

Research Article

Positive Solutions of Eigenvalue Problems for a Class of Fractional Differential Equations with Derivatives

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By establishing a maximal principle and constructing upper and lower solutions, the existence of positive solutions for the eigenvalue problem of a class of fractional differential equations is discussed. Some sufficient conditions for the existence of positive solutions are established.

1. Introduction

In this paper, we discuss the existence of positive solutions for the following eigenvalue problem of a class fractional differential equation with derivatives

$$\begin{aligned} -\mathfrak{D}_t^\alpha x(t) &= \lambda f\left(t, x(t), \mathfrak{D}_t^\beta x(t)\right), \quad t \in (0, 1), \\ \mathfrak{D}_t^\beta x(0) &= 0, \quad \mathfrak{D}_t^\gamma x(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^\gamma x(\xi_j), \end{aligned} \quad (1.1)$$

where λ is a parameter, $1 < \alpha \leq 2$, $\alpha - \beta > 1$, $0 < \beta \leq \gamma < 1$, $0 < \xi_1 < \xi_2 < \dots < \xi_{p-2} < 1$, $a_j \in [0, +\infty)$ with $c = \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1} < 1$, and \mathfrak{D}_t is the standard Riemann-Liouville derivative.

$f : (0, 1) \times (0, +\infty) \times (0, +\infty) \rightarrow [0, +\infty)$ is continuous, and $f(t, u, v)$ may be singular at $u = 0$, $v = 0$, and $t = 0, 1$.

As fractional order derivatives and integrals have been widely used in mathematics, analytical chemistry, neuron modeling, and biological sciences [1–6], fractional differential equations have attracted great research interest in recent years [7–17]. Recently, ur Rehman and Khan [8] investigated the fractional order multipoint boundary value problem:

$$\begin{aligned} \mathfrak{D}_t^\alpha y(t) &= f\left(t, y(t), \mathfrak{D}_t^\beta y(t)\right), \quad t \in (0, 1), \\ y(0) &= 0, \quad \mathfrak{D}_t^\beta y(1) - \sum_{i=1}^{m-2} \xi_i \mathfrak{D}_t^\beta y(\xi_i) = y_0, \end{aligned} \quad (1.2)$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \xi_i < 1$, $\xi_i \in [0, +\infty)$ with $\sum_{i=1}^{m-2} \xi_i \xi_i^{\alpha-\beta-1} < 1$. The Schauder fixed point theorem and the contraction mapping principle are used to establish the existence and uniqueness of nontrivial solutions for the BVP (1.2) provided that the nonlinear function $f : [0, 1] \times \mathbb{R} \times \mathbb{R}$ is continuous and satisfies certain growth conditions. But up to now, multipoint boundary value problems for fractional differential equations like the BVP (1.1) have seldom been considered when $f(t, u, v)$ has singularity at $t = 0$ and (or) 1 and also at $u = 0$, $v = 0$. We will discuss the problem in this paper.

The rest of the paper is organized as follows. In Section 2, we give some definitions and several lemmas. Suitable upper and lower solutions of the modified problems for the BVP (1.1) and some sufficient conditions for the existence of positive solutions are established in Section 3.

2. Preliminaries and Lemmas

For the convenience of the reader, we present here some definitions about fractional calculus.

Definition 2.1 (See [1, 6]). Let $\alpha > 0$ with $\alpha \in \mathbb{R}$. Suppose that $x : [a, \infty) \rightarrow \mathbb{R}$. Then the α th Riemann-Liouville fractional integral is defined by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds \quad (2.1)$$

whenever the right-hand side is defined. Similarly, for $\alpha \in \mathbb{R}$ with $\alpha > 0$, we define the α th Riemann-Liouville fractional derivative by

$$\mathfrak{D}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \quad (2.2)$$

where $n \in \mathbb{N}$ is the unique positive integer satisfying $n-1 \leq \alpha < n$ and $t > a$.

Remark 2.2. If $x, y : (0, +\infty) \rightarrow \mathbb{R}$ with order $\alpha > 0$, then

$$\mathfrak{D}_t^\alpha (x(t) + y(t)) = \mathfrak{D}_t^\alpha x(t) + \mathfrak{D}_t^\alpha y(t). \quad (2.3)$$

Lemma 2.3 (See [6]). *One has the following.*

(1) *If $x \in L^1(0, 1)$, $\nu > \sigma > 0$, then*

$$I^\nu I^\sigma x(t) = I^{\nu+\sigma} x(t), \quad \mathfrak{D}_t^\sigma I^\nu x(t) = I^{\nu-\sigma} x(t), \quad \mathfrak{D}_t^\sigma I^\sigma x(t) = x(t). \quad (2.4)$$

(2) *If $\nu > 0$, $\sigma > 0$, then*

$$\mathfrak{D}_t^\nu t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}. \quad (2.5)$$

Lemma 2.4 (See [6]). *Let $\alpha > 0$. Assume that $x \in C(0, 1) \cap L^1(0, 1)$. Then*

$$I^\alpha \mathfrak{D}_t^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad (2.6)$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \dots, n$), and n is the smallest integer greater than or equal to α .

Let

$$k_1(t, s) = \begin{cases} \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1} - (t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-\beta-1}(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.7)$$

$$k_2(t, s) = \begin{cases} \frac{(t(1-s))^{\alpha-\gamma-1} - (t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}, & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and for $t, s \in [0, 1]$, we have

$$k_i(t, s) \leq \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}, \quad i = 1, 2. \quad (2.8)$$

Lemma 2.5. *Let $h \in C(0, 1)$; If $1 < \alpha - \beta \leq 2$, then the unique solution of the linear problem*

$$-\mathfrak{D}_t^{\alpha-\beta} y(t) = h(t), \quad t \in (0, 1),$$

$$y(0) = 0, \quad \mathfrak{D}_t^{\gamma-\beta} y(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j) \quad (2.9)$$

is given by

$$y(t) = \int_0^1 K(t, s) h(s) ds, \quad (2.10)$$

where

$$K(t, s) = k_1(t, s) + \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) \quad (2.11)$$

is the Green function of the boundary value problem (2.9).

Proof. Applying Lemma 2.4, we reduce (2.9) to an equivalent equation:

$$y(t) = -I^{\alpha-\beta} h(t) + c_1 t^{\alpha-\beta-1} + c_2 t^{\alpha-\beta-2}, \quad c_1, c_2 \in \mathbb{R}. \quad (2.12)$$

From (2.12) and noting that $y(0) = 0$, we have $c_2 = 0$. Consequently the general solution of (2.9) is

$$y(t) = -I^{\alpha-\beta} h(t) + c_1 t^{\alpha-\beta-1}. \quad (2.13)$$

Using (2.13) and Lemma 2.3, we have

$$\begin{aligned} \mathfrak{D}_t^{\gamma-\beta} y(t) &= -\mathfrak{D}_t^{\gamma-\beta} I^{\alpha-\beta} h(t) + c_1 \mathfrak{D}_t^{\gamma-\beta} t^{\alpha-\beta-1} \\ &= -I^{\alpha-\gamma} h(t) + c_1 \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma)} t^{\alpha-\gamma-1} \\ &= -\int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} h(s) ds + c_1 \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma)} t^{\alpha-\gamma-1}. \end{aligned} \quad (2.14)$$

Thus,

$$\mathfrak{D}_t^{\gamma-\beta} y(1) = -\int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} h(s) ds + c_1 \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma)}, \quad (2.15)$$

and for $j = 1, 2, \dots, p-2$,

$$\mathfrak{D}_t^{\gamma-\beta} w(\xi_j) = -\int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} h(s) ds + c_1 \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma)} \xi_j^{\alpha-\gamma-1}. \quad (2.16)$$

Using $\mathfrak{D}_t^{\gamma-\beta} y(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j)$, (2.15), and (2.16), we obtain

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-\gamma-1} h(s) ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} (\xi_j-s)^{\alpha-\gamma-1} h(s) ds}{\Gamma(\alpha-\beta) \left(1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}\right)}. \quad (2.17)$$

So the unique solution of the problem (2.9) is

$$\begin{aligned}
 y(t) &= - \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s) ds + \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \\
 &\quad \times \left\{ \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} h(s) ds - \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} h(s) ds \right\} \\
 &= - \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s) ds + \int_0^1 \frac{(1-s)^{\alpha-\gamma-1} t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} h(s) ds \\
 &\quad + \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j \int_0^1 \frac{(1-s)^{\alpha-\gamma-1} \xi_j^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} h(s) ds \\
 &\quad - \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} h(s) ds \\
 &= \int_0^1 \left(k_1(t,s) + \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j,s) \right) h(s) ds \\
 &= \int_0^1 K(t,s) h(s) ds.
 \end{aligned} \tag{2.18}$$

The proof is completed. □

Lemma 2.6. The function $K(t, s)$ has the following properties.

- (1) $K(t, s) > 0$, for $t, s \in (0, 1)$
- (2) $t^{\alpha-\beta-1} \mathfrak{M}(s) \leq K(t, s) \leq M(1-s)^{\alpha-\gamma-1}$, for $t, s \in [0, 1]$,

where

$$\mathfrak{M}(s) = \frac{\sum_{j=1}^{p-2} a_j k_2(\xi_j, s)}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}}, \quad M = \frac{1 + \sum_{j=1}^{p-2} a_j (1 - \xi_j^{\alpha-\gamma-1})}{\Gamma(\alpha-\beta) (1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1})}. \tag{2.19}$$

Proof. It is obvious that (1) holds.

From (2.11), we obtain

$$K(t, s) \geq \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) = t^{\alpha-\beta-1} \mathfrak{M}(s). \tag{2.20}$$

From (2.8), we have

$$\begin{aligned}
 K(t, s) &= k_1(t, s) + \frac{t^{\alpha-\beta-1}}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \sum_{j=1}^{p-2} a_j k_2(\xi_j, s) \\
 &\leq \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} + \frac{\sum_{j=1}^{p-2} a_j}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)} \\
 &\leq \left(1 + \frac{\sum_{j=1}^{p-2} a_j}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \right) \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta)}.
 \end{aligned} \tag{2.21}$$

The proof is completed. \square

Consider the modified problem of the BVP (1.1):

$$\begin{aligned}
 -\mathfrak{D}_t^{\alpha-\beta} y(t) &= \lambda f\left(t, I^\beta y(t), y(t)\right), \\
 y(0) = 0, \quad \mathfrak{D}_t^{\gamma-\beta} y(1) &= \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j).
 \end{aligned} \tag{2.22}$$

Lemma 2.7. *Let $x(t) = I^\beta y(t)$ and $y(t) \in C[0, 1]$; then problem (1.1) is turned into (2.22). Moreover, if $y \in C([0, 1], [0, +\infty))$ is a solution of problem (2.22), then the function $x(t) = I^\beta y(t)$ is a positive solution of the problem (1.1).*

Proof. Substituting $x(t) = I^\beta y(t)$ into (1.1) and using Definition 2.1 and Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned}
 \mathfrak{D}_t^\alpha x(t) &= \frac{d^n}{dt^n} I^{n-\alpha} x(t) = \frac{d^n}{dt^n} I^{n-\alpha} I^\beta y(t) \\
 &= \frac{d^n}{dt^n} I^{n-\alpha+\beta} y(t) = \mathfrak{D}_t^{\alpha-\beta} y(t), \\
 \mathfrak{D}_t^\beta x(t) &= \mathfrak{D}_t^\beta I^\beta y(t) = y(t).
 \end{aligned} \tag{2.23}$$

Consequently, $\mathfrak{D}_t^\beta x(0) = y(0) = 0$. It follows from $\mathfrak{D}_t^\gamma x(t) = d^n/dt^n I^{n-\gamma} x(t) = (d^n/dt^n) I^{n-\gamma} I^\beta y(t) = (d^n/dt^n) I^{n-\gamma+\beta} y(t) = \mathfrak{D}_t^{\gamma-\beta} y(t)$ that $\mathfrak{D}_t^{\gamma-\beta} y(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j)$. Using $x(t) = I^\beta y(t)$, $y \in C[0, 1]$, we transform (1.1) into (2.22).

Now, let $y \in C([0, 1], [0, +\infty))$ be a solution for problem (2.22). Using Lemma 2.3, (2.22), and (2.23), one has

$$\begin{aligned}
 -\mathfrak{D}_t^\alpha x(t) &= -\frac{d^n}{dt^n} I^{n-\alpha} x(t) = -\frac{d^n}{dt^n} I^{n-\alpha} I^\beta y(t) = -\frac{d^n}{dt^n} I^{n-\alpha+\beta} y(t) = -\mathfrak{D}_t^{\alpha-\beta} y(t) \\
 &= \lambda f\left(t, I^\beta y(t), y(t)\right) = \lambda f\left(t, x(t), \mathfrak{D}_t^\beta x(t)\right), \quad 0 < t < 1.
 \end{aligned} \tag{2.24}$$

Noting

$$\mathfrak{D}_t^\beta x(t) = \mathfrak{D}_t^\beta I^\beta y(t) = y(t), \quad \mathfrak{D}_t^\gamma x(t) = \mathfrak{D}_t^{\gamma-\beta} y(t), \quad (2.25)$$

we have

$$\mathfrak{D}_t^\beta x(0) = 0, \quad \mathfrak{D}_t^\gamma x(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^\gamma x(\xi_j). \quad (2.26)$$

It follows from the monotonicity and property of I^β that

$$I^\beta y \in C([0, 1], [0, +\infty)). \quad (2.27)$$

Consequently, $x(t) = I^\beta y(t)$ is a positive solution of the problem (1.1). \square

Definition 2.8. A continuous function $\psi(t)$ is called a lower solution of the BVP (2.22), if it satisfies

$$\begin{aligned} -\mathfrak{D}_t^{\alpha-\beta} \psi(t) &\leq \lambda f(t, I^\beta \psi(t), \psi(t)), \\ \psi(0) &\geq 0, \quad \mathfrak{D}_t^{\gamma-\beta} \psi(1) \geq \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} \psi(\xi_j). \end{aligned} \quad (2.28)$$

Definition 2.9. A continuous function $\phi(t)$ is called an upper solution of the BVP (2.22), if it satisfies

$$\begin{aligned} -\mathfrak{D}_t^{\alpha-\beta} \phi(t) &\geq \lambda f(t, I^\beta \phi(t), \phi(t)), \\ \phi(0) &\leq 0, \quad \mathfrak{D}_t^{\gamma-\beta} \phi(1) \leq \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} \phi(\xi_j). \end{aligned} \quad (2.29)$$

By Lemmas 2.5 and 2.6, we have the maximal principle.

Lemma 2.10. *If $1 < \alpha - \beta \leq 2$ and $y \in C([0, 1], \mathbb{R})$ satisfies*

$$y(0) = 0, \quad \mathfrak{D}_t^{\gamma-\beta} y(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j), \quad (2.30)$$

and $-\mathfrak{D}_t^{\alpha-\beta} y(t) \geq 0$ for any $t \in (0, 1)$, then $y(t) \geq 0$, for $t \in [0, 1]$.

Set

$$\mathcal{G}(t) = t^{\alpha-\beta-1}, \quad \mathcal{L}(t) = I^\beta \mathcal{G}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} s^{\alpha-\beta-1} ds = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}. \quad (2.31)$$

To end this section, we present here two assumptions to be used throughout the rest of the paper.

(B1) $f \in C((0, 1) \times (0, \infty) \times (0, \infty), [0, +\infty))$ is decreasing in u and v , and for any $(u, v) \in (0, \infty) \times (0, \infty)$,

$$\lim_{\sigma \rightarrow +\infty} \sigma f(t, \sigma u, \sigma v) = +\infty \quad (2.32)$$

uniformly on $t \in (0, 1)$.

(B2) For any $\mu, \nu > 0$, $f(t, \mu, \nu) \neq 0$, and

$$\int_0^1 (1-s)^{\alpha-\gamma-1} f(s, \mu \mathcal{L}(s), \mu \mathcal{G}(s)) ds < +\infty. \quad (2.33)$$

3. Main Results

The main result is summarized in the following theorem.

Theorem 3.1. *Provided that (B1) and (B2) hold, then there is a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the problem (1.1) has at least one positive solution $x(t)$, which satisfies $x(t) \geq \mathcal{L}(t)$, $t \in [0, 1]$.*

Proof. Let $E = C[0, 1]$; we denote a set P and an operator T_λ in E as follows:

$$P = \{y \in E : \text{there exists positive number } l_y \text{ such that } y(t) \geq l_y \mathcal{G}(t), t \in [0, 1]\}, \quad (3.1)$$

$$(T_\lambda y)(t) = \lambda \int_0^1 K(t, s) f(s, I^\beta y(s), y(s)) ds, \quad \text{for any } y \in P. \quad (3.2)$$

Clearly, P is a nonempty set since $\mathcal{G}(t) \in P$. We claim that T_λ is well defined and $T_\lambda(P) \subset P$.

In fact, for any $\rho \in P$, by the definition of P , there exists one positive number l_ρ such that $\rho(t) \geq l_\rho \mathcal{G}(t)$ for any $t \in [0, 1]$. It follows from Lemma 2.6 and (B2) that

$$\begin{aligned} (T_\lambda \rho)(t) &= \lambda \int_0^1 K(t, s) f(s, I^\beta \rho(s), \rho(s)) ds \leq \lambda M \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, I^\beta \rho(s), \rho(s)) ds \\ &\leq \lambda M \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, l_\rho \mathcal{L}(s), l_\rho \mathcal{G}(s)) ds < +\infty. \end{aligned} \quad (3.3)$$

Setting $B = \max_{t \in [0,1]} \rho(t) > 0$, from (B2), we have $f(t, B/\Gamma(\beta + 1), B) \neq 0$. By the continuity of $f(t, u, v)$ on $(0, 1) \times (0, \infty) \times (0, \infty)$, we have $\int_0^1 \mathfrak{M}(s) f(s, B/\Gamma(\beta + 1), B) ds > 0$. On the other hand,

$$I^\beta B = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} B ds = \frac{Bt^\beta}{\beta\Gamma(\beta)} \leq \frac{B}{\Gamma(\beta + 1)},$$

$$\mathfrak{M}(s) = \frac{\sum_{j=1}^{p-2} a_j k_2(\xi_j, s)}{1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}} \leq \frac{\sum_{j=1}^{p-2} a_j (1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\beta) \left(1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}\right)}. \tag{3.4}$$

From (3.3), one has

$$0 < \int_0^1 \mathfrak{M}(s) f\left(s, \frac{B}{\Gamma(\beta + 1)}, B\right) ds \leq \int_0^1 \mathfrak{M}(s) f\left(s, I^\beta B, B\right) ds \leq \int_0^1 \mathfrak{M}(s) f\left(s, I^\beta \rho(s), \rho(s)\right) ds$$

$$\leq \frac{\sum_{j=1}^{p-2} a_j}{\Gamma(\alpha-\beta) \left(1 - \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1}\right)} \int_0^1 (1-s)^{\alpha-\gamma-1} f\left(s, I^\beta \rho(s), \rho(s)\right) ds < +\infty. \tag{3.5}$$

It follows from Lemma 2.6 and (3.3) that

$$(T_\lambda \rho)(t) \geq \lambda \mathcal{G}(t) \int_0^1 \mathfrak{M}(s) f\left(s, I^\beta \rho(s), \rho(s)\right) ds = l'_\rho \mathcal{G}(t), \tag{3.6}$$

where

$$l'_\rho = \lambda \int_0^1 \mathfrak{M}(s) f\left(s, I^\beta \rho(s), \rho(s)\right) ds. \tag{3.7}$$

Using (3.3) and (3.6), we know that T_λ is well defined and $T_\lambda(P) \subset P$.

Next we will focus on the upper and lower solutions of problem (2.22). From (B1) and (3.2), we know that the operator T_λ is decreasing in γ . Using

$$\int_0^1 K(t, s) f(s, \mathcal{L}(s), \mathcal{G}(s)) ds \geq \mathcal{G}(t) \int_0^1 \mathfrak{M}(s) f(s, \mathcal{L}(s), \mathcal{G}(s)) ds, \quad \forall t \in [0, 1], \tag{3.8}$$

and letting

$$\lambda_1 = \frac{1}{\int_0^1 \mathfrak{M}(s) f(s, \mathcal{L}(s), \mathcal{G}(s)) ds}, \tag{3.9}$$

we have

$$\lambda_1 \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds \geq \mathcal{G}(t), \quad \forall t \in [0, 1]. \quad (3.10)$$

On the other hand, letting $b(t) = \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds$, since $f(t, u, v)$ is decreasing with respect to u and v , for any $\lambda > \lambda_1$, we have

$$\begin{aligned} \int_0^1 K(t,s)f(s, \lambda I^\beta b(s), \lambda b(s))ds &\leq \int_0^1 K(t,s)f(s, \lambda_1 I^\beta b(s), \lambda_1 b(s))ds \\ &\leq \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds \leq M \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, \mathcal{L}(s), \mathcal{G}(s))ds \\ &< +\infty. \end{aligned} \quad (3.11)$$

From (3.2), (3.3), and (B1), for all $(u, v) \in (0, \infty) \times (0, \infty)$, we have

$$\lim_{\mu \rightarrow +\infty} \mu f(t, \mu u, \mu v) = +\infty \quad (3.12)$$

uniformly on $t \in (0, 1)$. Thus there exists large enough $\lambda^* > \lambda_1 > 0$, such that, for any $t \in (0, 1)$,

$$\lambda^* f(s, \lambda^* \mathcal{L}(s), \lambda^* \mathcal{G}(s)) \geq \frac{1}{\int_0^1 \mathfrak{M}(s)ds}. \quad (3.13)$$

From Lemma 2.6, one has

$$\begin{aligned} \lambda^* \int_0^1 K(t,s)f(s, \lambda^* \mathcal{L}(s), \lambda^* \mathcal{G}(s))ds &\geq \frac{\int_0^1 K(t,s)ds}{\int_0^1 \mathfrak{M}(s)ds} \\ &\geq \frac{\int_0^1 \mathcal{G}(t)\mathfrak{M}(s)ds}{\int_0^1 \mathfrak{M}(s)ds} = \mathcal{G}(t), \quad \forall t \in [0, 1]. \end{aligned} \quad (3.14)$$

Letting

$$\begin{aligned} \phi(t) &= \lambda^* \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds = \lambda^* b(t), \\ \psi(t) &= \lambda^* \int_0^1 K(t,s)f(s, \lambda^* I^\beta b(s), \lambda^* b(s))ds, \end{aligned} \quad (3.15)$$

and using Lemmas 2.3 and 2.7, we obtain

$$\begin{aligned}
 \phi(t) &= \lambda^* \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds \geq \mathcal{G}(t), \quad t \in [0, 1], \\
 \phi(0) &= 0, \quad \mathfrak{D}_t^{Y-\beta} \phi(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{Y-\beta} \phi(\xi_j), \\
 \psi(t) &= \lambda^* \int_0^1 K(t,s)f(s, \lambda^* I^\beta b(s), \lambda^* b(s))ds \geq \mathcal{G}(t), \quad t \in [0, 1], \\
 \psi(0) &= 0, \quad \mathfrak{D}_t^{Y-\beta} \psi(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{Y-\beta} \psi(\xi_j).
 \end{aligned} \tag{3.16}$$

Obviously, $\phi(t), \psi(t) \in P$. By (3.16), we have

$$\mathcal{G}(t) \leq \psi(t) = (T_{\lambda^*} \phi)(t), \quad \mathcal{G}(t) \leq \phi(t), \quad \forall t \in [0, 1], \tag{3.17}$$

which implies that

$$\begin{aligned}
 \psi(t) &= (T_{\lambda^*} \phi)(t) = \lambda^* \int_0^1 K(t,s)f(s, I^\beta \phi(s), \phi(s))ds \\
 &\leq \lambda^* \int_0^1 K(t,s)f(s, \mathcal{L}(s), \mathcal{G}(s))ds = \phi(t), \quad \forall t \in [0, 1].
 \end{aligned} \tag{3.18}$$

Consequently, it follows from (3.17)-(3.18) that

$$\begin{aligned}
 \mathfrak{D}_t \psi(t) + \lambda^* f(t, I^\beta \psi(t), \psi(t)) &= \mathfrak{D}_t (T_{\lambda^*} \phi)(t) + \lambda^* f(t, I^\beta (T_{\lambda^*} \phi)(t), (T_{\lambda^*} \phi)(t)) \\
 &\geq \mathfrak{D}_t (T_{\lambda^*} \phi)(t) + \lambda^* f(t, I^\beta \phi(t), \phi(t)) \\
 &= -\lambda^* f(t, I^\beta \phi(t), \phi(t)) + \lambda^* f(t, I^\beta \phi(t), \phi(t)) = 0,
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 \mathfrak{D}_t \phi(t) + \lambda^* f(t, I^\beta \phi(t), \phi(t)) &= -\lambda^* f(t, \mathcal{L}(t), \mathcal{G}(t)) + \lambda^* f(t, I^\beta \phi(t), \phi(t)) \\
 &\leq -\lambda^* f(t, \mathcal{L}(t), \mathcal{G}(t)) + \lambda^* f(t, \mathcal{L}(t), \mathcal{G}(t)) = 0.
 \end{aligned} \tag{3.20}$$

From (3.16) and (3.18)–(3.20), we know that $\psi(t)$ and $\phi(t)$ are upper and lower solutions of the problem (2.22), and $\psi(t), \phi(t) \in P$.

Define the function F and the operator A_{λ^*} in E by

$$F(t, y) = \begin{cases} f(t, I^\beta \psi(t), \psi(t)), & y < \psi(t), \\ f(t, I^\beta y(t), y(t)), & \psi(t) \leq y \leq \phi(t), \\ f(t, I^\beta \phi(t), \phi(t)), & y > \phi(t), \end{cases} \tag{3.21}$$

$$(A_{\lambda^*} y)(t) = \lambda^* \int_0^1 K(t,s)F(s, y(s))ds, \quad \forall y \in E.$$

It follows from (B1) and (3.21) that $F : (0, 1) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. Consider the following boundary value problem:

$$\begin{aligned} -\mathfrak{D}_t^{\alpha-\beta} y(t) &= \lambda^* F(t, y), \quad t \in [0, 1], \\ y(0) = 0, \quad \mathfrak{D}_t^{\gamma-\beta} y(1) &= \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} y(\xi_j). \end{aligned} \quad (3.22)$$

Obviously, a fixed point of the operator A_{λ^*} is a solution of the BVP (3.22). For all $y \in E$, it follows from Lemma 2.6, (3.21), and $\varphi(t) \geq \mathcal{G}(t)$ that

$$\begin{aligned} (A_{\lambda^*} y)(t) &\leq \lambda^* M \int_0^1 (1-s)^{\alpha-\gamma-1} F(s, y(s)) ds \\ &\leq \lambda^* M \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, I^\beta \varphi(s), \varphi(s)) ds \\ &\leq \lambda^* M \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, \mathcal{L}(s), \mathcal{G}(s)) ds \\ &< +\infty. \end{aligned} \quad (3.23)$$

So A_{λ^*} is bounded. From the continuity of $F(t, y)$ and $K(t, s)$, it is obviously that $A_{\lambda^*} : E \rightarrow E$ is continuous.

From the uniform continuity of $K(t, s)$ and the Lebesgue dominated convergence theorem, we easily get that $A_{\lambda^*}(\Omega)$ is equicontinuous. Thus from the Arzela-Ascoli theorem, $A_{\lambda^*} : E \rightarrow E$ is completely continuous. The Schauder fixed point theorem implies that A_{λ^*} has at least one fixed point w such that $w = A_{\lambda^*} w$.

Now we prove

$$\varphi(t) \leq w(t) \leq \phi(t), \quad t \in [0, 1]. \quad (3.24)$$

Let $z(t) = \phi(t) - w(t)$, $t \in [0, 1]$. Since $\phi(t)$ is the upper solution of problem (2.22) and w is a fixed point of A_{λ^*} , we have

$$w(0) = 0, \quad \mathfrak{D}_t^{\gamma-\beta} w(1) = \sum_{j=1}^{p-2} a_j \mathfrak{D}_t^{\gamma-\beta} w(\xi_j). \quad (3.25)$$

From (3.17), (3.18), and the definition of F , we obtain

$$\begin{aligned} f(t, I^\beta \phi(t), \phi(t)) &\leq F(t, y(t)) \leq f(t, I^\beta \varphi(t), \varphi(t)), \quad \forall y \in E, \\ f(t, \mathcal{L}(t), \mathcal{G}(t)) &\geq f(t, I^\beta \varphi(t), \varphi(t)), \quad \forall t \in [0, 1]. \end{aligned} \quad (3.26)$$

So

$$f\left(t, I^\beta \phi(t), \phi(t)\right) \leq F(t, y(t)) \leq f(t, \mathcal{L}(t), \mathcal{G}(t)), \quad \forall y \in E. \quad (3.27)$$

From (3.18) and (3.20), one has

$$\begin{aligned} \mathfrak{D}_t^{\alpha-\beta} z(t) &= \mathfrak{D}_t^{\alpha-\beta} \phi(t) - \mathfrak{D}_t^{\alpha-\beta} w(t) \\ &= -\lambda^* f(t, \mathcal{L}(t), \mathcal{G}(t)) + \lambda^* F(t, w(t)) \\ &\leq 0, \quad \forall t \in [0, 1]. \end{aligned} \quad (3.28)$$

By (3.27), (3.28), and Lemma 2.10, we get $z(t) \geq 0$ which implies that $w(t) \leq \phi(t)$ on $[0, 1]$. In the same way, we have $w(t) \geq \varphi(t)$ on $[0, 1]$. Thus we obtain

$$\varphi(t) \leq w(t) \leq \phi(t), \quad t \in [0, 1]. \quad (3.29)$$

Consequently, $F(t, w(t)) = f(t, I^\beta w(t), w(t))$, $t \in [0, 1]$. Then $w(t)$ is a positive solution of the problem (2.22). It thus follows from Lemma 2.7 that $x(t) = I^\beta w(t)$ is a positive solution of the problem (1.1).

Finally, by (3.29), we have

$$w(t) \geq \varphi(t) \geq \mathcal{G}(t). \quad (3.30)$$

Thus,

$$x(t) = I^\beta w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s) ds \geq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathcal{G}(s) ds = \mathcal{L}(t). \quad (3.31) \quad \square$$

Corollary 3.2. *Suppose that condition (B1) holds, and that for any $\mu, \nu > 0$, $f(t, \mu, \nu) \neq 0$, and*

$$\int_0^1 f(s, \mu \mathcal{L}(s), \nu \mathcal{G}(s)) ds < +\infty. \quad (3.32)$$

Then there exists a constant $\lambda^ > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the problem (1.1) has at least one positive solution $x(t)$, which satisfies $x(t) \geq \mathcal{L}(t)$, $t \in [0, 1]$.*

We consider some special cases in which $f(t, u, v)$ has no singularity at $u, v = 0$ or $t = 0, 1$.

We give the following assumption.

(B*1) $f \in C((0, 1) \times [0, \infty) \times [0, \infty), (0, +\infty))$ is decreasing in u, v .

Then, $f(t, u, v)$ is nonsingular at $u = v = 0$ and for all $u, v \geq 0$, $f(t, u, v) > 0$, $t \in (0, 1)$, which implies that $f(t, 0, 0) > 0$, $t \in (0, 1)$. Thus

$$\lim_{\mu \rightarrow +\infty} \mu f(t, 0, 0) = +\infty, \quad \text{uniformly for } t \in (0, 1) \quad (3.33)$$

naturally holds; we then have the following corollary.

Corollary 3.3. *If (B*1) holds and (B*2)*

$$\int_0^1 (1-s)^{\alpha-\gamma-1} f(s, 0, 0) ds < +\infty, \quad (3.34)$$

then there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the problem (1.1) has at least one positive solution $x(t)$, which satisfies $x(t) \geq \mathcal{L}(t)$, $t \in [0, 1]$.

Proof. In the proof of Theorem 3.1, we replace the set P by

$$P_1 = \{y \in E : y(t) \geq 0, \quad t \in [0, 1]\} \quad (3.35)$$

and the inequalities (3.18)–(3.20) by

$$0 \leq \varphi(t) = T_\lambda 0, \quad 0 \leq \phi(t) = T_\lambda \varphi(t) \leq T_\lambda 0 = \varphi(t). \quad (3.36)$$

Since $T_\lambda 0, T_\lambda \varphi(t) \in P$, we have

$$\begin{aligned} \mathfrak{D}_t^{\alpha-\beta} T_\lambda 0 + f\left(t, I^\beta T_\lambda, T_\lambda 0\right) &= -f(t, 0, 0) + f\left(t, I^\beta T_\lambda 0, T_\lambda 0\right) \leq 0, \\ \mathfrak{D}_t^{\alpha-\beta} T_\lambda \varphi(t) + f\left(t, I^\beta T_\lambda \varphi(t), T_\lambda \varphi(t)\right) &= -f\left(t, I^\beta \varphi(t), \varphi(t)\right) + f\left(t, I^\beta T_\lambda \varphi(t), T_\lambda \varphi(t)\right) \geq 0 \end{aligned} \quad , t \in [0, 1]. \quad (3.37)$$

The rest of the proof is similar to that of Theorem 3.1. □

If $f(t, u, v)$ is nonsingular at $u = 0$, $v = 0$ and $t = 0, 1$, we have the conclusion.

Corollary 3.4. *If $f(t, u, v) : [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow (0, +\infty)$ is continuous and decreasing in u and v , the problem (1.1) has at least one positive solution $x(t)$, which satisfies $x(t) \geq \mathcal{L}(t)$, $t \in [0, 1]$.*

Example 3.5. Consider the existence of positive solutions for the following eigenvalue problem of fractional differential equation:

$$\begin{aligned}
 -\mathfrak{D}_t^{3/2}x(t) &= \frac{\lambda}{e^t(1-t)^{1/8}} \left(x^{-1/2}(t) + \left(\mathfrak{D}_t^{1/8}x(t) \right)^{-1/8} \right), \\
 \mathfrak{D}_t^{1/8}x(0) &= 0, \quad \mathfrak{D}_t^{3/8}x(1) = 2\mathfrak{D}_t^{3/8}x\left(\frac{1}{2}\right) - \mathfrak{D}_t^{3/8}x\left(\frac{3}{4}\right).
 \end{aligned}
 \tag{3.38}$$

Let

$$f(t, u, v) = \frac{1}{e^t(1-t)^{1/8}} \left(u^{-1/2} + v^{-1/8} \right), \quad (t, u, v) \in (0, 1) \times (0, +\infty) \times (0, +\infty).
 \tag{3.39}$$

Then $f \in C((0, 1) \times (0, +\infty) \times (0, +\infty), (0, +\infty))$ is decreasing in u and v , and for any $(u, v) \in (0, \infty) \times (0, \infty)$,

$$\lim_{\sigma \rightarrow +\infty} \sigma f(t, \sigma u, \sigma v) = \lim_{\sigma \rightarrow +\infty} \frac{\sigma^{1/2}u^{-1/2} + \sigma^{7/8}v^{-1/8}}{e^t(1-t)^{1/8}} = +\infty,
 \tag{3.40}$$

uniformly on $t \in (0, 1)$. Thus (B1) holds.

On the other hand, for any $\mu, \nu > 0$ and $t \in (0, 1)$,

$$\begin{aligned}
 f(t, \mu, \nu) &= \frac{1}{e^t(1-t)^{1/8}} \left(\mu^{-1/2} + \nu^{-1/8} \right) \neq 0, \\
 \mathcal{L}(t) &= \int_0^t \frac{(t-s)^{-7/8} s^{3/8}}{\Gamma(1/8)} ds = \frac{\Gamma(11/8)}{\Gamma(3/2)} t^{1/2}.
 \end{aligned}
 \tag{3.41}$$

thus we have

$$\begin{aligned}
 \int_0^1 (1-s)^{\alpha-\gamma-1} f(s, \mu \mathcal{L}(s), \mu \mathcal{G}(s)) ds &= \int_0^1 (1-s)^{1/8} \left[\frac{1}{e^s(1-s)^{1/8}} \left(\mu^{-1/2} \mathcal{L}^{-1/2}(s) \right. \right. \\
 &\quad \left. \left. + \mu^{-1/8} \mathcal{G}^{-1/8}(s) \right) \right] \\
 &\leq \int_0^1 \left[\mu^{-1/2} \left(\frac{\Gamma(11/8)}{\Gamma(3/2)} t^{1/2} \right)^{-1/2} + \mu^{-1/8} s^{-3/64} \right] ds \\
 &= \int_0^1 \left[\mu^{-1/2} \left(\frac{\Gamma(11/8)}{\Gamma(3/2)} t^{-1/4} \right) + \mu^{-1/8} s^{-3/64} \right] ds < +\infty,
 \end{aligned}
 \tag{3.42}$$

which implies that (B2) holds. From Theorem 3.1, there is a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$ the problem (3.38) has at least one positive solution $x(t)$ and

$$x(t) \geq \mathcal{L}(t) = \frac{\Gamma(11/8)}{\Gamma(3/2)} t^{1/2} \approx 1.003t^{1/2}, \quad t \in [0, 1]. \quad (3.43)$$

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