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Best proximity point theorems for α-nonexpansive mappings in Banach spaces

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Abstract

In this paper, we discuss sufficient and necessary conditions for the existence of best proximity points for non-self-α-nonexpansive mappings in Banach spaces. We obtain convergence results under some assumptions, and we prove the existence of common best proximity points for a family of non-self-α-nonexpansive mappings.

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1 Introduction

Let (X, d) be a metric space and A a nonempty subset of X. The mapping $T : A \to X$ is said to have a fixed point in A if the fixed point equation $Tx = x$ has at least one solution. In metric terminology, we say that $x \in A$ is a fixed point of T if $d(x, Tx) = 0$. If the fixed point equation of a given mapping does not have a solution, then $d(x, Tx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense, and the x is said to be the best approximation of the fixed point of T. In this paper, we will study the best approximation theory and best proximity pair theorems.

Consider the following well-known best approximation theorem due to Ky Fan [1].

Theorem 1.1 [1] *Let A be a nonempty compact convex set in a normed linear space X. If T is a continuous map from A into X, then there exists a point x in A such that $\|x - Tx\| = \text{dist}(Tx, A)$.*

The point x in the theorem above is called a best approximation point of T in A. Note that if $x \in A$ is a best approximation point, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x \in A} \|x - Tx\|$ has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets A, B of a metric space X and a mapping $T : A \to B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq \text{dist}(A, B)$, it is of interest to find a point $x_0 \in A$ such that $d(x_0, Tx_0) = \text{dist}(A, B)$. This situation motivates the researchers to develop the notion called best proximity point theory. It is worth to note that the best proximity point theorems can be viewed as a generalization of fixed point theorems, since most fixed

point theorems can be derived as corollaries of best proximity point theorems. Some of the interesting results for best proximity points can be found in [2–18].

Let A, B be two nonempty subsets of a normed linear space X. Let us fix the following notation which will be used throughout this article:

$$A_0 = \{x \in A : \|x - y\| = \text{dist}(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : \|x - y\| = \text{dist}(A, B) \text{ for some } x \in A\}.$$

The following notion of weakly contractive mapping was introduced by Sankar Raj in [19].

Definition 1.1 [19] Let A, B be nonempty subsets of a metric space (X, d). A map $T : A \to B$ is said to be a weakly contractive mapping if

$$d(Tx, Ty) \leq d(x, y) - \psi\big(d(x, y)\big), \quad \forall x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive $(0, \infty)$, $\psi(0) = 0$, and $\lim_{t \to \infty} \psi(t) = \infty$. If A is bounded, then $\lim_{t \to \infty} \psi(t) = \infty$ can be omitted.

Note that $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) < d(x, y)$ if $x, y \in A$ with $x \neq y$. That is, T is a contractive map. In [19], Sankar Raj obtained the existence theorem of a best proximity point for weakly contractive mappings as follows.

Theorem 1.2 [19] *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the d-property. Then there exists a unique point x^* in A such that $d(x^*, Tx^*) = \text{dist}(A, B)$.*

In [20], Aoyama and Kohsaka introduced the self-α-nonexpansive mapping which is defined as follows.

Definition 1.2 [20] Let X be a Banach space, let A be a nonempty subset of X, and let α be a real number such that $\alpha < 1$. A mapping $T : A \to A$ is said to be α-nonexpansive if

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - y\|^2 + \alpha \|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in A$.

It is obvious that every nonexpansive mapping is 0-nonexpansive. We denote the fixed point set of a mapping T by $F(T)$. In [20], they proved the following result.

Theorem 1.3 [20] *Let X be a uniformly convex Banach space, let A be a nonempty, closed, and convex subset of X, and let $T : A \to A$ be an α-nonexpansive mapping for some real number α such that $\alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x \in A$ such that $\{T^n(x)\}$ is bounded.*

In [21] and [22], Abkar and Gabeleh studied the best proximity points of some non-self-mappings under appropriate conditions.

Motivated and inspired by the above mentioned work, in this article, we consider a map $T : A \to B$, where A and B are nonempty subsets of a Banach space X, which is non-self-α-nonexpansive in the sense of Definition 1.2. We attempt to study the sufficient and necessary conditions for the existence of a best proximity point for non-self-α-nonexpansive mappings and convergence results. Moreover, we discuss the existence of common best proximity points for a family of non-self-α-nonexpansive mappings. When the map T is considered to be a self-map, then our result reduces to the fixed point theorem of Aoyama and Kohsaka for α-nonexpansive mappings. Our results are generalization and improvement of the recent results obtained by many authors.

2 Preliminaries

Let X be a normed linear space and C be a nonempty subset of X. Then the metric projection operator $P_C : X \to C$ is defined by

$$P_C(x) = \left\{ y \in C : \|x - y\| = \mathrm{dist}(x, C) \right\} \quad \text{for all } x \in X.$$

It is well known that, if C is assumed to be a closed and convex subset of a strictly convex and reflexive Banach space X, then $P_C(x)$ is nonempty and single valued for all $x \in X$.

In [23], Kirk *et al.* proved the following lemma which guarantees the nonemptiness of A_0 and B_0.

Lemma 2.1 [23] *Let X be a reflexive Banach space and A be a nonempty, closed, bounded, and convex subset of X, and B be a nonempty, closed, and convex subset of X. Then A_0 and B_0 are nonempty and satisfy $P_B(A_0) \subseteq B_0$, $P_A(B_0) \subseteq A_0$.*

Also, in [24], Sadiq Basha and Veeramani proved that A_0 is contained in the boundary of A. It is easy to verify that A_0 and B_0 are closed convex subsets of A and B, respectively, if A and B are closed and convex.

The notion called the d-property was introduced in [25].

Definition 2.1 [25] A pair (A, B) of nonempty subsets of a normed linear space X is said to have the d-property if and only if

$$\left. \begin{array}{l} \|x_1 - y_1\| = \mathrm{dist}(A, B), \\ \|x_2 - y_2\| = \mathrm{dist}(A, B) \end{array} \right\} \quad \Rightarrow \quad \|x_1 - x_2\| = \|y_1 - y_2\|,$$

whenever $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 2.2 [25] A normed linear space X is said to have the d-property if and only if every pair (A, B) of nonempty and closed convex subsets of X has the d-property.

Lemma 2.2 [25] *X is strictly convex if and only if X has the d-property.*

Lemma 2.3 [25] *Let A, B be nonempty, closed, and convex subsets of a strictly convex space X such that A_0 is nonempty. Then the restriction of the metric projection mapping P_{A_0} to B_0 is an isometry. That is, $P_{A_0} : B_0 \to A_0$ is an isometry.*

Let A, B be nonempty, closed, and convex subsets of a strictly convex space X such that A_0 is nonempty. Consider the mapping $P : A \cup B \to A \cup B$ defined by

$$P(x) = \begin{cases} P_B(x), & \text{if } x \in A; \\ P_A(x), & \text{if } x \in B. \end{cases}$$

Let us define the notion of non-self-α-nonexpansive maps as follows.

Definition 2.3 Let A, B be nonempty subsets of a Banach space X and α a real number such that $\alpha < 1$. A map $T : A \to B$ is said to be an α-nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \alpha \|Tx - Py\|^2 + \alpha \|Ty - Px\|^2 + (1 - 2\alpha)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in A$.

Remark 2.1 We note that a non-self-nonexpansive mapping ($\|Tx - Ty\| \leq \|x - y\|$, $x, y \in A$) $T : A \to B$ is a non-self-0-nonexpansive mapping.

The following example shows that there is a discontinuous non-self-α-nonexpansive mapping.

Example 2.1 Let X be a uniformly convex Banach space, $A = \{x \in X : \|x - x_0\| \leq 2\}$, $B = \{x \in X : \|x\| \leq 1\}$, where $\|x_0\| \geq 3$. Let α and γ be real numbers such that $\frac{1}{2} \leq \alpha < 1$ and $0 < \gamma \leq \frac{1}{2}$. Then the mapping $F : A \to B$ defined by

$$F(x) = \begin{cases} \gamma P_B(x), & \text{if } x \in A_0; \\ 0, & \text{if } x \in A/A_0, \end{cases}$$

is non-self-α-nonexpansive and discontinuous.

Proof Since X is a uniformly convex Banach space and A, B are bounded closed convex subsets of X, we know that B_0 is a nonempty, closed, and convex subset in the boundary set of B and $P_B(x) = P_{B_0}(x)$, for all $x \in A_0$. For any $x, y \in A_0$, by Lemma 2.3, we have

$$\begin{aligned} \left\|F(x) - F(y)\right\|^2 &= \left\|\gamma P_B(x) - \gamma P_B(y)\right\|^2 \leq \left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 \\ &= \alpha \left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 + \alpha \left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 \\ &\quad + (1 - 2\alpha)\left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 \\ &= \alpha \left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 + \alpha \left\|P_{B_0}(x) - P_{B_0}(y)\right\|^2 + (1 - 2\alpha)\|x - y\|^2 \\ &= \alpha \left\|F(x) - P_{B_0}(y)\right\|^2 + \alpha \left\|P_{B_0}(x) - F(y)\right\|^2 + (1 - 2\alpha)\|x - y\|^2. \end{aligned}$$

Thus, in the case when either $x, y \in A_0$ or $x, y \in A/A_0$, we obviously have

$$\left\|F(x) - F(y)\right\|^2 \leq \alpha \left\|F(x) - P(y)\right\|^2 + \alpha \left\|P(x) - F(y)\right\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

In the case when $x \in A_0$ and $y \in A/A_0$, we have

$$\begin{aligned} &\alpha \left\|F(x) - P(y)\right\|^2 + \alpha \left\|P(x) - F(y)\right\|^2 + (1 - 2\alpha)\|x - y\|^2 \\ &\geq \alpha \left\|P(x) - F(y)\right\|^2 = \alpha \left\|P(x)\right\|^2 = \alpha \left\|P_{B_0}(x)\right\|^2 = \alpha \end{aligned}$$

and

$$\left\| F(x) - F(y) \right\|^2 = \left\| F(x) - F(y) \right\|^2 = \left\| \gamma P_B(x) \right\|^2 = \gamma^2 \left\| P_{B_0}(x) \right\|^2 = \gamma^2.$$

Then F is α-nonexpansive. $\qquad\square$

Lemma 2.4 *Let A, B be nonempty, closed, and convex subsets of a strictly convex space X such that A_0 is nonempty. Then $P^2(x) = x$ for all $x \in A_0 \cup B_0$.*

Proof For any $x \in A_0$, we have

$$\left\| x - P_{B_0}(x) \right\| = \mathrm{dist}(A, B).$$

From the above equality, we can obtain $x = P_{A_0} P_{B_0}(x)$, that is, $P^2(x) = x$. Similarly, we have $P^2(x) = x$ for all $x \in B_0$. $\qquad\square$

Remark 2.2 We note that there exists $x \in A_0$ such that $\|x - T(x)\| = \mathrm{dist}(A, B)$ if and only if x is the fixed point of $P_{A_0} T$.

3 Best proximity point theorems

Now let us use the above characterization of strictly convex spaces to prove the following best proximity point theorem of non-self-α-nonexpansive mappings.

Theorem 3.1 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X. Suppose that A_0 is nonempty and $T : A \to B$ is an α-nonexpansive mapping on A_0 for some real number α such that $0 < \alpha < 1$ and $T(A_0) \subseteq B_0$. Then T has at least one best proximity point if and only if there exists $x \in A_0$ such that $\{(PT)^n(x)\}$ is bounded. Moreover, if T is continuous and $\|(PT)^2(x) - x\| \le r\|(PT)(x) - x\|$ for all $x \in A_0$, where $0 < r < \sqrt{2}$, then $(PT)^n(x)$ converges to a proximity point for all $x \in A_0$.*

Proof First we remark that $P(x) = P_{B_0}(x)$, $P(y) = P_{B_0}(y)$ for any $x \in A_0$, $y \in B_0$. Let us consider the mapping $PT : A \to A$ on A_0, *i.e.* $P_{A_0} T : A_0 \to A_0$. Since $T : A \to B$ is an α-nonexpansive mapping, by Lemmas 2.3 and 2.4, for all $x, y \in A_0$, we have

$$
\begin{aligned}
\left\| (PT)(x) - (PT)(y) \right\|^2 \\
&= \left\| T(x) - T(y) \right\|^2 \\
&\le \alpha \left\| T(x) - P(y) \right\|^2 + \alpha \left\| T(y) - P(x) \right\|^2 + (1 - 2\alpha)\|x - y\|^2 \\
&= \alpha \left\| (P^2 T)(x) - P(y) \right\|^2 + \alpha \left\| (P^2 T)(y) - P(x) \right\|^2 + (1 - 2\alpha)\|x - y\|^2 \\
&= \alpha \left\| (PT)(x) - y \right\|^2 + \alpha \left\| (PT)(y) - x \right\|^2 + (1 - 2\alpha)\|x - y\|^2.
\end{aligned}
\tag{3.1}
$$

Thus $P_{A_0} T : A_0 \to A_0$ is α-nonexpansive. By Theorem 1.3, we see that $F(P_{A_0} T)$ is nonempty if and only if there exists $x \in A_0$ such that $\{(P_{A_0} T)^n(x)\}$ is bounded. Then T has at least one best proximity point if and only if there exists $x \in A_0$ such that $\{(PT)^n(x)\}$ is bounded.

Since $\|(PT)^2(x) - x\| \leq \sqrt{2}\|(PT)(x) - x\|$ for all $x \in A_0$ and T is α-nonexpansive, we have

$$
\begin{aligned}
&\big\|(PT)^{n+1}(x) - (PT)^n(x)\big\|^2 \\
&\quad = \big\|T(PT)^n(x) - T(PT)^{n-1}(x)\big\|^2 \\
&\quad \leq \alpha\big\|T(PT)^n(x) - P(PT)^{n-1}(x)\big\|^2 + \alpha\big\|P(PT)^n(x) - T(PT)^{n-1}(x)\big\|^2 \\
&\qquad + (1-2\alpha)\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 \\
&\quad = \alpha\big\|(PT)^{n+1}(x) - (PT)^{n-1}(x)\big\|^2 + \alpha\big\|(PT)^n(x) - (PT)^n(x)\big\|^2 \\
&\qquad + (1-2\alpha)\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 \\
&\quad = \alpha\big\|(PT)^{n+1}(x) - (PT)^{n-1}(x)\big\|^2 + (1-2\alpha)\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 \\
&\quad \leq r^2\alpha\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 + (1-2\alpha)\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 \\
&\quad = \big[1 - (2-r^2)\alpha\big]\big\|(PT)^n(x) - (PT)^{n-1}(x)\big\|^2 \\
&\quad \leq \big[1 - (2-r^2)\alpha\big]^2\big\|(PT)^{n-1}(x) - (PT)^{n-2}(x)\big\|^2 \\
&\quad \leq \cdots \leq \big[1 - (2-r^2)\alpha\big]^n\big\|(PT)(x) - x\big\|^2.
\end{aligned}
\tag{3.2}
$$

Consequently, since $1 - (2 - r^2)\alpha < 1$, one has

$$
\big\|(PT)^{n+p}(x) - (PT)^n(x)\big\|^2 \leq \frac{[1 - (2-r^2)\alpha]^n}{(2-r^2)\alpha}\big\|(PT)(x) - x\big\|^2.
\tag{3.3}
$$

Thus $\{(PT)^n(x)\}$ is a Cauchy sequence in A_0, and hence there exists $x^* \in A_0$ such that $\{(PT)^n(x)\} \to x^*$ for $n \to \infty$. Since T is continuous, we have $T(PT)^{n-1}(x) \to T(x^*)$. Thus

$$
\big\|T(x^*) - x^*\big\| = \lim_{n\to\infty}\big\|T(PT)^{n-1}(x) - (PT)^n(x)\big\| = \operatorname{dist}(A, B).
\tag{3.4}
$$

The assertion is proved. □

Remark 3.1 If $A = B$, Theorem 3.1 is the fixed point theorem of Aoyamma and Kohsaka [20] with convergence result.

Remark 3.2 If T is a weakly contractive mapping, by Theorem 3.1, we can obtain the best proximity point theorem of Sankar Raj [19] with convergence result.

Corollary 3.1 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X. Suppose that A is bounded and $T : A \to B$ is an α-nonexpansive mapping for some real number $\alpha < 1$ such that $T(A_0) \subseteq B_0$. Then T has at least one best proximity point.*

Proof Since X is a uniformly convex Banach space and A is bounded, we see that A_0 is nonempty and $\{(PT)^n(x)\}$ is bounded for any $x \in A_0$. By Theorem 3.1, T has at least one best proximity point. □

Corollary 3.2 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X, and let A_0 be nonempty. Suppose that $T(A_0) \subseteq B_0$ and $T : A \to B$*

is a nonexpansive mapping on A_0, i.e. $\|T(x) - T(y)\| \leq \|x - y\|$, for all $x, y \in A_0$. Then T has at least one best proximity point if and only if there exists $x \in A_0$ such that $\{(PT)^n(x)\}$ is bounded.

Proof It is well known that nonexpansive mappings are 0-nonexpansive. By Theorem 3.1, the assertion is proved. □

Let A, B be nonempty convex subsets of a normed linear space. A mapping $T : A \to B$ is said to be affine if $T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y)$, for all $x, y \in A$ and $\lambda \in (0, 1)$. For convenience, we define $F_A(T) = \{x \in A : \|x - T(x)\| = \text{dist}(A, B)\}$.

Theorem 3.2 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X. Suppose that A_0 is nonempty, bounded and $T : A \to B$, $S : A \to A$ satisfy the following conditions*:
  (i) *T, S are, respectively, non-self-α-nonexpansive and β-nonexpansive mappings on A_0 such that $\alpha, \beta < 1$;*
 (ii) *T is an affine and continuous mapping, and $T(A_0) \subseteq B_0$;*
(iii) *$P_{B_0}S(x) = TS(x)$ for all $x \in F_A(T)$.*
Then there exists at least one point $x^* \in A_0$ such that $\|S(x^*) - T(x^*)\| = \text{dist}(A, B)$.

Proof Since A_0 is bounded, by Theorem 3.1, we see that $F_A(T) \subseteq A_0$ is nonempty. Assume that $\{x_n\} \in F_A(T)$ and $x_n \to x$ for $n \to \infty$, then $\|x_n - T(x_n)\| = \text{dist}(A, B)$. Since T is continuous, we have $\|x - T(x)\| = \text{dist}(A, B)$. Therefore, $x \in F_A(T)$ and $F_A(T)$ is closed. As T is affine, take $x, y \in F_A(T)$ and $\lambda \in (0, 1)$, we have

$$\big\| \lambda x + (1 - \lambda)y - T\big(\lambda x + (1 - \lambda)y\big) \big\|$$
$$= \big\| \lambda\big(x - T(x)\big) + (1 - \lambda)\big(y - T(y)\big) \big\|$$
$$\leq \lambda \big\| x - T(x) \big\| + (1 - \lambda)\big\| y - T(y) \big\| = \text{dist}(A, B). \tag{3.5}$$

Thus $F_A(T)$ is convex. Now we consider S on $F_A(T)$. Take any $x \in F_A(T)$, by (iii), we have

$$\big\| S(x) - TS(x) \big\| = \text{dist}(A, B). \tag{3.6}$$

Hence S restricted to $F_A(T)$ is a self-map. That is $S : F_A(T) \to F_A(T)$. By Theorem 1.3, there exists $x^* \in F_A(T)$ such that $x^* = S(x^*)$. Furthermore, $\|S(x^*) - T(x^*)\| = \text{dist}(A, B)$. □

Remark 3.3 In Theorem 3.2, S does not need to be continuous.

Example 3.1 Consider the uniformly convex space $(\mathbb{R}^2, \|\cdot\|_2)$. Let

$$A = \big\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\big\} \quad \text{and}$$
$$B = \big\{(x, y) : 2 \leq x \leq 4, 0 \leq y \leq 1\big\}.$$

Then A and B are nonempty, closed, bounded, and convex subsets of $\mathbb{R}^2$ with $\text{dist}(A, B) = 1$. It is easy to verify that $A_0 = \{(1, y) : 0 \leq y \leq 1\}$ and $B_0 = \{(2, y) : 0 \leq y \leq 1\}$. Define a map

$T : A \to B$ by $T(x, y) = (x + 1, y)$ for all $(x, y) \in A$. Clearly, $T(A_0) \subseteq B_0$ and T is an α-nonexpansive mapping for any $\alpha < 1$. Define a map $S : A \to A$ by $S(x, y) = (x, \frac{1}{2}y^2)$ for all $(x, y) \in A$. It is easy to prove that S is $\frac{1}{2}$-nonexpansive on A_0 and $P_{B_0}S(x) = TS(x)$ for all $x \in A_0$. Note that $x^* = (1, 0)$ satisfies $\|S(x^*) - T(x^*)\| = \text{dist}(A, B)$.

4 Common best proximity points

In this section, we discuss sufficient conditions for the existence of common best proximity points for α-nonexpansive mappings.

Theorem 4.1 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X. Suppose that A_0 is nonempty and bounded, and $T : A \to B$ and $S : A \to B$ satisfy the following conditions:*

(i) *T, S are respectively α-nonexpansive and β-nonexpansive mappings on A_0 such that $\alpha, \beta < 1$;*

(ii) *T is an affine and continuous mapping, and $T(A_0) \subseteq B_0$, $S(A_0) \subseteq B_0$;*

(iii) *for any $x \in F_A(T)$, there exists $y \in F_A(T)$ such that $S(x) = T(y)$.*

Then there exists at least one point $x^* \in A_0$ such that $\|x^* - S(x^*)\| = \text{dist}(A, B)$ and $\|x^* - T(x^*)\| = \text{dist}(A, B)$.

Proof From the proof of Theorem 3.2, we know that $F_A(T)$ is a nonempty, closed, and convex subset of A_0. Now we consider $S : A \to B$ on $F_A(T)$. Take any $x \in F_A(T)$, by (iii), we have

$$\|y - S(x)\| = \|y - T(y)\| = \text{dist}(A, B). \tag{4.1}$$

Thus

$$\|S(x) - P_{F_A(T)}(S(x))\| \le \|y - S(x)\| = \text{dist}(A, B). \tag{4.2}$$

Therefore, $P_{F_A(T)}(S(x)) = P_{A_0}(S(x))$ for all $x \in F_A(T)$. Hence S is a β-nonexpansive mapping on $F_A(T)$. By Theorem 3.1, there exists at least one point $x^* \in F_A(T)$ such that $\|x^* - S(x^*)\| = \text{dist}(A, B)$. The assertion is proved. □

The following theorem guarantees the existence of a common best proximity point for a finite family $\Im = \{T_1, T_2, \ldots, T_n\}$ of affine, α-nonexpansive mappings.

Theorem 4.2 *Let X be a uniformly convex Banach space and A, B be nonempty, closed, and convex subsets of X. Suppose that A_0 is nonempty and bounded and $\Im = \{T_1, T_2, \ldots, T_n\}$ is a family mappings, where $T_i : A \to B$ ($i = 1, 2, \ldots, n$) satisfy the following conditions:*

(i) *T_i are respectively α_i-nonexpansive mappings on A_0 such that $\alpha_i < 1$, for all $i = 1, 2, \ldots, n$;*

(ii) *T_i are an affine and continuous mappings, and $T_i(A_0) \subseteq B_0$ for all $i = 1, 2, \ldots, n$;*

(iii) *for any $x \in \bigcap_{k=1}^{i-1} F_A(T_k)$, there exists $y \in \bigcap_{k=1}^{i-1} F_A(T_k)$ such that $T_i(x) = T_1(y)$ for all $i = 1, 2, \ldots, n$.*

Then there exists at least one point $x^* \in A_0$ such that $\|x^* - T_i(x^*)\| = \text{dist}(A, B)$ for all $i = 1, 2, \ldots, n$.

Proof From the proof of Theorem 3.2, we know that $F_A(T_i)$ is a nonempty, closed, and convex subset of A_0 for all $i = 1, 2, \ldots, n$. Theorem 4.1 implies that $F_A(T_1) \cap F_A(T_2)$ is a nonempty, closed, and convex set. Following a similar argument as Theorem 4.1, we can prove that T_3 on $F_A(T_1) \cap F_A(T_2)$ is an α_3-nonexpansive mapping. Thus there exists a best proximity point $z^* \in F_A(T_1) \cap F_A(T_2)$. By repeating the argument, we can prove that there exists $x^* \in \bigcap_{i=1}^{n} F_A(T_i)$ such that $\|x^* - T_i(x^*)\| = \mathrm{dist}(A, B)$ for all $i = 1, 2, \ldots, n$. □

Example 4.1 Consider the uniformly convex space $(\mathbb{R}^2, \|\cdot\|_2)$. Let

$$A = \big\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\big\},$$
$$B = \big\{(x, y) : 2 \leq x \leq 4, 0 \leq y \leq 1\big\}.$$

Then A and B are nonempty, closed, bounded, and convex subsets of $\mathbb{R}^2$ with $\mathrm{dist}(A, B) = 1$. It is easy to verify that $A_0 = \{(1, y) : 0 \leq y \leq 1\}$ and $B_0 = \{(2, y) : 0 \leq y \leq 1\}$. Define a map $T : A \to B$ by $T(x, y) = (x + 1, y)$ for all $(x, y) \in A$. Clearly, $T(A_0) \subseteq B_0$ and T is an α-non-expansive mapping for any $\alpha < 1$. Define a map $S : A \to B$ by

$$S(x) = \begin{cases} (x + 1, \frac{1}{2}y^2), & \text{if } (x, y) \in \{(x, y) : (x, y) \in A_0 \text{ such that } y \geq \frac{1}{2}\}; \\ (3x - 1, y), & \text{if } (x, y) \in \{(x, y) : (x, y) \in A_0 \text{ such that } y < \frac{1}{2}\}; \\ (2x + 2, y), & \text{if } (x, y) \in A/A_0. \end{cases}$$

It is easy to prove S is $\frac{1}{2}$-nonexpansive on A_0. Note that $x^* \in \{(x, y) : (x, y) \in A_0 \text{ such that } y < \frac{1}{2}\}$ satisfies $\|x^* - S(x^*)\| = \mathrm{dist}(A, B)$ and $\|x^* - T(x^*)\| = \mathrm{dist}(A, B)$.

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