GLUCK TWIST ON A CERTAIN FAMILY OF 2-KNOTS

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ABSTRACT. We show that by performing the Gluck twist along the 2-knot K^2_{pq} derived from two ribbon presentations of the ribbon 1-knot $K(p,q)$ we get the standard 4-sphere $\mathbb{S}^4$. In the proof we apply Kirby calculus.

1. Introduction

The paper of Freedman-Gompf-Morrison-Walker [5] about the potential application of Khovanov homology in solving the 4-dimensional smooth Poincaré conjecture (SPC4) revitalized this important subfield of topology. A sequence of papers appeared, some settling 30-year-old problems ([1, 7]), some introducing new potential exotic 4-spheres ([11]) and further works showing that the newly introduced examples are, in fact, standard [2, 12].

One underlying construction for producing examples of potential exotic 4-spheres is the Gluck twist along an embedded S^2 (a *2-knot*) in the standard 4-sphere $\mathbb{S}^4$. In this construction we remove the tubular neighbourhood of the 2-knot and glue it back with a specific diffeomorphism. (For a more detailed discussion, see Section 3.) In turn, any 2-knot in $\mathbb{S}^4$ admits a *normal form*, and hence can be described by an ordinary knot in S^3, together with two sets of ribbon bands (determining the 'southern' and 'northern' hemispheres of the 2-knot). Applying standard ideas of Kirby calculus (see, for example, [8]) the complement of a 2-knot, and from there the result of the Gluck twist, can be explicitly drawn. From such a presentation then we derive the following result.

Theorem 1.1. *Consider the knot $K(p,q)$ depicted by Figure 1, and use the bands b_1 and b_2 to construct the southern and northern hemispheres of a 2-knot $K^2_{pq} \subset \mathbb{S}^4$. Then Gluck twist along the 2-knot K^2_{pq} provides the 4-sphere with its standard smooth structure.*

Remark 1.2. For certain choices of p and q the 1-knot $K(p,q)$ can be identified more familiarly: for instance, $K(0,0)$ is isotopic to $F\#F = F\#m(F)$, where F is the Figure-8 knot (isotopic to its mirror image $m(F)$), $K(1,-1)$ is the 8_9 knot, while $K(1,1)$ is 10_{155} in the standard knot tables. Notice that in [3] the knot 8_9 defines the 2-knot along which the Gluck twist is performed, although the bands used in [3] are potentially different from b_1 and b_2 used in the theorem above, cf. [3, Fig. 16].

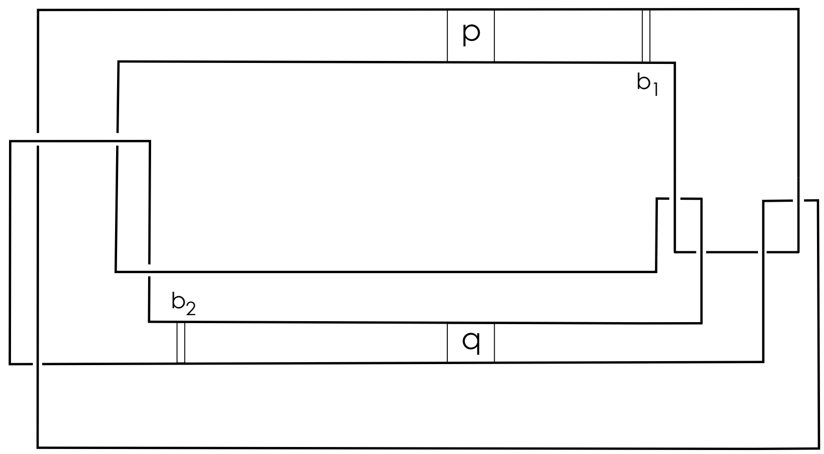

FIGURE 1. The knot $K(p,q)$ with the two ribbon bands b_1 and b_2, giving rise to the 2-knot $K^2_{pq} \subset \mathbb{S}^4$.

Before we prove the above result, in Sections 2 and 3 we briefly invoke basic facts about 2-knots, the Gluck twist, and the derivation of a Kirby diagram for the result of the Gluck twist along a 2-knot given by a ribbon 1-knot and two sets of ribbons. In Section 4 then a simple Kirby calculus argument provides the proof of Theorem 1.1. (A slightly different argument, still within Kirby calculus, for the same result is given in an Appendix.)

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2. Ribbon 2-disks and related 2-knots

Every 2-knot is equivalent to one in normal form [4], that is, for a 2-knot $K \subset \mathbb{S}^4$ there is an ambiently isotopic $K' \approx S^2 \subset \mathbb{R}^4$ (i.e. $\mathbb{S}^4 \setminus \infty$) with a projection $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ such that p restricted to K' gives a Morse function with the properties:

1. $K' \subset \mathbb{R}^3 \times [-c, c]$ some $c > 0$,
2. all index-0 critical points are in $K' \cap \mathbb{R}^3 \times \{-c\}$,
3. all index-1 critical points with negative p-value give fusion bands within $K' \cap \mathbb{R}^3 \times (-c, 0)$,
4. $K' \cap \mathbb{R}^3 \times \{0\}$ is a single 1-knot k,
5. all index-1 critical points with positive p-value give fission bands within $K' \cap \mathbb{R}^3 \times (0, c)$, and
6. all index-2 critical points are in $K' \cap \mathbb{R}^3 \times \{c\}$.

In particular, this means that any 2-knot K is formed from the union of two *ribbon 2-disks* glued together along their boundaries which is the same (ribbon) 1-knot k for both. Since such a (ribbon disk) hemisphere D of a 2-knot has a handlebody with only 0- and 1-handles, we can construct a Kirby diagram for any ribbon disk complement in the 4-disk D^4, and from there for any 2-knot in $\mathbb{S}^4$ as follows (cf. [8, Chapter 6]).

Lemma 2.1. *Let K be a 2-knot given by the union of two ribbon disks with equatorial 1-knot k, lower hemisphere ribbon presentation $\mathcal{B}_1 = \{b_1, b_2, \dots, b_m\}$, and upper hemisphere ribbon presentation $\mathcal{B}_2 = \{b'_1, b'_2, \dots, b'_n\}$, as above. Then a handlebody for $\mathbb{S}^4 \setminus K$ can be constructed by the following algorithm:*

(1) *at each $\mathcal{B}_1$ ribbon, split from k into a dotted circle component and add a 2-handle as in the left diagram of Figure 2,*
(2) *at each $\mathcal{B}_2$ ribbon, add the 2-handle as in the right diagram of Figure 2, and*
(3) *add a 3-handle for each ribbon of $\mathcal{B}_2$ and then a single 4-handle.*

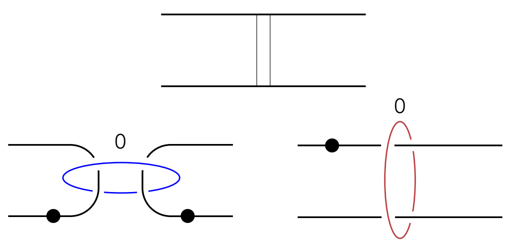

FIGURE 2. Handles from a ribbon move in the lower hemisphere (left) and upper hemisphere (right).

Proof. Let us start with describing the complement of one ribbon disk D; let $X = D^4 \setminus D$ denote the ribbon disk complement. Its handlebody starts with a 0-handle (four ball) X_0. Then, for each 0-handle of D which is carved out, a (4-dimensional) 1-handle is added to X_0 to form the 1-handlebody X_1. Finally, the ribbons (or 1-handles) of D each yield 2-handles in the complement, and these are attached along curves formed from the union of push-offs of the core 1-disk of the ribbons (cf. [8, Section 6.2]). The attaching circles have 0-framings since push-offs of the core do not link. Therefore the result can be easily presented from a diagram of the equatorial knot k by locally replacing the bands with the diagram presented on the left of Figure 2.

Next consider the special case when K is the 2-knot which we get by doubling the disk D, i.e. $K = D \cup \overline{D}$. In this case a Kirby diagram for the knot exterior $Y = \mathbb{S}^4 - (D \cup \overline{D})$ can be built up easily from the handlebody decomposition of D. This amounts to taking the above disk complement X and adding a second "upside-down" copy of X (relative to the carved out 2-disk D, so that the result is still a manifold with boundary). For each ribbon in the upper hemisphere, again we add a 2-handle to the complement. However with D "turned upside-down" in the upper hemisphere, the ribbons have cores and co-cores opposite to their counterparts in the lower hemisphere. Consequently, the 2-handles added in the upper hemisphere's complement have attaching curves formed from the union of two co-cores of the original ribbons of D. This is shown in figure 3, where a pair of 0-handles of K and a 1-handle fusing them together gives the handlebody configuration in the complement on the right, with the vertical 2-handle depicting the "upside-down" copy coming from the upper hemisphere of K (and "dotted circles" depicting the 4-dimensional 1-handles). Moreover, for each of the upper hemisphere 2-handles (corresponding to 0-handles of D), we get a 3-handle, and then finally a 4-handle to complete the description of the complement of K. For a similar discussion see [8, Exercise 6.2.11(b)].

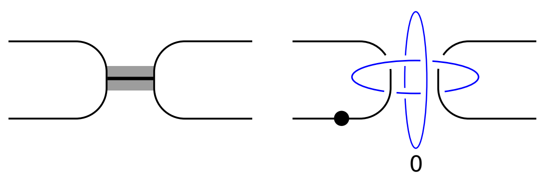

FIGURE 3. Handles in Y from a 1-handle in (each copy of) D.

Finally consider the general case, when the 2-knot K is formed from two disks D_1 and D_2 (as 'southern' and 'northern' hemispheres); i.e. $K = D_1 \cup \overline{D_2}$. The recipe above provides a diagram for $D^4 - D_1$ and for $\mathbb{S}^4 - (D_2 \cup \overline{D_2})$. We only need to replace $D^4 - D_2$ with $D^4 - D_1$ to get a diagram for $\mathbb{S}^4 - K$. This simply amounts to finding a diffeomorphism between the boundaries of $D^4 - D_1$ and $D^4 - D_2$, and then pulling back the attaching circles of the 2-handles of $D^4 - \overline{D_2}$ to the diagram of $D^4 - D_1$. By converting the dots to 0-framings, sliding the (once dotted, now 0-framed) circles on each other and then canceling (in the 3-dimensional sense) the obvious handle pairs, we see that both $\partial(D^4 - D_1)$ and $\partial(D^4 - D_2)$ are diffeomorphic to the result of 0-surgery along the equatorial knot k. Using this diffeomorphism, the pull-back provides the attaching circle given in the statement, concluding the proof. $\square$

3. THE GLUCK TWIST AND KIRBY DIAGRAMS

Suppose that $K \subset \mathbb{S}^4$ is a given 2-knot in the 4-sphere. Remove a normal neighborhood νK of $K \subset \mathbb{S}^4$ from the 4-sphere and reglue $S^2 \times D^2$ by the diffeomorphism of the boundary $\partial(S^2 \times D^2) \approx \partial(\mathbb{S}^4 \setminus \nu K) \approx S^2 \times S^1$

$$\mu : S^2 \times S^1 \longrightarrow S^2 \times S^1,$$

given by $(x, \theta) \overset{\mu}{\mapsto} (rot_\theta(x), \theta)$ for rot_θ the rotation with angle θ of the 2-sphere about the axis through its poles.

Definition 3.1. The above construction is called the *Gluck twist* along the 2-knot $K \subset \mathbb{S}^4$. The result of the Gluck twist along the 2-knot $K \subset \mathbb{S}^4$ will be denoted by $\Sigma(K)$.

Since the result $\Sigma(K)$ of a Gluck twist is simply connected, Freedman's celebrated theorem implies that $\Sigma(K)$ is homeomorphic to $\mathbb{S}^4$.

For a 2-sphere K embedded in the 4-sphere, a handlebody for νK consists of a 0-handle plus one 2-handle attached along a 0-framed unknot. This can also be built "upside down" from its boundary $S^2 \times S^1$ by attaching the (dualized) 2-handle h_K along any meridian $\{\text{pt.}\} \times S^1$ of the sphere, and then attaching the dualized 0-handle as a 4-handle. Therefore, if a handlebody diagram for the knot exterior $Y = \mathbb{S}^4 \setminus \nu K$ is given, then one can reconstruct $\mathbb{S}^4$ by attaching the 2-handle h_K along a 0-framed meridian of any 1-handle h corresponding to a 0-handle of K. The homotopy sphere $\Sigma(K)$ resulting from the Gluck twist on K then can be formed from Y by attaching the 2-handle h_K with ± 1-framing along the same meridional circle of the 1-handle h (see also [8, Exercise 6.2.4]). Note that all the further attaching circles of 2-handles linking the 1-handle h can be slid off h by the use of h_K, and then h and h_K can be cancelled against each other. Therefore, in practice the presentation of the Gluck twist along K amounts to blowing down one of the dotted circles corresponding to a 0-handle of K as if the dotted circle was a (-1)-framed (or a $(+1)$-framed, up to our choice) unknot. Notice also that in the preceding section we presented a diagram for Y which admits a 4-handle. Since in gluing $S^2 \times D^2$ back we add a further 4-handle, one of them can be cancelled against a 3-handle.

In [6] a further alternative of the effect of the Gluck twist is presented. Since the rotation rot_θ involved in the gluing map fixes both poles N and S of the 2-sphere, there are two resulting fixed circles $\{N\} \times S^1$ and $\{S\} \times S^1$ of μ. Presenting $S^2 \times D^2$ as the union of a 0-handle, a 1-handle and two 2-handles (or in the upside down picture two 2-handles h_K and h'_K, a 3- and a 4-handle), one can construct $\Sigma(K)$ from Y by attaching the two 2-handles h_K and h'_K (one along $\{N\} \times S^1$ and one along $\{S\} \times S^1$ with framings $(+1)$ and (-1), respectively) and a 3- and 4-handle, where the two attaching circles are meridional circles of two dotted circles (corresponding to two 0-handles of K). Once again, since the 2-handles can be slid over h_K and h'_K, and then these 2-handles can cancel the corresponding dotted circles, in practice the Gluck twist along K amounts to simply blowing down two dotted circles as if one were a $(+1)$-, the other a (-1)-framed unknot (and then adding a 3- and a 4-handle). As before, one 3-handle cancels one of the two 4-handles appearing in the decomposition.

In conclusion, if a 2-knot K in normal form is given in $\mathbb{S}^4$ by a ribbon knot k with two sets of bands $\mathcal{B}_1$ and $\mathcal{B}_2$, then the above description provides a simple algorithmic way of producing a handle decomposition of the result $\Sigma(K)$ of the Gluck twist along K. Notice furthermore that if $|\mathcal{B}_1| = 1$ (or

$|\mathcal{B}_2| = 1$) then the resulting decomposition can be chosen not to contain any 1-handles.

Remark 3.2. For some special classes of 2-knots K, the diffeomorphism type of $\Sigma(K)$ is well-understood: in [9] Gordon proved that $\Sigma(K)$ is diffeomorphic to $\mathbb{S}^4$ for any twist-spun 2-knot K. Then in [10] Melvin showed that every ribbon 2-knot K has $\Sigma(K)$ standard as well. Additionally, since any ribbon 2-knot is the double of a ribbon 2-disk, [8, Exercise 6.2.11(b)] gives an alternate proof of this second result.

4. A family of 2-knots

For p, q (possibly non-distinct) integers let $K(p, q)$ be the knot of Figure 1. This is a ribbon knot of 1-fusion, that is, there is a ribbon presentation of $K(p, q)$ such that performing the indicated single ribbon move transforms the knot into a two-component unlink. In fact, there are two apparent choices for the single ribbon move (or two apparently distinct ribbon presentations). These are indicated by the fine-lined bands b_i $(i = 1, 2)$ of Figure 1. Either of these ribbon presentations corresponds to a ribbon 2-disk which we will denote by $D(p, q)_i$ $(i = 1, 2)$.

Definition 4.1. Define the 2-knot K^2_{pq} as the union $(D^4, D(p, q)_1) \cup \overline{(D^4, D(p, q)_2)}$.

Following the recipe of Section 3 we exhibit a handlebody description of the result $\Sigma(K^2_{pq})$ of the Gluck twist along K^2_{pq}. In doing so, first we present a diagram for $Y_{pq} = \mathbb{S}^4 - \nu K^2_{pq}$ in Figure 4 below:

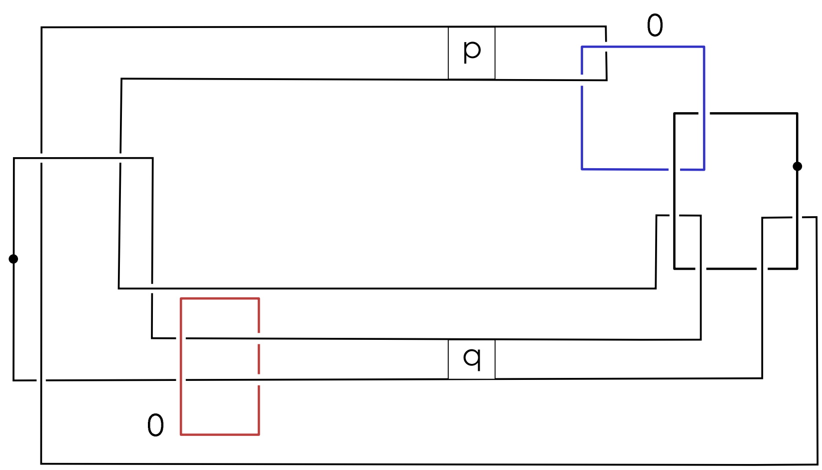

Figure 4. Kirby diagram for Y_{pq} minus two 3-handles and one 4-handle.

Figures 5 through 8 demonstrate an isotopy of the above diagram of Y_{pq} into a form where the 1-handles are visibly separated. In Figure 5 we have the result of undoing the p-twist in the first 1-handle and then starting to isotope the 2-handle through the second 1-handle. In Figure 6, the q-twist of the second 1-handle is undone by twisting the indicated four strands of the 2-handle. Further isotopies then finally produce Figure 8, where the 1-handles are conveniently separated.

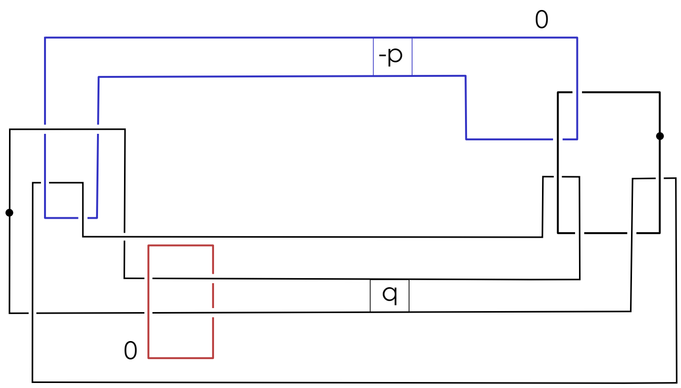

FIGURE 5. Transferring the p-twist from the dotted circle to the 0-framed unknot.

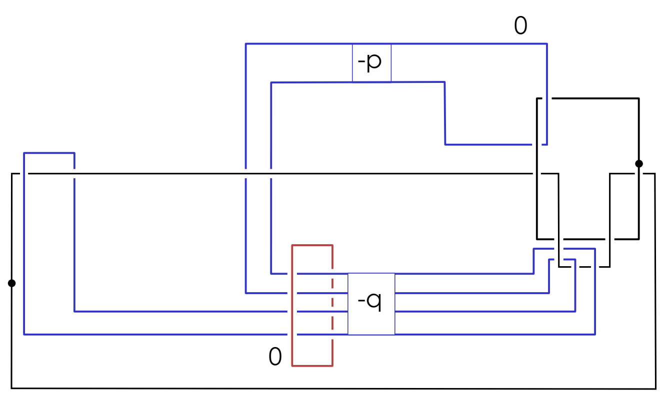

FIGURE 6. A further isotopy of the diagram of Figure 5.

With this handlebody depiction of Y_{pq} we can begin to analyze the 2-knot K^2_{pq}, compute its knot group directly and show the following:

Proposition 4.2. *The two 2-knots K^2_{pq} and K^2_{rs} are distinct provided that the parities of the pairs $\{p, q\}$, $\{r, s\}$ (up to reordering within a pair) are distinct.*

Proof. Choosing orientations on generators of π_1 as in Figure 8, we obtain $\pi_1(Y_{pq}) \cong \langle x, y \mid r_{pq}\rangle$, where the relation r_{pq} takes one of the following four

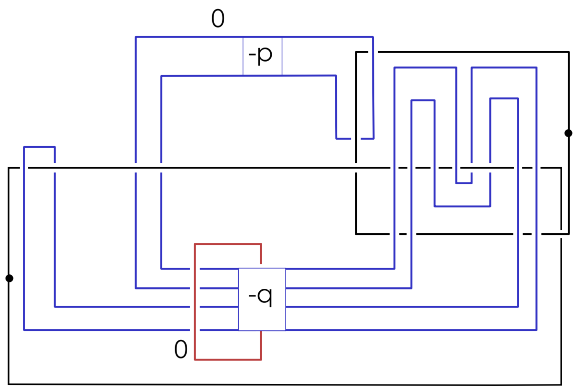

FIGURE 7. Isotopy to separate the dotted circles.

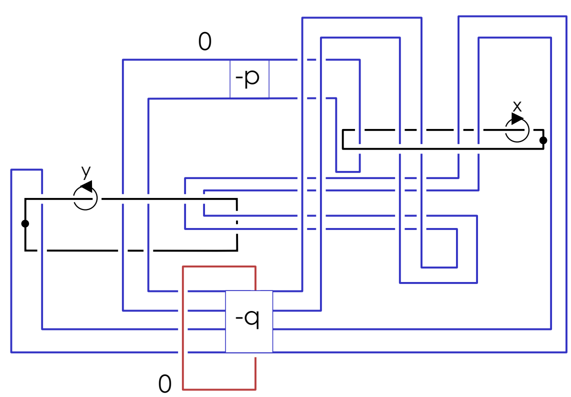

FIGURE 8. The knot complement Y_{pq} (minus two 3-handles and one 4-handle) with 1-handles separated (and generators of π_1 labeled).

forms:

p even, q even	$r_{pq}:\ xyxy^{-1}x^{-1}yxyx^{-1}y^{-1}$
p odd, q odd	$r_{pq}:\ xyxyx^{-1}y^{-1}xy^{-1}x^{-1}y^{-1}$
p odd, q even	$r_{pq}:\ xyxy^{-1}x^{-1}y^{-1}xyx^{-1}y^{-1}$
p even, q odd	$r_{pq}:\ xyxyx^{-1}yxy^{-1}x^{-1}y^{-1}$

Using Fox calculus it is easy to see that each 2-knot does have a principal first elementary ideal and hence, an Alexander polynomial:

p even, q even	$\Delta(t) = -t^2 + 3t - 1,$
p odd, q odd	$\Delta(t) = 1 - t + 2t^2 - t^3,$
p odd, q even	$\Delta(t) = 2 - 2t + t^2,$
p even, q odd	$\Delta(t) = 2t^2 - 2t + 1.$

This gives three clearly distinguished cases for a pair $\{p, q\}$. In particular, K^2_{pq} and K^2_{rs} have distinct Alexander polynomials if the pairs of parities are distinct. □

Remark 4.3. In the three cases other than p, q both even, Δ_K is asymmetric and therefore it is not the Alexander polynomial of a 1-knot. Consequently, K^2_{pq} cannot be a spun knot if p, q are not both even.

Now we are ready to provide a proof of the main result of the paper:

Proof of Theorem 1.1. After isotoping the small 2-handle until it becomes parallel to the rightmost dotted circle and blowing down the two dotted circles (as the implementation of the Gluck twist demands) we arrive at Figure 9. Finally, figure 10 (obtained by performing the indicated handle slide in 9) unravels to give a pair of disjoint 0-framed 2-handles which cancel against the 3-handles to give $\mathbb{S}^4$. □

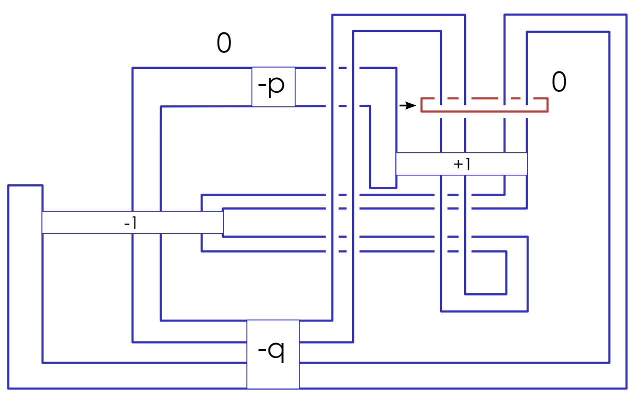

FIGURE 9. The diagram of $\Sigma(K^2_{pq})$ (minus two uniquely attached 3-handles and one 4-handle).

APPENDIX: ALTERNATE PROOF OF THEOREM 1.1

For the particular case of the 2-knot K^2_{pq} there is, in fact, a way to see that the Gluck twist leaves $\mathbb{S}^4$ standard without separating the 1-handles first.

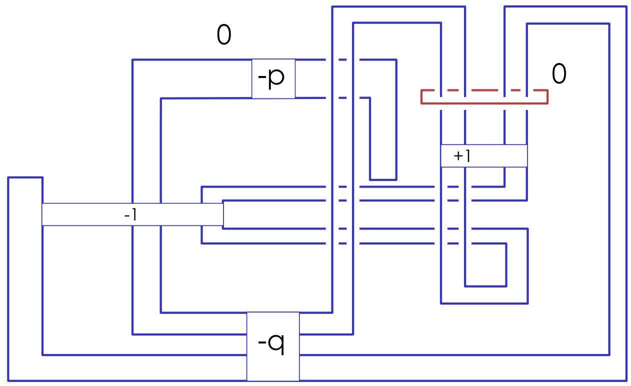

FIGURE 10. After sliding one 2-handle over the other one (as instructed by the arrow in Figure 9), we are left with two unlinked 2-handles, which can be seen to cancel against the 3-handles.

Proof. Starting in Y_{pq} (cf. Figure 4), realize the Gluck twist on K^2_{pq} by adding a (-1)-framed 2-handle to a 1-handle (instead of just immediately blowing down the dotted 1-handle). Slide the lower 0-framed 2-handle over the upper 1-handle and off of the q-twisted 1-handle to get Figure 11. Now in Figure 11 slide the (-1)-framed 2-handle over the left-most 0-framed 2-handle and off of its 1-handle and the right-most 2-handle. Next, in Figure 12 slide the 0-framed 2-handle on the left over the (-1)-framed 2-handle (which changes its own framing to -1)

and use the remaining 0-framed 2-handle to unhook the other two 2-handles from each other. This results in a collection of Hopf links, and standard handle cancellations then show that $\Sigma(K^2_{pq})$ is, indeed, diffeomorphic to the standard 4-sphere $\mathbb{S}^4$.

□

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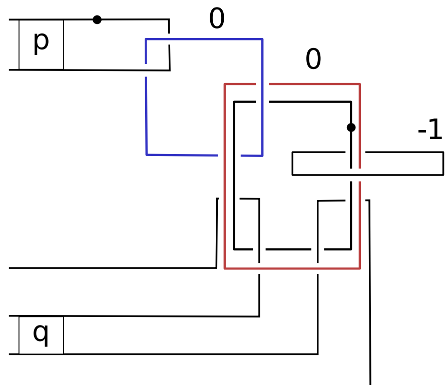

FIGURE 11. The relevant portion of the diagram of $\Sigma(K^2_{pq})$ (again, minus 3- and 4-handles) after one handle slide.

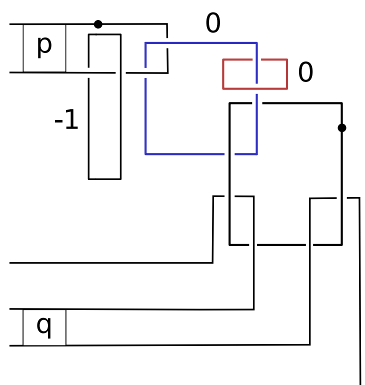

FIGURE 12. The relevant portion of the diagram after sliding the (-1)-framed 2-handle and isotoping the right-most 0-framed component.

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