On the $\overline{\mu}$ invariant of rational surface singularities

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Abstract We show that for rational surface singularities with odd determinant the $\overline{\mu}$ invariant defined by W. Neumann is an obstruction for the link of the singularity to bound a rational homology 4–ball. We identify the $\overline{\mu}$ invariant with the corresponding correction term in Heegaard Floer theory.

AMS Classification

Keywords

1 Introduction

Smoothings of surface singularities play a prominent role in constructing new and interesting smooth (and symplectic) 4–manifolds. It is of particular interest when the singularity at hand admits a rational homology 4–ball smoothing. Such smoothings led to the discovery of the rational blow–down procedure [2, 20], which in turn provided a major tool for finding exotic 4–manifolds. Restrictions for a singularity to admit rational homology 4–ball smoothing have been found recently in [22].

A topological obstruction for a $\mathbb{Z}_2$–homology 3–sphere (that is, a 3–manifold Y with $H_*(Y;\mathbb{Z}_2) = H_*(S^3;\mathbb{Z}_2)$) to bound a spin rational homology 4–ball is its μ–invariant, defined modulo 16. An integral lift $\overline{\mu}$ of μ has been defined by Neumann in [14] (cf. also [21]) for plumbed $\mathbb{Z}_2$–homology 3–spheres, but it was unclear whether this integer valued invariant obstructs the 3–manifold to bound a spin rational homology 4–ball. Special cases, like Seifert fibered 3–manifolds, have been considered by Saveliev [21]. More recently, based on work of Ozsváth and Szabó [16, 17, 18] the *correction term* of spin^c 3–manifolds (stemming from gradings on the Ozsváth–Szabó homology groups) provided further obstructions. For applications of these invariants along similar lines see [7].

In fact, in [14] the $\overline{\mu}$–invariant is defined for any spin rational homology 3–sphere which can be given by plumbing spheres along a tree (i.e., the assumption on the parity of the determinant of the plumbing graph can be relaxed). By identifying $\overline{\mu}$ of a spin 3–manifold $(Y, \mathbf{s})$ which is a link of a rational surface singularity with the appropriate correction term, we show

Theorem 1.1 *Suppose that Y_Γ is given as a plumbing of spheres along a negative definite tree Γ, defining a rational surface singularity.*

- *For a spin structure $\mathbf{s} \in Spin(Y_\Gamma)$ the invariant $\overline{\mu}(Y_\Gamma, \mathbf{s}) \in \mathbb{Z}$ is an obstruction for the existence of a spin^c rational homology 4–ball $(X, \mathbf{t})$ with boundary $(Y_\Gamma, \mathbf{s})$.*
- *If $\Pi_{\mathbf{s} \in Spin(Y_\Gamma)} \overline{\mu}(Y_\Gamma, \mathbf{s}) \neq 0$ then the rational singularity does not bound a spin rational homology 4–ball.*
- *Specifically, if $\det \Gamma$ is odd and $\overline{\mu}(Y_\Gamma) \neq 0$ then Y_Γ is not the boundary of a rational homology 4–ball. Consequently the corresponding singularity does not admit rational homology 4–ball smoothing.*

Corollary 1.2 *Suppose that S_Γ is a normal surface singularity with $\det \Gamma$ odd. If $\overline{\mu}(Y_\Gamma) \neq 0$ then S_Γ does not admit a rational homology 4–ball smoothing.*

Proof If S_Γ is not a rational singularity then it does not admit rational homology 4–ball smoothing. If $\det \Gamma$ is odd, then for rational surface singularities Theorem 1.1 concludes the proof. □

We hope that this obstruction will be useful in completing the characteriztion of surface singularities with rational homology 4–ball smoothing, along the line initiated in [22].

Remark 1.3 The assumption on the parity of $\det \Gamma$ cannot be relaxed in general, since for example the singularity with resolution graph having a single vertex of weight (-4) has two spin structures with $\overline{\mu}$ invariants -3 and $+1$, but the link of the singularity is the boundary of a rational homology 4–ball: the complement of a quadric in the complex projective plane. In fact, this rational homology 4–ball can be given as a smoothing of the singularity. In accordance with Theorem 1.1 the spin structures on the link of the singularity do not extend to the rational homology 4–ball.

As it was indicated earlier, the proof of Theorem 1.1 above rests on the following, more technical statement. Here the invariant $d(Y, \mathbf{s})$ of a spin^c 3–manifold $(Y, \mathbf{s})$ is the correction term in Heegaard Floer theory. (For more on Heegaard Floer theory see Section 4.)

Theorem 1.4 *Suppose that Γ is a negative definite plumbing tree of spheres, giving rise to a rational surface singularity. Let $\mathbf{s}$ be a given spin structure on the associated 3–manifold Y_Γ. Then $\overline{\mu}(Y_\Gamma, \mathbf{s}) = -4d(Y_\Gamma, \mathbf{s})$.*

2 The μ and $\overline{\mu}$ invariants

Suppose that Y is a rational homology 3–sphere, and the rank $|H_1|$ of its first homology is odd. Then $H_1(Y;\mathbb{Z}_2) = H^1(Y;\mathbb{Z}_2) = 0$, hence Y admits a unique spin structure. Consider a spin 4–manifold X with $\partial X = Y$. The classical definition of Rokhlin's μ–invariant is

$$\mu(Y) \equiv \sigma(X) \mod 16,$$

where $\sigma(X)$ is the signature of the 4–manifold X. The invariance of this quantity is a simple consequence of Rokhlin's famous result on the divisibility of the signature of a closed spin 4–manifold by 16. (If Y is an integral homology sphere, that is, $H_1(Y;\mathbb{Z}) = 0$ also holds, then the signature $\sigma(X)$ of a spin 4–manifold X with $\partial X = Y$ is divisible by 8, and in this case sometimes Rokhlin's invariant is defined as $\frac{\sigma(X)}{8} \in \mathbb{Z}_2$.)

It is not hard to see that if X is a spin rational homology 4–ball (i.e., $H_*(X;\mathbb{Q}) = H_*(D^4;\mathbb{Q})$) with $\partial X = Y$ and $H_1(Y;\mathbb{Z}_2) = 0$ then $\mu(Y) = 0$. Consequently, the μ–invariant of a $\mathbb{Z}_2$–homology sphere Y provides an obstruction for Y to bound a spin rational homology 4–ball. (The spin assumption on X is important, since for example the Brieskorn sphere $\Sigma(2,3,7)$ has $\mu = 1$ and bounds a nonspin rational homology 4–ball, cf. [3].) Since μ is defined only mod 16, it is typically less effective than an integer valued invariant. Interest in integral lifts (or related obstructions) was motivated also by a result of Galewski and Stern [4] about higher dimensional (simplicial) triangulation theory.

In [14] Walter Neumann defined a lift $\overline{\mu} \in \mathbb{Z}$ of μ for spin 3–manifolds given by the plumbing construction along a weighted tree. Before giving the definition of this invariant we shortly review a few basic facts about plumbing trees. For a general reference see [14].

Suppose that Γ is a weighted tree with nonzero determinant. Let X_Γ denote the 4–manifold defined by plumbing disk bundles over spheres according to the weighted tree Γ, and define Y_Γ as ∂X_Γ. As it is described in [14], the mod 2 homology $H_1(Y_\Gamma;\mathbb{Z}_2)$ can be determined by a simple algorithm, which we outline below. Consider a leaf v of Γ, connected to the vertex w.

- **Move 1**: If the weight on v is even, then erase v and w from Γ.

- **Move 2**: If the weight of v is odd, then erase v and change the parity of the weight on w.

This procedure stops once we reach a graph Γ' with no edges. Suppose that Γ' contains p vertices, q of them with even weights.

Lemma 2.1 *The dimension of the vector space $H_1(Y_\Gamma;\mathbb{Z}_2)$ over $\mathbb{Z}_2$ is equal to q.*

Proof Denote the set of vertices of the given weighted plumbing tree Γ with nonzero determinant by $V=\{v_1,\ldots,v_n\}$. It is known (cf. [5, Proposition 5.3.11]) that the homology group $H_1(Y_\Gamma;\mathbb{Z})$ admits a presentation by taking elements of V as generators, and equations

$$n_i\cdot v_i+\sum_{j\neq i}\langle v_j,v_i\rangle\cdot v_j=0$$

as relations ($i=1,\ldots,n$), with the convention that n_i is the weight on v_i, and $\langle v_j,v_i\rangle$ is one or zero depending on whether v_j and v_i (as vertices of the tree Γ) are connected or not. These relations follow easily from the existence of Seifert surfaces for the components of the surgery link. The mod 2 reduction of the relations (with the same generators) provide a presentation for $H_1(Y_\Gamma;\mathbb{Z}_2)$. Now the moves for simplifying the graph (until it becomes a disjoint union of some vertices) obviously correspond to base changes and expressions of generators in terms of others. Indeed, when **Move 1** applies to v and w then the relation for v shows $w=0$, while the relation for w expresses v in terms of the other neighbours of w. In the situation of **Move 2** the relation for v simply asserts that $v=w$ (mod 2). From this observation the statement easily follows: a single point with odd weight gives rise to a 3–manifold with vanishing first mod 2 homology, while with even weight the first mod 2 homology is 1-dimensional. □

Recall that an oriented 3–manifold Y always admits a spin structure, and the space of spin structures is parametrized by the first mod 2 cohomology $H^1(Y;\mathbb{Z}_2)(\cong H_1(Y;\mathbb{Z}_2))$ of Y. A convenient parametrization of the set of spin structures on the rational homology 3–sphere Y_Γ is given as follows. First we define a set of subsets of the vertex set for every plumbing graph Γ. We start with a graph Γ' having no edges: in that case consider the subsets of the vertices which contain all vertices with odd weights. Every such subset will give rise to a unique subset $S\subset V$ for the original graph Γ as follows. We describe the change of S under one step in the process giving Γ' from Γ. Suppose that Γ'

is given by **Move 1** from Γ (via erasing $v = v_i$ and $w = v_j$), and a set $S' \subset V'$ is specified for Γ'. Now we define the set $S \subset V$ by taking it to be equal to S' or $S' \cup \{v_i\}$ according as the number of indices in S' adjacent to $w = v_j$ have the same parity as n_j or $n_j - 1$. If Γ' is derived from Γ by **Move 2** (via erasing v_i) then let S be equal to S' or $S' \cup \{v_i\}$ depending on whether v_j was in S' or not. It is not hard to see from this algorithm that if $v_i, v_j \in S$ then v_i and v_j are not connected by an egde in Γ.

Suppose now that $S \subset V$ is a subset defined as above. Consider the submanifold $\Sigma_S \subset X_\Gamma$ defined as the union of the spheres corresponding to the vertices in S. Notice that since by construction S does not contain adjacent vertices, the above surface is a disjoint union of embedded spheres. Let $c_S \in H^2(X_\Gamma; \mathbb{Z})$ denote the Poincaré dual of Σ_S. The inductive definition (and the starting condition) shows that c_S is a *characteristic element*, that is, for every surface $\Sigma_v \subset X_\Gamma$ defined by a vertex v we have

$$c_S(\Sigma_v) \equiv n_v \mod 2.$$

On the simply connected 4–manifold X_Γ a characteristic cohomology class uniquely specifies a spin^c structure $\mathbf{t}_S$, which restricts to a spin^c structure $\mathbf{s}_S$ on the boundary Y_Γ. Since $PD(c_S) = \bigcup_v \Sigma_v = \Sigma_S$ is in $H_2(X_\Gamma; \mathbb{Z})$, on the boundary the spin^c structure $\mathbf{s}_S = \mathbf{t}_S|_{\partial X_\Gamma}$ has vanishing first Chern class, therefore it is a spin structure on Y_Γ. Hence every subset S constructed above defines a spin structure $\mathbf{s}_S$ on Y_Γ; the set S is called the *Wu set* of the corresponding spin structure. Since this construction provides a spin structure on the complement $X - \Sigma_S$, it is obvious that two different sets induce different spin structures: if S_1 and S_2 differ on the vertex v of even weight (in the disconnected graph our construction started with) then only the spin structure corresponding to the Wu set not containing v will extend to the cobordism we get by the appropriate handle attachment along v. In conclusion, we get an identification of $H_1(Y_\Gamma; \mathbb{Z}) (\cong H^1(Y_\Gamma; \mathbb{Z}))$ with the set of spin structures on Y_Γ: take the characteristic function of S on the starting disconnected graph Γ' (which by the above said determines S), and associate to it the corresponding first mod 2 cohomology class. Now the definition of the $\overline{\mu}$ invariant of Neumann (cf. also [14]) is as follows.

Definition 2.2 For a spin structure $\mathbf{s}$ on Y_Γ consider the corresponding Wu set S and embedded Wu surface $\Sigma_S \subset X_\Gamma$. Define $\overline{\mu}(Y_\Gamma, \mathbf{s}) \in \mathbb{Z}$ as the difference

$$\overline{\mu}(Y_\Gamma, \mathbf{s}) = \sigma(X_\Gamma) - [\Sigma_S]^2.$$

By applying the handle calculus developed in [15] together with the Wu set S, the proof of the following statement easily follows.

Proposition 2.3 *([14, Theorem 4.1]) The quantity $\overline{\mu}(Y_\Gamma, \mathbf{s})$ is an invariant of the spin 3–manifold $(Y_\Gamma, \mathbf{s})$ and is independent of the choices made in the definition.* □

3 Rational singularities

Consider the plumbing tree Γ and suppose that Γ is negative definite. According to a classical result of Grauert [6], for any negative definite plumbing graph there exists a normal surface singularity such that the plumbing along the given graph is diffeomorphic to a resolution of the singularity.

Definition 3.1 A normal surface singularity S_Γ is called *rational* if its geometric genus $p_g = 0$. A negative definite plumbing graph Γ is *rational* if there is a rational singularity S_Γ with resolution diffeomorphic to X_Γ.

Although the singularity corresponding to a plumbing graph might not be unique, it is known that rationality is a topological property and can be fairly easily read off from the plumbing graph through Laufer's algorithm. Namely, consider the homology class

$$K_0 = \sum_{v \in \Gamma} [\Sigma_v] \in H_2(X_\Gamma; \mathbb{Z}).$$

In the i^{th} step, consider the product $K_i \cdot \Sigma_{v_j} = \langle PD(K_i), [\Sigma_{v_j}] \rangle$. If it is at least 2 then the algorithm stops and the singularity is not rational. If the product is nonpositive, move to the next vertex. Finally, if the product is 1 for some $v \in \Gamma$, then replace K_i with $K_{i+1} = K_i + [\Sigma_v]$ and start checking the value of the product for all vertices of Γ again. If all products are nonpositive, the algorithm stops and the graph gives rise to a rational singularity.

Lemma 3.2 *A rational plumbing graph is always a (negative definite) tree of spheres, and the link is a rational homology 3–sphere. In addition, for any vertex $v_i \in \Gamma$ the sum of its weight n_i and the number d_i of its neighbours is at most 1.* □

Notice that in a rational graph a vertex with weight (-1) has degree $d \leq 2$, hence can be blown down by keeping Γ a plumbing tree. For this reason, we might assume that $n_i \leq -2$ for all vertices $v_i \in \Gamma$.

4 Heegaard Floer groups

In [17, 18] a set of very powerful invariants, the Ozsváth–Szabó homology groups $\widehat{HF}(Y,\mathbf{s}), HF^{\pm}(Y,\mathbf{s})$ and $HF^{\infty}(Y,\mathbf{s})$ of a spin^c 3–manifold $(Y,\mathbf{s})$ were introduced. In the following we will use these groups and relations among them; for a more thorough introduction see [17, 18, 10]. Recall that a rational homology 3–sphere Y is an *L–space* if $\widehat{HF}(Y,\mathbf{s})=\mathbb{Z}_2$ for every spin^c structure $\mathbf{s}\in Spin^c(Y)$. (In the version of the theory we are about to apply, we use $\mathbb{Z}_2$–coefficients.) In this case we can label the unique nonzero element of $\widehat{HF}(Y,\mathbf{s})$ by the corresponding spin^c structure $\mathbf{s}$. Recall also that for a rational homology 3–sphere Y the groups are equipped with a natural $\mathbb{Q}$–grading. The grading of the unique nontrivial element of $\widehat{HF}(Y,\mathbf{s})$ for an L–space Y is called the *correction term* $d(Y,\mathbf{s})$ of the spin^c 3–manifold $(Y,\mathbf{s})$. For the proof of the next proposition, see for example [8, Theorem 2.3].

Proposition 4.1 *Suppose that $d(Y,\mathbf{s})\neq 0$. Then there is no spin^c rational homology 4–ball $(X,\mathbf{t})$ with $\partial(X,\mathbf{t})=(Y,\mathbf{s})$.* □

Proposition 4.2 *Suppose that $\det\Gamma$ is odd. If $d(Y_\Gamma,\mathbf{s})\neq 0$ for the unique spin structure $\mathbf{s}$ then Y_Γ does not bound any rational homology 4–ball.*

Proof Suppose that $Y_\Gamma=\partial X$ for a rational homology 4–ball X. Let $\varphi\colon Y_\Gamma\to X$ denote the embedding of the boundary, inducing the map φ_* on homology. Since $|H_1(Y_\Gamma;\mathbb{Z})|$ is odd, the size of the subgroup Im φ_* is also odd. This implies that an odd number of spin^c structures in $Spin^c(Y_\Gamma)$ extend to X. Since $\mathbf{s}\in Spin^c(Y_\Gamma)$ and its conjugate $\overline{\mathbf{s}}$ extend at the same time, we conclude that the spin structure $\mathbf{s}=\overline{\mathbf{s}}$ of Y_Γ extends to X as a spin^c structure, therefore Proposition 4.1 concludes the proof. □

A relation between the singularity's holomorphic structure and its Heegaard Floer theoretic behaviour was found by A. Némethi:

Theorem 4.3 (Némethi, [13]) *Suppose that the negative definite plumbing tree Γ gives rise to a rational singularity. Then Y_Γ is an L–space.* □

5 A relation between $\overline{\mu}(Y_\Gamma,\mathbf{s})$ and $d(Y_\Gamma,\mathbf{s})$

The proof of our main result about the $\overline{\mu}$–invariant relies on the identification of it with the appropriate multiple of the d–invariant of the spin 3–manifold at hand.

Proof of Theorem 1.4 Let Γ be a given negative definite rational plumbing tree with a spin structure $\mathbf{s}$ (represented by its Wu set $S \subset V$). Let $m_{\Gamma,S}$ denote the number of those vertices $v_i \in \Gamma$ which are not in S but $-n_i$ of the neighbours of v_i are in S. (Notice that by the rationality of Γ this means that v_i has $-n_i$ or $-n_i+1$ neighbours and either all or all but one neighbours are in S.)

The proof of the theorem will proceed by induction on $m_{\Gamma,S}$. Let us start with the easy case when $m_{\Gamma,S}=0$, that is, for any vertex v_i in Γ we have

$$c_S(\Sigma_{v_i}) < -n_i. \tag{5.1}$$

For $v_i \in S$ we have $c_S(\Sigma_{v_i}) = n_i$, while if v_i is not in S then $c_S(\Sigma_{v_i})$ is the number of neighbours of v_i which are in S. In particular, $0 \leq c_S(\Sigma_{v_i}) \leq d_i$ holds for all v_i not in S. Since c_S is characteristic, Inequality (5.1) actually means that $c_S(\Sigma_{v_i}) \leq -n_i - 2$. In conclusion, c_S satisfies $n_i \leq c_S(\Sigma_{v_i}) \leq -n_i - 2$ for all vertices, hence c_S is a *terminal vector* in the sense of [19]. By subtracting twice the Poincaré duals of the homology classes represented by surfaces corresponding to vertices in S, eventually we get a path back to a vector $K \in H^2(X_\Gamma;\mathbb{Z})$ which satisfies $K(\Sigma_{v_i}) = -n_i$ for $v_i \in S$ and $K(\Sigma_{v_i}) \geq -d_i \geq n_i + 2$ if v_i is not in S. This means that K is an *initial vector*, hence c_S is in a *full path* (again, in the terminology of [19]). By the identification of [13] this implies that c_S gives rise to a Heegaard Floer homology element in $\widehat{HF}(Y,\mathbf{s})$ of degree $\frac{1}{4}(c_S^2 - 3\sigma(X_\Gamma) - 2\chi(X_\Gamma))$. (Here, as costumary in Heegaard Floer theory, $\chi(X_\Gamma)$ is understood as the Euler characteristic of the cobordism we get from S^3 to Y_Γ by deleting a point from X_Γ.) Since Y_Γ is an L–space, this degree must be equal to $d(Y,\mathbf{s})$. On the other hand, since Γ is negative definite, $\chi(X_\Gamma) = -\sigma(X_\Gamma)$, hence the above formula for the degree shows that $-\overline{\mu}(Y_\Gamma,\mathbf{s}) = c_S^2 - \sigma(X_\Gamma)$ is equal to $4d(Y_\Gamma,\mathbf{s})$.

Next we assume that the statement is proved for graphs (Γ, S) with $m_{\Gamma,S} \leq m-1$. In the inductive step we will utilize the exact triangle for Heegaard Floer homologies, proved for a surgery triple, see [18, 9]. To this end, fix a graph Γ with Wu set S and corresponding spin structure $\mathbf{s} \in Spin(Y_\Gamma)$ having $m_{\Gamma,S} = m > 0$ and let v denote a vertex with $-n_i$ neighbours in S. (Consequently v is not in S.) Consider the following plumbing graphs (with spin structures specified by the various Wu sets):

- Let Γ', Γ'' denote the same graphs as Γ with the alteration of the framing on v from n_i to $n_i - 2$ and $n_i - 4$, resp. It is easy to see that S still provides Wu sets S', S'' (and hence spin structures $\mathbf{s}', \mathbf{s}''$) for Γ' and Γ''. Notice that $m_{\Gamma',S'} = m_{\Gamma'',S''} = m_{\Gamma,S} - 1$. In addition, since v was not in the Wu set S, we see directly that $\overline{\mu}(Y_\Gamma,\mathbf{s}) = \overline{\mu}(Y_{\Gamma'},\mathbf{s}') = \overline{\mu}(Y_{\Gamma''},\mathbf{s}'')$.

Laufer's algorithm shows that Γ', Γ'' are also rational.

- Let Γ_1 be the disjoint union of Γ' and the graph on a single vertex w with framing (-2). The set S_1 is chosen as $S \cup \{w\}$. Simple computation shows that $\overline{\mu}(Y_{\Gamma_1}, \mathbf{s}_1) = \overline{\mu}(Y_\Gamma, \mathbf{s}) + 1$. In the surgery picture for Y_{Γ_1} resulting from the plumbing let K denote the unknot linking the unknot corresponding to $v \in \Gamma$ chosen above and the new (-2)–framed circle (corresponding to w) once.

Attach a 4–dimensional 2–handle to the 3–manifold Y_{Γ_1} along K with framing (-1). The resulting coboridsm will be denoted by X.

Lemma 5.1 *The result of the above surgery is Y_Γ, and the spin structure $\mathbf{s}_1$ on Y_{Γ_1} defined by S_1 extends as a spin structure to provide a spin cobordism $(X, \mathbf{t})$ from $(Y_{\Gamma_1}, \mathbf{s}_1)$ to $(Y_\Gamma, \mathbf{s})$.*

Proof By sliding K and the handle corresponding to w down, the first statement is obvious. The extension follows from the fact that for the graph containing Γ_1 together with K, the vertex corresponding to K is not in S_1. $\square$

Notice that by induction on $m_{\Gamma,S}$ the statement of the theorem holds for Γ_1 and Γ', hence we have that $-4d(Y_{\Gamma_1}, \mathbf{s}_1) = \overline{\mu}(Y_{\Gamma_1}, \mathbf{s}_1) = \overline{\mu}(Y_\Gamma, \mathbf{s}) + 1$ and $-4d(Y_{\Gamma'}, \mathbf{s}') = \overline{\mu}(Y_{\Gamma'}, \mathbf{s}') = \overline{\mu}(Y_\Gamma, \mathbf{s})$.

If the spin cobordism $(X, \mathbf{t})$ of Lemma 5.1 between $(Y_{\Gamma_1}, \mathbf{s}_1)$ and $(Y_\Gamma, \mathbf{s})$ induces a nontrivial map on the Ozsváth–Szabó homology groups, we can easily conclude the argument: since a negative definite spin cobordism with $\chi = 1$ and $\sigma = -1$ shifts degree for Ozsváth–Szabó homologies by $\frac{1}{4}$, the unique nontrivial element of $\widehat{HF}(Y_{\Gamma_1}, \mathbf{s}_1)$ maps to the unique nontrivial element of $\widehat{HF}(Y_\Gamma, \mathbf{s})$ of degree $d(Y_{\Gamma_1}, \mathbf{s}_1) + \frac{1}{4} = d(Y_\Gamma, \mathbf{s})$, reducing the proof to elementary arithmetics. The nontriviality of the map $F_{X,\mathbf{t}}$ is, however, not so obvious. Let us set up the exact triangle defined by the surgery triple $(Y_{\Gamma_1}, Y_\Gamma, Y_{\Gamma''})$ along the knot $K \subset Y_{\Gamma_1}$:

$$\begin{array}{ccc} \widehat{HF}(Y_{\Gamma_1}) & \xrightarrow{F_X} & \widehat{HF}(Y_\Gamma) \\ {\scriptstyle F_V} \nwarrow & & \swarrow {\scriptstyle F_U} \\ & \widehat{HF}(Y_{\Gamma''}) & \end{array}$$

for the identification of the two manifolds $Y_\Gamma, Y_{\Gamma''}$ simple Kirby calculus arguments are needed. Recall that the map F_X is the sum of all $F_{X,\mathbf{u}}$ for $\mathbf{u} \in Spin^c(X)$.

We claim first that $F_X(\mathbf{s}_1)$ has nonzero $\mathbf{s}$–component. Since U is not negative definite, the map F_U^∞ vanishes, and since Y_Γ is an L–space, this implies the same for the maps F_U^+ and F_U. In particular, by exactness we get that F_V is injective and F_X is surjective. Suppose that $F_X(\mathbf{s}_1)$ has zero $\mathbf{s}$–component. Then $F_X(\mathbf{s}_1) = a + \overline{a}$ for some $a \in \widehat{HF}(Y_\Gamma)$, where a is a formal sum of some spin^c structures on Y_Γ and $\overline{a}$ denotes the sum of the conjugate spin^c structures, cf. [10]. By surjectivity now there is $x \in \widehat{HF}(Y_{\Gamma_1})$ with $F_X(x) = a$, hence $\mathbf{s}_1 + x + \overline{x}$ is in the kernel of F_X, so in the image of F_V. If $F_V(y) = \mathbf{s}_1 + x + \overline{x}$ then the same holds for $\overline{y}$, hence by the injectivity of F_V the element y satisfies $\overline{y} = y$. In order $F_V(y)$ to have spin component, y must have a spin component, hence we have found some spin and spin^c structures $\mathbf{z} \in Spin(Y_{\Gamma''})$ and $\mathbf{t}' \in Spin^c(V)$ with $F_{V,\mathbf{t}'}(\mathbf{z}) = \mathbf{s}_1$. By the uniqueness of extensions this $\mathbf{z}$ must be $\mathbf{s}''$, and the spin^c cobordism $(V, \mathbf{t}')$ connecting $\mathbf{z} = \mathbf{s}''$ and $\mathbf{s}_1$ must be spin. Therefore the grading shift between the elements $\mathbf{s}''$ and $\mathbf{s}_1$ is $\frac{1}{4}$. This implies

$$d(Y_{\Gamma''}, \mathbf{s}'') + \frac{1}{4} = d(Y_{\Gamma_1}, \mathbf{s}_1). \tag{5.2}$$

Recall that

$$\overline{\mu}(Y_{\Gamma''}, \mathbf{s}'') = \overline{\mu}(Y_\Gamma, \mathbf{s}) = \overline{\mu}(Y_{\Gamma_1}, \mathbf{s}_1) - 1. \tag{5.3}$$

Since by induction for the spin 3–manifolds $(Y_{\Gamma_1}, \mathbf{s}_1)$ and $(Y_{\Gamma''}, \mathbf{s}'')$ the invariant $\overline{\mu}$ actually computes the correction term, that is, $-4d(Y_{\Gamma_1}, \mathbf{s}_1) = \overline{\mu}(Y_{\Gamma_1}, \mathbf{s}_1)$ and $-4d(Y_{\Gamma'}, \mathbf{s}') = \overline{\mu}(Y_{\Gamma'}, \mathbf{s}')$, Equations (5.2) and (5.3) contradict each other. Therefore the element $F_X(\mathbf{s}_1)$ has nontrivial $\mathbf{s}$–component, verifying our claim.

The nontriviality of F_X between $\mathbf{s}_1$ and $\mathbf{s}$, however, implies that there is a connecting spin structure $\mathbf{t}$ with $F_{X,\mathbf{t}}(\mathbf{s}_1) = \mathbf{s}$, cf. [10, Lemma 3.3]. Consequently the degree shift given by $F_{X,\mathbf{t}}$ is $\frac{1}{4}$, hence the inductive step concludes the proof of Theorem 1.4. □

Proof of Theorem 1.1 Combining Propositions 4.1 and 4.2 with the identification of Theorem 1.4 the proof follows at once. □

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