Hao et al. *Boundary Value Problems* (2015) 2015:125
DOI 10.1186/s13661-015-0393-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Iterative solution to singular *n*th-order nonlocal boundary value problems

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Abstract

By using the cone theory and the Banach contraction mapping principle, we study the existence and uniqueness of an iterative solution to the singular *n*th-order nonlocal boundary value problems.

Keywords: iterative solution; singular higher-order differential equation; nonlocal boundary conditions; cone theory

1 Introduction

The boundary value problems (BVPs for short) for nonlinear differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory. The nonlocal BVPs have been studied extensively. The methods used therein mainly depend on the fixed-point theorems, degree theory, upper and lower techniques, and monotone iteration. Many existence, uniqueness, and multiplicity results have been obtained. For instance, see [1–19] and the references therein.

The purpose of this paper is to investigate the existence and uniqueness of iterative solution to the following nth-order nonlocal BVP:

$$\begin{cases} x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-2)}(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x^{(n-2)}(1) = \int_0^1 x^{(n-2)}(s)\,dA(s), \end{cases} \tag{1.1}$$

where $f \in C((0,1) \times R^{n-1}, R)$, $\Gamma := \int_0^1 t\,dA(t) \neq 1$. $\int_0^1 x^{(n-2)}(s)\,dA(s)$ denotes the Riemann-Stieltjes integral, where A is of bounded variation.

In BVP (1.1), $\int_0^1 x^{(n-2)}(s)\,dA(s)$ denotes the Riemann-Stieltjes integral with a signed measure. This includes as special cases the two-point, three-point, multi-point problems and integral problems. Let us remark that the idea of using a Riemann-Stieltjes integral in the boundary conditions is quite old, see for example the review by Whyburn in [1]. The BVP (1.1) used to model various nonlinear phenomena in physics, chemistry and biology. Over the past decades, great efforts have been devoted to nonlinear nth-order nonlocal BVP (1.1) and its particular and related cases, and many results of the existence of solutions have been obtained by several authors; see [8, 9, 11, 14, 16–19] and references therein. For example, when $n = 2$, $A(t) \equiv 0$, the BVP (1.1) becomes the second-order two-point BVP

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) = 0, \qquad x(1) = 0. \end{cases} \tag{1.2}$$

BVP (1.2) is the well-known second-order Dirichlet BVP, which has been extensively studied and has important applications in physical sciences. When $n = 2$, $\int_0^1 x(s)\,dA(s) = \alpha x(\eta)$, the BVP (1.1) reduces to the second-order three-point BVP

$$\begin{cases} x''(t) + f(t, x(t)) = 0, & t \in (0,1), \\ x(0) = 0, & x(1) = \alpha x(\eta). \end{cases}$$

When $n = 4$, the BVP (1.1) reduces to the fourth-order nonlocal BVP

$$\begin{cases} x''''(t) + f(t, x(t), x'(t), x''(t)) = 0, & t \in (0,1), \\ x(0) = x'(0) = x''(0) = 0, & x''(1) = \int_0^1 x''(s)\,dA(s). \end{cases} \tag{1.3}$$

In material mechanics, the BVP (1.3) describes the deflection or deformation of an elastic beam whose the ends are controlled.

Motivated by the works mentioned above, in this paper, we consider the nth-order nonlocal BVP (1.1). The existence and uniqueness of iterative solution is established by applying the cone theory and the Banach contraction mapping principle. In comparison with previous works, this paper has several new features. Firstly, the nonlinearity f is allowed to depend on higher derivatives of unknown function $x(t)$ up to $n-2$ order, and we allow f to be singular at $t = 0, 1$. The second new feature is that the nonlinearity f is not monotone or convex, the conclusions and the proof used in this paper are different from the known papers. Thirdly, the scope of Γ is not limited to $0 \le \Gamma < 1$, therefore, we do not need to suppose that the Green function $G(t,s)$ is nonnegative.

2 The preliminary lemmas

Lemma 2.1 ([3]) *For any $y \in L[0,1]$, the BVP*

$$\begin{cases} -x''(t) = y(t), & t \in (0,1), \\ x(0) = 0, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

has a unique solution $x(t) = \int_0^1 G(t,s)y(s)\,ds$, where

$$G(t,s) = G_0(t,s) + \frac{t}{1-\Gamma}\mathcal{G}_A(s), \quad t, s \in [0,1],$$

$$G_0(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \qquad \mathcal{G}_A(s) = \int_0^1 G_0(t,s)\,dA(t).$$

Denote $I = [0,1]$, $J = (0,1)$, and for any $x \in C(I)$, $t \in I$, define

$$(I_1 x)(t) = \int_0^t x(s)\,ds, \qquad (I_i x)(t) = \int_0^t \frac{(t-s)^{i-1}}{(i-1)!} x(s)\,ds, \quad i = 2, \ldots, n-2,$$

and

$$(Fx)(t) = \int_0^1 G(t,s) f\big(s, (I_{n-2}x)(s), \ldots, (I_1 x)(s), x(s)\big)\,ds.$$

By Lemma 2.1 and routine calculations, we have the following lemma.

Lemma 2.2

(i) *If* $x \in C^{n-2}(I)$ *is a solution of BVP* (1.1), *then* $y(t) = x^{(n-2)}(t) \in C(I)$ *is a fixed point of the operator* F.

(ii) *If* $x \in C(I)$ *is a fixed point of the operator* F, *then* $y(t) = (I_{n-2}x)(t) = \int_0^t \frac{(t-s)^{n-3}}{(n-3)!} x(s)\,ds \in C^{n-2}(I)$ *is a solution of BVP* (1.1).

Let

$$\begin{aligned} h(t,s) &= \frac{1}{s(1-s)}\big|G(t,s)\big|, \\ e_1(t) &= \int_0^1 h(t,s)\,ds, \\ e_{m+1}(t) &= \max\left\{\int_0^1 h(t,s)e_m(s)\,ds, \int_0^1 h(t,s)\int_0^s e_m(\tau)\,d\tau\,ds\right\}, \quad m = 1,2,\ldots, \\ \overline{G} &= \lim_{m\to\infty}\Big[\sup_{t\in J} e_m(t)\Big]^{-\frac{1}{m}}. \end{aligned}$$

It is easy to see that $\overline{G} > 0$.

Lemma 2.3 ([20]) *P is a generating cone in the Banach space $(E, \|\cdot\|)$ if and only if there exists a constant $\tau > 0$ such that every element $x \in E$ can be represented in the form $x = y - z$, where $y, z \in P$ and $\|y\| \leq \tau\|x\|$, $\|z\| \leq \tau\|x\|$.*

3 Main results

Consider the Banach space $C(I)$ of the usual real-valued continuous functions $u(t)$ defined on I with the norm $\|u\| = \sup_{t\in I}|u(t)|$ for all $u \in C(I)$. Let $P = \{u \in C(I) \mid u(t) \geq 0, \forall t \in I\}$. Obviously, P is a normal solid cone of $C(I)$, by Lemma 2.1.2 in [21], we see that P is a generating cone in $C(I)$.

Theorem 3.1 *Suppose that $f(t, x_0, x_1, \ldots, x_{n-2}) = g(t, x_0, x_0, x_1, x_1, \ldots, x_{n-2}, x_{n-2})$, and there exist positive constants $B_0, C_0, B_1, C_1, \ldots, B_{n-2}, C_{n-2}$ with $B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!} < \overline{G}$, such that for any $t \in J$, $a_{10}, b_{10}, a_{20}, b_{20}, a_{11}, b_{11}, a_{21}, b_{21}, \ldots, a_{1,n-2}, b_{1,n-2}, a_{2,n-2}, b_{2,n-2} \in R$ with $a_{10} \leq b_{10}, a_{20} \geq b_{20}, a_{11} \leq b_{11}, a_{21} \geq b_{21}, \ldots, a_{1,n-2} \leq b_{1,n-2}, a_{2,n-2} \geq b_{2,n-2}$,*

$$\begin{aligned} &-B_{n-2}(b_{1,n-2} - a_{1,n-2}) - C_{n-2}(a_{2,n-2} - b_{2,n-2}) - \cdots \\ &\qquad - B_1(b_{11} - a_{11}) - C_1(a_{21} - b_{21}) - B_0(b_{10} - a_{10}) - C_0(a_{20} - b_{20}) \\ &\quad \leq t(1-t)\big[g(t, a_{1,n-2}, a_{2,n-2}, \ldots, a_{11}, a_{21}, a_{10}, a_{20}) \\ &\qquad - g(t, b_{1,n-2}, b_{2,n-2}, \ldots, b_{11}, b_{21}, b_{10}, b_{20})\big] \\ &\quad \leq B_{n-2}(b_{1,n-2} - a_{1,n-2}) + C_{n-2}(a_{2,n-2} - b_{2,n-2}) + \cdots \\ &\qquad + B_1(b_{11} - a_{11}) + C_1(a_{21} - b_{21}) + B_0(b_{10} - a_{10}) + C_0(a_{20} - b_{20}), \end{aligned} \tag{3.1}$$

and there exist $x_0, y_0 \in C^{n-2}(I)$ such that

$$t(1-t)g\big(t, x_0(t), y_0(t), x_0'(t), y_0'(t), \ldots, x_0^{(n-2)}(t), y_0^{(n-2)}(t)\big) \in L^1[0,1].$$

Then BVP (1.1) *has a unique solution* $I_{n-2}x^*$ *in* $C^{n-2}(I)$. *Moreover, for any* $\overline{x}_0 \in C(I)$, *the iterative sequence*

$$x_1(t) = \int_0^1 G(t,s)f\bigl(s,(I_{n-2}\overline{x}_0)(s),\ldots,(I_1\overline{x}_0)(s),\overline{x}_0(s)\bigr)\,ds,$$
$$x_m(t) = \int_0^1 G(t,s)f\bigl(s,(I_{n-2}x_{m-1})(s),\ldots,(I_1x_{m-1})(s),x_{m-1}(s)\bigr)\,ds \quad (m = 2,3,\ldots)$$

converges to x^* *in* $C(I)$.

Proof By $t(1-t)g(t,x_0(t),y_0(t),x_0'(t),y_0'(t),\ldots,x_0^{(n-2)}(t),y_0^{(n-2)}(t)) \in L^1[0,1]$, it is easy to see that for any $t \in J$,

$$\int_0^1 G(t,s)g\bigl(s,x_0(s),y_0(s),x_0'(s),y_0'(s),\ldots,x_0^{(n-2)}(s),y_0^{(n-2)}(s)\bigr)\,ds$$

is well defined. Set $p(t) = x_0^{(n-2)}(t)$, $q(t) = y_0^{(n-2)}(t)$, then

$$\int_0^1 G(t,s)g\bigl(s,(I_{n-2}p)(s),(I_{n-2}q)(s),\ldots,(I_1p)(s),(I_1q)(s),p(s),q(s)\bigr)\,ds < +\infty.$$

For any $x,y \in C(I)$, let $u(t) = |p(t)| + |x(t)|$, $v(t) = -|q(t)| - |y(t)|$, then $u \geq p$, $v \leq q$. By (3.1), we have

$$\begin{aligned}
&-B_{n-2}(I_{n-2}u - I_{n-2}p)(t) - C_{n-2}(I_{n-2}q - I_{n-2}v)(t) - \cdots \\
&\quad - B_1(I_1u - I_1p)(t) - C_1(I_1q - I_1v)(t) - B_0(u-p)(t) - C_0(q-v)(t) \\
&\leq t(1-t)g\bigl(t,(I_{n-2}u)(t),(I_{n-2}v)(t),\ldots,(I_1u)(t),(I_1v)(t),u(t),v(t)\bigr) \\
&\quad - t(1-t)g\bigl(t,(I_{n-2}p)(t),(I_{n-2}q)(t),\ldots,(I_1p)(t),(I_1q)(t),p(t),q(t)\bigr) \\
&\leq B_{n-2}(I_{n-2}u - I_{n-2}p)(t) + C_{n-2}(I_{n-2}q - I_{n-2}v)(t) + \cdots \\
&\quad + B_1(I_1u - I_1p)(t) + C_1(I_1q - I_1v)(t) + B_0(u-p)(t) + C_0(q-v)(t),
\end{aligned}$$

then

$$\begin{aligned}
&\bigl|G(t,s)g\bigl(s,(I_{n-2}u)(s),(I_{n-2}v)(s),\ldots,(I_1u)(s),(I_1v)(s),u(s),v(s)\bigr) \\
&\quad - G(t,s)g\bigl(s,(I_{n-2}p)(s),(I_{n-2}q)(s),\ldots,(I_1p)(s),(I_1q)(s),p(s),q(s)\bigr)\bigr| \\
&\leq h(t,s)\bigl[B_{n-2}\bigl|(I_{n-2}u - I_{n-2}p)(s)\bigr| + C_{n-2}\bigl|(I_{n-2}q - I_{n-2}v)(s)\bigr| + \cdots \\
&\quad + B_1\bigl|(I_1u - I_1p)(s)\bigr| + C_1\bigl|(I_1q - I_1v)(s)\bigr| + B_0\bigl|(u-p)(s)\bigr| + C_0\bigl|(q-v)(s)\bigr|\bigr] \\
&\leq h(t,s)\bigl[(B_{n-2} + \cdots + B_1 + B_0)\|u-p\| + (C_{n-2} + \cdots + C_1 + C_0)\|q-v\|\bigr].
\end{aligned}$$

Following the former inequality, we can easily get

$$\begin{aligned}
&\int_0^1 G(t,s)\bigl[g\bigl(s,(I_{n-2}u)(s),(I_{n-2}v)(s),\ldots,(I_1u)(s),(I_1v)(s),u(s),v(s)\bigr) \\
&\quad - g\bigl(s,(I_{n-2}p)(s),(I_{n-2}q)(s),\ldots,(I_1p)(s),(I_1q)(s),p(s),q(s)\bigr)\bigr]\,ds
\end{aligned}$$

is convergent, and then

$$\begin{aligned}
&\int_0^1 G(t,s)g\bigl(s,(I_{n-2}u)(s),(I_{n-2}v)(s),\ldots,(I_1u)(s),(I_1v)(s),u(s),v(s)\bigr)\,ds \\
&\quad = \int_0^1 G(t,s)g\bigl(s,(I_{n-2}p)(s),(I_{n-2}q)(s),\ldots,(I_1p)(s),(I_1q)(s),p(s),q(s)\bigr)\,ds \\
&\qquad + \int_0^1 G(t,s)\bigl[g\bigl(s,(I_{n-2}u)(s),(I_{n-2}v)(s),\ldots,(I_1u)(s),(I_1v)(s),u(s),v(s)\bigr) \\
&\qquad - g\bigl(s,(I_{n-2}p)(s),(I_{n-2}q)(s),\ldots,(I_1p)(s),(I_1q)(s),p(s),q(s)\bigr)\bigr]\,ds
\end{aligned}$$

is convergent. Similarly, by $u \geq x$, $v \leq y$, we get

$$\int_0^1 G(t,s)g\bigl(s,(I_{n-2}x)(s),(I_{n-2}y)(s),\ldots,(I_1x)(s),(I_1y)(s),x(s),y(s)\bigr)\,ds < +\infty.$$

Define the operator $A : C(I) \times C(I) \to C(I)$ by

$$A(x,y)(t) = \int_0^1 G(t,s)g\bigl(s,(I_{n-2}x)(s),(I_{n-2}y)(s),\ldots,(I_1x)(s),(I_1y)(s),x(s),y(s)\bigr)\,ds.$$

Then $I_{n-2}x$ is the solution of BVP (1.1) if and only if $x = A(x,x)$. Let

$$\begin{aligned}
&(K_0x)(t) = B_0\int_0^1 h(t,s)x(s)\,ds, \\
&(K_ix)(t) = B_i\int_0^1 h(t,s)(I_ix)(s)\,ds, \quad i = 1,2,\ldots,n-2, \\
&(M_0y)(t) = C_0\int_0^1 h(t,s)y(s)\,ds, \\
&(M_iy)(t) = C_i\int_0^1 h(t,s)(I_iy)(s)\,ds, \quad i = 1,2,\ldots,n-2, \\
&(Kx)(t) = (K_0x + K_1x + \cdots + K_{n-2}x)(t), \\
&(My)(t) = (M_0y + M_1y + \cdots + M_{n-2}y)(t).
\end{aligned}$$

By (3.1), for any $x_1,x_2,y_1,y_2 \in C(I)$, $x_1 \geq x_2$, $y_1 \leq y_2$, we have

$$-K(x_1-x_2) - M(y_2-y_1) \leq A(x_1,y_1) - A(x_2,y_2) \leq K(x_1-x_2) + M(y_2-y_1) \tag{3.2}$$

and

$$\begin{aligned}
&(K+M)x(t) \\
&\quad = \int_0^1 h(t,s)\bigl[B_0x + C_0x + B_1(I_1x) + C_1(I_1x) + \cdots + B_{n-2}(I_{n-2}x) + C_{n-2}(I_{n-2}x)\bigr](s)\,ds \\
&\quad \leq \left[B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!}\right]\|x\|e_1(t), \\
&(K+M)^2x(t)
\end{aligned}$$

$$= \int_0^1 h(t,s)\big[B_0(K+M)x + C_0(K+M)x + B_1 I_1(K+M)x + C_1 I_1(K+M)x$$
$$+ \cdots + B_{n-2} I_{n-2}(K+M)x + C_{n-2} I_{n-2}(K+M)x\big](s)\,ds$$
$$\le \left[B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!}\right]^2 \|x\| e_2(t).$$

By the method of mathematical induction, for any positive integer m and $t \in J$,

$$(K+M)^m x(t) \le \left[B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!}\right]^m \|x\| e_m(t).$$

Then

$$\big\|(K+M)^m\big\| \le \left[B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!}\right]^m \sup_{t\in J} e_m(t)$$

and

$$r(K+M) \le \frac{B_0 + C_0 + B_1 + C_1 + B_2 + C_2 + \cdots + \frac{B_{n-2}+C_{n-2}}{(n-3)!}}{\overline{G}} < 1.$$

Hence, we can choose a $\beta > 0$ such that

$$\lim_{m\to\infty} \big\|(K+M)^m\big\|^{\frac{1}{m}} = r(K+M) < \beta < 1.$$

So, there exists a positive integer m_0 such that

$$\big\|(K+M)^m\big\| < \beta^m < 1, \quad m \ge m_0. \tag{3.3}$$

Since P is a generating cone in $C(I)$, from Lemma 2.3, there exists $\tau > 0$ such that every element $x \in C(I)$ can be represented in the form

$$x = y - z, \quad \text{where } y, z \in P \text{ and } \|y\| \le \tau\|x\|, \|z\| \le \tau\|x\|, \tag{3.4}$$

which implies

$$-(y+z) \le x \le y + z. \tag{3.5}$$

Let

$$\|x\|_0 = \inf\big\{\|u\| \mid u \in P, -u \le x \le u\big\}, \tag{3.6}$$

by (3.5) we know that $\|x\|_0$ is well defined for any $x \in C(I)$. It is easy to verify that $\|\cdot\|_0$ is a norm in $C(I)$. By (3.4)-(3.6), we get

$$\|x\|_0 \le \|y+z\| \le 2\tau\|x\|, \quad \forall x \in C(I). \tag{3.7}$$

On the other hand, for any $u \in P$ which satisfies $-u \le x \le u$, we have $\theta \le x + u \le 2u$, then $\|x\| \le \|x+u\| + \|-u\| \le (2N+1)\|u\|$, where N denotes the normal constant of P.

Since u is arbitrary, we have

$$\|x\| \leq (2N+1)\|x\|_0, \quad \forall x \in C(I). \tag{3.8}$$

It follows from (3.7) and (3.8) that the norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent.

Now, for any $x, y \in C(I)$ and $u \in P$ which satisfies $-u \leq x - y \leq u$, we set

$$u_1 = \frac{1}{2}(x + y - u), \qquad u_2 = \frac{1}{2}(x - y + u), \qquad u_3 = \frac{1}{2}(-x + y + u),$$

then $x \geq u_1$, $y \geq u_1$, and $x - u_1 = u_2$, $y - u_1 = u_3$, $u_2 + u_3 = u$. It follows from (3.2) that

$$-Ku_2 \leq A(x,x) - A(u_1,x) \leq Ku_2, \tag{3.9}$$

$$-Ku_3 - Mu_2 \leq A(y,u_1) - A(u_1,x) \leq Ku_3 + Mu_2, \tag{3.10}$$

$$-Mu_3 \leq A(y,u_1) - A(y,y) \leq Mu_3. \tag{3.11}$$

By (3.9)-(3.11), we have

$$-(K+M)u \leq A(x,x) - A(y,y) \leq (K+M)u.$$

Let $\widetilde{A}(x) = A(x,x)$, then we obtain

$$-(K+M)u \leq \widetilde{A}(x) - \widetilde{A}(y) \leq (K+M)u.$$

As K and M are both positive linear bounded operators, so $K + M$ is a positive linear bounded operator, and therefore $(K + M)u \in P$. Hence, by mathematical induction, it is easy to see that for the natural number m_0 in (3.3), we have

$$-(K+M)^{m_0}u \leq \widetilde{A}^{m_0}(x) - \widetilde{A}^{m_0}(y) \leq (K+M)^{m_0}u, \quad (K+M)^{m_0}u \in P.$$

Since $(K + M)^{m_0}u \in P$, we see that

$$\left\|\widetilde{A}^{m_0}(x) - \widetilde{A}^{m_0}(y)\right\|_0 \leq \left\|(K+M)^{m_0}\right\| \|u\|,$$

which implies by virtue of the arbitrariness of u that

$$\left\|\widetilde{A}^{m_0}x - \widetilde{A}^{m_0}y\right\|_0 \leq \left\|(K+M)^{m_0}\right\| \|x-y\|_0 \leq \beta^{m_0}\|x-y\|_0.$$

By $0 < \beta < 1$, we have $0 < \beta^{m_0} < 1$. Thus the Banach contraction mapping principle implies that $\widetilde{A}^{m_0}$ has a unique fixed point x^* in $C(I)$, and so $\widetilde{A}$ has a unique fixed point x^* in $C(I)$. By the definition of $\widetilde{A}$, A has a unique fixed point x^* in $C(I)$, then by Lemma 2.2, $I_{n-2}x^*$ is the unique solution of BVP (1.1). And, for any $\overline{x}_0 \in C(I)$, let $x_1 = A(\overline{x}_0, \overline{x}_0)$, $x_m = A(x_{m-1}, x_{m-1})$ $(m = 2, 3, \ldots)$, we have $\|x_m - x^*\|_0 \to 0$ $(m \to \infty)$. By the equivalence of $\|\cdot\|_0$ and $\|\cdot\|$ again, we get $\|x_m - x^*\| \to 0$ $(m \to \infty)$. This completes the proof. □

Remark 3.1 For the case $n = 3$,

$$A(t) = \begin{cases} 0, & t \in [0, \eta), \\ \alpha, & t \in [\eta, 1], \end{cases}$$

if $f(t,x,y) = f(t,x)$, Theorem 3.1 is reduced to Theorem 3.1 in [19], if $1 < \alpha < \frac{1}{\eta}$ and $f(t,x,y) = h(t)f(t,x)$, the existence results of nontrivial solutions are given by means of the topological degree theory in [11]. So our results extend the corresponding results of [11, 19] to some degree.

Example 3.1 To illustrate the applicability of our results, we consider the BVP (1.1) with $n = 3$ and

$$f(t,x,y) = \frac{1}{t(1-t)}\left[\sin\frac{x}{n_1} + \cos\frac{x}{n_2} + \frac{\ln(1+y^2)}{n_3} + \sin\frac{y}{n_4}\right],$$
$$A(t) = \begin{cases} 0, & t \in [0, \frac{1}{4}), \\ 2, & t \in [\frac{1}{4}, 1], \end{cases}$$

where n_1, n_2, n_3, n_4 are positive integral numbers. Then $\Gamma = \int_0^1 t\,dA(t) = 2 \times \frac{1}{4} = \frac{1}{2}$, and BVP (1.1) becomes the singular third-order three-point BVP

$$\begin{cases} x'''(t) + \frac{1}{t(1-t)}[\sin\frac{x(t)}{n_1} + \cos\frac{x(t)}{n_2} + \frac{\ln(1+(x'(t))^2)}{n_3} + \sin\frac{x'(t)}{n_4}] = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, \qquad x'(1) = 2x'(\frac{1}{4}). \end{cases} \tag{3.12}$$

Let

$$G(t,s) = 2t\left[(1-s) - 2\left(\frac{1}{4} - s\right)\chi_{[0,\frac{1}{4}]}(s)\right] - (t-s)\chi_{[0,t]}(s),$$

where χ is the characteristic function, *i.e.*

$$\chi_{[a,b]}(t) = \begin{cases} 1, & t \in [a,b], \\ 0, & t \notin [a,b] \end{cases}$$

and

$$(Fx)(t) = \int_0^1 G(t,s)f\bigl(s,(I_1x)(s),x(s)\bigr)\,ds, \quad t \in (0,1).$$

By Lemma 2.2, if $x \in C(I)$ is a fixed point of the operator F, then $y(t) = (I_1x)(t) = \int_0^t x(s)\,ds \in C^1(I)$ is a solution of BVP (3.12). Let $f(t,x,y) = g(t,x,x,y,y)$, then for any $t \in J$, $a_{10}, b_{10}, a_{20}, b_{20}, a_{11}, b_{11}, a_{21}, b_{21} \in R$ with $a_{10} \le b_{10}$, $a_{20} \ge b_{20}$, $a_{11} \le b_{11}$, $a_{21} \ge b_{21}$, we have

$$\begin{aligned} &-\frac{1}{n_1}(b_{11} - a_{11}) - \frac{1}{n_2}(a_{21} - b_{21}) - \frac{1}{n_3}(b_{10} - a_{10}) - \frac{1}{n_4}(a_{20} - b_{20}) \\ &\quad \le t(1-t)\bigl[g(t,a_{11},a_{21},a_{10},a_{20}) - g(t,b_{11},b_{21},b_{10},b_{20})\bigr] \\ &\quad = \sin\frac{a_{11}}{n_1} + \cos\frac{a_{21}}{n_2} + \frac{\ln(1+a_{10}^2)}{n_3} + \sin\frac{a_{20}}{n_4} \end{aligned}$$

$$- \left[\sin\frac{b_{11}}{n_1} + \cos\frac{b_{21}}{n_2} + \frac{\ln(1+b_{10}^2)}{n_3} + \sin\frac{b_{20}}{n_4} \right]$$
$$\le \frac{1}{n_1}(b_{11}-a_{11}) + \frac{1}{n_2}(a_{21}-b_{21}) + \frac{1}{n_3}(b_{10}-a_{10}) + \frac{1}{n_4}(a_{20}-b_{20}).$$

By Theorem 3.1, BVP (3.12) has a unique solution $I_1x^* \in C^1(I)$ provided $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} < \overline{G}$. Moreover, for any $x_0 \in C(I)$, the iterative sequence

$$x_m(t) = \int_0^1 G(t,s) f\bigl(s, (I_1 x_{m-1})(s), x_{m-1}(s)\bigr)\,ds \quad (m = 1, 2, 3, \ldots)$$

converges to x^* $(m \to \infty)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors wrote, read, and approved the final manuscript.

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Acknowledgements
The authors would like to thank the referees for their pertinent comments and valuable suggestions. This work is supported by the National Natural Science Foundation of China (11371221, 11201260), the Natural Science Foundation of Shandong Province of China (ZR2015AM022, ZR2013AQ014), the Specialized Research Fund for the Doctoral Program of Higher Education (20123705120004, 20123705110001) and Foundation of Qufu Normal University (BSQD20100103).

Received: 16 March 2015 Accepted: 16 July 2015 Published online: 31 July 2015

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