A DIOPHANTINE PROBLEM CONCERNING POLYGONAL NUMBERS

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Dedicated to the memory of Brindza Béla and Alf van der Poorten

Abstract. Motivated by some earlier Diophantine works on triangular numbers by Ljunggren and Cassels, we consider similar problems for general polygonal numbers.

1. Introduction and the main results

Ljunggren [16] and Cassels [8] proved that the only triangular numbers that are the squares of triangular numbers are $0, 1$ and 36. In other words, using different methods they resolved the Diophantine equation

$$\frac{x(x+1)}{2} = \left(\frac{y(y+1)}{2}\right)^2 \tag{1}$$

for integers x and y (see Chapter 28 of the classical book by Mordell [18]). As $1 + 2 + \ldots + x = \frac{x(x+1)}{2}$ and $1^3 + 2^3 + \ldots + y^3 = \left(\frac{y(y+1)}{2}\right)^2$ we can give another interpretation of (1) related to the common values of power sums. For a generalization of this problem we refer to [6] and [3].

Triangular numbers are a well-known special case of polygonal numbers. Let

$$\mathrm{Pol}_x^m = \frac{x\left((m-2)x + 4 - m\right)}{2}$$

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be the polygonal numbers with integral parameters $x \geq 1$ and $m \geq 3$. These figurate numbers and their relatives including pyramidal numbers have an extensive literature, see the monographs of Dickson [10] and Deza and Deza [9]. For some recent Diophantine results in this topic we refer to [15],[7], [14], and [19].

The purpose of our note is to generalize the problem mentioned above. Let m, n be fixed integers with $m \geq 3, n \geq 3$. Now consider the equation

$$\mathrm{Pol}_x^m = \left(\mathrm{Pol}_y^n\right)^k \tag{2}$$

for the unknown integers $x > 1, y > 1$ and $k \geq 2$.

Theorem 1.1. *Suppose that $m \neq 4$. Then equation (2) possesses only finitely many solutions in $x > 1, y > 1$, and $k \geq 2$. Further, $\max(x, y, k) < c_1$, where c_1 is an effectively computable constant depending on m and n.*

For $m = 4$, we have $\mathrm{Pol}_x^4 = x^2$, so our problem leads to a trivial equation. For (very) small values of m we will resolve (2). More precisely, we prove the following

Theorem 1.2. *For $m = 3, 5, 6, 8$ and 20, all the solutions of the equation*

$$Pol_x^m = z^k$$

for positive integers x, z, k with $x > 1, z > 1$ and $k \geq 3$ are

$$(m, x, z, k) = (8, 2, 2, 3), (20, 8, 2, 9), (20, 8, 8, 3).$$

Further, for $k = 2$ and $3 \leq m, n \leq 12, m \neq 4$, the solutions (x, y) to (2) are

$$(m, n, x, y) = (3, 3, 8, 3), (3, 5, 49, 5), (3, 6, 8, 2), (3, 9, 288, 8),$$

$$(3, 10, 9800, 42), (6, 5, 25, 5), (7, 4, 6, 3), (7, 9, 6, 2), (8, 3, 9, 5),$$

$$(8, 6, 9, 3), (9, 3, 2, 2), (9, 3, 49, 13), (9, 6, 49, 7), (9, 12, 18, 3),$$

$$(11, 3, 81, 18), (12, 3, 25, 10), (12, 7, 25, 5), (12, 8, 4, 2).$$

It would be preferable to extend the previous theorem for larger values of m, as in the case of pyramidal numbers, see for example [12] and [11], however, it seems well beyond the reach of our techniques, see the remark after the proof of Theorem 1.2.

2. Auxiliary results

In this section, we give some results from the modern theory of Diophantine equations.

Lemma 2.1. *Let $f(X)$ be a polynomial with rational coefficients and suppose that it has at least two distinct zeros in the field of complex numbers $\mathbb{C}$. Then the equation $f(x) = y^k$ for integers $x, |y| > 1$ and $k \geq 2$ implies $k < C_1$, where C_1 is an effectively computable constant depending on the parameters of f.*

Proof. See [20]. □

Our next lemma is a special case of a general theorem concerning the superelliptic equations proved by Brindza [5].

Lemma 2.2. *Let $f(X)$ be a polynomial with rational coefficients and k be a fixed integer with $k \geq 3$. Assume that $f(X)$ possesses at least two simple zeros (over $\mathbb{C}$). Then, the equation $f(x) = y^k$ for integers x and y implies $\max\{|x|, |y|\} < C_2$, where C_2 is an effectively computable constant depending on the parameters of f and k.*

Proof. See [5]. □

Another corollary of Brindza's result [5] is as follows.

Lemma 2.3. *Let $f(X)$ be a polynomial with rational coefficients and suppose that it has at least three simple zeros (over $\mathbb{C}$). Then the hyperelliptic equation $f(x) = y^2$ for integers x and y implies $\max\{|x|, |y|\} < C_3$, where C_3 is an effectively computable constant depending on the parameters of f.*

To prove our second theorem we need the following lemma.

Lemma 2.4. *If m, t, α, β, y and n are nonnegative integers with $n \geq 3$ and $y \geq 1$, then the only solutions to the equation*

$$m(m + 2^t) = 2^\alpha 3^\beta y^n$$

are those with $m \in \{2^t, 2^{t\pm1}, 3 \cdot 2^t, 2^{t\pm3}\}$.

Proof. The proof of this auxiliary result is based on the modular method, see [1]. For similar results on the product of two consecutive integers, we refer to [2] and [13]. □

3. Proofs

Proof of Theorem 1.1. Let m, n be fixed rational integers with $m \geq 3, n \geq 3$ and $m \neq 4$. For $y > 1$, the polygonal number $Pol_y^n > 1$, and for $m \neq 4$, Pol_x^m is a quadratic polynomial in x with rational coefficients and two distinct zeros. Thus Lemma 2.1 gives an effective upper bound for the exponent k depending only on m. In the sequel, we can fix k and first suppose that $k \geq 3$. From Lemma 2.2 we have an upper bound for $\max(x, Pol_y^n)$ depending only on m and this yields that $\max(x, y)$ is bounded by an effectively computable constant depending on m and n. If $k = 2$, then we get

$$(2(m-2)x + 4 - m)^2 = 8(m-2)\left(\frac{y\left((n-2)y + 4 - n\right)}{2}\right)^2 + (4-m)^2,$$

and, by Lemma 2.3, it is enough to guarantee that the quartic polynomial (in Y)

$$8(m-2)\left(\frac{Y\left((n-2)Y + 4 - n\right)}{2}\right)^2 + (4-m)^2 \tag{3}$$

has only simple zeros, or equivalently, its discriminant is a nonzero number for every value of $m \geq 3, m \neq 4$ and $n \geq 3$. An easy calculation shows that the discriminant of this polynomial is

$$256(n-2)^4(m-2)^3(m-4)^4 D(m,n),$$

where

$$D(m,n) = mn^4 - 2n^4 - 16mn^3 + 8m^2n^2 - 32nm^2 + 32m^2 + 32mn^2 + 32n^3 - 64n^2.$$

We can check that

$$D(m,n) = n^3(m-2)(n-16) + 8nm^2(n-4) + 32n^2(m-2) + 32m^2.$$

For $n \geq 16$ and $m \geq 3$, $D(m,n)$ is positive, further, if $n < 16$, then the equation $D(m,n) = 0$ gives $m = n = 4$. Thus, we have proved that the discriminant of (3) is nonzero for every $m \geq 3, m \neq 4$, and $n \geq 3$. $\square$

Proof of Theorem 1.2**.** From the equation

$$\mathrm{Pol}_x^m = z^k$$

we have

$$((m-2)x)\,((m-2)x + 4 - m) = 2(m-2)z^k.$$

Now we can apply Lemma 2.4 to this equation when

$$2(m-2) = 2^\alpha 3^\beta, |m-4| = 2^t,$$

that is $m = 3, 5, 6, 8$ and 20 and $t = 0, 0, 1, 2$ and 4, respectively. Indeed, for $m = 3, 5$ we have $t = 0$. For $m > 5$, our system of equations is

$$m - 2 = 2^{\alpha-1}3^\beta \quad \text{and} \quad m - 4 = 2^4,$$

and it leads to the equation

$$2^{\alpha-2}3^\beta - 2^{t-1} = 1.$$

If $t = 1$, then $\alpha = 3, \beta = 0$. For $t > 1$, we obtain $\alpha = 2$ and thus we have to solve the equation

$$3^\beta - 2^{t-1} = 1. \tag{4}$$

Applying a cannon to kill a fly, by Mihailescu's result [17] on the solution of Catalan's conjecture, we get that all the solutions to (4) are $(\beta, t) = (1,2), (2,4)$. Lemma 2.4 gives the following (essentially two) solutions

$$m = 8, x = z = 2, k = 3,$$

$$m = 20, x = 8, z = 2, k = 9,$$

and

$$m = 20, x = z = 8, k = 3.$$

For $k = 2$ and small values of m and n, we can find the integral points on the corresponding quartic hyperelliptic curve using MAGMA [4], with the subroutine `IntegralQuarticPoints`. □

Remark. For general m the equation $\mathrm{Pol}_x^m = z^k$ leads to several binomial Thue equations of the type

$$Ax_1^k - Bx_2^k = C$$

in the unknown integers $k \geq 3, x_1, x_2$. As the original problem has a solution $x = z = 1$ we cannot apply the local method to all of these Thue equations. The present of this trivial solution means that the application of the modular method is also a great challenge.

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