On the reconstruction of combinatorial structures from line-graphs

Péter L. Erdős *

ABSTRACT: We present some generalizations of reconstruction results of H. Whitney, C. Berge, J. C. Fournier and L. Lovász. Certain classes of hypergraphs are described which are determined (up to isomorphism) by their line-graphs. There are Hamming schemes, t-designs and finite vector spaces among the classes described.

1. Introduction

Let $G = (E_i : i \in M)$ and $G' = (F_i : i \in M)$ be two connected simple graphs with $|M| > 2$ edges. H. Whitney ([Wh]) had

Theorem 1. (H. Whitney) *Whenever the minimum valencies in the above graphs are at least 4 then*

$$|E_i \cap E_j| = |F_i \cap F_j| \qquad (i, j \in M)$$

implies that $G \simeq G'$. ∎

In other words: the line-graphs with the degree condition are isomorphic if and only if the graphs themselves are isomorphic.

This paper presents some new analogous results for hypergraphs. (For the notins of hypergraph theory not defined here we follow Berge [B2]).

Definition. The hypergraphs $\mathcal{H} = (E_i : i \in M)$ and $\mathcal{H}' = (F_i : i \in M)$ are *isomorphic* if there exists a bijection α between their vertex sets and a permutation π of M such that $\alpha(E_i) = F_{\pi(i)}$ for each $i \in M$. Denote the set of the *automorphisms* of the hypergraph $\mathcal{H}$ by $\mathrm{Aut}(\mathcal{H})$.
For every $I \subseteq M$ we put

$$E_I = \bigcup_{i \in I} E_i.$$

The *line-graph* $\mathcal{L}(\mathcal{H})$ of the hypergraph $\mathcal{H}$ is defined as follows: the underlying set of $\mathcal{L}(\mathcal{H})$ is the set of edges of $\mathcal{H}$ and the pair (E_i, E_j) is an edge of $\mathcal{L}(\mathcal{H})$ ($E_i \neq E_j$, $i, j \in M$) iff $E_i \cap E_j \neq \emptyset$. Every automorphism $\alpha \in \mathrm{Aut}(\mathcal{H})$ *induces* an automorphism a_α of $\mathcal{L}(\mathcal{H})$ in a natural way, namely

$$a_\alpha(E_i) = \{\alpha(x) : x \in E_i\} \qquad (i \in M.)$$

Finally let K_n^r denote the r-uniform complete hypergraph on n vertices.

The following result was proved by Lovász ([Lo]):

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Theorem 2. *When $a \in \mathrm{Aut}(\mathcal{L}(K_n^r))$ where $n > 2r$ then there exists an automorphism $\alpha \in \mathrm{Aut}(K_n^r)$ which induces the automorphism a i.e. $a = a_\alpha$.*

Lovász strengthened an earlier theorem of C. Berge:

Corollary 3. ([B1]) *Let $\mathcal{H} = (E_i : i \in M)$ be isomorphic to K_n^r where $n > 2r$. Suppose that*

$$|E_i \cap E_j| = |F_i \cap F_j| \qquad (i,\ j \in M).$$

Then $\mathcal{H}'$ is isomorphic to $\mathcal{H}$. ∎

(In fact, Berge proved a bit more, namely that this isomorphism is *strong*, but from the Lovász theorem this can be derived easily.) In the same year J.C. Fournier proved the analogous result under the condition $n < 2r$ ([F1]). Two years later Fournier found a common proof for both results ([F2]).

2. A general reconstruction principle

First of all we give a very simple proof for Theorem 2. The original proof applied counting to prove the existence of the wanted permutation α. The following proof was developed by Z. Füredi and the author ([EF]) to construct the prescribed permutation. A very similar proof is contained in the paper of Poljak and Rödl ([PR]), also.

This proof is based on a well-known theorem of Paul Erdős, Chao Ko and Richard Rado ([EKR]). Let a hypergraph be called *intersecting* if the pairwise intersections of the edges are not empty.

Erdős - Ko - Rado Theorem. *If $\mathcal{H} = (E_i : i \in M)$ is an r-uniform intersecting hypergraph, where $|E_M| > 2r$, then*

$$|M| < \binom{|E_M| - 1}{r - 1}$$

and the equality holds iff there is a point $x \in E_M$ for which $\cap\mathcal{H} = \{x\}$. In other words:

$$\mathcal{H} = \mathcal{H}_x = \{E \subset E_M : |E| = r, x \in E\}. ∎$$

Proof of the Theorem 2. ([EF]) Let $\mathcal{C}_x$ denote the system of edges of $\mathcal{H}$ containing the vertex x ($x \in X = \cup K_n^r$), i. e. let

$$\mathcal{C}_x = \{E \in K_n^r : x \in E\}.$$

Then $\mathcal{C}_x$ is an intersecting set system, and $|\mathcal{C}_x| = \binom{n-1}{r-1}$. Since a is an automorphism of the line-graph $\mathcal{L}(\mathcal{H})$ therefore $\{a(E) : E \in \mathcal{C}_x\} = \mathcal{C}'_x$ is intersecting furthermore due to the Erdős - Ko - Rado Theorem, there exists a point $\alpha(x) \in X$ for which $\mathcal{C}'_x = a(\mathcal{C}_x) = \mathcal{H}_{\alpha(x)}$. One can prove easily that the map $\alpha : X \longrightarrow X$ is a bijection and $a = a_\alpha$. ∎

The previous proof is just a special case of a rather general reconstruction principle. This is the following:

EKR reconstruction principle: Let S be a finite underlying set and $\mathcal{F}$ be the system of all subsets of S of a certain kind. Let the notion of *pairwise intersection* be defined in $\mathcal{F}$ (not necessarily by intersection in S). We say, that the pair $(S, \mathcal{F})$ satisfies the *EKR-property* with the function $f(S, \mathcal{F})$ if every subfamily $\mathcal{G}$ of $\mathcal{F}$ with pairwise intersecting elements and with cardinality $|\mathcal{G}| \geq f(S, \mathcal{F})$ satisfies the condition $|\bigcap \mathcal{G}| = 1$.

Metatheorem. *Let $\mathcal{H}$ be a subhypergraph of $\mathcal{F}$ satisfying the following* valency condition *in every vertex $x \in S$:*

$$v_{\mathcal{H}}(x) := |\{E \in \mathcal{H} : x \in E\}| \geq f(S, \mathcal{F}).$$

Let $a \in \mathrm{Aut}(L(\mathcal{H}))$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{H})$ such that $a = a_\alpha$.

Metaproof. We just have to repeat the previous proof. Let C_x denote the "star" of the vertex x in the hypergraph $\mathcal{H}$. The image $a(C_x)$ is a pairwise intersecting subhypergraph of $\mathcal{F}$. Since $\mathcal{F}$ satisfies the EKR-property due to the valency condition of $\mathcal{H}$, therefore the vertex $\alpha(x) := \bigcap a(C_x)$ is well defined, and one can easily that the map $\alpha : X \longrightarrow X$ is a bijection and $a = a_\alpha$. ∎

Hereafter we examine several structures which satisfy the EKR-property and prove analogues reconstruction results for them. For any structure at first we list the known EKR-type result, and then determine the reconstruction result. The proofs will contain the required "extra" facts, only.

In the remaining part of this section we apply the EKR reconstruction principle to improve Fournier's theorem. (This improvement for Berge's theorem was done in the paper [EF].) At first we remark that, due to Hilton and Milner ([HM]), the family $\mathcal{F}$ of all r-element subsets of the n-element set X satisfies the EKR-property with the function

$$v(n, r) = \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 2.$$

Theorem 4. *Let $\mathcal{H} = (E_i : i \in M)$ be an r-uniform hypergraph with $|E_M| = 2r - k$ points ($0 < k < r$). Suppose that for any point $x \in E_M$ the condition*

$$|M| - v_{\mathcal{H}}(x) > \binom{2r-k-1}{r} - \binom{r-1}{k} + 1$$

holds. Let $a \in \mathrm{Aut}(\mathcal{L}(\mathcal{H}))$ for which

$$|E_i \cap E_j| = |a(E_i) \cap a(E_j)|$$

whenever $i, j \in M$. Then there exists an $\alpha \in \mathrm{Aut}(\mathcal{H})$ for which $a = a_\alpha$.

Proof. Let $\bar{\mathcal{H}} = (\bar{E}_i : i \in M)$ where $\bar{E}_i = E_M \setminus E_i$ $(i \in M)$. The $\bar{\mathcal{H}}$ is an $(r-k)$-uniform hypergraph with $|\bar{E}_M| = 2r - k$ points. Let $\bar{a}$ be the permutation of : $\{\bar{E}_i\}$ for which

$$\bar{a}(\bar{E}) = E_M \setminus a(E).$$

By the intersection condition of the automorphism a:

$$\begin{aligned}|\bar{E}_i \cap \bar{E}_j| &= |(E_M \setminus E_i) \cap (E_M \setminus E_j)| = |E_M \setminus (E_i \cup E_j)| = \\ &= |E_M| - 2r + |E_i \cap E_j| = |E_M| - 2r + |a(E_i) \cap a(E_j)| = \\ &= |(E_M \setminus a(E_i)) \cap (E_M \setminus a(E_j))| = |\bar{a}(\bar{E}_i) \cap \bar{a}(\bar{E}_j)|\end{aligned}$$

for any $i, j \in M$, that is $\bar{a} \in \mathrm{Aut}(\mathcal{L}(\bar{\mathcal{H}}))$. Furthermore

$$v_{\bar{\mathcal{H}}}(x) \geq \binom{2r-k-1}{r} - \binom{r-1}{k} + 2$$

holds for each points of $\bar{\mathcal{H}}$, since $(x \in \bar{E} \in \bar{\mathcal{H}}) \Longleftrightarrow (x \notin E \in \mathcal{H})$. So we can apply Metatheorem (with the Hilton-Milner condition) for the hypergraph $\bar{\mathcal{H}}$ and the automorphism $\bar{a} \in \mathrm{Aut}(\mathcal{L}(\bar{\mathcal{H}}))$. Consequently there exists an automorphism $\alpha \in \mathrm{Aut}(\bar{\mathcal{H}})$ for which $\bar{a}(\bar{E}) = \{\alpha(x) : x \in \bar{E}\}$ (when $\bar{E} \in \bar{\mathcal{H}}$). This permutation $\alpha : E_M \longrightarrow E_M$ is also a suitable automorphism of the hypergraph $\mathcal{H}$. We must show, that the condition

$$a(E) = \{\alpha(x) : x \in E\}$$

holds for every edge $E \in \mathcal{H}$. But we know, that for every point $x \notin E$ the relation $\alpha(x) \in \bar{a}(\bar{E}) = E_M \setminus a(E)$ holds, that is $\alpha(x) \notin a(E)$. Hence $\alpha(\bar{E}) = E_M \setminus a(E)$ therefore the equality $\alpha(E) = a(E)$ holds for the bijection α. ∎

Remark 5. We know, that any pair of edges of the r-uniform hypergraph $\mathcal{H}$ is automatically intersecting because $|E_M| = 2r - k$. Furthermore the proof used only the condition $\bar{E}_i \cap \bar{E}_j = \emptyset \Longleftrightarrow \bar{a}(\bar{E}_i) \cap \bar{a}(\bar{E}_j) = \emptyset$. Therefore the map a must be a permutation of $\{E_i\}$ which satisfies the condition

$$E_i \cup E_j = E_M \Longleftrightarrow a(E_i) \cup a(E_j) = E_M \qquad (i, j \in M).$$

3. Hamming schemes $H(r,q)$

Let $Q = \{1,\ 2,\ \ldots,\ q\}$ and $Q^r = \{x = (x_1,\ \ldots,\ x_r) :\ x_i \in Q\}$. We define the *distance* $d(x,y)$ $(x, y \in Q^r)$ as follows:

$$d(x,y) = |\{i \in \{1,\ 2,\ \ldots,\ r\} : x_i \neq y_i\}|.$$

The structure (Q^r, d) is called *Hamming scheme* $H(r,q)$. Two elements of $H(r,q)$ is called *t-intersecting* if their distance $\leq r - t$.

For convenience we reformulate these notions.

Let $X_1,\ X_2,\ \ldots,\ X_r$ be sets that are pairwise disjoint. Let $|X_i| = q$ $(1 \leq i \leq r)$. Let $X = \bigcup_i X_i$. Let n denote the cardinality $|X| = rq$. Let $K_{r,q}$ be the following r-class hypergraph:

$$K_{r,q} = \{E \subset X :\ \ |E \cap X_i| = 1 \text{ if } 0 \leq i \leq r\}.$$

By now the distance of any two elements of $H(r,q)$ is at most $r - t$ iff the corresponding edges of hypergraph $K_{r,q}$ are t-intersecting. In the case $t = 1$ and $q \geq 3$ the pair $(X, K_{r,q})$ satisfies EKR-property with the function $f(X, K_{r,q}) = q^{r-1}$. (See Livingston [Li], Frankl and Füredi [FF] or Moon [Mo].)

Theorem 6. *Let $q \geq 3$. For every automorphism $a \in \mathrm{Aut}(L(K_{r,q}))$ there exists an automorphism $\alpha \in \mathrm{Aut}(K_{r,q})$ which induces the automorphism a, i.e. $a = a_\alpha$.*

Proof. Just apply the Metatheorem. ∎

We note that there is a permutation $\pi : \{1, 2, \ldots, r\} \longrightarrow \{1, 2, \ldots, r\}$ for which

$$\alpha(X_i) = X_{\pi(i)}$$

if $i = 1, \ldots, r$. To prove it is enough to realize that the points $x, y \in X$ are in same classe iff there is no edge in $K_{r,q}$ which contains both of them. But $\alpha(x)$ and $\alpha(y)$ also satisfy this condition.

4. t-Designs

A $t - (n, r, \lambda)$ *design* is an r-uniform $(B_i : i \in M)$ hypergraph for which $|B_M| = n$ and every t-element subset of B_M is covered by exactly λ *blocks* B_i. To avoid degenerate cases it is assumed, that $0 < t \leq r \leq n$. It is a well-known fact that every s-element $(s \leq t)$ subset are contained by exactly b_s blocks. This number depends on the parameters of the design and the number s, only. Namely:

$$b_s = \lambda \binom{n-s}{t-s} \Big/ \binom{r-s}{t-s}.$$

The t-designs also satisfy the EKR-property.

Rands' Theorem. ([Ra]) *There exists a function $f(r, t, s)$ with the following property: Let $\mathcal{B}$ an arbitrary $t - (n, r, \lambda)$ design with $n \geq f(r, t, s)$ points. Let $\mathcal{H}$ be a system of s-intersecting blocks of $\mathcal{B}$. Then*

$$|\mathcal{H}| \leq b_s$$

and in the case of equality $\mathcal{H}$ can be described as follows: there exists an $X_0 \subset B_M$, $|X_0| = s$ that

$$\mathcal{H} = \{B \in \mathcal{B} : X_0 \subset B\}. \blacksquare$$

The following estimation for the function f is known.

$$f(r, t, s) \leq \begin{cases} s + \binom{r}{s}(r - s + 1)(r - s) & \text{if } s < t - 1 \\ s + (r - s)\binom{r}{s}^2 & \text{if } s = t - 1. \end{cases}$$

This theorem generalizes the Erdős - Ko - Rado theorem since all r-element subsets of an n-element underlying set form an $r - (n, r, 1)$ design.

The following result is proven by our Metatheorem.

Theorem 7. *Let $\mathcal{B}$ be an arbitrary $t-(n,r,\lambda)$ design. Let $n \geq f(r,t,1)$. Furthermore let $a \in \mathrm{Aut}(\mathcal{L}(\mathcal{B}))$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{B})$ such that $a = a_\alpha$.* ∎

5. Finite vector spaces

Let V denote the n-dimensional vector space over the q-element finite field. Furthermore let V^r denote the set of all r-dimensional subspace of the vector space V. It is a well-known fact that the number of r-dimensional subspaces which contains a prescribed t-dimensional subspace is the Gaussian q-binomial coefficient $\begin{bmatrix} n-t \\ r-t \end{bmatrix}_q$.

Two subspace of V is called intersecting if the dimension of their intersection is at least 1. As Hsieh proved ([Hs] or [FW]) if $n \geq 2r+2$ or $n \geq 2r+1$ and $q \geq 3$ then the family V^r of all r-dimensional subspaces as a hypergraph of all 1-dimensional subspaces satisfies the EKR-property with the function $\begin{bmatrix} n-t \\ r-t \end{bmatrix}_q$.

Theorem 8. *Let $n \geq 2r+2$ or $n \geq 2r+1$ and $q \geq 3$. Let $a \in \mathrm{Aut}(\mathcal{L}(V^r))$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(V^r)$ such that $a = a_\alpha$.*

Proof. The application of Metatheorem shows that there exists an automorphism $\bar{\alpha} \in \mathrm{Aut}(V^1)$ which induces the automorphism a. But the map $\bar{\alpha}$ can be extended to a bijection $\alpha : V \longrightarrow V$ easily. Namely for every $X \in V^1$ let α_X be any bijection $X \longrightarrow \bar{\alpha}(X)$ which preserves the origin. (This bijection exists because the cardinality of the non-zero elements in any one dimensional subspace is constant.) Finally let $\alpha(y) = \alpha_X(y)$ if $y \in X$. Easy to see, that the map α is an automorphism of the hypergraph V^r and $a = a_\alpha$. ∎

Remark. As L. Babai pointed out ([Ba]) the transformation α can be defined to be linear.

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Péter L. Erdős
Mathematical Institute of the Hungarian Academy of Science, Budapest and
Institute of Operations Research, University of Bonn, Germany