite elimination base. Theorem 1.2 applies to all of these structures, cf. Proposition 2.11.

We shall see in Proposition 4.6 that in the case when $\mathcal{A}$ is strongly minimal, the conditions of Theorem 1.2 may be simplified (in fact, we need to assume a weak version of (a) only, and do not need to assume (b)). We note that the structures in Examples C, D, E and F above are not strongly minimal, but satisfy the conditions of Theorem 1.2.

[1] For this notion we refer to [5, Chapter 2.7, p. 67] where it is called an elimination set.

The proof of Theorem 1.2 is divided into two parts. First we establish some basic properties of finite substructures of a structure satisfying conditions (a) and (b). Then we examine a method to find isomorphisms between ultraproducts acting "coordinatewise". This method is related to (but does not depend on) the results of [4, 9, 11]. To establish further investigations of finitary generalizations of Morley's theorem, we are trying to be rather general. We offer a variety of notions which perhaps may be used in related investigations. Some of them may seem rather technical, or complicated. However, we hope, these notions will be useful to find more natural finitary generalizations of Morley's Theorem.

1.2 Organization of this paper

At the end of this section we are summing up our system of notation. We also recall some facts which we shall use throughout the paper. We believe that these facts are well known results in classical model theory. In §2 we present some basic observations about $\aleph_1$-categorical structures also satisfying some variants of the conditions of Theorem 1.2. Subsection 2.1 contains the definitions needed in later sections; Subsection 2.2 is devoted to establishing connections between definitions given in Subsection 2.1 and traditional model theoretic notions. These investigations (combined with the examples given above) may illustrate how general our results are. Subsection 2.2 is inserted to the paper for completeness, we do not use its results in later sections. Readers, who would prefer to see our main results rather than the brief analysis of the notions involved, may simply skip Subsection 2.2.

§3 makes some preliminary observations on stable structures. In §4 we are dealing with ultraproducts of finite structures. This section contains the technical cornerstones of our construction. Here *decomposable* sets play a central role: a subset R of an ultraproduct $A = \prod_{i\in I} \mathcal{A}_i/\mathcal{F}$ is decomposable iff for every $i \in I$ there is $R_i \subseteq A_i$ such that $R = \prod_{i\in I} R_i/\mathcal{F}$, for more details, cf. [4, 9, 11]. As another tool, we also will use basics of stability theory. In general, our strategy is as follows: to obtain results about finite structures first we study an infinite ultraproduct of them. A similar approach may be found in [10, 14].

The main goal of §4 is to prove Theorem 4.2 which claims that Theorem 1.2 (the main result of the paper) is true if we add to our assumptions that there exists a $\varnothing$-definable strongly minimal set. §4 is divided into three subsections.

In Subsection 4.1 we are dealing with strongly minimal structures. Here the goal is to establish the Finite Morley Property for certain strongly minimal structures. This is achieved in Proposition 4.6.

In Subsection 4.2 We assume that our structures contain a $\varnothing$-definable strongly minimal set. Using Zilber's ladder theorem (which will be recalled at the beginning of Subsection 4.2), in Theorem 4.19 we show that certain decomposable elementary mappings defined on a $\varnothing$-definable strongly minimal set can be extended to a decomposable elementary embedding.

In Subsection 4.3 we combine the results of the previous two subsections to obtain Theorem 4.2; as we already mentioned, this theorem establishes the Finite Morley Property for $\aleph_1$-categorical structures containing a $\varnothing$-definable strongly minimal set and satisfying (a) and (b) of Theorem 1.2.

On the basis of these results, in §5 we present the main result of the paper: we show that the assumption about the existence of a $\varnothing$-definable strongly minimal set may be omitted. Thus, under some additional technical conditions, Morley's Categoricity Theorem may be extended to the finite. For the details, cf. Theorem 1.2. Finally, at the end of §5 we mention further related questions which remained open.

1.3 Notation

Sets. Throughout, ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, \ldots, n-1\}$. Let A and B be sets. Then ${}^{A}B$ denotes the set of functions from A to B, $|A|$ denotes the cardinality of A, $[A]^{<\omega}$ denotes the set of finite subsets of A and if κ is a cardinal then $[A]^{\kappa}$ denotes the set of subsets of A of cardinality κ. Sequences of variables or elements will be denoted by overlining, i.e., for example, $\bar{x}$ denotes a sequence of variables $x_0, x_1, \ldots$ Let f be a function. Then $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ denote the domain and range of f, respectively. If A is a set, $f : A \to A$ is a unary partial function and $\bar{x}$ is a sequence of elements of A then, for simplicity, by a slight abuse of notation, we shall write $\bar{x} \in A$ in place of $\mathrm{ran}(\bar{x}) \subseteq A$. Particularly, $\bar{x} \in \mathrm{dom}(f)$ expresses that f is defined on every member of $\bar{x}$, i.e., $\mathrm{ran}(\bar{x}) \subseteq \mathrm{dom}(f)$.

Structures. We shall use the following conventions. Models are denoted by calligraphic letters and the universe of a given model is always denoted by the same latin letter.

If $\mathcal{A}$ is a model for a language L and $R_0, \dots, R_{n-1}$ are relations on A, then $\langle \mathcal{A}, R_0, \dots, R_{n-1}\rangle$ denotes the expansion of $\mathcal{A}$, whose similarity type is expanded by n new relation symbols (with the appropriate arities) and the interpretation of the new symbols are $R_0, \dots, R_{n-1}$ respectively. The set of formulas of a language L is denoted by $\mathsf{Form}(L)$. Throughout L will be fixed so we may simply write Form instead. If X is a set (of parameters), then by Form_X we understand the set of formulas in the language extended with constant symbols for $x \in X$.

Throughout, we denote the relation defined by the formula φ in $\mathcal{A}$ by $\|\varphi\|^{\mathcal{A}}$, i.e.,

$$\|\varphi\|^{\mathcal{A}} = \{\bar{a} \in A : \mathcal{A} \models \varphi(\bar{a})\}.$$

If $\mathcal{A}$ is clear from the context, we omit it.

We shall rely on the following natural convention. If $\mathcal{M}$ is a structure and $X \subseteq M$ can be defined with a formula φ and $\mathcal{A}$ is any structure then by $X^{\mathcal{A}}$ we understand $\|\varphi\|^{\mathcal{A}}$. In particular if $\mathcal{A} = \prod_{i\in\omega} \mathcal{A}_i/\mathcal{F}$ then every definable subset of $\mathcal{A}$ is decomposable (for a definition of a decomposable relation we refer to Definition 2.1 below) and hence

$$X^{\mathcal{A}} = \|\varphi\|^{\mathcal{A}} = \prod_{i\in\omega} \|\varphi\|^{\mathcal{A}_n}/\mathcal{F} = \prod_{i\in\omega} X^{\mathcal{A}_n}/\mathcal{F}$$

in this case. If $\mathcal{A}$ is a φ-elementary substructure of $\mathcal{M}$ then $X^{\mathcal{A}} = A \cap X^{\mathcal{M}}$. Sometimes, when it is clear from the context, we omit the superscript.

1.4 Facts

Here we collect some important and presumably well known facts of model theory which we are going to use without any reference later on.

Ultraproducts. According to Łoś's lemma [1, Theorem 4.1.9] (also called the Fundamental Theorem of Ultraproducts), if $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i/\mathcal{F}$ is an ultraproduct and φ is a formula (in the language of $\mathcal{A}$) then $\mathcal{A} \models \varphi$ if and only if $\{i \in I : \mathcal{A}_i \models \varphi\} \in \mathcal{F}$. As usual in the literature we may express this latter fact by saying 'φ holds *for almost all $i \in I$*' or 'φ holds *in a big set of indices*'. In this paper **each ultrafilter is supposed to be non-trivial**.

By [1, Theorem 6.1.4], there are countably incomplete $|I|^+$-good ultrafilters over every infinite set I; in addition, by [1, Theorem 6.1.8], ultraproducts modulo countably incomplete, κ-good ultrafilters are κ-saturated (also cf. [1, Thm 6.1.1] when I is countable). We note that this remains true if we extend the language of our structures with finitely many relation symbols and interpret them by decomposable relations. Also recall that κ-saturated infinite structures are κ^+-universal and κ-homogeneous (cf. [1, 5.1.14]).

$\aleph_1$-categorical structures and strongly minimal sets. If $\mathcal{M}$ is an uncountable, $\aleph_1$-categorical structure then it is $\aleph_0$-stable [6, Corollary 5.2.10], moreover, if $X \subseteq M$ is an infinite, and definable subset, then $\mathcal{M}$ is prime (and atomic) over X, cf. [6, Theorem 6.1.14].

In any strongly minimal set one can define a notion of independence [6, Definition 6.1.5], in particular, algebraic closure defines a pregeometry and hence it is meaningful to speak about dimension and basis in this context [6, Definition 6.1.10].

2 Basic definitions and preliminary observations

This section is devoted to study the conditions occurring in the main result (Theorem 1.2) of the paper. In Subsection 2.1 we present our basic definitions; in Subsection 2.2 we provide a brief analysis for them. As we already mentioned, later sections do not depend on Subsection 2.2, so it may be skipped if the reader would prefer doing so. Recall that we are working with a fixed finite first order language L.

2.1 Definitions and some explanations for them

Let $\mathcal{A}$ be a first order structure and let $X \subseteq A$ be arbitrary. Then $\mathsf{acl}^{\mathcal{A}}(X)$ denotes the *algebraic closure of X in* $\mathcal{A}$. When $\mathcal{A}$ is clear from the context we omit it. Recall that acl_Δ was defined in Definition 1.1.

By a *partial isomorphism* we mean a partial function $f : A \to A$ such that if $\bar{a}, b \in \mathrm{dom}(f)$ then for every relation symbol R and function symbol g we have

$$\mathcal{A} \models R(\bar{a}) \quad \text{iff} \quad \mathcal{A} \models R(f(\bar{a})) \quad \text{and}$$

$$\mathcal{A} \models g(\bar{a}) = b \quad \text{iff} \quad \mathcal{A} \models g(f(\bar{a})) = f(b).$$

We remark that f is a partial isomorphism if and only if it is elementary with respect to the set of unnested atomic formulas (for the definition of an unnested atomic formula cf. [5, p. 58]).

Next, we recall the definition of decomposability. Decomposable relations were introduced in [9] and further studied in [4] and [11].

Definition 2.1 Let I be any set and $\mathcal{F} \subseteq \mathcal{P}(I)$ an ultrafilter over I. Suppose A_i are sets for $i \in I$ and let $A = \prod_{i\in I} A_i/\mathcal{F}$ be the ultraproduct of these sets. A k-ary relation $R \subseteq {}^kA$ is defined to be *decomposable in A* iff for every $i \in I$ there exists a k-ary relation $R_i \subseteq {}^kA_i$ such that

$$R = \left\{ \langle s_0/\mathcal{F}, \ldots, s_{k-1}/\mathcal{F}\rangle \in {}^k\!\left(\prod_{i\in I} A_i/\mathcal{F}\right) :\ \{i \in I :\ \langle s_0(i), \ldots, s_{k-1}(i)\rangle \in R_i\} \in \mathcal{F} \right\}.$$

In this case we say that R can be decomposed to $\langle R_i : i \in I\rangle$ or $\langle R_i : i \in I\rangle$ is a decomposition of R and sometimes write $R = \langle R_i : i \in I\rangle/\mathcal{F}$.

In order to simplify notation we shall identify ${}^k(\prod_{i\in I} A_i)$ with $\prod_{i\in I} {}^kA_i$ in the natural way, i.e., k-tuples of sequences are identified with single sequences whose terms are k-tuples. We shall use this obvious identification without any further warning. According to what has been said, R is decomposable in A if and only if

$$\langle A, R\rangle = \prod_{i\in I} \langle A_i, R_i\rangle/\mathcal{F}.$$

In general, if R is any relation then R is said to be *decomposable* if it is of the form of an ultraproduct (after an eventual identification of k-tuples of sequences and sequences of k-tuples, see above), i.e., $R = \prod_{i\in I} R_i/\mathcal{F}$ for some sets R_i (and I, $\mathcal{F}$). If $k = 1$ then we may say decomposable set instead of decomposable relation.

Observe that a relation $R \subseteq {}^kA$ is decomposable in A if and only if it is decomposable (in the general sense in the definition above). This motivates to use the two notions decomposable in A and decomposable freely.

Suppose $A = \prod_{i\in I} A_i/\mathcal{F}$ and $B = \prod_{i\in I} B_i/\mathcal{F}$ are two ultraproducts and $f : A \to B$ is a function. Then viewing f as a relation $f \subseteq A \times B$ it makes sense to speak about decomposable functions. Accordingly, f is called a *decomposable function* if there exist functions $f_i \subseteq A_i \times B_i$ for $i \in I$ such that

$$f = \left\{ \langle s_0/\mathcal{F}, s_1/\mathcal{F}\rangle \in \left(\prod_{i\in I} A_i/\mathcal{F}\right) \times \left(\prod_{i\in I} B_i/\mathcal{F}\right) :\ \{i \in I :\ \langle s_0(i), s_1(i)\rangle \in A_i \times B_i\} \in \mathcal{F} \right\}$$

which, after the natural identification, can be written as $f = \prod_{i\in I} f_i/\mathcal{F}$.

This is equivalent to saying that $f \subseteq \prod_{i\in I}(A_i \times B_i)/\mathcal{F}$ (here we used again the identification of $\prod_{i\in I}(A_i \times B_i)/\mathcal{F}$ and $\prod_{i\in I} A_i/\mathcal{F} \times \prod_{i\in I} B_i/\mathcal{F}$) is decomposable *in* $\prod_{i\in I}(A_i \times B_i)/\mathcal{F}$.

Examples of decomposable relations are the finite ones: a trivial application of Łoś's lemma shows that if $R \subseteq {}^k(\prod_{i\in I} A_i/\mathcal{F})$ is a finite set then there exist $R_i \in A_i$ (with $|R_i| = |R|$ almost everywhere) such that $R = \prod_{i\in I} R_i/\mathcal{F}$. Assuming $\mathcal{F}$ is non-principal, no countably infinite $R \subseteq \prod_{i\in I} A_i/\mathcal{F}$ can be decomposable as $\prod_{i\in I} R_i/\mathcal{F}$ is either finite or uncountably infinite (cf. [1, Prop. 4.3.7]).

Decomposable elementary maps "act coordinate-wise" in the sense of the following proposition (which is a straightforward application of Łoś's lemma):

Proposition 2.2 *Let $f = \langle f_i : i \in I\rangle/\mathcal{F} : \prod_{i\in I} \mathcal{A}_i/\mathcal{F} \to \prod_{i\in I} \mathcal{B}_i/\mathcal{F}$ be a decomposable elementary mapping. Then for every formula φ there exists $J = J(\varphi) \in \mathcal{F}$ such that f_i preserves φ for all $i \in J$.*

P r o o f. By way of contradiction suppose that $\{i \in I : f_i \text{ preserves } \varphi\} \notin \mathcal{F}$. It follows that for every $i \in I$ there is $\bar{a}_i \in \mathrm{dom}(f_i)$ such that

$$\big\{i \in I : \mathcal{A}_i \models \varphi(\bar{a}_i) \not\Leftrightarrow \mathcal{B}_i \models \varphi\big(f_i(\bar{a}_i)\big)\big\} \in \mathcal{F}$$

and without loss of generality we may assume

$$\big\{i \in I : \mathcal{A}_i \models \varphi(\bar{a}_i) \text{ and } \mathcal{B}_i \models \neg\varphi\big(f_i(\bar{a}_i)\big)\big\} \in \mathcal{F}.$$

Let now $\bar{a} = \langle \bar{a}_i : i \in I\rangle/\mathcal{F}$ and observe that $f(\bar{a}) = \langle f_i(\bar{a}_i) : i \in I\rangle/\mathcal{F}$ (this is a consequence of f being decomposable). Applying Łoś's lemma for the $\mathcal{A}_i$'s and for the $\mathcal{B}_i$'s we get that $\mathcal{A} \models \varphi(\bar{a})$ and $\mathcal{B} \models \neg\varphi(f(\bar{a}))$ and this contradicts the fact that f is elementary. □

Next, let us recall, for completeness, the notion of Cantor-Bendixson rank:

Definition 2.3 Suppose that $\mathcal{M}$ is a structure, $A \subseteq M$ and $\varphi(v)$ is a formula with parameters from A. We recall the usual definition of $\mathsf{CB}_A^{\mathcal{M}}(\varphi)$, the *Cantor-Bendixson rank* of φ in $\mathcal{M}$. First, we inductively define $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \alpha$ for α an ordinal.

(i) $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq 0$ if and only if $\|\varphi\|^{\mathcal{M}}$ is nonempty.
(ii) if α is a limit ordinal, then $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \alpha$ if and only if $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \beta$ for all $\beta < \alpha$.
(iii) for any ordinal α, $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \alpha + 1$ if and only if there is a sequence $\langle \psi_i(v, \bar{a}_i) : i \in \omega\rangle$ of formulas with parameters $\bar{a}_i \in A$ such that $\langle \|\psi_i(v, \bar{a}_i)\|^{\mathcal{M}} : i \in \omega\rangle$ forms an infinite family of pairwise disjoint subsets of $\|\varphi(\bar{v})\|^{\mathcal{M}}$ and $\mathsf{CB}_A^{\mathcal{M}}(\psi_i) \geq \alpha$ for all i.

If $\|\varphi\|^{\mathcal{M}}$ is empty, then $\mathsf{CB}_A^{\mathcal{M}}(\varphi) = -1$. If $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \alpha$ but $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \not\geq \alpha + 1$, then $\mathsf{CB}_A^{\mathcal{M}}(\varphi) = \alpha$. If $\mathsf{CB}_A^{\mathcal{M}}(\varphi) \geq \alpha$ for all ordinals α, then $\mathsf{CB}_A^{\mathcal{M}}(\varphi) = \infty$.

If $\mathsf{CB}_A^{\mathcal{M}}(\varphi) = \alpha$ for all finite sets $A \subseteq M$ then we write $\mathsf{CB}^{\mathcal{M}}(\varphi) = \alpha$. If $\mathcal{M}$ or A is clear from the context we may omit them.

Definition 2.4 Let $\mathcal{M}$ be a structure and let $E \subseteq M$, $\bar{e} \in E$. Then we say that $\varphi(x, \bar{e})$ is an *E-atom* if $\|\varphi(x, \bar{e})\|^{\mathcal{M}}$ is an atom of the Boolean-algebra of E-definable relations of $\mathcal{M}$. Similarly if a subset A is defined by an E-atom $\varphi(x, \bar{e})$ then we may simply write A is an E-atom.

As we mentioned in the introduction, if $X \subseteq M$ then Form_X denotes the set of formulas that may contain parameters from X. Now we turn to discuss condition (b) of Theorem 1.2.

Definition 2.5 Let E be an infinite subset of M definable by parameters from $X \subseteq M$. Then a function $\partial_E : \mathsf{Form}_X \to \mathsf{Form}_X$ is defined to be an *atom defining schema for E over $\mathcal{M}$* if $\|\varphi(x, \bar{e})\|$ is an E-atom if and only if $\mathcal{M} \models \partial_E\varphi(\bar{e})$ and $\mathsf{CB}_X(\partial_E\varphi) = 0$.

We say that the structure $\mathcal{M}$ has an atom defining schema if for all infinite definable subsets E there exist the corresponding function ∂_E. Further, when E is clear from the context, we may simply write ∂ instead of ∂_E.

Having an atom defining schema expresses that for a fixed infinite, definable relation E and formula φ, the fact that $\varphi(v, \bar{d})$ defines an atom in the Boolean-algebra of E-definable relations of $\mathcal{A}$ is a first order property of $\bar{d}$. Particularly, $\varphi(v, \bar{d})$ is an atom if and only if $\mathcal{A} \models \partial_E\varphi(\bar{d})$ for a first order formula $\partial_E\varphi$. We also require the Cantor-Bendixson rank of $\partial_E\varphi$ to be equal to zero. This condition expresses that whenever $\varphi(v, \bar{d})$ isolates a type in the Stone space $\mathsf{S}(E)$, the type $\mathsf{tp}(\bar{d}/\varnothing)$ is also an isolated point of $\mathsf{S}_n(\varnothing)$ (where n is the length of $\bar{d}$). In this point of view, our condition can be seen as a transfer principle stating that utilizing φ, isolated points of $\mathsf{S}(E)$ may be obtained from isolated points of $S_n(\varnothing)$, only. We shall see in Proposition 2.10 that $\aleph_0$-categoricity implies the existence of an atom defining schema.

Next, we analyze condition (a) of Theorem 1.2.

Definition 2.6 A structure $\mathcal{A}$ is said to have the *extension property* if the following holds. For any finite set ε of formulas there exists another finite set $\Delta \supseteq \varepsilon$ of formulas such that whenever $\Delta' \supseteq \Delta$ is finite and g is a Δ'-elementary mapping then there exists a Δ-elementary mapping h such that $h \supseteq g$ and such that the following hold:

$$\mathrm{dom}(h) = \mathsf{acl}_{\Delta'}(\mathrm{dom}(g)) \quad \text{and}$$
$$\mathrm{ran}(h) = \mathsf{acl}_{\Delta'}(\mathrm{ran}(g)).$$

As we mentioned in the Introduction, every elementary mapping f can be extended to an elementary mapping to $\mathsf{acl}(\mathrm{dom}(f))$; this fact will be called 'extension property for elementary mappings' (EPE, for short). Definition 2.6 above is a finitary version of EPE. Let $f : X \to Y$ be a function that we would like to extend to another function f'. To get a finitary version of EPE it is useful to isolate three hidden parameters occurring in it:

1. which formulas are preserved by f;
2. which formulas are preserved by f' (the extension of f);
3. what is the relationship between $\mathrm{dom}(f)$ and $\mathrm{dom}(f')$.

Roughly, our extension property expresses that if ε is a finite set of formulas, and Δ' is another large enough finite set of formulas then an ε-elementary function f can be extended to $\mathsf{acl}_{\Delta'}(\mathrm{dom}(f))$ and the extension remains elementary enough. If we do not require finiteness of ε, Δ and Δ', and letting them equal to the set of all formulas, then clause (a) reduces to the original notion of EPE. We shall see shortly that if the theory of $\mathcal{A}$ has a finite elimination base for quantifiers, (particularly, if a countable elementary substructure of $\mathcal{A}$ is isomorphic to the Fraïsse limit of its age), then $\mathcal{A}$ has the extension property.

We shall also deal with a special weaker form of the extension property, mainly in Subsection 4.1, which we call the *weak extension property*. We shall see in Theorem 4.7 that for strongly minimal structures this weaker property already implies the Finite Morley Property.

Definition 2.7 The structure $\mathcal{A}$ satisfies the *weak extension property* if and only if $(*)$ below holds for it.

$(*)$ There exists a finite set Δ of formulas such that whenever $\Delta' \supseteq \Delta$ is a finite set of formulas and f is a Δ'-elementary mapping then there exists a partial isomorphism f' extending f so that $\mathrm{dom}(f') = \mathsf{acl}_{\Delta'}(\mathrm{dom}(f))$ and $\mathrm{ran}(f') = \mathsf{acl}_{\Delta'}(\mathrm{ran}(f))$.

We note that this condition is somewhat weaker than the condition obtained from the extension property by letting ε in it to be the set of unnested atomic formulas.

We shall see in Proposition 2.11 that the presence of a finite elimination base implies the extension property.

In what follows, in §5, we are going to prove our two main results: Theorems 2.8 and 2.9 which we recall here. In the next two statements, we suppose that a covering sequence $\langle \Delta_n \in [\mathsf{Form}]^{<\omega} : n \in \omega \rangle$ of formulas (cf. Definition 4.1 below) is fixed arbitrarily.

Theorem 2.8 *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure satisfying the extension–property and having an atom-defining schema. Suppose $\mathcal{A}_n$, $\mathcal{B}_n$ are equinumerous finite, Δ_n-elementary substructures of $\mathcal{M}$. Then*

$$\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\} \in \mathcal{F}$$

for any non-principal ultrafilter $\mathcal{F}$ (i.e., the set $\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\}$ is co-finite).

Theorem 2.9 *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure satisfying the extension property and having an atom-defining schema. Then there exists $N \in \omega$ such that for any $n \geq N$ and $k \in \omega$ (counting up to isomorphisms) $\mathcal{M}$ has at most one Δ_n-elementary substructure of size k.*

As we shall show in §5 below, Theorem 2.9 may be quickly derived from Theorem 2.8: if for infinitely many n there existed non-isomorphic, finite, equinumerous, Δ_n-elementary substructures $\mathcal{A}_n$, $\mathcal{B}_n$ of $\mathcal{M}$, then taking an ultraproduct of the $\mathcal{A}_n$'s and $\mathcal{B}_n$'s and applying Theorem 2.8, one could conclude that $\mathcal{A}_n$ and $\mathcal{B}_n$ should be isomorphic for all, but finitely many n; this is a contradiction. For further details check the proofs in §5 below.

2.2 Connections with traditional notions

We start by providing sufficient conditions that imply the extension property and the existence of an atom defining schema.

Proposition 2.10 *Suppose $\mathcal{A}$ is $\aleph_0$-categorical and let E be an infinite X-definable subset of A for some finite $X \subseteq A$. Then there is an atom-defining schema ∂_E for E in $\mathcal{A}$.*

Proof. Suppose $\varphi(v, \bar{d})$ defines an E-atom. Then this is a property of $\bar{d}$, which is invariant under those elements of $\mathsf{Aut}(\mathcal{A})$ that fix X pointwise. Hence $\mathsf{tp}^{\mathcal{A}}(\bar{d}/X)$ determines it. But $\mathcal{A}$ is $\aleph_0$-categorical, thus this type can be described with one single formula. Let $\partial_E\varphi$ be this formula.

To see $\mathsf{CB}_X(\partial_E\varphi) = 0$ we need to prove that $\|\partial_E\varphi\|$ cannot split into infinitely many parts using a fixed finite set P of parameters. But this follows immediately from the fact that after adjoining P as constant symbols to the language of $\mathcal{A}$, the resulting structure is still $\aleph_0$-categorical and hence there are only finitely many non-equivalent formulas having one free variable. □

Proposition 2.11 *Suppose $\mathcal{A}$ has a finite elimination base. Then $\mathcal{A}$ satisfies the extension property and has an atom defining schema.*

Proof. If $\mathcal{A}$ has a finite elimination base then it is $\aleph_0$-categorical whence, by Proposition 2.10 it has an atom defining schema.

To show $\mathcal{A}$ has the extension property suppose Δ is a finite set of formulas which forms an elimination base, i.e., any formula is equivalent to a Boolean combination of formulas in Δ. Then if f is Δ-elementary then it is elementary, as well, consequently it can be extended to $\mathsf{acl}(\mathrm{dom}(f))$ as an elementary function (cf., e.g., [5]), thus extension property easily follows. □

3 Stability and categoricity

In this section our main goal is to prove Lemma 3.4. That lemma provides a method of extending elementary maps between certain uncountably categorical structures. For this, we shall make use of some technical lemmas (Lemmas 3.2 and 3.3) about stability and splitting chains.

3.1 Splitting chains

We start by recalling the definition of splitting (c.f., [13, Definition I.2.6]).

Definition 3.1 Let $p \in \mathsf{S}_n^{\mathcal{A}}(X)$ and $Y \subseteq X$. Then p *splits over* Y if there exist $\bar{a}, \bar{b} \in X$ and $\varphi \in \mathsf{Form}$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{a}/Y) = \mathsf{tp}^{\mathcal{A}}(\bar{b}/Y)$, but $\varphi(v, \bar{a}) \in p$ and $\neg\varphi(v, \bar{b}) \in p$.

Lemma 3.2 *Suppose $\mathcal{A}$ is a λ-stable structure, $D \subset A$ and $\langle \mathcal{A}, D\rangle$ is λ^+-saturated. Then there exist $A_D \subseteq D$, $p_D \in \mathsf{S}(A_D)$, and $a_D \in A \setminus D$, such that $|A_D| \leq \lambda$, a_D realizes p_D, and if $c \in A \setminus D$ realizes p_D then $\mathsf{tp}^{\mathcal{A}}(c/D)$ does not split over A_D.*

Proof. We apply transfinite recursion. Let $a_0 \in A \setminus D$ be arbitrary, $A_0 = \varnothing$ and $p_0 = \mathsf{tp}^{\mathcal{A}}(a_0/A_0)$. Let $\beta < \lambda$ be an ordinal and suppose for all $\alpha < \beta$ that a_α, $A_\alpha \subseteq D$, and p_α are already defined, such that $p_\alpha \in \mathsf{S}(A_\alpha)$, $|A_\alpha| \leq |\alpha| + \aleph_0$, and a_α realizes p_α.

I. Suppose that β is a successor, say $\beta = \alpha + 1$. First, suppose there exists $c \in A \setminus D$ which realizes p_α but $\mathsf{tp}^{\mathcal{A}}(c/D)$ splits over A_α (it may happen that $c = a_\alpha$). Then by definition there exist $\bar{d}_0, \bar{d}_1 \in D$ and φ such that $\mathsf{tp}^{\mathcal{A}}(\bar{d}_0/A_\alpha) = \mathsf{tp}^{\mathcal{A}}(\bar{d}_1/A_\alpha)$, but $\varphi(v, \bar{d}_0) \in \mathsf{tp}^{\mathcal{A}}(c/D)$ and $\varphi(v, \bar{d}_1) \notin \mathsf{tp}^{\mathcal{A}}(c/D)$. Let $A_\beta = A_\alpha \cup \{\bar{d}_0, \bar{d}_1\}$, $p_\beta = \mathsf{tp}^{\mathcal{A}}(c/A_\beta)$, and $a_\beta = c$. If there are no such $c \in A \setminus D$ with $\mathsf{tp}^{\mathcal{A}}(c/D)$ splitting over A_α, then A_β, p_β and a_β are undefined, and the transfinite construction is complete.
II. Suppose that β is a limit ordinal. Let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and $p_\beta = \bigcup_{\alpha<\beta} p_\alpha$. By assumption $\langle \mathcal{A}, D\rangle$ is λ^+-saturated hence there exists $a_\beta \in A \setminus D$ which realizes p_β.
III. Clearly, for each α, $p_{\alpha+1}$ splits over A_α, hence by [13, Lemma I.2.7] this construction stops at a level $\beta < \lambda$. Let $A_D = A_\beta$, $p_D = p_\beta$, and $a_D = a_\beta$. □

Lemma 3.3 *Let $\mathcal{A}$ be λ-stable, and $D \subseteq A$ such that $\langle \mathcal{A}, D\rangle$ is a λ^+-saturated structure. Then there exist $a \in A \setminus D$ and sets $A(a) \subseteq B(a) \subseteq D$ such that*

(1) $|A(a)| \le \lambda$ *and* $\mathsf{tp}^{\mathcal{A}}(a/D)$ *does not split over* $A(a)$*;*
(2) $|B(a)| \le \lambda$ *and every type over* $A(a)$ *can be realized in* $B(a)$*;*
(3) forall$b \in A \setminus D$ *the following holds:*

$$\mathsf{tp}^{\mathcal{A}}(a/B(a)) = \mathsf{tp}^{\mathcal{A}}(b/B(a)) \Longrightarrow \mathsf{tp}^{\mathcal{A}}(a/D) = \mathsf{tp}^{\mathcal{A}}(b/D).$$

Proof.

(1) Let A_D, p_D and a_D be as in Lemma 3.2, and let $A(a) = A_D$ and $a = a_D$. Then $\mathsf{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$.
(2) Choose an arbitrary realization of each type over $A(a)$, and let their collection be $B(a)$. By (1) we have $|A(a)| \le \lambda$, hence by stability

$$|B(a)| \le \aleph_0 \cdot \left| \bigcup_{i \in \omega} \mathsf{S}_i^{\mathcal{A}}(A(a)) \right| \le \aleph_0^2 \lambda = \lambda.$$

Clearly $A(a) \subseteq B(a)$, and every type over $A(a)$ can be realized in $B(a)$.
(3) We prove that $B(a)$ fulfills (3). Suppose $\mathsf{tp}^{\mathcal{A}}(a/B(a)) = \mathsf{tp}^{\mathcal{A}}(b/B(a))$ and $\varphi(v, \bar{d}) \in \mathsf{tp}^{\mathcal{A}}(a/D)$. We have to show $\varphi(v, \bar{d}) \in \mathsf{tp}^{\mathcal{A}}(b/D)$. By (2) there exists $\bar{d}' \in B(a)$ such that $\mathsf{tp}^{\mathcal{A}}(\bar{d}/A(a)) = \mathsf{tp}^{\mathcal{A}}(\bar{d}'/A(a))$. By (1) $\mathsf{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$ hence

$$\varphi(v, \bar{d}') \in \mathsf{tp}^{\mathcal{A}}(a/B(a)) = \mathsf{tp}^{\mathcal{A}}(b/B(a)).$$

Since b realizes p_D, Lemma 3.2 implies that $\mathsf{tp}^{\mathcal{A}}(b/D)$ does not split over $A(a)$ as well. Therefore $\varphi(v, \bar{d}) \in \mathsf{tp}^{\mathcal{A}}(b/D)$, as desired. □

3.2 Elementary extension in the $\aleph_1$-categorial case

Lemma 3.4 *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are elementarily equivalent, their common theory is uncountably categorical,* $f : A \to B$ *is an elementary mapping such that* $D = \mathrm{dom}(f) \ne A$, $R = \mathrm{ran}(f) \ne B$ *and* $\langle \mathcal{A}, D \rangle$, $\langle \mathcal{B}, R \rangle$ *are* $\aleph_1$*-saturated. Then there exists an elementary mapping* f' *strictly extending* f*.*

It is well known that every saturated structure $\mathcal{A}$ is strongly homogeneous: every elementary mapping f of $\mathcal{A}$ with $|f| < |A|$ can be extended to an automorphism of $\mathcal{A}$; for more details, we refer to [1, Proposition 5.1.9]. The basic idea of the proof of this theorem is that by saturation, if $f : A \to A$ is a "small" elementary mapping, and $a \notin \mathrm{dom}(f)$, then the type $f[\mathsf{tp}^{\mathcal{A}}(a/\mathrm{dom}(f))]$ can be realized outside of $\mathrm{ran}(f)$. In our case the problem is that it is not only the "small" mappings which we would like to extend. For instance, if $\mathcal{A}$ is an ultraproduct and f is decomposable then $|f|$ might be as big as $|A|$, and since $\mathcal{A}$ can not be $|A|^+$-saturated we can not hope anything like above. The point here is that our statement may also apply to cases when $|\mathrm{dom}(f)| = |A|$, so ordinary saturation cannot be used.

Proof. We distinguish two cases.

Case 1: $D = \mathrm{dom}(f)$ is not an elementary substructure of $\mathcal{A}$. Then by the Łoś-Vaught test, there is a formula ψ, and constants $\bar{d} \in D$, such that $\mathcal{A} \models \exists v \psi(v, \bar{d})$, but there is no such $v \in D$. Since $\mathcal{A}$ is uncountably categorical, it is $\aleph_0$-stable. Hence, the isolated types over D are dense in $\mathsf{S}_1^{\mathcal{A}}(D)$. Consequently, there is an isolated type $p \in \mathsf{S}_1^{\mathcal{A}}(D)$ containing $\psi(v, \bar{d})$. Let $a \in A$ be a realization of p (such a realization exists since p is isolated). Then $\mathcal{A} \models \psi(a, \bar{d})$, so $a \notin D$. Let $b \in B$ be a realization of $f[p]$ in $\mathcal{B}$. Again, since $f[p]$ is isolated, b exists. Finally let $f' = f \cup \{\langle a, b \rangle\}$. Clearly, f' is an elementary mapping strictly extending f.

Case 2: $\mathcal{D} \prec \mathcal{A}$ is an elementary substructure. Let $a \in A \setminus D$, $A(a) \subseteq B(a) \subseteq D$ as in Lemma 3.3. It is enough to show that $p = f[\mathsf{tp}^{\mathcal{A}}(a/B(a))]$ can be realized in $B \setminus \mathrm{ran}(f)$ because if b realizes p in $B \setminus \mathrm{ran}(f)$ then $f' = f \cup \langle \{a, b\} \rangle$ is the required elementary mapping strictly extending f. Note that $\mathcal{A}$ and $\mathcal{B}$ are $\aleph_1$-categorical, hence they are $\aleph_0$-stable. Consequently, Lemma 3.3 (2) ensures $|B(a)| \le \aleph_0$.

Adjoin a new relation symbol R to the language of $\mathcal{B}$ and interpret it in $\mathcal{B}$ as $\mathrm{ran}(f)$. By saturation it is enough to show that each $\varphi \in p$ can be realized in $B \setminus R$. Let $\varphi \in p$ be arbitrary, but fixed. By assumption, $\mathcal{D}$ is an elementary substructure of $\mathcal{A}$, so it follows that a is not algebraic over D. Hence, because of f is elementary, the relation defined by φ in $\mathcal{B}$ is infinite as well. In addition, $\mathcal{B}$ is uncountably categorical, consequently $\langle \mathcal{B}, f[\mathcal{D}] \rangle$ is not a Vaughtian pair (cf., e.g., [6, Theorem 6.1.18]). Thus the relation defined by φ in $\mathcal{B}$ can be realized in $B \setminus R$ therefore $\neg R(v) \wedge \varphi(v)$ can be satisfied in $\mathcal{B}$, for all $\varphi \in p$. □

4 Extending decomposable mappings

In this section we are presenting a method for constructing so called *decomposable* isomorphisms between certain ultraproducts. We briefly recall (for more detail, cf. Definition 2.1 above) that a relation R in an ultraproduct $\prod_{i \in I} \mathcal{A}_i/\mathcal{F}$ is defined to be decomposable iff for all $i \in I$ there are relations R_i on A_i such that $R = \prod_{i \in I} R_i/\mathcal{F}$. Similarly, a function $f : \prod_{i \in I} \mathcal{A}_i/\mathcal{F} \to \prod_{i \in I} \mathcal{B}_i/\mathcal{F}$ is called decomposable iff "f acts coordinatewise", i.e., iff for all $i \in I$ there are functions $f_i : A_i \to B_i$ such that $f = \prod_{i \in I} f_i/\mathcal{F}$.

Our method is similar in spirit to [14]: in order to prove certain properties of finite structures, we are dealing with infinite ultraproducts of them. As we already mentioned, to establish further applications, we are trying to present our construction in a rather general way.

Definition 4.1 A sequence $\langle \Delta_n \in [\mathsf{Form}]^{<\omega} : n \in \omega \rangle$ is defined to be a *covering sequence of formulas* if the following properties hold for it.

1. The sequence is increasing: $\Delta_i \subseteq \Delta_j$ whenever $i \leq j \in \omega$;
2. For all $n \in \omega$ the finite set of formulas Δ_n is closed under subformulas;
3. $\bigcup \{\Delta_n : n \in \omega\} = \mathsf{Form}$, i.e., the sequence covers Form.

If $\mathcal{M}$ is a structure and $\mathcal{A}_n \leq \mathcal{M}$ is a Δ_n-elementary substructure then $\prod_{n \in \omega} \mathcal{A}_n/\mathcal{F}$ is elementarily equivalent to $\mathcal{M}$.

From now on, unless otherwise stated, $\langle \Delta_n : n \in \omega \rangle$ **denotes an arbitrary covering sequence of formulas.**

Our aim in this section is to prove the following theorem.

Theorem 4.2 *Let $\mathcal{M}$ be an $\aleph_1$-categorical structure with an atom–defining schema, having the extension property. Suppose that there is a $\varnothing$-definable strongly minimal subset M_0 of M and suppose for each $n \in \omega$ the finite structures $\mathcal{A}_n$ and $\mathcal{B}_n$ are equinumerous, Δ_n–elementary substructures of $\mathcal{M}$. Then there is a decomposable isomorphism*

$$f = \langle f_n : n \in \omega \rangle/\mathcal{F} : \prod_{n \in \omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n \in \omega} \mathcal{B}_n/\mathcal{F}.$$

We split the proof into three parts: each part is contained in a different subsection. We sketch here the main line of the proof. If $\mathcal{M}$ is an $\aleph_1$-categorical structure with $M_0 \subseteq M$ being a $\varnothing$-definable strongly minimal subset then by Zilber's Ladder Theorem [16, Theorem 0.1, Chapter V] there exists a finite increasing sequence

$$M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{z-1} = M$$

of subsets of M such that M_ℓ is $\varnothing$-definable for all $\ell \in z$ (and certain remarkable properties hold which will be recalled later).

First, in Subsection 4.1 we extend certain decomposable elementary mappings to the whole of M_0 (cf. Proposition 4.8). Then, in Subsection 4.2 we continue to extend the mapping along Zilber's ladder to M (cf. Theorem 4.19). Finally, in Subsection 4.3, we combine our results obtained so far to get Theorem 4.2.

From now on, throughout this section $\mathcal{M}$ is a fixed uncountable, $\aleph_1$-categorical structure satisfying the extension property and having an atom-defining schema. Further, we assume that $M_0 \subseteq M$is a $\varnothing$-definable strongly minimal subset of$M$.

For completeness, we note that, we do not need all these properties in all of our steps. To be more concrete, in Subsection 4.1 we need $\mathcal{M}$ to be $\aleph_1$-categorical satisfying the extension property, and in Subsection 4.2 we need $\mathcal{M}$ to be $\aleph_1$-categorical having an atom-defining schema for $\varnothing$-definable infinite relations.

4.1 The strongly minimal case

We shall deal first with strongly minimal structures $\mathcal{N}$ and we provide a method to extend certain decomposable mappings in this case (Proposition 4.6). Then we move on to the case when the whole structure is not strongly minimal (Proposition 4.8). We shall need several Lemmas.

Lemma 4.3 *Let $\mathcal{A}$ be a structure and let $M \subseteq A$ be $\varnothing$-definable and strongly minimal. Then there exists a function $\varepsilon : [\mathsf{Form}]^{<\omega} \to \omega$ such that for all $\Delta \in [\mathsf{Form}]^{<\omega}$ if $\mathcal{B} \leq \mathcal{A}$ is a Δ-algebraically closed substructure with $B \subseteq M$ and $|B| \geq \varepsilon(\Delta)$ then $\mathcal{B}$ is a Δ-elementary substructure of $\mathcal{A}$.*

Proof. By strong minimality, for any formula φ either $\|\varphi\| \cap M$ or $(A \setminus \|\varphi\|) \cap M$ is finite, i.e., φ is algebraic or transcendental, respectively. Let Δ be a finite set of formulas and let $\mathcal{B}$ be a Δ-algebraically closed substructure of $\mathcal{A}$ with $B \subseteq M$. Let Δ' be the smallest set of formulas containing Δ and closed under subformulas. We shall define the number $\varepsilon(\Delta)$ so that if $|B| \geq \varepsilon(\Delta)$ then $\mathcal{B}$ is a Δ-elementary substructure. Pick $\varphi \in \Delta$ and $\bar{b} \in B$.

Case 1. Suppose $\varphi(x, \bar{b})$ is algebraic and suppose $\mathcal{A} \models \varphi(a, \bar{b})$ for some $a \in A$. Then $a \in B$ because $\mathcal{B}$ is Δ-algebraically closed. In this case let $\mathsf{n}(\varphi) = 0$.

Case 2. Suppose $\varphi(x, \bar{b})$ is transcendental. By compactness, there exists $\mathsf{n}(\varphi)$, depending on φ only, such that $\left|M \setminus \|\varphi(x, \bar{b})\|\right| \leq \mathsf{n}(\varphi)$. Thus if $|B| > \mathsf{n}(\varphi)$ then there must exists $c \in B$ such that $\mathcal{B} \models \varphi(c, \bar{b})$.

Setting $\varepsilon(\Delta) = \max\{\mathsf{n}(\varphi) + 1 : \varphi \in \Delta'\}$, a straightforward induction on the complexity of elements of Δ' completes the proof. □

The next lemma can be regarded as a kind of converse of Lemma 4.3.

Lemma 4.4 *Let $\mathcal{A}$ be strongly minimal and let $\mathcal{B}$ be a substructure of $\mathcal{A}$. Then for every finite set ε of formulas there exists a finite set δ of formulas such that if $\mathcal{B}$ is a δ-elementary substructure then $\mathcal{B}$ is $\mathsf{acl}_\varepsilon^{\mathcal{A}}$-closed.*

Proof. For all $\varphi \in \varepsilon$, by compactness, there is a natural number $\mathsf{n}(\varphi)$ (depending only on φ) such that if $\varphi(v, \bar{b})$ is algebraic for some $\bar{b} \in B$, then $\varphi(v, \bar{b})$ can have at most $\mathsf{n}(\varphi)$ pairwise distinct realizations in A (else, there would exists an infinite–co-infinite definable subset in some elementary extension, contradicting strong minimality). Let $\varphi_n(\bar{y})$ denote the next formula:

$$\varphi_n(\bar{y}) = \exists_n x \varphi(x, \bar{y}) = \text{“}\varphi(x, \bar{y}) \text{ has exactly } n \text{ realizations”}.$$

Clearly φ_n can be made a strict first order formula, for all fixed $n \in \omega$. Put

$$\delta = \{\varphi_n : n \leq \mathsf{n}(\varphi), \varphi \in \varepsilon\} \cup \varepsilon.$$

Clearly, if $\mathcal{B}$ is δ-elementary then it is $\mathsf{acl}_\varepsilon^{\mathcal{A}}$-closed. □

Lemma 4.5 *Let $\Delta \in [\mathsf{Form}]^{<\omega}$ be closed under subformulas. Let $\mathcal{B}, \mathcal{C}$ be Δ-elementary substructures of $\mathcal{A}$. If $f : \mathcal{B} \to \mathcal{C}$ is an isomorphism then f is a Δ-elementary mapping of $\mathcal{A}$.*

Proof. A straightforward induction on the complexity of the formulas in Δ; the details are left to the reader. □

Let $\mathcal{N}$ be a fixed strongly minimal (hence $\aleph_1$-categorical) structure with the weak extension property (Definition 2.7). Recall that by the weak extension property there exists a finite set Δ of formulas satisfying $(*)$ of Definition 2.7. Let Φ be a set of formulas such that if $X = \mathsf{acl}_\Phi(X)$ then X is a substructure. Such Φ exists and can be chosen to be finite because our language is finite. *Fix a covering sequence of formulas*

$\langle \Delta_n \in [\mathsf{Form}]^{<\omega} : n \in \omega\rangle$ in a way that $\Phi, \Delta \subseteq \Delta_n$ for all $n \in \omega$. By Lemma 4.4, after eventual rescaling, we may assume that

$$\mathcal{A}_n \text{ and } \mathcal{B}_n \text{ are } \mathsf{acl}^{\mathcal{N}}_{\Delta_n}\text{–closed substructures of } \mathcal{N}. \qquad (**)$$

Proposition 4.6 can be considered as the strongly minimal case of Theorem 4.2.

Proposition 4.6 *Let $\mathcal{N}$ be a strongly minimal structure with the weak extension property. Suppose that for each $n \in \omega$ the finite structures $\mathcal{A}_n$ and $\mathcal{B}_n$ are Δ_n-elementary (hence, by $(**)$, acl_{Δ_n}-closed) substructures of $\mathcal{N}$ with $|A_n| \le |B_n|$. Let*

$$g = \langle g_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$$

be a decomposable elementary mapping with

$$\big\{n \in \omega : g_n \text{ is } \Delta_n\text{-elementary and } |\mathrm{dom}(g_n)| \ge \varepsilon(\Delta_n)\big\} \in \mathcal{F},$$

where ε comes from Lemma 4.3. Then g can be extended to a decomposable elementary embedding.

We remark that if $|A_n| = |B_n|$ for all (in fact, almost all) n, then the resulting extension is a decomposable isomorphism.

Proof. Let $\mathcal{A} = \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F}$ and $\mathcal{B} = \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$. Note that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent with $\mathcal{N}$ because the increasing sequence Δ_n covers Form. By transfinite recursion we construct a sequence $\langle f^\alpha : \alpha \le \kappa\rangle$ such that for $\alpha \le \kappa$ the following properties hold:

(P1) $f^\alpha = \big\langle f^\alpha_n : n \in \omega\big\rangle\big/\mathcal{F} : A \to B$ is a decomposable elementary mapping;
(P2) $f^\gamma_n \subseteq f^\nu_n$ for $\gamma < \nu \le \kappa$ and all $n \in \omega$;
(P3) $\mathrm{dom}\big(f^\alpha_n\big)$ is an $\mathsf{acl}^{\mathcal{N}}_{\Delta_n}$-closed substructure of $\mathcal{A}_n$ for all $n \in \omega$;
(P4) $\mathrm{ran}\big(f^\alpha_n\big)$ is an $\mathsf{acl}^{\mathcal{N}}_{\Delta_n}$-closed substructure of $\mathcal{B}_n$ for all $n \in \omega$;
(P5) f^α_n is Δ_n-elementary for all $n \in \omega$.

If $\mathrm{dom}(f^\kappa) = A$ then we are done, because since each A_i and B_i are finite, it follows that f^κ is a decomposable elementary embedding. As it is usual in transfinite recursions, $\kappa < |A|^+$ but we do not have any other estimates of κ, so it may well happen that $|A| < \kappa$ (as an ordinal). It would be of interest if one could guarantee $\kappa = |A|$, for example.

Now we construct the first element f^0 of the sequence. By assumption

$$J = \big\{n \in \omega : g_n \text{ is } \Delta_n\text{-elementary and } |\mathrm{dom}(g_n)| \ge \varepsilon(\Delta_n)\big\} \in \mathcal{F}.$$

Because Δ in the weak extension property is contained in each Δ_n, it follows that for all $n \in J$ there exists a partial isomorphism h_n extending g_n, with $\mathrm{dom}(h_n) = \mathsf{acl}^{\mathcal{N}}_{\Delta_n}(\mathrm{dom}(g_n))$ and $\mathrm{ran}(h_n) = \mathsf{acl}^{\mathcal{N}}_{\Delta_n}(\mathrm{ran}(g_n))$. Note that because $\mathcal{A}_n$ is Δ_n-algebraically closed, it follows that $\mathrm{dom}(h_n) \subseteq A_n$. Therefore $\mathrm{dom}(h_n)$ is a substructure of $\mathcal{N}$ (hence of $\mathcal{A}_n$, too). By $|\mathrm{dom}(h_n)| \ge \varepsilon(\Delta_n)$ and by Lemma 4.3 we get $\mathrm{dom}(h_n)$ is a Δ_n-elementary substructure of ($\mathcal{N}$ and hence of) $\mathcal{A}_n$. Similarly $\mathrm{ran}(h_n)$ is a Δ_n-elementary substructure of $\mathcal{B}_n$. But then Lemma 4.5 applies: h_n is also a Δ_n-elementary mapping. Let

$$f^0_n = \begin{cases} h_n & \text{if } n \in J,\\ \varnothing & \text{otherwise,}\end{cases}$$

and $f^0 = \big\langle f^0_n : n \in \omega\big\rangle\big/\mathcal{F}$. Then properties (P1)-(P5) hold.

Now suppose $\langle f^\alpha : \alpha < \beta\rangle$ has already been defined for some $\beta \le \kappa$. Then we define f^β as follows.

I. Successor case Suppose $\beta = \alpha + 1$. We may assume $A \setminus \mathrm{dom}(f^\alpha) \ne \varnothing$, since otherwise the construction would stop. Because f^α is decomposable we have

$$\langle \mathcal{A}, \mathrm{dom}(f^\alpha)\rangle = \prod_{n\in\omega} \big\langle \mathcal{A}_n, \mathrm{dom}\big(f^\alpha_n\big)\big\rangle\big/\mathcal{F}$$

and thus $\langle \mathcal{A}, \mathrm{dom}(f^\alpha)\rangle$ is $\aleph_1$-saturated (and similarly with $\langle \mathcal{B}, \mathrm{ran}(f^\alpha)\rangle$). Consequently Lemma 3.4 applies: there exist $a \in A \setminus \mathrm{dom}(f^\alpha)$, $b \in B \setminus \mathrm{ran}(f^\alpha)$ such that $f = f^\alpha \cup \{\langle a, b\rangle\}$ is an elementary mapping. If $a = \langle a_n : n \in \omega\rangle/\mathcal{F}$ and $b = \langle b_n : n \in \omega\rangle/\mathcal{F}$ then

$$I = \left\{n \in \omega : a_n \notin \mathrm{dom}\left(f_n^\alpha\right),\ b_n \notin \mathrm{ran}\left(f_n^\alpha\right)\right\} \in \mathcal{F}.$$

Thus if

$$f_n = \begin{cases} f_n^\alpha \cup \{\langle a_n, b_n\rangle\} & \text{if } n \in I,\\ f_n^\alpha & \text{otherwise,}\end{cases}$$

then $f = \langle f_n : n \in \omega\rangle/\mathcal{F}$. As f is an elementary mapping by Łoś's lemma (or rather by Proposition 2.2)

$$J = \{n \in \omega : f_n \text{ is } \Delta\text{-elementary}\} \in \mathcal{F}.$$

We claim that for each $n \in J$, f_n is not only Δ-elementary but Δ_n-elementary. To see this, let $\varphi \in \Delta_n$, $\bar{d} \in \mathrm{dom}(f_n)$ and suppose $\mathcal{A}_n \models \varphi(\bar{d})$. We have to show that $\mathcal{B}_n \models \varphi(f_n(\bar{d}))$. Let us replace all the occurrences of a_n in $\bar{d}$ with a variable v and denote this sequence by $v^\frown\bar{d}'$. Then $\bar{d}' \in \mathrm{dom}(f_n^\alpha)$ and $a_n \in \|\varphi(v, \bar{d}')\|^{\mathcal{A}_n}$. Since $\mathrm{dom}(f_n^\alpha)$ is $\mathsf{acl}^{\mathcal{N}}_{\Delta_n}$-closed (by (P3)), it follows that $\varphi(v, \bar{d}')$ is not a Δ_n-algebraic formula since else it would imply $a_n \in \mathrm{dom}(f_n^\alpha)$. Since $\mathcal{N}$ is strongly minimal, exactly one of $\varphi(v, \bar{d}')$ or $\neg\varphi(v, \bar{d}')$ is algebraic, thus if $\varphi(v, \bar{d}')$ is not algebraic then $\varphi\left(v, f_n^\alpha(\bar{d}')\right)$ is not algebraic, too. The same is the situation in $\mathcal{B}_n$, hence $b_n \notin \left\|\neg\varphi\left(v, f_n^\alpha(\bar{d}')\right)\right\|^{\mathcal{B}_n}$, and thus $b_n \in \left\|\varphi\left(v, f_n^\alpha(\bar{d}')\right)\right\|^{\mathcal{B}_n}$, as needed.

So, f_n is Δ_n-elementary and $\Delta \subseteq \Delta_n$ hence by the weak extension property, for all $n \in J$ there exists a partial isomorphism h_n extending f_n with $\mathrm{dom}(h_n) = \mathsf{acl}^{\mathcal{N}}_{\Delta_n}(\mathrm{dom}(f_n))$. Then by Lemma 4.3, $\mathrm{dom}(h_n)$ is a Δ_n-elementary substructure of $\mathcal{A}_n$ (similarly $\mathrm{ran}(h_n)$ is a Δ_n-elementary substructure of $\mathcal{B}_n$) and hence by Lemma 4.5, h_n is a Δ_n-elementary mapping. Let us define f_n^β as follows:

$$f_n^\beta = \begin{cases} h_n & \text{if } n \in J,\\ f_n^\alpha & \text{otherwise.}\end{cases}$$

Set $f^\beta = \left\langle f_n^\beta :\ n \in \omega\right\rangle\big/\mathcal{F}$. Then clearly, stipulations (P1)-(P5) hold for f^β.

II. Limit case Suppose β is a limit ordinal. Set $f_n^\beta = \bigcup_{\alpha<\beta} f_n^\alpha$ for all $n \in \omega$, and let $f^\beta = \left\langle f_n^\beta : n \in \omega\right\rangle\big/\mathcal{F}$. Then (P2)-(P4) are true for f^β and for (P1) we only have to show that f^β is still elementary. For this it is enough to prove that f_n^β preserves Δ_n for all $n \in \omega$, i.e., f_n^β is a Δ_n-elementary mapping. But this is exactly (P5) which property is preserved under chains of Δ_n-elementary mappings. □

As an immediate corollary of the results established so far, in Theorem 4.7 below, we prove that a strongly minimal structure with the weak extension property can be obtained in an essentially unique way, as an ultraproduct of certain finite substructures.

Theorem 4.7 (First Unique Factorization Theorem). *Let $\mathcal{N}$ be a strongly minimal structure having the weak extension property (Definition 2.7). Suppose $\mathcal{A}_n$, $\mathcal{B}_n$ are equinumerous finite, acl_{Δ_n}-closed substructures of $\mathcal{N}$ for all $n \in \omega$ such that* $\sup\{|A_n| : n \in \omega\}$ *is infinite. Then*

$$\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\} \text{ is co-finite,}$$

and thus belongs to any non-principal ultrafilter $\mathcal{F}$.

Proof. By way of contradiction suppose that $X = \{n \in \omega :\ \mathcal{A}_n \ncong \mathcal{B}_n\}$ is infinite. Since $\sup\{|A_n| : n \in \omega\}$ is infinite by assumption, it follows that for all $n \in \omega$ there exists $\gamma(n) \in \omega$ such that $|A_{\gamma(n)}| \geq \varepsilon(\Delta_n)$, where ε comes from Lemma 4.3. Hence the structure $\mathcal{A}_{\gamma(n)}$ is a Δ_n-elementary substructure of $\mathcal{N}$. By possibly increasing $\gamma(n)$, without loss of generality we can assume that $\gamma(n) \in X$, as well. For simplicity, to avoid ugly notation, by replacing $\mathcal{A}_n$ with $\mathcal{A}_{\gamma(n)}$ we may suppose $\mathcal{A}_n$ and $\mathcal{B}_n$ are non-isomorphic, equinumerous, Δ_n-elementary, finite substructures of $\mathcal{N}$. For an arbitrary non-principal ultrafilter $\mathcal{F}$ let $\mathcal{A} = \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F}$ and $\mathcal{B} = \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$. The increasing sequence Δ_n covers Form, hence $\mathcal{A}$ and $\mathcal{B}$ are both elementarily equivalent with $\mathcal{N}$. By universality, taking a large enough ultrapower $\mathcal{A}'$ of $\mathcal{A}$, $\mathcal{N}$ can be elementarily embedded into $\mathcal{A}'$. Hence $\mathcal{A}_n$ is a Δ_n-elementary substructure of $\mathcal{A}'$ as well. Now taking an elementary substructure of $\mathcal{A}'$ of power $|A|$ containing (the image of)

$\mathcal{A}_n$ it is isomorphic to $\mathcal{A}$ by categoricity. Hence we may assume that $\mathcal{A}_n$ is a Δ_n-elementary substructure of $\mathcal{A}$ for all $n \in \omega$. By a similar argument we may also assume that $\mathcal{B}_n$ is a Δ_n-elementary substructure of $\mathcal{B}$.

For all $n \in \omega$ because $\mathcal{A}_n$ is finite, by Łoś's lemma, there exists $n \leq \beta(n) \in \omega$ such that $\mathcal{A}_{\beta(n)}$ and $\mathcal{B}_{\beta(n)}$ contains an isomorphic copy of $\mathcal{A}_n$. Consequently there exist partial isomorphisms $g_{\beta(n)} : \mathcal{A}_{\beta(n)} \to \mathcal{B}_{\beta(n)}$ whose domains are the $\mathcal{A}_n$'s. By $\Delta_n \subseteq \Delta_{\beta(n)}$ we get that $\mathcal{A}_{\beta(n)}$ and $\mathcal{B}_{\beta(n)}$ are also Δ_n-elementary substructures. Applying Lemma 4.5 these partial isomorphisms are Δ_n-elementary mappings.

Let $\mathcal{A}^* = \prod_{n\in\omega} \mathcal{A}_{\beta(n)}/\mathcal{F}$ and $\mathcal{B}^* = \prod_{n\in\omega} \mathcal{B}_{\beta(n)}/\mathcal{F}$. Then $g = \langle g_{\beta(n)} : n \in \omega\rangle/\mathcal{F} : \mathcal{A}^* \to \mathcal{B}^*$ is a decomposable elementary mapping which, by Proposition 4.6 and the comment following it, extends to a decomposable isomorphism $f = \langle f_n : n \in \omega\rangle/\mathcal{F} : \mathcal{A}^* \to \mathcal{B}^*$. Now, applying Proposition 2.2 it follows that for almost all coordinates f_n is an isomorphism (clearly, being an isomorphism can be expressed by a first order formula if the language is finite). But this is a contradiction because components of $\mathcal{A}^*$ and $\mathcal{B}^*$ were non-isomorphic (by the choice of γ). □

Now we turn to the case when the whole structure is not necessarily strongly minimal. Recall that $\mathcal{M}$ is supposed to be a fixed uncountable, $\aleph_1$-categorical structure satisfying the extension property and M_0 is a $\varnothing$-definable strongly minimal subset of M (see the boldface quote at page 14).

Proposition 4.8 *Suppose for each $n \in \omega$ the finite structures $\mathcal{A}_n, \mathcal{B}_n$ are Δ_n-elementary substructures of $\mathcal{M}$ such that*

$$\left\{n \in \omega : \left|M_0^{\mathcal{A}_n}\right| \leq \left|M_0^{\mathcal{B}_n}\right|\right\} \in \mathcal{F}.$$

Let $g = \langle g_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$ be a decomposable elementary mapping with $\mathrm{dom}(g_n) \subseteq M_0^{\mathcal{A}_n}$ *and* $\mathrm{ran}(g_n) \subseteq M_0^{\mathcal{B}_n}$ *for all $n \in \omega$. Assume that*

$$\{n \in \omega : g_n \text{ is } \Delta_n\text{-elementary and } |\mathrm{dom}(g_n)| \geq \varepsilon(\Delta_n)\} \in \mathcal{F}$$

where ε comes from Lemma 4.3. Then g can be extended to a decomposable elementary mapping $g^+ = \langle g_n^+ : n \in \omega\rangle/\mathcal{F}$ such that $\mathrm{dom}(g_n^+) = M_0^{\mathcal{A}_n}$ *and* $\mathrm{ran}(g_n^+) \subseteq M_0^{\mathcal{B}_n}$ *(almost everywhere).*

We note that if $\left|M_0^{\mathcal{A}_n}\right| = \left|M_0^{\mathcal{B}_n}\right|$ almost everywhere, then we get $\mathrm{dom}(g_n^+) = M_0^{\mathcal{A}_n}$ and $\mathrm{ran}(g_n^+) = M_0^{\mathcal{B}_n}$ for almost all n.

P r o o f. We intend to use Proposition 4.6. To do so we have to ensure that M_0 is not just a strongly minimal set but a structure. In general this cannot be guaranteed in the original language of $\mathcal{M}$. Our plan is to apply Proposition 4.6 for a sequence of strongly minimal structures defined in terms of relations of M_0.

Since we shall use different first order languages in this proof, let us denote by $L(\mathcal{M})$ the language of $\mathcal{M}$. For each $L(\mathcal{M})$-formula φ let us associate a relation symbol R_φ whose arity equals to the number of free variables in φ. Let $L(R)$ be the language that consists of these new relation symbols:

$$L(R) = \left\{R_\varphi : \varphi \in \mathsf{Form}(L(\mathcal{M}))\right\}.$$

Next, we turn $\mathcal{M}$ into an $L(R)$-structure as follows: if $\varphi(\bar{x})$ is an $L(\mathcal{M})$-formula then interpret R_φ in $\mathcal{M}$ as follows:

$$R_\varphi^{\mathcal{M}} = \|\varphi\|^{\mathcal{M}} \cap {}^{|\bar{x}|}M_0.$$

It is easy to see that relations definable with $L(R)$-formulas (in $\mathcal{M}$) are also definable with $L(\mathcal{M})$-formulas. In fact by an obvious induction on the complexity of formulas of $L(R)$ one can easily check that there is a function $\iota : \mathsf{Form}(L(R)) \to \mathsf{Form}(L(\mathcal{M}))$ such that for any formula $\psi \in \mathsf{Form}(L(R))$ we have

$$\|\psi\|^{\mathcal{M}} = R_{\iota(\psi)}^{\mathcal{M}}.$$

For a set Δ of $L(\mathcal{M})$-formulas we write

$$R(\Delta) = \{R_\varphi : \varphi \in \Delta\}.$$

Let us enumerate $\mathsf{Form}(L(\mathcal{M}))$ as

$$\mathsf{Form}(L(\mathcal{M})) = \langle \varphi_n : n \in \omega \rangle.$$

For $\ell \in \omega$ let us define a structure $\mathcal{N}_\ell$ as follows.

By the extension property of $\mathcal{M}$, for $\varepsilon_\ell = \{\varphi_0, \dots, \varphi_{\ell-1}\}$ there exists a corresponding finite set of formulas D_ℓ (here we use letter D because Δ is already in use). Let

$$\mathcal{N}_\ell = \langle M_0, R_\varphi^{\mathcal{M}} \rangle_{\varphi \in D_\ell}.$$

Thus the language $L(\mathcal{N}_\ell)$ consists of the relation symbols $\{R_\varphi : \varphi \in D_\ell\}$. We have the next few auxiliary claims.

(1) $\mathcal{N}_\ell$ is strongly minimal: To see this, let $\psi \in \mathsf{Form}(L(\mathcal{N}_\ell))$ be any formula. Then $\|\psi\|^{\mathcal{M}} = R^{\mathcal{M}}_{\iota(\psi)} = \|\iota(\psi)\|^{\mathcal{M}} \cap M_0$ which is either finite or cofinite (because $\iota(\psi) \in \mathsf{Form}(L(\mathcal{M}))$).

(2) $\mathcal{N}_\ell$ has the weak extension property described in Definition 2.7: We have to find a set Δ (a finite set of $L(\mathcal{N}_\ell)$-formulas) such that whenever $\Delta' \supseteq \Delta$ and f is a Δ'-elementary mapping then it can be extended to a partial isomorphism to $\mathsf{acl}_{\Delta'}(\mathrm{dom}(f))$. Now we claim that $\Delta = R(D_\ell)$ works. To see this, suppose $\Delta' \supseteq \Delta$ and f is a Δ'-elementary mapping. We have to extend f in a way that the extension preserves all the formulas in $R(D_\ell)$ (this would mean that the extension is a partial isomorphism in the language $L(\mathcal{N}_\ell)$).

 (i) Observe first that we may assume that $\iota[R(D_\ell)] = D_\ell$, because the formulas in the two sides of the equation define the same relations in M_0.
 (ii) Clearly, we have $\iota[\Delta'] \supseteq D_\ell$.
 (iii) If f preserves an $L(R)$-formula ψ then it preserves $\iota(\psi)$ as well. Therefore f is $\iota[\Delta']$-elementary.

 Consequently, by the extension property of $\mathcal{M}$, there is a D_ℓ-elementary (in the language $L(\mathcal{M})$) extension f' of f whose domain and range are respectively $\mathsf{acl}_{\iota[\Delta']}(\mathrm{dom}(f))$ and $\mathsf{acl}_{\iota[\Delta']}(\mathrm{ran}(f))$. Clearly, if f' preserves D_ℓ then it also preserves $R(D_\ell)$. Thus f' is a partial isomorphism in the language $L(\mathcal{N}_\ell)$, as desired.

(3) Let $i \in \omega$ be arbitrary. Then there exists ℓ such that $\Delta_i \subseteq \{\varphi_k : k \in \ell\}$. Since $\mathcal{A}_i$ is a Δ_i-elementary substructure of $\mathcal{M}$, it follows that $M_0^{\mathcal{A}_i}$ (which equals $A_i \cap M_0$ if i is large enough) is the underlying set of an $R(\Delta_i)$-elementary substructure of $\mathcal{N}_\ell$. If $\langle \Delta_i : i \in \omega \rangle$ is a covering sequence of $\mathsf{Form}(L(\mathcal{M}))$ then $\langle R(\Delta_i) : i \in \omega \rangle$ can be considered as a covering sequence of $\mathsf{Form}(L(R))$: note that for each $\psi \in \mathsf{Form}(L(R))$ we have $\|\psi\|^{\mathcal{M}} = R^{\mathcal{M}}_{\iota(\psi)}$ and $\iota(\psi) \in \Delta_i$ for large enough i. Observe that $R(\Delta_i)$ and Δ_i define the same relations in M_0 and since $\mathrm{dom}(g_n)$ and $\mathrm{ran}(g_n)$ are subsets of $M_0^{\mathcal{A}_n}$ and $M_0^{\mathcal{B}_n}$, respectively, it follows that by the assumption of the present proposition g_n is $R(\Delta_n)$-elementary for almost all $n \in \omega$. By (2) above, $\mathcal{N}_\ell$ has the weak extension property and hence putting together that g_n is $R(\Delta_n)$-elementary for almost all $n \in \omega$ and that $R(\Delta_i)$ and Δ_i define the same relations in M_0 we conclude that $\varepsilon(R(\Delta_i))$ and $\varepsilon(\Delta_i)$ in Lemma 4.3 are equal. Consequently, the conditions of Proposition 4.6 are satisfied.

By Proposition 4.6 for all $\ell \in \omega$ there exists a decomposable elementary embedding $g^\ell = \langle g_n^\ell : n \in \omega \rangle / \mathcal{F}$ (it is elementary in the language $L(\mathcal{N}_\ell)$) extending g, with $\mathrm{dom}(g_n^\ell) = M_0^{\mathcal{A}_n}$ and $\mathrm{ran}(g_n^\ell) \subseteq M_0^{\mathcal{B}_n}$. That is, we know that

$$X_\ell = \{ n \in \omega : \ g_n^\ell \text{ is } \Delta_n \text{ -elementary and } \mathrm{dom}(g_n^\ell) = M_0^{\mathcal{A}_n}, \ \mathrm{ran}(g_n^\ell) \subseteq M_0^{\mathcal{B}_n} \} \in \mathcal{F}.$$

Let $\langle I_n : n \in \omega \rangle$ be a decreasing sequence with $I_n \in \mathcal{F}$, $I_0 = \omega$ and $\cap_{n\in\omega} I_n = \varnothing$. Write

$$J_n = \{ i \in I_n : g_i^n \text{ is } \Delta_i \text{ -elementary and dom } (g_i^n) = M_0^{\mathcal{A}_i}, \ \mathrm{ran}(g_i^n) \subseteq M_0^{\mathcal{B}_i} \}.$$

Clearly, $J_n = I_n \cap X_n$ thus $J_n \in \mathcal{F}$ for all $n \in \omega$ and for a fixed i the set $\{n : i \in J_n\}$ is finite. Let

$$\nu(i) = \max\{n \in \omega : i \in J_n\}$$

and put

$$g^+ = \langle g_i^{\nu(i)} : i \in \omega \rangle / \mathcal{F}.$$

Then g^+ is the desired extension. □

4.2 Climbing Zilber's ladder

Recall that $\mathcal{M}$ is a fixed uncountable, $\aleph_1$-categorical structure with an atom–defining schema ∂ for $\varnothing$-definable infinite relations (Definition 2.5). By Zilber's Ladder Theorem [16, Theorem 0.1, Chapter V], if $\mathcal{M}$ is $\aleph_1$-categorical and $M_0 \subseteq M$ is $\varnothing$-definable and strongly minimal then there exists a finite increasing sequence

$$M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{z-1} = M$$

of subsets of M such that for all $\ell \in z$ we have

1. $M_{\ell+1}$ is $\varnothing$-definable;
2. $\mathsf{Gal}(A, M_\ell)$ is $\varnothing$-definable together with its action on A for all M_ℓ-atoms $A \subseteq M_{\ell+1}$. Moreover $\mathsf{Gal}(A, M_\ell) \subseteq \mathsf{dcl}(M_\ell)$.

Here by $\mathsf{Gal}(A, M_\ell)$ we understand the group of all M_ℓ-elementary automorphisms of the set A: following Zilber [16] an M_ℓ-elementary automorphism of A is defined as a permutation of A which preserves M_ℓ-definable sets, i.e., if X is defined using parameters from M_ℓ and $g \in \mathsf{Gal}(A, M_\ell)$ then for any tuple $\bar{a}$ in A we have $\bar{a} \in X$ iff $g(\bar{a}) \in X$.

We note that $\mathsf{Gal}(A, M_\ell)$ acts transitively on A because A is an atom (the group G acts transitively on X if for any $x, y \in X$ there is $g \in G$ with $gx = y$). We fix this ladder and z will denote its length.

Strictly speaking, elements of $\mathsf{Gal}(A, M_\ell)$ are imaginary elements in the sense of, e.g., [5, §4.3]. In more detail, the proof of [16, Lemma 2.3, Chapter V] describes how one can identify elements of $\mathsf{Gal}(A, M_\ell)$ by certain imaginary elements (in fact, $\mathsf{Gal}(A, M_\ell)$ is a sort of $\mathcal{A}^{\mathrm{eq}}$).

Therefore it makes sense to speak about $\mathsf{Gal}(A, M_\ell) \subseteq \mathsf{dcl}(M_\ell)$ in item 2 above, because such statements are expressible in the language of $\mathcal{A}^{\mathrm{eq}}$. Moreover, according to [5, Theorem 4.3.3 (c)] each first order formula φ of $\mathcal{A}^{\mathrm{eq}}$ can be translated back to a formula $\varphi^{\downarrow}$ in the language of $\mathcal{A}$ such that for any $\bar{a} \in A$ we have

$$\mathcal{A}^{\mathrm{eq}} \models \varphi(\bar{a}) \quad \text{iff } \mathcal{A} \models \varphi^{\downarrow}(\bar{a}).$$

Below, we shall work in $\mathcal{A}$ only but, in order to keep notation simpler, we shall formalize properties φ in the language of $\mathcal{A}^{\mathrm{eq}}$ and will refer to those properties as first order formulas $\varphi^{\downarrow}$ in the language of $\mathcal{A}$. One can check that all such statements can be translated back to the first order language of $\mathcal{A}$.

The main proposition in this subsection is Theorem 4.19. In order to prove it we make use of the following Lemmas.

Lemma 4.9 *Suppose $\mathcal{M}$ has an atom-defining schema. Then for all infinite, definable E and formula φ there exists a finite set $T_\varphi \subseteq \mathsf{S}^{\mathcal{M}}(\varnothing)$ of types such that if $\varphi(v, \bar{e})$ defines an E-atom, then $\mathsf{tp}^{\mathcal{M}}(\bar{e}) \in T_\varphi$.*

Informally we shall refer to this fact as "the formula φ has finitely many atom-types over E".

Proof. Suppose, seeking a contradiction, that $\{e_i \in \|\partial_E \varphi\| : i \in \omega\}$ is such that

$$H = \{\mathsf{tp}^{\mathcal{M}}(e_i) : i \in \omega\}$$

is infinite. Then $H \subseteq \mathsf{S}^{\mathcal{M}}(\varnothing)$ is an infinite topological subspace of $\mathsf{S}^{\mathcal{M}}(\varnothing)$, hence it has an infinite strongly discrete subspace: there is an injective function $s : \omega \to \omega$ and there are pairwise disjoint basic open sets $U_i \subseteq \mathsf{S}^{\mathcal{M}}(\varnothing)$ such that $\mathsf{tp}^{\mathcal{M}}(e_{s(i)}) \in U_j$ if and only if $i = j$. Thus there are pairwise contradictory formulas $\{\gamma_i : i \in \omega\}$ (γ_i corresponds to U_i) such that $\|\gamma_i\| \subseteq \|\partial_E \varphi\|$ and $\gamma_i \in \mathsf{tp}^{\mathcal{M}}(e_{s(i)})$. Then $\mathsf{CB}(\partial_E \varphi) > 0$ which is a contradiction.

Note that here the γ_i's are parameter-free formulas. □

Lemma 4.10 *M_ℓ-atoms cover $M_{\ell+1} \setminus M_\ell$ for all $\ell \in z$, i.e., every element $m \in M_{\ell+1} \setminus M_\ell$ is contained in a (unique) M_ℓ-atom. In general, if E is an infinite, definable subset of $\mathcal{M}$ then each $a \in M$ is contained in a (unique) E-atom.*

Proof. Since $\mathcal{M}$ is uncountable and $\aleph_1$-categorical it is prime over M_ℓ, hence atomic over M_ℓ (cf. [6, Theorem 6.1.17]). Consequently, only isolated types are realized. Therefore for all $m \in M_{\ell+1}$ the type $\mathsf{tp}^{\mathcal{M}}(m/M_\ell)$ is isolated by some formula φ_m. Clearly φ_m defines an M_ℓ-atom in which m is contained.

The proof of the general statement of the lemma is the same: $\mathcal{M}$ is atomic over E and then for all $a \in M$ the type $\mathsf{tp}^{\mathcal{M}}(a/E)$ is isolated by an appropriate formula. □

Lemma 4.11 *Let E be an infinite, definable subset of $\mathcal{M}$. Then there exists a finite set Γ of formulas such that any E-atom can be defined by a formula $\psi \in \Gamma$. In more detail, if $\varphi(x, \bar{e})$ defines an E-atom in $\mathcal{M}$, then $\|\varphi(x, \bar{e})\|^{\mathcal{M}} = \|\psi(x, \bar{e}')\|^{\mathcal{M}}$ for some formula $\psi(x, \bar{y}) \in \Gamma$ and parameters $\bar{e}' \in E$.*

Proof. Suppose the contrary. Then for all finite Γ there is an E-atom which cannot be defined by a formula from Γ, in particular, there is an element a_Γ such that whenever $\psi(v, \bar{e})$ defines an E-atom, where $\psi \in \Gamma$ and $\bar{e} \in E$ then $a_\Gamma \notin \|\psi(v, \bar{e})\|^{\mathcal{M}}$.

Since E is definable and $\mathcal{M}$ has an atom defining schema, this fact can be expressed by a first order formula. In fact, the formula

$$\theta_\Gamma(v) = \bigwedge_{\psi \in \Gamma} \forall \bar{e}\big(E(\bar{e}) \wedge \partial_E \psi(\bar{e}) \to \neg\psi(v, \bar{e})\big)$$

is realized by a_Γ.

Therefore the set $H = \{\theta_\Gamma : \Gamma \in [\mathsf{Form}]^{<\omega}\}$ is finitely satisfiable and since $\mathcal{M}$ is uncountable and $\aleph_1$-categorical it is saturated so H is realized by some $a \in M$. But then a cannot be contained in any E-atom which contradicts to the second claim of Lemma 4.10. □

Lemma 4.12 *The action of the group $\mathsf{Gal}(A, M_\ell)$ is regular (in other words, $\mathsf{Gal}(A, M_\ell)$ is sharply transitive) for each $\ell \in z$, i.e., if A is an M_ℓ-atom and $a, b \in A$ then there is a unique $g \in \mathsf{Gal}(A, M_\ell)$ such that $g(a) = b$.*

Proof. The group $G = \mathsf{Gal}(A, M_\ell)$ acts transitively on A because A is an M_ℓ-atom. Suppose $g(a) = h(a) = b$ for some elements $g, h \in G$ and $a, b \in A$. We shall prove $g = h$. Consider the set

$$H = \{x \in A : g^{-1}h(x) = x\}.$$

Then $a \in H$, so $H \neq \varnothing$. But A is an M_ℓ-atom and H is definable over A (in fact, the previous line can be turned into a formal definition of H as both A and the action of G is definable). It follows that $H = A$, whence $g^{-1}h = \mathsf{id}$, consequently $g = h$. □

If $A \subseteq M$ is a subset and $\bar{d} \in M \setminus A$ is a *finite* set of parameters then by $\Theta(\bar{d})$ we denote the equivalence relation on A where

$$(a, b) \in \Theta(\bar{d}) \text{ if and only if } \mathsf{tp}^{\mathcal{M}}(a/\bar{d}) = \mathsf{tp}^{\mathcal{M}}(b/\bar{d}).$$

$\Theta(\bar{d})$ is called a *cut* with parameters $\bar{d}$. By a *partition* of $\Theta(\bar{d})$ we understand an equivalence class of it. $\Theta(\bar{d}')$ is defined to be a *refinement* of $\Theta(\bar{d})$ iff each partition of the prior is contained in a partition of the latter; we denote this fact by

$$\Theta(\bar{d}') \leq \Theta(\bar{d}).$$

Clearly, if $\bar{d} \subseteq \bar{d}'$ then $\Theta(\bar{d}')$ is a refinement of $\Theta(\bar{d})$. We say $\Theta(\bar{d})$ is *minimal* if no further refinement can be made by increasing $\bar{d}$, i.e., for all $\bar{d}' \supseteq \bar{d}$ we have $\Theta(\bar{d}') = \Theta(\bar{d})$.

If A happens to be an M_ℓ-atom then for each cut $\Theta(\bar{d})$ we define $G(\bar{d})$ to be the subgroup of $\mathsf{Gal}(A, M_\ell)$ containing those permutations of $\mathsf{Gal}(A, M_\ell)$ which preserve each partitions of $\Theta(\bar{d})$.

Lemma 4.13 *Every M_ℓ-atom has minimal cuts, in more detail, if A is an M_ℓ-atom, then there exists a finite $\bar{d} \in M \setminus A$ such that $\Theta(\bar{d})$ is minimal.*

Proof. Let A be an M_ℓ-atom defined by the formula ψ with parameters $\bar{e} \in M_\ell$. Starting from $\bar{d}_0 = \bar{e}$ we build a chain of refinements

$$\Theta(\bar{d}_0) \gneqq \Theta(\bar{d}_1) \gneqq \ldots \gneqq \Theta(\bar{d}_i) \gneqq \ldots,$$

in such a way that $\bar{d}_i \subsetneq \bar{d}_j$ for all $i < j$.

Claim. For any finite $\bar{d}$ containing $\bar{e}$, partitions of $\Theta(\bar{d})$ and orbits of $G(\bar{d})$ coincide. In other words, the following are equivalent:

(i) $\mathsf{tp}^{\mathcal{M}}(a/\bar{d}) = \mathsf{tp}^{\mathcal{M}}(b/\bar{d})$;
(ii) a and b are in the same orbit according to the action of $G(\bar{d})$.

P r o o f o f c l a i m. Direction (ii)⇒(i) is easy, so we prove (i)⇒(ii). Assume (i) holds. By saturation of $\mathcal{M}$ there exists an automorphism $\alpha \in \mathsf{Aut}(\mathcal{M})$ which fixes $\bar{d}$ and maps a onto b. Then $\alpha \restriction A$ is M_ℓ-elementary because of the following. Let $x \in A$ and observe that $\alpha(A) = A$ because $\bar{e} = \bar{d}_0 \subseteq \bar{d}$ is fixed by α. Therefore, since A is an M_ℓ-atom, $\mathsf{tp}(x/M_\ell) = \mathsf{tp}(\alpha(x)/M_\ell)$. Hence $\alpha \restriction A \in G(\bar{d})$. □

We recall that by [6, Theorem 7.1.2] any descending chain of definable subgroups of an $\aleph_0$-stable group is of finite length. We claim that $G(\bar{d})$ is a definable subgroup of $\mathsf{Gal}(A, M_\ell)$ (which is $\aleph_0$-stable since it is definable in $\mathcal{M}$). For a formula ψ let $C_\psi(\bar{d})$ be the subgroup defined as

$$C_\psi(\bar{d}) = \big\{g \in \mathsf{Gal}(A, M_\ell) : \forall a \in A\ (\mathcal{M} \models \psi(a, \bar{d}) \longleftrightarrow \psi(g(a), \bar{d}))\big\}.$$

Then

$$G(\bar{d}) = \bigcap_\psi C_\psi(\bar{d}).$$

This intersection gives rise to a chain of definable subgroups which must stop after finitely many steps. Consequently, $G(\bar{d})$ can be defined using those finitely many formulas appeared in the chain.

It is easy to see that if $\Theta(\bar{d}_i) \gneqq \Theta(\bar{d}_j)$ is a proper refinement, then $G(\bar{d}_i) \gneqq G(\bar{d}_j)$ (cf. the definition of $G(\bar{d})$ and the auxiliary claim), and we just have seen that each group $G(\bar{d})$ is a definable subgroup of $\mathsf{Gal}(A, M_\ell)$. Thus for our chain of refinements $\Theta(\bar{d}_0) \gneqq \Theta(\bar{d}_1) \gneqq \dots$ there exist a corresponding (proper) descending chain of subgroups

$$\mathsf{Gal}(A, M_\ell) = G(\bar{d}_0) \gneqq G(\bar{d}_1) \gneqq \dots \gneqq G(\bar{d}_i) \gneqq \dots .$$

Again, by of [6, Theorem 7.1.2] any descending chain of definable subgroups of an $\aleph_0$-stable group is of finite length, hence, our chain of cuts above stops in finitely many steps. The last member of the chain is minimal. □

Lemma 4.14 *Let A be an M_ℓ-atom and let $\Theta(\bar{d})$ be a minimal cut with the corresponding subgroup $G = G(\bar{d})$. Then G has finitely many orbits, or equivalently, the cut is finite: it has finitely many partitions.*

P r o o f. Since $\bar{d}$ is finite, by $\aleph_0$-stability there are at most $\aleph_0$ many types over $\bar{d}$, hence, applying the Auxiliary Claim in the proof of Lemma 4.13 above, we get that G has at most $\aleph_0$ many orbits. Suppose, seeking a contradiction, that G has infinitely many orbits, say $\langle O_i : i \in \omega\rangle$. For each i fix $o_i \in O_i$ and let $\varphi_i(v)$ be the formula expressing "$v \in A$, but $v \notin O_i$". Using the parameter o_i and that $\mathsf{Gal}(A, M_\ell)$ is definable, φ_i is indeed a first order formula. Then $\{\varphi_n : n \in \omega\}$ is finitely satisfiable, hence by $\aleph_1$-saturation of $\mathcal{M}$ it can be realized. But this is a contradiction, therefore G has finitely many orbits. □

Let us introduce the finitary analogue dcl_Γ of dcl, in a similar spirit as we defined acl_Γ (in our investigations below the parameter Γ will be a finite set of formulas).

Definition 4.15 If $\mathcal{M}$ is a structure $X \subseteq M$ and Γ is a set of formulas then by $\mathsf{dcl}_\Gamma^{\mathcal{M}}(X)$ we understand those points of $\mathsf{dcl}^{\mathcal{M}}(X)$ which are witnessed by a formula in Γ, i.e.,

$$\mathsf{dcl}_\Gamma^{\mathcal{M}}(X) = \{a \in M : \mathcal{M} \models \exists! v\varphi(v, \bar{x}) \wedge \varphi(a, \bar{x}) \text{ for some } \bar{x} \in X \text{ and } \varphi \in \Gamma\}.$$

We stress the difference between the definitions of dcl_Γ and acl_Γ.

Lemma 4.16 *Suppose $g = \langle g_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$ is a decomposable elementary mapping. Then there exists a decomposable elementary mapping $g^+ = \langle g_n^+ : n \in \omega\rangle/\mathcal{F}$ extending g such that* $\mathrm{dom}(g^+) \supseteq \mathsf{dcl}(\mathrm{dom}(g))$.

We note that $\mathsf{dcl}(\mathrm{dom}(g))$ is not necessarily decomposable.

P r o o f. Our plan is to find two sequences Γ_n and Φ_n of formulas in such a manner that (1) we can extend g_n to g_n^+ defined on $\mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$ so that (2) this extension is Φ_n-elementary and (3) for any formulas φ and ρ we have $\{n : \varphi \in \Gamma_n\} \in \mathcal{F}$ and $\{n : \rho \in \Phi_n\} \in \mathcal{F}$. Then by (3), because

$$\prod_{n\in\omega} \mathsf{dcl}_{\Gamma_n}\mathrm{dom}(g_n)/\mathcal{F} \ \supseteq\ \mathsf{dcl}(\mathrm{dom}(g)),$$

we get the desired decomposable elementary mapping extending g by setting

$$g^+ = \langle g_n^+ : n \in \omega\rangle / \mathcal{F}.$$

(1) To obtain an extension, for each $a \in \mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$ we have to find a pair $b \in B_n$ which will be the g_n^+-image of a. Observe that the fact that a belongs to $\mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$ can be expressed by a first order formula $\varphi \in \Gamma_n$ with parameters $\bar{y} \in \mathrm{dom}(g_n)$:

$$\mathcal{A}_n \models \exists! x\varphi(x, \bar{y}) \wedge \varphi(a, \bar{y}).$$

In order to find the pair b for a we have to ensure that there is a unique element $b \in B_n$ realizing $\varphi(x, g_n(\bar{y}))$. This means that g_n has to preserve $\exists! x\varphi(x, \bar{y})$ and consequently we shall define Γ_n as the set of those formulas φ for which g_n preserves $\exists! x\varphi(x, \bar{y})$.

(2) We would like g_n^+ to be Φ_n-elementary. We shall see that for a formula ρ there exists another formula (or rather set of formulas) ρ^{Γ_n} so that if g_n preserves ρ^{Γ_n} then g_n^+ preserves ρ (Γ_n is in the superscript because g_n^+ depends on Γ_n). Therefore we collect in Φ_n those formulas ρ for which ρ^{Γ_n} is preserved by g_n.

Let us see the proof in more detail. Let $\rho(x_0, \dots, x_n)$ be any formula and let Φ be a finite set of formulas. We write

$$\rho^\Phi = \left\{\forall x_0 \dots \forall x_n (\varphi_{i_0}(x_0, \bar{y}_0) \wedge \dots \wedge \varphi_{i_n}(x_n, \bar{y}_n) \to \rho(x_0, \dots, x_n)) : \{\varphi_{i_0}, \dots, \varphi_{i_n}\} \subseteq \Phi\right\}.$$

Then ρ^Φ is a finite set.

Let us now define the sets Γ_n and Φ_n as follows.

$$\Gamma_n = \left\{\varphi(x, \bar{y}) : g_n \text{ preserves } \exists! x\varphi(x, \bar{y})\right\}, \text{ and}$$

$$\Phi_n = \left\{\rho : g_n \text{ preserves } \rho^{\Gamma_n}\right\}.$$

Then it is easy to see (cf., e.g., Proposition 2.2) that for any formulas φ and ρ we have

$$\{n : \varphi \in \Gamma_n\} \in \mathcal{F} \quad \text{and} \quad \{n : \rho \in \Phi_n\} \in \mathcal{F}.$$

Now we claim that g_n can be extended to g_n^+, defined on $\mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$ in such a way that g_n^+ is Φ_n-elementary. First we give the extension. If $a \in \mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$ then there is a formula $\varphi \in \Gamma_n$ witnessing this: there are parameters $\bar{y} \in \mathrm{dom}(g_n)$ such that

$$\mathcal{A}_n \models \exists! x\varphi(x, \bar{y}) \wedge \varphi(a, \bar{y}).$$

Since $\varphi \in \Gamma_n$, g_n preserves $\exists! x\varphi(x, \bar{y})$ thus we have $\mathcal{B}_n \models \exists! x\varphi(x, g_n(\bar{y}))$. Let $b_a \in B_n$ be this unique element and put

$$g_n^+ = g_n \cup \left\{\langle a, b_a\rangle : a \in \mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))\right\}.$$

We claim that g_n^+ is Φ_n-elementary: if g_n preserves ρ^{Γ_n} then g_n^+ preserves ρ. For, suppose $\mathcal{A}_n \models \rho(\bar{a})$ for $\bar{a} = \langle a_0, \dots, a_k\rangle \in \mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$. Then there are formulas $\varphi_i \in \Gamma_n$ and parameters $\bar{y}_i \in \mathrm{dom}(g_n)$ witnessing $\bar{a} \in \mathsf{dcl}_{\Gamma_n}(\mathrm{dom}(g_n))$, particularly,

$$\mathcal{A}_n \models \varphi_0(a_0, \bar{y}_0) \wedge \dots \wedge \varphi_k(a_k, \bar{y}_k),$$

and

$$\mathcal{A}_n \models \exists! x_0\varphi_0(x_0, \bar{y}_0) \wedge \dots \wedge \exists! x_k\varphi_k(x_k, \bar{y}_k),$$

hence

$$\mathcal{A}_n \models \forall x_0 \dots \forall x_k \big(\varphi_0(x_0, \bar{y}_0) \wedge \dots \wedge \varphi_k(x_k, \bar{y}_k) \to \rho(\bar{x})\big).$$

But the formula

$$\forall x_0 \dots \forall x_k \big(\varphi_0(x_0, \bar{y}_0) \wedge \dots \wedge \varphi_k(x_k, \bar{y}_k) \to \rho(\bar{x})\big)$$

is an element of Φ_n, therefore it is preserved by g_n, thus

$$\mathcal{B}_n \models \forall x_0 \dots \forall x_k \big(\varphi_0(x_0, g_n(\bar{y}_0)) \wedge \dots \wedge \varphi_k(x_k, g_n(\bar{y}_k)) \to \rho(\bar{x})\big).$$

Finally, we have chosen $\bar{b}_{\bar{a}} = \bar{b} = \langle b_0, \ldots, b_k \rangle$ in a way that $\bar{b}$ is the unique element satisfying

$$\mathcal{B}_n \models \varphi_0(b_0, g_n(\bar{y}_0)) \wedge \ldots \wedge \varphi_k(b_k, g_n(\bar{y}_k)),$$

consequently

$$\mathcal{B}_n \models \rho(\bar{b}).$$

□

Lemma 4.17 *Suppose* $g = \langle g_n : n \in \omega \rangle / \mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$ *is a decomposable elementary mapping with* $\mathrm{dom}(g_n) = M_\ell^{\mathcal{A}_n}$ *and* $\mathrm{ran}(g_n) \subseteq M_\ell^{\mathcal{B}_n}$ *for a fixed* $0 \le \ell < z-1$, *where* $\mathcal{A}_n$ *and* $\mathcal{B}_n$ *are finite,* Δ_n*–elementary substructures of* $\mathcal{M}$. *Then* g *can be extended to a decomposable elementary mapping* $h = \langle h_n : n \in \omega \rangle / \mathcal{F}$ *with* $\mathrm{dom}(h_n) = M_{\ell+1}^{\mathcal{A}_n}$ *and* $\mathrm{ran}(h_n) \subseteq M_{\ell+1}^{\mathcal{B}_n}$. *Particularly,* $\left|M_{\ell+1}^{\mathcal{A}_n}\right| \le M_{\ell+1}^{\mathcal{B}_n}$.

As in Propositions 4.6 and 4.8, we note that if $\left|M_{\ell+1}^{\mathcal{A}_n}\right| = \left|M_{\ell+1}^{\mathcal{B}_n}\right|$ then $\mathrm{ran}(h_n) = M_{\ell+1}^{\mathcal{B}_n}$.

Proof. Let us denote by $\mathcal{A}$ and $\mathcal{B}$ the structures $\prod_{n\in\omega} \mathcal{A}_n/\mathcal{F}$ and $\prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}$, respectively. By a slight abuse of notation (or rather for the sake of keeping superscripts in a bearable level) we shall have $\mathcal{A} = \mathcal{M}$ in mind. Since $\mathcal{A} \equiv \mathcal{M}$ everything which was said about $\mathcal{M}$ is true for $\mathcal{A}$. So from now on every such notion like M_ℓ, $\mathsf{Gal}(A, M_\ell)$, atom, which are definable, are to be meant in $\mathcal{A}$. E.g. from now on $\mathsf{Gal}(A, M_\ell)$ denotes $\mathsf{Gal}^{\mathcal{A}}(A^{\mathcal{A}}, M_\ell^{\mathcal{A}})$, etc. Note that here A is an M_ℓ-atom and *not* the universe of $\mathcal{A}$.

Using Lemma 4.16 there is a decomposable elementary extension $g^+ = \langle g_n^+ : n \in \omega \rangle / \mathcal{F}$ of g such that $\mathrm{dom}(g^+) \supseteq \mathsf{dcl}(\mathrm{dom}(g))$. In order to keep notation simpler, from now on denote g^+ by g.

We show first that there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ which is an extension of g (but f is not necessarily decomposable). By $\aleph_0$-stability, there are elementary substructures $\mathcal{A}^*$ and $\mathcal{B}^*$ of $\mathcal{A}$ and $\mathcal{B}$, respectively which are constructible over $\mathrm{dom}(g)$ and $\mathrm{ran}(g)$ (cf. [6, Lemma 6.4.2] for the existence of constructible elementary submodels). Because of $M_\ell^{\mathcal{A}}$ is infinite, definable and is contained in $\mathrm{dom}(g)$, by a standard two cardinals theorem (cf., e.g., [1, Theorem 3.2.9]) $\mathcal{A}^* = \mathcal{A}$ and similarly, $\mathcal{B}^* = \mathcal{B}$. Since they are constructible, they are atomic over $M_\ell^{\mathcal{A}}$ and hence there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ extending g.

By Lemma 4.10, M_ℓ-atoms cover $M_{\ell+1} \setminus M_\ell$, so fix an enumeration of M_ℓ-atoms $\langle A^\lambda : \lambda < \kappa \rangle$. By Lemma 4.13 for all atom A^λ there is a minimal cut Θ^λ and by Lemma 4.14 this cut has finitely many partitions, say $\mathsf{n}(\lambda)$ many. For each $\lambda < \kappa$ and $i < \mathsf{n}(\lambda)$ let us adjoin a new relation symbol $R_{\lambda,i}$ to our language and interpret it in $\mathcal{A}$ as the corresponding partition of A_λ. So $R_{\lambda,i}^{\mathcal{M}}$ is the i^{th} partition of the λ^{th} atom. We denote this extended language by L^+ and let us denote the set of new relation symbols by $\mathcal{R}$:

$$\mathcal{R} = \{R_{\lambda,i} : \lambda < \kappa, i < \mathsf{n}(\lambda)\}.$$

Each $R \in \mathcal{R}$ is a partition of a minimal cut of an atom, hence R is definable by a formula with parameters. It follows that each $R \in \mathcal{R}$ is decomposable (by Łoś's Lemma) and so it is meaningful to speak about $R^{\mathcal{A}_n}$ for $R \in \mathcal{R}$ and $n \in \omega$.

Define the interpretation of these relations in $\mathcal{B}$ as

$$R_{\lambda,i}^{\mathcal{B}} = f[R_{\lambda,i}^{\mathcal{A}}],$$

for all λ and i. Observe that f is an elementary mapping in the extended language L^+ because it is an isomorphism. In addition, a restriction of an elementary mapping is still elementary, therefore g is also elementary in the language L^+.

For a formula $\varphi(v, \bar{y})$ let

$$\varphi' = \{\forall v(R(v) \to \varphi(v, \bar{y})) : R \in \mathcal{R}\} \text{ and let}$$
$$\varphi^+ = \{\forall \bar{y}(\exists x(R(x) \wedge \varphi(x, \bar{y})) \to \forall x(R(x) \to \varphi(x, \bar{y}))) : R \in \mathcal{R}\}.$$

We emphasize that φ' and φ^+ are possibly infinite sets of formulas. Observe first that $\mathcal{A}, \mathcal{B} \models \varphi^+$ for all formula φ and thus by Łoś's lemma for any $\vartheta \in \varphi^+$ we have

$$\{n \in \omega : \mathcal{A}_n, \mathcal{B}_n \models \vartheta\} \in \mathcal{F}.$$

What is more, we claim that formulas in φ^+ are "simultaneously" decomposable:

Claim 4.18 *For any formula φ the following hold:*

$$\{n \in \omega : \mathcal{A}_n,\ \mathcal{B}_n \models \varphi^+\} \in \mathcal{F}.$$

P r o o f. We present the proof for $\{n \in \omega : \mathcal{A}_n \models \varphi^+\} \in \mathcal{F}$, the similar statement for the $\mathcal{B}_n$'s can be done the very same way.

Suppose the contrary, i.e., for almost all $n \in \omega$ there is some $R_n \in \mathcal{R}$ and $\bar{y}_n$ such that

$$R_n^{\mathcal{A}_n} \cap \|\varphi(v, \bar{y}_n)\|^{\mathcal{A}_n} \neq \varnothing \quad \text{and} \quad R_n^{\mathcal{A}_n} \setminus \|\varphi(v, \bar{y}_n)\|^{\mathcal{A}_n} \neq \varnothing.$$

According to Lemmas 4.11 and 4.9, there is a finite set $S \subseteq \mathsf{S}(\mathcal{M})$ of types such that if a sequence $\bar{e}$ defines an atom (say, with a formula $\psi \in \Gamma$, where Γ comes from Lemma 4.11), then $\mathtt{tp}(\bar{e}) \in S$. Consequently for almost all n, R_n's are partitions of a minimal cut of the same type of atom, and since every minimal cut has finitely many partitions, R_n's are defined with the same formula ϑ almost everywhere (of course with potentially different parameters). So for some sequences $\bar{c}_n$ in a big set of indices we have

$$\|\vartheta(v, \bar{c}_n)\|^{\mathcal{A}_n} \cap \|\varphi(v, \bar{y}_n)\|^{\mathcal{A}_n} \neq \varnothing \quad \text{and} \quad \|\vartheta(v, \bar{c}_n)\|^{\mathcal{A}_n} \setminus \|\varphi(v, \bar{y}_n)\|^{\mathcal{A}_n} \neq \varnothing.$$

Considering the ultraproduct we get

$$\|\vartheta(v, \bar{c})\|^{\mathcal{A}} \cap \|\varphi(v, \bar{y})\|^{\mathcal{A}} \neq \varnothing \quad \text{and} \quad \|\vartheta(v, \bar{c})\|^{\mathcal{A}} \setminus \|\varphi(v, \bar{y})\|^{\mathcal{A}} \neq \varnothing,$$

which is impossible, because by construction $\|\vartheta(v, \bar{c})\|$ defines a partition of a minimal cut. □

Recall that by "g preserves φ" we mean that for all $\bar{d} \in \mathrm{dom}(g)$ the following is true:

$$\text{if } \mathcal{A} \models \varphi(\bar{d}) \quad \text{then} \quad \mathcal{B} \models \varphi(g(\bar{d})).$$

Similarly, by "g preserves φ'" we mean that all the formulas in φ' are preserved by g. For $\varphi(v, \bar{y}) \in \mathsf{Form}$ we define $I(\varphi) \in \mathcal{F}$ follows.

$$I(\varphi) = \{n \in \omega : g_n \text{ preserves } \{\varphi\} \cup \varphi' \text{ and } \mathcal{A}_n, \mathcal{B}_n \models \varphi^+\}$$

We claim that $I(\varphi) \in \mathcal{F}$. On the one hand, by Proposition 2.2 we have $\{n :\ g_n \text{ preserves } \varphi\} \in \mathcal{F}$ and on the other hand we just have showed (Claim 4.18) that $\{n \in \omega : \mathcal{A}_n,\ \mathcal{B}_n \models \varphi^+\} \in \mathcal{F}$. So it remains to prove $\{n :\ g_n$ preserves $\varphi'\} \in \mathcal{F}$. Similarly as we showed that formulas of φ^+ are simultaneously decomposable, it is also true that

$$\{n \in \omega : g_n \text{ preserves } \vartheta \text{ for all } \vartheta \in \varphi'\} \in \mathcal{F}. \tag{$\star$}$$

To see this, suppose, seeking a contradiction, that for almost all n there is $\vartheta_n \in \varphi'$ which is not preserved by g_n. In more detail, this means that g_n does not preserve a formula of the form

$$\vartheta_n = \forall v(R_n(v) \to \varphi(v, \bar{y}_n)).$$

In a similar manner as above, by Lemmas 4.11 and 4.9 there is a big set of indices such that R_n's are defined with the same parametric formula ϑ. Then considering the ultraproduct we get that f, which is an extension of g, doesn't preserve the formula

$$\forall v(\vartheta(v) \to \varphi(v, \bar{y})).$$

But this is impossible because f is an isomorphism. So ($\star$) above has been established.

Next we define sets ∇_n of formulas for $n \in \omega$ as follows:

$$\nabla_n = \{\varphi : n \in I(\varphi)\}.$$

Then as we saw $I(\varphi) \in \mathcal{F}$ and for all formulas φ we have

$$\{n \in \omega : \varphi \in \nabla_n\} \in \mathcal{F}.$$

We divide the rest of the proof into two steps. In the first step, we extend g so that it will meet every atom in at least one point, then in the second step we continue the extension to the remaining parts of the atoms.

Step 1: We proceed by transfinite recursion. Let $g_n^0 = g_n$ for all $n \in \omega$. We construct a sequence of mappings $\langle g_n^\lambda : n \in \omega, \lambda \le \kappa \rangle$ in such a way that the following stipulations hold.

(S1) $g^\lambda = \langle g_n^\lambda : n \in \omega \rangle / \mathcal{F}$ is elementary;
(S2) $g_n^\varepsilon \subseteq g_n^\delta$ for all $\varepsilon \le \delta \le \kappa$ and $n \in \omega$;
(S3) $A^\varepsilon \cap \mathrm{dom}(g^\lambda) \neq \varnothing$ for all $\varepsilon < \lambda$;
(S4) g_n^λ is ∇_n-elementary for $\lambda \le \kappa$ and $n \in \omega$.

Note that (S1) is a consequence of (S4) and that g^0 satisfies (S1)–(S4), particularly, (S4) holds for g_n^0 by definition of ∇_n: $\varphi \in \nabla_n$ iff $n \in I(\varphi)$ and then g_n^0 preserves φ. Suppose that g_n^ε has already been defined for $n \in \omega$ and $\varepsilon < \delta \le \kappa$.

If δ is limit then, similarly as in the proof of Proposition 4.8, we take the coordinatewise union, i.e., $g_n^\delta = \bigcup_{\varepsilon<\delta} g_n^\varepsilon$ for $n \in \omega$.

Suppose δ is a successor, say $\delta = \varepsilon + 1$, and $A^\delta \cap \mathrm{dom}(g^\varepsilon) = \varnothing$. First, observe that A^δ is definable by parameters from M_ℓ and g^ε is elementary, hence $(A^\delta)^{\mathcal{B}} \cap \mathrm{ran}(g^\varepsilon) = \varnothing$ as well. Pick an arbitrary $a \in A^\delta$. There is a unique $R \in \mathcal{R}$ such that $a \in R^{\mathcal{A}}$. Since $R^{\mathcal{A}}$ is non-empty and f is an isomorphism, $R^{\mathcal{B}}$ is also non-empty. So pick any $b \in R^{\mathcal{B}}$. Note that $R^{\mathcal{A}} \subseteq A^\delta$ and hence $\mathcal{A} \models \forall v(R(v) \to A^\delta(v))$ (and similarly with $\mathcal{B}$). Take representatives $a = \langle a_n : n \in \omega \rangle / \mathcal{F}$ and $b = \langle b_n : n \in \omega \rangle / \mathcal{F}$. If

$$I_{\notin} = \{ n \in \omega : a_n \notin \mathrm{dom}(g_n^\varepsilon) \text{ and } b_n \notin \mathrm{ran}(g_n^\varepsilon) \},$$
$$I_{\mathcal{R}} = \{ n \in \omega : a_n \in R^{\mathcal{A}_n}, b_n \in R^{\mathcal{B}_n} \text{ and } R^{\mathcal{A}_n} \subseteq (A^\delta)^{\mathcal{A}_n}, R^{\mathcal{B}_n} \subseteq (A^\delta)^{\mathcal{B}_n} \}$$

then clearly $I_{\notin} \cap I_{\mathcal{R}} \in \mathcal{F}$. Set $g^\delta = \langle g_n^\delta : n \in \omega \rangle / \mathcal{F}$ where

$$g_n^\delta = \begin{cases} g_n^\varepsilon \cup \{\langle a_n, b_n \rangle\} & \text{if } n \in I_{\notin} \cap I_{\mathcal{R}}, \\ g_n^\varepsilon & \text{otherwise.} \end{cases}$$

We claim that g^δ satisfies properties (S1)–(S4). Here (S2) and (S3) are obvious. Moreover, as we already mentioned, (S1) is a consequence of (S4), therefore it is enough to deal with the latter one.

Let $n \in I_{\notin} \cap I_{\mathcal{R}}$ be arbitrary but fixed, and suppose $\varphi(v, \bar{y}) \in \nabla_n$. We have to prove that g_n^δ preserves φ.

Since $\varphi \in \nabla_n$ we have $n \in I(\varphi)$ hence, g_n preserves φ', in particular, g_n preserves $\forall v(R(v) \to \varphi(v, \bar{y}))$. By construction $\mathcal{A}_n, \mathcal{B}_n \models \varphi^+$. Suppose $a_n \in \|\varphi(v, \bar{d})\|^{\mathcal{A}_n}$ for some $\bar{d} \in \mathrm{dom}(g_n)$. Then because $\mathcal{A}_n \models \varphi^+$ and $a_n \in R^{\mathcal{A}_n}$ we get

$$\mathcal{A}_n \models \forall v(R(v) \to \varphi(v, \bar{d})).$$

This last formula belongs to φ', hence it is preserved by g_n, therefore

$$\mathcal{B}_n \models \forall v(R(v) \to \varphi(v, g_n(\bar{d}))).$$

Since $b_n \in R^{\mathcal{B}_n}$, we get $b_n \in \|\varphi(v, g_n(\bar{d}))\|^{\mathcal{B}_n}$, consequently g_n^δ preserves φ, as desired.

Step 2: What we get so far from the transfinite recursion is a function g^κ satisfying (S1)–(S4) above. We claim that every atom A^λ is contained in $\mathsf{dcl}(\mathrm{dom}(g^\kappa))$. To prove this let A be an M_ℓ-atom and let $a \in A \cap \mathrm{dom}(g^\kappa)$. Such an element a exists by (S3). Now, by Lemma 4.12 (sharp transitivity of $\mathsf{Gal}(A, M_\ell)$) for any $x \in A$ there is a unique group element $g_x \in \mathsf{Gal}(A, M_\ell)$ with $g_x(a) = x$. Since $\mathsf{Gal}(A, M_\ell) \subseteq \mathsf{dcl}^{\mathcal{A}^{\mathrm{eq}}}(\mathrm{dom}(g))$ it follows that $g_x \in \mathsf{dcl}^{\mathcal{A}^{\mathrm{eq}}}(\mathrm{dom}(g^\kappa))$, hence $x \in \mathsf{dcl}^{\mathcal{A}}(\mathrm{dom}(g^\kappa))$. Therefore every element of the atom A can be defined from $\mathrm{dom}(g^\kappa)$. Applying Lemma 4.16 to g^κ one can finish the proof.

For completeness we note that $\mathsf{dcl}(\mathrm{dom}(g^\kappa)) = M_{\ell+1}$ which is definable, hence decomposable, cf. the remark before the proof of Lemma 4.16. The last sentence of the statement of Lemma 4.17 follows, because h is a decomposable elementary mapping. □

Theorem 4.19 *Suppose $\mathcal{A}_n$, $\mathcal{B}_n$ are finite Δ_n-elementary substructures of $\mathcal{M}$. Let $g = \langle g_n : n \in \omega \rangle / \mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n / \mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n / \mathcal{F}$ be a decomposable elementary mapping with $\mathrm{dom}(g_n) = M_0^{\mathcal{A}_n}$, $\mathrm{ran}(g_n) \subseteq M_0^{\mathcal{B}_n}$. Then g can be extended to a decomposable elementary embedding.*

We have the usual remark: if we assume $\left|M_\ell^{\mathcal{A}_n}\right| = \left|M_\ell^{\mathcal{B}_n}\right|$ for all $0 \le \ell < z - 1$ and $n \in \omega$, and $\mathrm{ran}(g_n) = M_0^{\mathcal{B}_n}$, then the resulting extension is a decomposable isomorphism.

P r o o f. Straightforward iteration of Lemma 4.17. □

4.3 The general case

We put the result of Subsections 4.1 and 4.2 together. Recall that $\mathcal{M}$ is an uncountable, $\aleph_1$-categorical structure with an atom–defining schema for $\varnothing$-definable infinite relations, having the extension property. Also, we assume that there is a $\varnothing$-definable strongly minimal subset $M_0 \subseteq M$.

As earlier in this chapter, throughout $\langle \Delta_n : n \in \omega \rangle$ is a covering sequence of formulas (Definition 4.1).

Lemma 4.20 *For each $n \in \omega$ let $\mathcal{A}_n$, $\mathcal{B}_n$ be finite, Δ_n-elementary substructures of $\mathcal{M}$. Then for any $k, m \in \omega$ there exists $N \in \omega$ such that $m \le N$ and whenever $n \ge N$ then there is a Δ_m-elementary mapping $g_n : \mathcal{A}_n \to \mathcal{B}_n$ such that* $\mathrm{dom}(g_n) \subseteq M_0^{\mathcal{A}_n}$, $\mathrm{ran}(g_n) \subseteq M_0^{\mathcal{B}_n}$ *and* $|\mathrm{dom}(g_n)| \ge k$.

P r o o f. Let $k, m \in \omega$ be fixed and for each $n \in \omega$ let $\bar{a}_n \in M_0^{\mathcal{A}_n}$ and $\bar{b}_n \in M_0^{\mathcal{B}_n}$ be bases in $\mathcal{A}_n$ and $\mathcal{B}_n$, respectively. We emphasize that acl and algebraic dependence is always computed in the infinite structure $\mathcal{M}$. We distinguish three cases.

Case 1: Suppose $I = \{n \in \omega : |\bar{a}_n| < k\}$ is infinite. Observe that $A_n \cap M_0 = M_0^{\mathcal{A}_n}$ for large enough n, because M_0 is definable by an element of Δ_n. Since $\sup\{|A_n \cap M_0| : n \in \omega\}$ is infinite, it follows that $\sup\{|\mathsf{acl}(\bar{a}_n) \cap M_0| : n \in \omega\}$ is infinite, as well. Hence, for all but finitely many $n \in I$ (denote this infinite set by I') there exists $\gamma(n) \in \omega$ with

$$\left|\mathsf{acl}_{\Delta_{\gamma(n)}}(\bar{a}_n) \cap M_0\right| \ge k.$$

Let $N_0 \in I'$ and let $N \ge \max\{\gamma(N_0), m\}$ be such that M_0 is definable by a formula in Δ_N and the existential closure of the type

$$p = \mathsf{tp}_{\Delta_m}\big(\mathsf{acl}_{\Delta_{\gamma(N_0)}}(\bar{a}_{N_0}) \cap M_0\big)$$

is in Δ_N. By the existential closure of the type p we understand the set which consists of the existential closures of formulas in p, in symbols:

$$\exists \bar{x}\, p = \big\{\exists \bar{x} \varphi : \ \varphi \in p,\ \exists \bar{x} \varphi \text{ has no free variables}\big\}.$$

Clearly $\exists \bar{x}\, p$ is a finite set of closed formulas and $\exists \bar{x}\, p \in \Delta_N$. Thus if $n \ge N$ then we also have $\exists \bar{x}\, p \in \Delta_n$ because the Δ_n's increase. As $\mathcal{A}_n$ and $\mathcal{B}_n$ are Δ_n-elementary substructures it follows that $\mathcal{A}_n, \mathcal{B}_n \models \exists \bar{x}\, p$ provided $n \ge N$. Consequently, p can be realized in $\mathcal{A}_n$ and $\mathcal{B}_n$ for any $n \ge N$. A bijection g_n between these realizations is a Δ_m-elementary mapping, so g_n satisfies the conclusion of the lemma.

Case 2: Suppose $I = \{n \in \omega : |\bar{b}_n| < k\}$ is infinite. Swapping $\mathcal{A}_n$ and $\mathcal{B}_n$, one can apply case one above.

Case 3: Suppose, there is an $N_0 \in \omega$ such that $n \ge N_0$ implies $|\bar{a}_n|, |\bar{b}_n| \ge k$. Then choose N so that $N \ge \max\{N_0, m\}$. If $n \ge N$ then let g_n be a bijection mapping the first k elements of $\bar{a}_n$ onto the first k elements of $\bar{b}_n$. Since $\bar{a}_n$ and $\bar{b}_n$ are bases, $g_n : M_0 \cap A_n \to M_0 \cap B_n$ is an elementary mapping, hence $g_n : A_n \to B_n$ is Δ_m-elementary, as desired.

□

Lemma 4.21 *Suppose $\mathcal{A}_n$ and $\mathcal{B}_n$ are finite, Δ_n-elementary substructures of $\mathcal{M}$ such that $\left|M_0^{\mathcal{A}_n}\right| = \left|M_0^{\mathcal{B}_n}\right|$ for almost all $n \in \omega$. Then $|A_n| = |B_n|$ almost everywhere.*

A converse of this statement is presented in Lemma 4.25.

P r o o f. Suppose, seeking a contradiction, that

$$I = \{n \in \omega : |A_n| < |B_n|\} \in \mathcal{F}. \qquad (*)$$

Let m be arbitrary. Applying Lemma 4.20 with $k = \varepsilon(\Delta_n)$ we get a Δ_m-elementary function

$$g_m : M_0^{\mathcal{A}_{n(m)}} \to M_0^{\mathcal{B}_{n(m)}},$$

where $m \leq n(m) \in I$ such that $|\mathrm{dom}(g_m)| \geq \varepsilon(\Delta_m)$ (where ε comes from Lemma 4.3). Applying Proposition 4.8 to $\mathcal{A}_{n(m)}$ and $\mathcal{B}_{n(m)}$, we obtain a decomposable elementary mapping

$$g^+ = \langle g_m^+ : m \in \omega\rangle/\mathcal{F} : \prod_{m\in\omega} \mathcal{A}_{n(m)}/\mathcal{F} \to \prod_{m\in\omega} \mathcal{B}_{n(m)}/\mathcal{F}$$

with $\mathrm{dom}(g_m^+) = M_0^{\mathcal{A}_{n(m)}}$ and $\mathrm{ran}(g_m^+) = M_0^{\mathcal{B}_{n(m)}}$ (here equality holds because we assumed $|M_0^{\mathcal{A}}| = |M_0^{\mathcal{B}}|$). By Theorem 4.19, g^+ can be extended to a decomposable elementary embedding

$$g^{++} : \prod_{m\in\omega} \mathcal{A}_{n(m)}/\mathcal{F} \to \prod_{m\in\omega} \mathcal{B}_{n(m)}/\mathcal{F}.$$

On the one hand $g^{++}[M_0^{\mathcal{A}}] = M_0^{\mathcal{B}}$, on the other hand, g^{++} is not surjective (this is because g^{++} is decomposable and by the indirect assumption $(*)$). Thus,

$$g^{++}\left[\prod_{m\in\omega} \mathcal{A}_{n(m)}/\mathcal{F}\right] \quad \text{and} \quad \prod_{m\in\omega} \mathcal{B}_{n(m)}/\mathcal{F}$$

forms a Vaughtian pair for the $\aleph_1$-categorical theory of $\mathcal{M}$ – which is a contradiction ($\aleph_1$-categorical theories cannot have Vaughtian pairs, cf. [6, Corollary 4.3.39] or [6, Theorem 6.1.18]). □

Remark 4.22 *If M_0 is strongly minimal, then, by compactness, for all formulas φ there is a natural number $\mathsf{n}(\varphi)$ (not depending on parameters in φ) such that if $M_0 \cap \|\varphi(v,\bar{c})\|$ is infinite then $|M_0 \setminus \|\varphi(v,\bar{c})\|| \leq \mathsf{n}(\varphi)$. This we used once in the proof of Lemma 4.4. Next, we utilize another variant of this idea.*

Lemma 4.23 *Let $\mathcal{M}$ be $\aleph_1$-categorical and let $M_0 \subseteq M$ be a $\varnothing$-definable, strongly minimal subset. Then for all finite set ε of formulas there exists another finite set δ of formulas such that if $\mathcal{A}$ is a δ-elementary substructure of $\mathcal{M}$ and $\varphi \in \varepsilon$, $\bar{c} \in A$ and $M_0 \cap \|\varphi(v,\bar{c})\|^{\mathcal{M}}$ is finite, then $M_0 \cap \|\varphi(v,\bar{c})\|^{\mathcal{M}} \subseteq M_0^{\mathcal{A}}$.*

P r o o f. For all $\varphi \in \varepsilon$ let $\varphi_n(\bar{y})$ denote the next formula:

$$\varphi_n(\bar{y}) = \text{"}\varphi(x,\bar{y}) \text{ has exactly } n \text{ realizations"}.$$

For all fixed $n \in \omega$, φ_n can be made a strict first order formula and it is sometimes denoted as $\exists_n x\varphi(x,\bar{y})$. Let χ be a formula defining M_0. Put

$$\delta = \varepsilon \cup \{\chi\} \cup \{\varphi_n : n \leq \mathsf{n}(\neg\varphi), \varphi \in \varepsilon\}.$$

A simple argument shows that δ fulfills our purposes. □

Lemma 4.24 *For a formula φ, let $\mathsf{n}(\varphi)$ be as in Remark 4.22. For all (large enough) finite set ε of formulas there is another finite set $\delta \supset \varepsilon$ of formulas such that if $\mathcal{A}$ is a δ-elementary substructure of $\mathcal{M}$ with*

$$|M_0^{\mathcal{A}}| > \max\{\mathsf{n}(\varphi) : \varphi \in \delta\}$$

and $\bar{b} \in M_0$ is arbitrary then $A \cup \{\bar{b}\}$ is a universe of an ε-elementary substructure $\mathcal{A}'$ of $\mathcal{M}$ and $\mathcal{A}$ is an ε-elementary substructure of $\mathcal{A}'$.

P r o o f. For a formula $\varphi(v,\bar{y})$ let $\hat{\varphi}$ be the formula expressing

$$\hat{\varphi}(\bar{y}) = \text{"there are at most } \mathsf{n}(\varphi) \text{ many elements} x \text{ of } M_0 \text{ such that } \neg\varphi(x,\bar{y})\text{"}.$$

Since M_0 is definable and $\mathsf{n}(\varphi)$ is finite, this can be made a first order formula for each φ.

For ε let δ be the smallest set of formulas closed under subformulas and containing the union of ε, $\{\hat{\varphi} : \varphi \in \varepsilon\}$ and the set of formulas δ in Lemma 4.23 (corresponding to ε). We prove this choice is suitable. We apply the Łoś-Vaught test. Let $\varphi \in \varepsilon$, $\bar{c} \in A$ and suppose $\varphi(v,\bar{c})$ is realized by $a \in A'$. If $a \in A$ then there is nothing to prove, so assume $a \notin A$. Then by construction $a \in M_0 \setminus A$.

If $M_0 \cap \|\varphi(v, \bar{c})\|^{\mathcal{M}}$ is finite then by Lemma 4.23, $a \in M_0^{\mathcal{A}} \subseteq A$ would follow, which contradicts to $a \in M_0 \setminus A$. So we have $M_0 \cap \|\varphi(v, \bar{c})\|^{\mathcal{M}}$ is infinite. Then, since M_0 is strongly minimal, each but finitely many elements of M_0 realizes $\varphi(v, \bar{c})$. But $\left|M_0^{\mathcal{A}}\right| > \mathsf{n}(\varphi)$ is large enough, consequently there is an $a' \in A$ realizing $\varphi(v, \bar{c})$. This proves that $\mathcal{A}$ is a φ-elementary substructure of $\mathcal{A}'$.

Next, we prove that $\mathcal{A}'$ is an ε-elementary substructure of $\mathcal{M}$. Let $\varphi \in \varepsilon$, $\bar{c} \in A'$ and assume $\mathcal{M} \models \varphi(\bar{c})$. We proceed by induction on $|\bar{c} \setminus A|$.

If $|\bar{c} \setminus A| = 0$ then $\bar{c} \in A$ and since $\mathcal{A}$ is a δ-elementary substructure, it follows that $\mathcal{A} \models \varphi(\bar{c})$. We have already proved that $\mathcal{A}$ is an ε-elementary substructure of $\mathcal{A}'$, hence $\mathcal{A}' \models \varphi(\bar{c})$.

If $|\bar{c} \setminus A| > 0$ then $\bar{c} = d^\frown \bar{c}_0$ for some $d \in \bar{c} \setminus A$, $d \in \bar{b} \subseteq M_0$. We claim that $\mathcal{M} \models \hat{\varphi}(\bar{c}_0)$ which means that there are at most $\mathsf{n}(\varphi)$ many elements x of M_0 such that $\neg\varphi(x, \bar{c}_0)$ holds, i.e., $M_0 \cap \|\neg\varphi(v, \bar{c}_0)\|^{\mathcal{M}} \leq \mathsf{n}(\varphi)$. For if not, using that M_0 is strongly minimal and the definition of $\mathsf{n}(\varphi)$, we get that $M_0 \cap \|\neg\varphi(v, \bar{c}_0)\|^{\mathcal{M}}$ is infinite, consequently $M_0 \cap \|\varphi(v, \bar{c}_0)\|^{\mathcal{M}}$ is finite. Applying Lemma 4.23 we get $M_0 \cap \|\varphi(v, \bar{c}_0)\|^{\mathcal{M}} \subseteq M_0^{\mathcal{A}}$, particularly, $d \in M_0^{\mathcal{A}} \subseteq A$ which is a contradiction.

So far, we got

$$\mathcal{M} \models \hat{\varphi}(\bar{c}_0).$$

Because $\mathcal{A}$ is δ-elementary it follows that

$$\mathcal{A} \models \hat{\varphi}(\bar{c}_0),$$

and by the inductive hypothesis ($|\bar{c}_0| < |\bar{c}|$) we get

$$\mathcal{A}' \models \hat{\varphi}(\bar{c}_0).$$

By Lemma 4.23, if $x \in M_0$ is such that $\mathcal{M} \models \neg\varphi(x, c_0)$, then $x \in A \cap A'$. Therefore $\mathcal{A}' \models \varphi(d, c_0)$, as desired. □

Lemma 4.25 *Suppose for each $n \in \omega$ the finite $\mathcal{A}_n$ and $\mathcal{B}_n$ are equinumerous, Δ_n-elementary substructures of $\mathcal{M}$. Then for all, but finitely many $n \in \omega$ we have*

$$\left|M_0^{\mathcal{A}_n}\right| = \left|M_0^{\mathcal{B}_n}\right|.$$

Proof. Let δ_n be the finite set of formulas guaranteed by Lemma 4.24 for $\varepsilon_n = \Delta_n$. Since the sequence Δ_n is monotone increasing, we may assume, by a possible re-scaling of this sequence, that $\mathcal{A}_n$ and $\mathcal{B}_n$ are also δ_n-elementary substructures of $\mathcal{M}$.

We may suppose, seeking a contradiction, that $\left|M_0^{\mathcal{A}_n}\right| < \left|M_0^{\mathcal{B}_n}\right|$ for all n. For each n chose $\bar{b}_n \in M_0$ such that

$$\left|M_0^{\mathcal{A}_n} \cup \{\bar{b}_n\}\right| = \left|M_0^{\mathcal{B}_n}\right|.$$

Let $\mathcal{A}'_n$ be the substructure in Lemma 4.24 whose underlying set is $M_0^{\mathcal{A}_n} \cup \{\bar{b}_n\}$. Then, by Lemma 4.24, $\mathcal{A}_n$ is a Δ_n-elementary substructure of $\mathcal{A}'_n$, hence $\mathcal{A}'_n$ is a Δ_n-elementary substructure of $\mathcal{M}$. Further, $\left|M_0^{\mathcal{A}'_n}\right| = \left|M_0^{\mathcal{B}_n}\right|$ and $|A'_n| > |B_n|$. But this contradicts to Lemma 4.21. □

Theorem 4.26 *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure with an atom–defining schema, having the extension property. Suppose that there is a $\varnothing$-definable strongly minimal subset M_0 of M and suppose for each $n \in \omega$ the finite structures $\mathcal{A}_n$ and $\mathcal{B}_n$ are equinumerous, Δ_n–elementary substructures of $\mathcal{M}$. Then there is a decomposable isomorphism*

$$f = \langle f_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}.$$

Proof. By Lemma 4.25 we have $\left|M_0^{\mathcal{A}_n}\right| = \left|M_0^{\mathcal{B}_n}\right|$. Since $\Delta_n \subseteq \Delta_{n+1}$ is an increasing sequence, by Lemma 4.20 there is a decomposable elementary mapping

$$g = \langle g_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F},$$

such that (after a suitable rescaling) the following stipulations hold for almost all $n \in \omega$:

1. $\mathrm{dom}(g_n) \subseteq M_0^{\mathcal{A}_n}$ and $\mathrm{ran}(g_n) \subseteq M_0^{\mathcal{B}_n}$,
2. g_n is Δ_n-elementary,
3. $|\mathrm{dom}(g_n)| \geq \varepsilon(\Delta_n)$ (where ε comes from Lemma 4.3).

Then Proposition 4.8 applies: g can be extended to a decomposable elementary mapping $g^+ = \langle g_n^+ : n \in \omega\rangle/\mathcal{F}$ such that $\mathrm{dom}(g_n^+) = M_0^{\mathcal{A}_n}$ and $\mathrm{ran}(g_n^+) = M_0^{\mathcal{B}_n}$.

Finally, applying Theorem 4.19, one can obtain the desired decomposable isomorphism. □

We close this subsection with the following observation. The extension property is only needed in order to be able to take the first step of the extension, namely to extend $\varnothing$ to the trace of M_0 in the A_i's. Without the extension property one can prove the following theorem.

Theorem 4.27 *Let $\mathcal{M}$ be an $\aleph_1$-categorical structure with an atom–defining schema. Suppose that there is a $\varnothing$-definable strongly minimal subset M_0 of M and suppose for each $n \in \omega$ the finite structures $\mathcal{A}_n$ and $\mathcal{B}_n$ are equinumerous, Δ_n–elementary substructures of $\mathcal{M}$ such that*

$$\mathsf{tp}^{\mathcal{M}}(M_0 \cap A_n/\varnothing) = \mathsf{tp}^{\mathcal{M}}(M_0 \cap B_n/\varnothing)$$

hold for almost all $n \in \omega$. Then there is a decomposable isomorphism

$$f = \langle f_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}.$$

P r o o f. By Lemma 4.25 we have $\left|M_0^{\mathcal{A}_n}\right| = \left|M_0^{\mathcal{B}_n}\right|$ and by assumption there is an elementary bijection $g_n : M_0^{\mathcal{A}_n} \to M_0^{\mathcal{B}_n}$. Applying Theorem 4.19 completes the proof. □

5 Categoricity in finite cardinals

In this section we show that finite fragments of certain $\aleph_1$-categorical theories T are also categorical in the following sense: for all finite subsets Σ of T there exists a finite extension Σ' of Σ, such that up to isomorphism, Σ' can have at most one n-element model Σ'-elementarily embeddable into models of T, for all $n \in \omega$. For details, cf. Theorem 1.2, which is the main theorem of the paper.

We start by two theorems stating that (under some additional technical conditions) an $\aleph_1$-categorical structure can be uniquely decomposed to ultraproducts of its finite substructures.

Recall that $\langle \Delta_n : n \in \omega\rangle$ is a covering sequence of formulas (Definition 4.1).

Theorem 5.1 (Second Unique Factorization Theorem) *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure satisfying the extension–property and having an atom-defining schema. Suppose $\mathcal{A}_n$, $\mathcal{B}_n$ are equinumerous finite, Δ_n-elementary substructures of $\mathcal{M}$. Then*

$$\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\} \in \mathcal{F}$$

for any non-principal ultrafilter $\mathcal{F}$ (i.e., the set $\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\}$ is co-finite).

P r o o f. We would like to apply Theorem 4.2. Recall that by [6, Lemma 6.1.13] there is a strongly minimal subset $M_0 \subseteq M$ which is definable in $\mathcal{M}$ with parameters $\bar{c} \in M$. Consider the structure $\mathcal{M}' = \langle \mathcal{M}, \bar{c}\rangle$. Then there is a $\varnothing$-definable strongly minimal subset of $\mathcal{M}'$. Furthermore, $\mathcal{M}'$ inherits the extension property and the atom-defining schema from $\mathcal{M}$. Particularly, in $\mathcal{M}'$ every $\varnothing$-definable infinite relation has an atom-defining schema. Also, the appropriate expansions of $\mathcal{A}_n$ and $\mathcal{B}_n$ are Δ_n-elementary substructures of $\mathcal{M}'$, as well (possibly, after a rescaling of the sequence Δ_n).

It follows that all the conditions of Theorem 4.2 are satisfied in $\mathcal{M}'$, whence there is a decomposable isomorphism

$$f = \langle f_n : n \in \omega\rangle/\mathcal{F} : \prod_{n\in\omega} \mathcal{A}_n/\mathcal{F} \to \prod_{n\in\omega} \mathcal{B}_n/\mathcal{F}.$$

Then the statement follows from Proposition 2.2 (being an isomorphism can be expressed by a first order formula). □

Theorem 5.2 (Finite Morley Theorem) *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure satisfying the extension property and having an atom-defining schema. Then there exists $N \in \omega$ such that for any $n \geq N$ and $k \in \omega$ (counting up to isomorphisms) $\mathcal{M}$ has at most one Δ_n-elementary substructure of size k.*

Proof. By way of contradiction, suppose for all $N \in \omega$ there exist $l \geq N$, $k \in \omega$ and (at least) two non-isomorphic finite models $\mathcal{A}_N$, $\mathcal{B}_N$ of cardinality k which are Δ_l-elementary substructures of $\mathcal{M}$. Then Theorem 2.8 implies that $\{n \in \omega : \mathcal{A}_n \cong \mathcal{B}_n\}$ is infinite, which contradicts to the choices of $\mathcal{A}_N$, $\mathcal{B}_N$. □

Finally, we present a theorem, in which we do not assume the extension–property and still obtain uniqueness of Δ-elementary substructures having a fixed finite cardinality. This result may be a basis for further investigations, when instead of proving their uniqueness, one would like to estimate the number of pairwise non-isomorphic Δ-elementary substructures of $\mathcal{M}$ having a given finite cardinality. In this respect, we refer to Problem 5.7 below.

Theorem 5.3 *Let $\mathcal{M}$ be an uncountable, $\aleph_1$-categorical structure with an atom-defining schema. Let M_0 be a strongly minimal subset of $\mathcal{M}$ definable by parameters. Then there exists $N \in \omega$ such that for any $n \geq N$ and $k \in \omega$, if $\mathcal{A}$ and $\mathcal{B}$ are Δ_n-elementary substructures of $\mathcal{M}$ of cardinality k, and $\mathtt{tp}(M_0 \cap A/\varnothing) = \mathtt{tp}(M_0 \cap B/\varnothing)$ then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Proof. Similarly to Theorem 2.8, assume M_0 is definable by parameters $\bar{c}$. Adjoining $\bar{c}$ to the language, it still has an atom defining schema. Then the proof can be completed similarly to the proof of Theorem 1.2: assume, seeking a contradiction, that for all $N \in \omega$ there exists $n > N$ and non-isomorphic, equinumerous Δ_n-elementary substructures $\mathcal{A}_n$ and $\mathcal{B}_n$ of $\mathcal{M}$ with

$$\mathtt{tp}(M_0 \cap A_n/\varnothing) = \mathtt{tp}(M_0 \cap B_n/\varnothing)$$

and apply Theorem 4.27. □

We finish the paper by posing some problems which remained open.

Open problems

Conjecture 5.4 *If the language L contains only at most binary relation symbols, T is an L-theory and $\mathsf{S}_2(T)$ is finite, then T has the extension property.*

We have an idea to prove this conjecture but it seems that providing a proof needs a certain amount of further work. Hence we postpone to examine the details.

Problem 5.5 *Provide equivalent conditions for a theory to have the Finite Morley Property.*

Problem 5.6 *We assumed that the Cantor-Bendixson rank of each $\partial\varphi$ in an atom-defining schema is zero. Can Theorem 1.2 be proved without this assumption, or from the weaker assumption that this rank is finite?*

Let k be a natural number. As we mentioned before Theorem 5.3, instead of proving uniqueness of k-sized Δ-elementary substructures of an $\aleph_1$-categorical structure, one can try to estimate the number of pairwise non isomorphic such structures, or one can try to describe all of them. To be more specific, in this direction we offer the following problem.

Problem 5.7 *Let $\mathcal{M}$ be an $\aleph_1$-categorical structure with an atom-defining schema. Continuing investigations initiated in Theorem 5.3, characterize (or give upper estimations for the number of) equinumerous Δ_n-elementary, pairwise non-isomorphic finite substructures of $\mathcal{M}$, by using their trace on a strongly minimal subset. Perhaps, such a characterization or estimation may be obtained in terms of pre-geometries induced by the algebraic closure operation.*

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