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THE CONTROL PARAMETERIZATION METHOD FOR NONLINEAR OPTIMAL CONTROL: A SURVEY

Qun Lin, Ryan Loxton and Kok Lay Teo

Department of Mathematics and Statistics, Curtin University
GPO Box U1987 Perth, Western Australia 6845, Australia

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Abstract. The control parameterization method is a popular numerical technique for solving optimal control problems. The main idea of control parameterization is to discretize the control space by approximating the control function by a linear combination of basis functions. Under this approximation scheme, the optimal control problem is reduced to an approximate nonlinear optimization problem with a finite number of decision variables. This approximate problem can then be solved using nonlinear programming techniques. The aim of this paper is to introduce the fundamentals of the control parameterization method and survey its various applications to non-standard optimal control problems. Topics discussed include gradient computation, numerical convergence, variable switching times, and methods for handling state constraints. We conclude the paper with some suggestions for future research.

1. **Introduction.** The field of *optimal control* is concerned with determining control strategies for manipulating real-world systems in the best possible manner. Optimal control has applications in almost every area of science and engineering, including aquaculture modelling [80], cancer chemotherapy [51], switching power converters [48], spacecraft attitude control [1], and underwater vehicles [15]. The main theoretical tools for solving optimal control problems analytically are the famous *Pontryagin minimum principle* and the *Hamilton-Jacobi-Bellman equation*. Practical problems, however, are usually too complex to solve using analytical techniques alone, and thus numerical methods are indispensable for applications.

One such numerical method is the *control parameterization technique*—the topic of this paper. The control parameterization technique involves approximating the control function by a linear combination of basis functions, where the coefficients in the linear combination are decision variables to be chosen optimally. Applying this approximation scheme yields a finite-dimensional approximation of the original optimal control problem. By designing special algorithms for computing the gradients of the cost and constraint functions, the approximate problem can be solved using standard gradient-based optimization techniques such as conjugate gradient methods and sequential quadratic programming [50, 53].

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Control parameterization is most commonly applied using a piecewise-constant approximation scheme, with the control values as decision variables. The popularity of the piecewise-constant approximation scheme is due to its simplicity: implementing a piecewise-constant control signal—i.e. a finite sequence of constant input values—is often much easier than implementing a continuously-changing control signal. Moreover, in many practical control problems (e.g. bang-bang control problems), the true optimal control is indeed a piecewise-constant function. Other advantages of the piecewise-constant approximation scheme include its strong convergence properties and its versatility at handling non-standard optimal control problems. The MISER 3.3 optimal control software [21], which is based on the control parameterization method, allows the user to choose between piecewise-constant and piecewise-linear approximation schemes.

The fundamentals of the control parameterization method are comprehensively described in the 1991 monograph *A Unified Computational Approach to Optimal Control Problems* by Teo, Goh, and Wong (reference [65]). However, this monograph is now out of print, and many important results have appeared since its publication in the early 1990s. Reference [56] surveys the key developments in the control parameterization method up until 1999, but this reference is also now out of date and difficult to access. Thus, given the many significant advances in the area of control parameterization over the past decade, it seems apt to write a new survey paper discussing the latest developments in this area. This is the aim of our current paper, which is designed to be both a tutorial introducing the fundamental concepts, as well as a survey of recent results.

The paper is organized as follows. We first describe the formulation of a standard optimal control problem in Section 2. Next, in Section 3, we introduce the traditional control parameterization approach in which the control is approximated by a piecewise-constant function with variable control heights and fixed switching times. In Section 4, we consider the superior approach in which *both* the control heights and the switching times are decision variables to be chosen optimally. We give a complete description of the computational difficulties caused by variable switching times, thereby justifying the use of the so-called *time-scaling transformation* for circumventing these difficulties. We are not aware of any similar analysis in the literature; although many papers allude to the computational difficulties caused by variable switching times, the exact nature of these difficulties is still poorly understood. In Section 5, we discuss two methods for handling continuous inequality constraints—the *constraint transcription method* and the recently-developed *exact penalty method*—and in Section 6, we discuss applications of the control parameterization method to non-standard optimal control problems. Finally, in Section 7, we conclude the paper with a discussion of future research opportunities.

2. **Standard optimal control problems.**

The classical optimal control problem involves a dynamic system of the following form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [0, T], \tag{1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}^0, \tag{2}$$

where t denotes time, $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the *state* of the system at time t, $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the *control signal* at time t, $\boldsymbol{x}^0 \in \mathbb{R}^n$ is a given initial state, $T > 0$ is a given *terminal time*, and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a given function. The interval $[0, T]$ is called the *time horizon* for the system.

The control signal in (1)-(2) is an input variable to be chosen optimally. The system evolves under the influence of the control signal according to equation (1), which expresses the rate of change of the system's state as a function of the current time, the current state, and the current value of the control signal. Thus, the control signal drives the system's evolution from $t = 0$ to $t = T$.

Typically, bound constraints are imposed on the control signal:

$$a_i \leq u_i(t) \leq b_i, \quad t \in [0, T], \quad i = 1, \ldots, r, \tag{3}$$

where a_i and b_i, $i = 1, \ldots, r$, are given real numbers such that $a_i < b_i$. These bound constraints often express physical limitations (the control signal cannot be unbounded in a real system).

Any Borel measurable function $\boldsymbol{u} : [0, T] \to \mathbb{R}^r$ satisfying (3) almost everywhere is called an *admissible control*. Let $\mathcal{U}$ denote the class of all such admissible controls.

The *solution* or *state trajectory* of (1)-(2) is a continuous function $\boldsymbol{x} : [0, T] \to \mathbb{R}^n$ satisfying the dynamics (1) almost everywhere and the initial condition (2). Since $\boldsymbol{u}$ influences the state's evolution through equation (1), the state trajectory clearly depends on $\boldsymbol{u}$. Thus, different controls produce different state trajectories. When the function $\boldsymbol{f}$ on the right-hand side of (1) satisfies some mild conditions, a unique state trajectory exists for each admissible control (see [2, 3, 65] for more details). We assume throughout this paper that system (1)-(2) admits a unique state trajectory for each $\boldsymbol{u} \in \mathcal{U}$. Let $\boldsymbol{x}(\cdot|\boldsymbol{u})$ denote this state trajectory.

Operational requirements on system (1)-(2) can be modelled by the following *canonical constraints*:

$$g_j(\boldsymbol{u}) := \Phi_j(\boldsymbol{x}(T|\boldsymbol{u})) + \int_0^T \mathcal{L}_j(t, \boldsymbol{x}(t|\boldsymbol{u}), \boldsymbol{u}(t))dt \begin{cases} = 0 \\ \geq 0 \end{cases} \qquad j = 1, \ldots, m, \tag{4}$$

where $\Phi_j : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{L}_j : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, $j = 1, \ldots, m$, are given functions. Note that each canonical constraint may be of either equality or inequality type.

The canonical form (4) encapsulates many of the common constraints arising in practice. For example, terminal state constraints of the form $\boldsymbol{x}(T) = \boldsymbol{x}^f$, where $\boldsymbol{x}^f$ is a desired terminal state, can be modelled by a canonical equality constraint with $\Phi_j := |\boldsymbol{x}(T) - \boldsymbol{x}^f|^2$ and $\mathcal{L}_j := 0$, where $|\cdot|$ denotes the Euclidean norm. Furthermore, *continuous inequality constraints* of the form $h(t, \boldsymbol{x}(t)) \geq 0$, where $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given continuous function, can (in theory) be modelled by a canonical equality constraint with $\Phi_j := 0$ and $\mathcal{L}_j := \min\{h(t, \boldsymbol{x}(t)), 0\}^2$.[1]

Any admissible control $\boldsymbol{u} \in \mathcal{U}$ satisfying the canonical constraints (4) is called a *feasible control*. Let $\mathcal{F}$ denote the class of all such feasible controls.

Our goal is to find a feasible control that minimizes the following measure of system cost:

$$g_0(\boldsymbol{u}) := \Phi_0(\boldsymbol{x}(T|\boldsymbol{u})) + \int_0^T \mathcal{L}_0(t, \boldsymbol{x}(t|\boldsymbol{u}), \boldsymbol{u}(t))dt, \tag{5}$$

where $\Phi_0 : \mathbb{R}^n \to \mathbb{R}$ defines the *terminal cost* and $\mathcal{L}_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ defines the *running cost*. Equation (5) is called the *cost function* or *objective function*. Note that the cost function is in the same form as the canonical constraints.

Our standard optimal control problem can now be stated as follows: *Choose a feasible control $\boldsymbol{u} \in \mathcal{F}$ to minimize the cost function* (5).

[1]This approach for handling continuous inequality constraints is known to cause numerical difficulties. A superior approach is to use the *constraint transcription method* to approximate $h(t, \boldsymbol{x}(t)) \geq 0$ by a canonical inequality constraint. See Section 5 for details.

3. **Control parameterization.** The control parameterization method involves approximating the control by a linear combination of basis functions, thereby yielding an approximate optimization problem with a finite number of decision variables. By exploiting special formulae for the cost and constraint gradients, this approximate problem can be solved using standard gradient-based optimization techniques. In this paper, we focus on *piecewise-constant basis functions*, as these are the most widely used in practice. Other basis functions can be handled similarly.

3.1. **Piecewise-constant control approximation.** With piecewise-constant basis functions, the control signal $\boldsymbol{u}$ is approximated as follows:

$$\boldsymbol{u}(t) \approx \boldsymbol{u}^p(t) = \boldsymbol{\xi}^k, \quad t \in [\tau_{k-1}, \tau_k), \quad k = 1, \ldots, p, \tag{6}$$

where $p \geq 1$ is the number of control subintervals, $[\tau_{k-1}, \tau_k)$ is the kth control subinterval, and $\boldsymbol{\xi}^k$ is the value of the control on the kth subinterval. This approximation scheme is illustrated in Figure 1.

In (6), the constant control values $\boldsymbol{\xi}^k$, $k = 1, \ldots, p$, are decision variables to be chosen optimally, while τ_k, $k = 0, \ldots, p$, are pre-fixed knot points satisfying

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_{p-1} < \tau_p = T.$$

For more accurate results, the knot points can also be taken as decision variables. However, in this section, we focus solely on the traditional control parameterization method in which the knot points are fixed. Variable knot points are discussed in the next section.

The approximate piecewise-constant control can be written as follows:

$$\boldsymbol{u}^p(t|\boldsymbol{\xi}) := \sum_{k=1}^{p} \boldsymbol{\xi}^k \chi_{[\tau_{k-1},\tau_k)}(t), \quad t \in [0, T], \tag{7}$$

where

$$\boldsymbol{\xi} = [(\boldsymbol{\xi}^1)^\top, \ldots, (\boldsymbol{\xi}^p)^\top]^\top \in \mathbb{R}^{pr} \tag{8}$$

and $\chi_{[\tau_{k-1},\tau_k)} : \mathbb{R} \to \mathbb{R}$ is the *characteristic function* defined by

$$\chi_{[\tau_{k-1},\tau_k)}(t) := \begin{cases} 1, & \text{if } t \in [\tau_{k-1}, \tau_k), \\ 0, & \text{if } t \notin [\tau_{k-1}, \tau_k). \end{cases}$$

Note that $\boldsymbol{u}^p(\cdot|\boldsymbol{\xi})$ switches value at times $t = \tau_k$, $k = 1, \ldots, p-1$. These times are called *switching times*.

Equation (7) does not define the value of $\boldsymbol{u}^p(\cdot|\boldsymbol{\xi})$ at $t = T$. Although it is often natural to define $\boldsymbol{u}^p(T|\boldsymbol{\xi}) := \boldsymbol{u}^p(T^-|\boldsymbol{\xi})$, the value of $\boldsymbol{u}^p(T|\boldsymbol{\xi})$ is actually irrelevant, as it does not affect the evolution of the state trajectory. In fact, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are two admissible controls such that $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for almost all $t \in [0, T]$, then $\boldsymbol{x}(t|\boldsymbol{u}) = \boldsymbol{x}(t|\boldsymbol{v})$ for all $t \in [0, T]$. Thus, changing the control on a set of measure zero does not change the state trajectory.

From (3), we obtain the following bound constraints on $\boldsymbol{\xi}$:

$$a_i \leq \xi_i^k \leq b_i, \quad k = 1, \ldots, p, \quad i = 1, \ldots, r. \tag{9}$$

Let Ξ^p denote the set of all $\boldsymbol{\xi} \in \mathbb{R}^{pr}$ satisfying (8) and (9).

Substituting (7) into the dynamic system (1)-(2) yields

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{\xi}^k), \quad t \in [\tau_{k-1}, \tau_k), \quad k = 1, \ldots, p, \tag{10}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}^0. \tag{11}$$

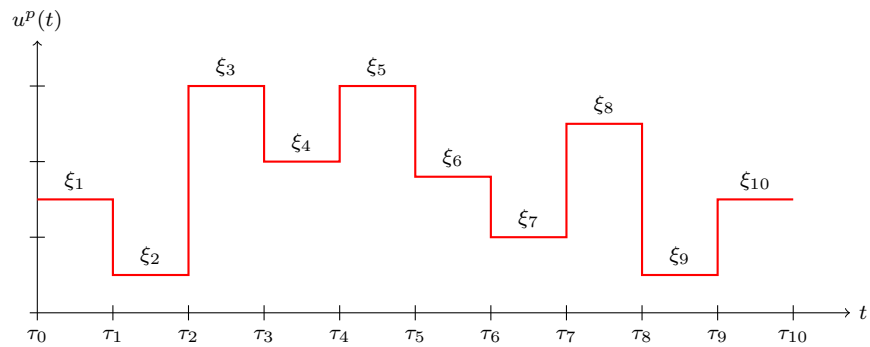


FIGURE 1. Piecewise-constant control approximation with equidistant knot points ($p = 10$ subintervals).

Let $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi})$ denote the solution of (10)-(11) corresponding to $\boldsymbol{\xi} \in \Xi^p$. Then clearly,

$$\boldsymbol{x}^p(t|\boldsymbol{\xi}) = \boldsymbol{x}(t|\boldsymbol{u}^p(\cdot|\boldsymbol{\xi})), \quad t \in [0, T].$$

Under the control approximation (7), the canonical constraints (4) become

$$g_j^p(\boldsymbol{\xi}) := \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi})) + \sum_{k=1}^{p} \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^k)dt \begin{cases} = 0 \\ \geq 0 \end{cases} \qquad j = 1, \ldots, m, \quad (12)$$

where $g_j^p(\boldsymbol{\xi}) := g_j(\boldsymbol{u}^p(\cdot|\boldsymbol{\xi}))$.

Let Γ^p denote the set of all $\boldsymbol{\xi} \in \Xi^p$ satisfying (12). Furthermore, let $\mathcal{F}^p$ denote the set of all $\boldsymbol{u}^p(\cdot|\boldsymbol{\xi})$ defined by (7) corresponding to elements of Γ^p. Then

$$\boldsymbol{\xi} \in \Gamma^p \quad \Longleftrightarrow \quad \boldsymbol{u}^p(\cdot|\boldsymbol{\xi}) \in \mathcal{F}^p.$$

The cost function (5) becomes

$$g_0^p(\boldsymbol{\xi}) := g_0(\boldsymbol{u}^p(\cdot|\boldsymbol{\xi})) = \Phi_0(\boldsymbol{x}^p(T|\boldsymbol{\xi})) + \sum_{k=1}^{p} \int_{\tau_{k-1}}^{\tau_k} \mathcal{L}_0(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^k)dt. \quad (13)$$

We now define the following approximate problem: *Choose a vector $\boldsymbol{\xi} \in \Gamma^p$ to minimize the cost function* (13).

This approximate problem only involves a finite number of decision variables. Thus, it should be much easier to solve than the original optimal control problem, which involves determining the value of a *function* at an infinite number of time points (every point in the interval $[0, T]$). Two important questions that arise are:

(i) How can the approximate problem be solved?
(ii) Does the solution of the approximate problem converge in some sense to the solution of the original problem?

We discuss these issues in the remainder of this section.

3.2. **Gradient computation.** The approximate problem defined above is a nonlinear optimization problem in which a finite number of decision variables need to be chosen to minimize a cost function subject to a set of constraints. For this approximate problem, the cost and constraint functions are implicit—rather than explicit—functions of the decision vector $\boldsymbol{\xi}$. Thus, it is not obvious how to determine the gradients of the cost and constraint functions.

Standard algorithms for nonlinear optimization (e.g. sequential quadratic programming) exploit the cost and constraint gradients to generate search directions that lead to profitable areas of the search space [50, 53]. To apply such algorithms to the approximate problem, we need to be able to compute the partial derivatives of g_j^p, $j = 0, \ldots, m$, at each $\boldsymbol{\xi} \in \mathbb{R}^{pr}$. There are two methods for doing this: the *variational method* (also called the *sensitivity method*) [26, 44, 47, 72] and the *costate method* [19, 65]. We describe these methods below, starting with the variational method.

Define the *Kronecker delta function* as follows:

$$\delta_{kl} := \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{otherwise.} \end{cases}$$

In the variational method, we define the following *variational system* for each $k = 1, \ldots, p$ and $i = 1, \ldots, r$:

$$\dot{\boldsymbol{\psi}}^{ki}(t) = \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\psi}^{ki}(t) + \delta_{kl} \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^l)}{\partial u_i}, \quad t \in [\tau_{l-1}, \tau_l), \quad l = k, \ldots, p, \tag{14}$$

with the initial condition

$$\boldsymbol{\psi}^{ki}(t) = \mathbf{0}, \quad t \in [0, \tau_{k-1}), \tag{15}$$

where $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi})$ is the solution of (10)-(11) corresponding to $\boldsymbol{\xi}$.

Let $\boldsymbol{\psi}^{ki}(\cdot|\boldsymbol{\xi})$ denote the solution of (14)-(15). It can be shown (see [44, 47]) that $\boldsymbol{\psi}^{ki}(t|\boldsymbol{\xi})$ is the partial derivative of $\boldsymbol{x}^p(t|\boldsymbol{\xi})$ with respect to the decision variable ξ_i^k:

$$\frac{\partial \boldsymbol{x}^p(t|\boldsymbol{\xi})}{\partial \xi_i^k} = \boldsymbol{\psi}^{ki}(t|\boldsymbol{\xi}), \quad t \in [0, T], \quad k = 1, \ldots, p, \quad i = 1, \ldots, r. \tag{16}$$

This partial derivative is called the *state variation* with respect to ξ_i^k.

In view of (16), the cost and constraint gradients can be computed using the chain rule of differentiation:

$$\begin{aligned} \frac{\partial g_j^p(\boldsymbol{\xi})}{\partial \xi_i^k} = {} & \frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi}))}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}^p(T|\boldsymbol{\xi})}{\partial \xi_i^k} + \sum_{l=1}^{p} \int_{\tau_{l-1}}^{\tau_l} \frac{\partial \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}^p(t|\boldsymbol{\xi})}{\partial \xi_i^k} dt \\ & + \int_{\tau_{k-1}}^{\tau_k} \frac{\partial \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^k)}{\partial u_i} dt, \quad j = 0, \ldots, m. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial g_j^p(\boldsymbol{\xi})}{\partial \xi_i^k} = {} & \frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi}))}{\partial \boldsymbol{x}} \boldsymbol{\psi}^{ki}(T|\boldsymbol{\xi}) + \sum_{l=1}^{p} \int_{\tau_{l-1}}^{\tau_l} \frac{\partial \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\psi}^{ki}(t|\boldsymbol{\xi}) dt \\ & + \int_{\tau_{k-1}}^{\tau_k} \frac{\partial \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^k)}{\partial u_i} dt, \quad j = 0, \ldots, m. \end{aligned} \tag{17}$$

Using these formulae, the cost and constraint gradients are computed in the same way. This is the essence of the so-called *unified computational approach* advocated in the monograph by Teo et al. [65]. Since the variational systems depend on the solution of the state system, the state and variational systems can be combined to form an enlarged system of differential equations. This enlarged system can then be solved numerically to yield $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi})$ and $\boldsymbol{\psi}^{ki}(\cdot|\boldsymbol{\xi})$, after which (17) can be evaluated. This gradient computation method can be integrated with a nonlinear programming

algorithm to solve the approximate problem numerically. More details are given in [26, 44, 47, 72] and the references cited therein.

An alternative method for computing the cost and constraint gradients is the so-called *costate method*. In the costate method, we define the *Hamiltonian function* as follows:

$$H_j(t, \boldsymbol{x}, \boldsymbol{\xi}^l, \boldsymbol{\lambda}) := \mathcal{L}_j(t, \boldsymbol{x}, \boldsymbol{\xi}^l) + \boldsymbol{\lambda}^\top \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\xi}^l).$$

Consider the following *costate system* for each $j = 0, \ldots, m$:

$$\dot{\boldsymbol{\lambda}}^j(t) = -\left[\frac{\partial H_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^l, \boldsymbol{\lambda}^j(t))}{\partial \boldsymbol{x}}\right]^\top, \quad t \in [\tau_{l-1}, \tau_l), \quad l = 1, \ldots, p, \tag{18}$$

with the terminal condition

$$\boldsymbol{\lambda}^j(T) = \left[\frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi}))}{\partial \boldsymbol{x}}\right]^\top. \tag{19}$$

The solution of (18)-(19), denoted by $\boldsymbol{\lambda}^j(\cdot|\boldsymbol{\xi})$, is called the *costate*.

The cost and constraint gradients can be expressed as follows:

$$\frac{\partial g_j^p(\boldsymbol{\xi})}{\partial \xi_i^k} = \int_{\tau_{k-1}}^{\tau_k} \frac{\partial H_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}), \boldsymbol{\xi}^k, \boldsymbol{\lambda}^j(t|\boldsymbol{\xi}))}{\partial u_i} dt, \quad j = 0, \ldots, m.$$

See Chapters 5 and 6 of [65] for a proof of this expression.

Note that the costate system (18)-(19) contains a terminal condition rather than an initial condition. Hence, (18)-(19) must be integrated backwards in time, starting from $t = T$. Consequently, unlike the variational system, the costate system cannot be solved simultaneously with the state system—instead, the state system must be solved first, after which the solution of the state system can be used to solve the costate system. This makes the costate method slightly more complicated to implement than the variational method. However, one advantage of the costate method is that it only involves $m + 1$ auxiliary systems of differential equations (one for the cost, one for each of the constraints), whereas the variational method involves pr auxiliary systems (pr is normally much larger than $m + 1$).

Both the variational method and the costate method will produce the same results. By combining one of these gradient computation methods with a nonlinear optimization algorithm, the approximate problem can be solved efficiently.

3.3. **Convergence results.** Convergence is a key issue for any numerical method and control parameterization is no exception. In fact, one of the main virtues of the control parameterization method is its strong convergence properties. We briefly discuss these properties below.

Let $\boldsymbol{\xi}^* \in \Gamma^p$ be a solution of the approximate optimization problem, where

$$\boldsymbol{\xi}^* = [(\boldsymbol{\xi}^{1,*})^\top, \ldots, (\boldsymbol{\xi}^{p,*})^\top]^\top.$$

Then a corresponding *suboptimal control* for the original optimal control problem can be defined according to (7) as follows:

$$\boldsymbol{u}^p(t|\boldsymbol{\xi}^*) = \sum_{k=1}^{p} \boldsymbol{\xi}^{k,*} \chi_{[\tau_{k-1}, \tau_k)}(t).$$

An obvious question to ask is: how close is this suboptimal control to the true optimal control? It is natural to expect that the quality of the approximation will

improve as the number of subintervals increases, and this is indeed the case. More specifically, it can be shown that

$$\lim_{p\to\infty} g_0(\boldsymbol{u}^p(\cdot|\boldsymbol{\xi}^*)) = g_0(\boldsymbol{u}^*), \tag{20}$$

where $\boldsymbol{u}^*$ is the true optimal control. This result assumes that the subintervals are chosen so that all knot points for a particular value of p are also knot points for $p+1$ (i.e. successive partitions of the time horizon are *refinements* of each other).

Equation (20) shows that the cost of the suboptimal control produced by control parameterization converges to the true optimal cost as the number of subintervals approaches infinity. This result is proved in Chapter 6 of [65]. It also proved in Chapter 6 of [65] that if the sequence of suboptimal controls converges to a function almost everywhere on $[0,T]$, then this function must be an optimal control. There is no guarantee, however, that the sequence of suboptimal controls will converge.

The convergence results described above pertain to the general optimal control problem with nonlinear dynamics and nonlinear cost and constraints. In the special case when the dynamic system is linear, the cost function is convex, and the state variables are subject only to linear terminal inequality constraints, stronger convergence results are available. More specifically, it can be shown that in this case, the sequence of suboptimal controls produced by control parameterization has a subsequence converging to the true optimal control in the weak* topology. See Chapter 7 of [65] for more details.

Based on the convergence results described above, practical problems are often solved using the following iterative strategy:

1. Choose a small value for p and solve the approximate problem.
2. Increase p and re-solve the approximate problem, using the solution from the previous value of p as the initial guess.
3. Keep repeating Step 2 until the difference in optimal cost values in successive iterations is sufficiently small.

For the classical control parameterization scheme described in this section, the number of subintervals required before the cost improvements start to taper off is normally quite large. This is because the knot points are fixed, so there is not much flexibility to modify the control. A better approach is to consider the knot points as decision variables, along with the control heights. This will allow for *adaptive* selection of optimal switching instants. This issue is discussed in Section 4.

3.4. **Almost smooth controls.** The piecewise-constant control (7) is convenient for implementation in many practical applications. However, in some systems, discontinuous control signals such as (7) should be avoided because abrupt changes in the control signal could excite the system and cause instabilities. Furthermore, in some systems, it may not even be possible to implement a discontinuous control signal. For example, in the glider control problems in [36, 38], the control variable is the glider's angle of attack, which must be varied continuously.

Fortunately, it is easy to produce continuous control signals using the control parameterization approach. This is done by introducing a new control variable $\boldsymbol{v}$, where $\boldsymbol{v}$ is the *rate of change* of the original control variable $\boldsymbol{u}$:

$$\dot{\boldsymbol{u}}(t) = \boldsymbol{v}(t), \quad t \in [0,T]. \tag{21}$$

We now consider $\boldsymbol{u}$ as a state variable, with $\boldsymbol{v}$ the control function to be determined optimally. Note that $\boldsymbol{u}$ will be continuous and piecewise-linear if $\boldsymbol{v}$ is approximated

by a piecewise-constant function. The initial condition for $\boldsymbol{u}$ is

$$\boldsymbol{u}(0) = \boldsymbol{\zeta}, \tag{22}$$

where $\boldsymbol{\zeta}$ is a vector of system parameters to be determined optimally. In practice, bounds are often imposed on $\boldsymbol{v}$ to limit the rate of change of $\boldsymbol{u}$:

$$a_i \leq v_i(t) \leq b_i, \quad t \in [0, T], \quad i = 1, \ldots, r. \tag{23}$$

The new optimal control problem with (21)-(23) can be solved using an extension of the techniques described above. More details are given in Chapter 9 of [65].

4. **Variable switching times.** Suppose that the control switching times in (7) are decision variables to be chosen optimally, along with the control values. Then to solve the approximate problem using nonlinear optimization techniques, we need a method for computing the partial derivatives of g_j^p, $j = 0, \ldots, m$, with respect to the switching times. The gradient formulae in Section 3.2, however, only provide the partial derivatives with respect to the control values.

It is widely claimed in the optimal control literature that the partial derivatives of the cost and constraint functions with respect to variable switching times are "discontinuous" (see [29, 30, 62, 78]) and "not effective for numerical implementation" (see [17, 33, 44, 49]). However, tracing the literature back, we cannot find any mathematical justification for these claims. Despite this, it has become customary to avoid dealing with variable switching times by invoking the so-called *time-scaling transformation* (see Section 4.2), which maps variable switching times to fixed points in a new time horizon. The resulting optimization problem can then be solved using the computational techniques described in Section 3.

But why not optimize the switching times directly? After all, formulae for computing the partial derivatives of the cost and constraint functions with respect to the switching times do exist; see Chapter 5 of [65] for details. According to numerous papers over the past two decades, these gradient formulae are difficult to implement numerically. But there is no explanation in the literature as to why this is the case. We now proceed to investigate this issue more closely.

4.1. **Gradient computation.** Since the switching times are variable, we now denote the solution of (10)-(11) by $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ (i.e. the state trajectory now depends on *both* the control heights and the switching times).

For each $k = 1, \ldots, p-1$, consider the following *variational system*:

$$\dot{\boldsymbol{\phi}}^k(t) = \frac{\partial \boldsymbol{f}(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau}), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\phi}^k(t), \quad t \in [\tau_{l-1}, \tau_l), \quad l = k+1, \ldots, p, \tag{24}$$

with jump condition

$$\boldsymbol{\phi}^k(\tau_k^+) = \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau}), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau}), \boldsymbol{\xi}^{k+1}) \tag{25}$$

and initial condition

$$\boldsymbol{\phi}^k(t) = \boldsymbol{0}, \quad t \in [0, \tau_k). \tag{26}$$

In (25), we set $\tau_k^+ = T$ if $\tau_k = T$. Let $\boldsymbol{\phi}^k(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ denote the unique right-continuous solution of (24)-(26).

The following result gives the *right partial derivative* of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ with respect to τ_k. The proof of this result, which is given in Appendix A, is similar in essence to the proofs of Theorem 2 in [12] and Theorem 1 in [47].

Theorem 4.1. *Let $k \in \{1, \ldots, p-1\}$ and suppose that $\tau_k < \tau_{k+1}$. Then for all time points $t \neq \tau_k$,*

$$\lim_{\epsilon \to 0+} \frac{\boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau} + \epsilon \boldsymbol{e}^k) - \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau})}{\epsilon} = \boldsymbol{\phi}^k(t|\boldsymbol{\xi}, \boldsymbol{\tau}), \tag{27}$$

where $\boldsymbol{e}^k$ is the kth unit basis vector in $\mathbb{R}^{p-1}$. Furthermore, for $t = \tau_k$,

$$\lim_{\epsilon \to 0+} \frac{\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau} + \epsilon \boldsymbol{e}^k) - \boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau})}{\epsilon} = \boldsymbol{0}. \tag{28}$$

Theorem 4.1 shows that the solution of the variational system (24)-(26) coincides with the right partial derivative of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ (with respect to τ_k) at all times $t \neq \tau_k$. At $t = \tau_k$, there is a jump in the variational system and the corresponding right partial derivative. Note that the condition $\tau_k < \tau_{k+1}$ in Theorem 4.1 is essential; if $\tau_k = \tau_{k+1}$, then the perturbed switching time vector $\boldsymbol{\tau} + \epsilon \boldsymbol{e}^k$ will not be feasible for the approximate problem because $\epsilon > 0$. Thus, if $\tau_k = \tau_{k+1}$, then the right partial derivative of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ with respect to τ_k does not exist.

In our next result, we give the *left partial derivative* of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ with respect to τ_k. The proof of this result is given in Appendix B.

Theorem 4.2. *Let $k \in \{1, \ldots, p-1\}$ and suppose that $\tau_{k-1} < \tau_k$. Then for all time points $t \in [0, T]$,*

$$\lim_{\epsilon \to 0-} \frac{\boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau} + \epsilon \boldsymbol{e}^k) - \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau})}{\epsilon} = \boldsymbol{\phi}^k(t|\boldsymbol{\xi}, \boldsymbol{\tau}), \tag{29}$$

where $\boldsymbol{e}^k$ is the kth unit basis vector in $\mathbb{R}^{p-1}$.

Comparing Theorems 4.1 and 4.2, we see that the right partial derivative of $\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau})$ with respect to τ_k is the *left limit* of $\boldsymbol{\phi}^k(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ at $t = \tau_k$, and the left partial derivative of $\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi}, \boldsymbol{\tau})$ with respect to τ_k is the *right limit* of $\boldsymbol{\phi}^k(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$ at $t = \tau_k$. The following result, which follows immediately from Theorems 4.1 and 4.2, gives the state variation with respect to the switching times.

Theorem 4.3. *Let $k \in \{1, \ldots, p-1\}$ and suppose that $\tau_{k-1} < \tau_k < \tau_{k+1}$. Then for all time points $t \neq \tau_k$,*

$$\frac{\partial \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau})}{\partial \tau_k} = \lim_{\epsilon \to 0} \frac{\boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau} + \epsilon \boldsymbol{e}^k) - \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau})}{\epsilon} = \boldsymbol{\phi}^k(t|\boldsymbol{\xi}, \boldsymbol{\tau}).$$

Unlike the variational system for the control values, the variational system for the switching times involves a jump condition at $t = \tau_k$. Consequently, the left and right partial derivatives differ at $t = \tau_k$, and thus the corresponding state variation does not exist. This is why $t = \tau_k$ is deliberately excluded from Theorem 4.3.

Note that the technical caveats in Theorem 4.3 ($t \neq \tau_k$ and $\tau_{k-1} < \tau_k < \tau_{k+1}$) are not needed for the state variation with respect to the control values. Thus, optimizing the switching times is much more difficult than optimizing the control values. This is one reason for the popularity of the time-scaling transformation, which allows one to circumvent the difficulties caused by variable switching times (see Section 4.2).

Using Theorem 4.3, we can derive the partial derivatives of the cost and constraint functions with respect to the switching times (assuming that all switching times are distinct). First, recall that the cost and constraint functions are defined by

$$g_j^p(\boldsymbol{\xi}, \boldsymbol{\tau}) := \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi}, \boldsymbol{\tau})) + \sum_{l=1}^{p} \int_{\tau_{l-1}}^{\tau_l} \mathcal{L}_j(t, \boldsymbol{x}^p(t|\boldsymbol{\xi}, \boldsymbol{\tau}), \boldsymbol{\xi}^l) dt, \quad j = 0, \ldots, m.$$

Using the chain rule of differentiation, we obtain

$$\frac{\partial}{\partial \tau_k}\big\{\Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi},\boldsymbol{\tau}))\big\} = \frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi},\boldsymbol{\tau}))}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}^p(T|\boldsymbol{\xi},\boldsymbol{\tau})}{\partial \tau_k}. \tag{30}$$

Furthermore, using the Leibniz rule for differentiating integrals, we obtain

$$\begin{aligned}\frac{\partial}{\partial \tau_k}&\left\{\sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l}\mathcal{L}_j(t,\boldsymbol{x}^p(t|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^l)dt\right\}\\ &= \mathcal{L}_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^k) - \mathcal{L}_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^{k+1})\\ &\quad+\sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l}\frac{\partial \mathcal{L}_j(t,\boldsymbol{x}^p(t|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^l)}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}^p(t|\boldsymbol{\xi},\boldsymbol{\tau})}{\partial \tau_k}dt.\end{aligned} \tag{31}$$

Applying the Leibniz rule to interchange the order of differentiation and integration is valid here because the partial derivative of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi},\boldsymbol{\tau})$ with respect to τ_k exists at all points in the interior of the interval $[\tau_{l-1},\tau_l]$. Recall that the Leibniz rule does not require differentiability at the end points.

Combining (30) and (31) with Theorem 4.3 yields the following gradient formulae:

$$\begin{aligned}\frac{\partial g_j^p(\boldsymbol{\xi},\boldsymbol{\tau})}{\partial \tau_k} &= \frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi},\boldsymbol{\tau}))}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\xi},\boldsymbol{\tau})\\ &\quad+\mathcal{L}_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^k) - \mathcal{L}_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^{k+1})\\ &\quad+\sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l}\frac{\partial \mathcal{L}_j(t,\boldsymbol{x}^p(t|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^l)}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(t|\boldsymbol{\xi},\boldsymbol{\tau})dt.\end{aligned} \tag{32}$$

These gradient formulae are based on the variational system. In Chapter 5 of [65], alternative gradient formulae are derived using the costate method. The derivation in [65] is rather complex and requires introducing generalized functions. We will give a simpler derivation.

As in Section 3.2, we consider the following *costate system* for each $j = 0,\ldots,m$:

$$\dot{\boldsymbol{\lambda}}^j(t) = -\left[\frac{\partial H_j(t,\boldsymbol{x}^p(t|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^l,\boldsymbol{\lambda}^j(t))}{\partial \boldsymbol{x}}\right]^\top, \quad t\in[\tau_{l-1},\tau_l), \quad l=1,\ldots,p, \tag{33}$$

with the terminal condition

$$\boldsymbol{\lambda}^j(T) = \left[\frac{\partial \Phi_j(\boldsymbol{x}^p(T|\boldsymbol{\xi},\boldsymbol{\tau}))}{\partial \boldsymbol{x}}\right]^\top, \tag{34}$$

where H_j is the *Hamiltonian* defined by

$$H_j(t,\boldsymbol{x},\boldsymbol{\xi}^l,\boldsymbol{\lambda}) := \mathcal{L}_j(t,\boldsymbol{x},\boldsymbol{\xi}^l) + \boldsymbol{\lambda}^\top \boldsymbol{f}(t,\boldsymbol{x},\boldsymbol{\xi}^l).$$

Let $\boldsymbol{\lambda}^j(\cdot|\boldsymbol{\xi},\boldsymbol{\tau})$ denote the solution of (33)-(34). Gradient formulae for the cost and constraints with respect to the switching times are derived below.

Theorem 4.4. *Let $k\in\{1,\ldots,p-1\}$ and suppose that $\tau_{k-1}<\tau_k<\tau_{k+1}$. Then*

$$\begin{aligned}\frac{\partial g_j^p(\boldsymbol{\xi},\boldsymbol{\tau})}{\partial \tau_k} &= H_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^k,\boldsymbol{\lambda}^j(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}))\\ &\quad- H_j(\tau_k,\boldsymbol{x}^p(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau}),\boldsymbol{\xi}^{k+1},\boldsymbol{\lambda}^j(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau})).\end{aligned} \tag{35}$$

Proof. For simplicity, we write $\boldsymbol{x}^p$ instead of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi},\boldsymbol{\tau})$. Let $\boldsymbol{v}:[0,T]\to\mathbb{R}^n$ be any absolutely continuous function. Then we can write g_j^p as follows:

$$\begin{aligned} g_j^p(\boldsymbol{\xi},\boldsymbol{\tau}) &= \Phi_j(\boldsymbol{x}^p(T)) + \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l} \mathcal{L}_j(t,\boldsymbol{x}^p(t),\boldsymbol{\xi}^l)dt \\ &= \Phi_j(\boldsymbol{x}^p(T)) + \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l} H_j(t,\boldsymbol{x}^p(t),\boldsymbol{\xi}^l,\boldsymbol{v}(t))dt - \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l} \boldsymbol{v}(t)^\top \dot{\boldsymbol{x}}^p(t)dt. \end{aligned}$$

Using integration by parts,

$$\begin{aligned} g_j^p(\boldsymbol{\xi},\boldsymbol{\tau}) &= \Phi_j(\boldsymbol{x}^p(T)) + \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l} H_j(t,\boldsymbol{x}^p(t),\boldsymbol{\xi}^l,\boldsymbol{v}(t))dt \\ &\qquad - \sum_{l=1}^{p}\left\{\boldsymbol{v}(\tau_l)^\top\boldsymbol{x}^p(\tau_l) - \boldsymbol{v}(\tau_{l-1})^\top\boldsymbol{x}^p(\tau_{l-1}) - \int_{\tau_{l-1}}^{\tau_l}\dot{\boldsymbol{v}}(t)^\top\boldsymbol{x}^p(t)dt\right\} \\ &= \Phi_j(\boldsymbol{x}^p(T)) - \boldsymbol{v}(T)^\top\boldsymbol{x}^p(T) + \boldsymbol{v}(0)^\top\boldsymbol{x}^p(0) \\ &\qquad + \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l}\left\{H_j(t,\boldsymbol{x}^p(t),\boldsymbol{\xi}^l,\boldsymbol{v}(t)) + \dot{\boldsymbol{v}}(t)^\top\boldsymbol{x}^p(t)\right\}dt. \end{aligned}$$

Using the Leibniz rule, this equation can be differentiated with respect to τ_k to give

$$\begin{aligned} \frac{\partial g_j^p(\boldsymbol{\xi},\boldsymbol{\tau})}{\partial\tau_k} &= \frac{\partial\Phi_j(\boldsymbol{x}^p(T))}{\partial\boldsymbol{x}}\frac{\partial\boldsymbol{x}^p(T)}{\partial\tau_k} - \boldsymbol{v}(T)^\top\frac{\partial\boldsymbol{x}^p(T)}{\partial\tau_k} \\ &\quad + H_j(\tau_k,\boldsymbol{x}^p(\tau_k),\boldsymbol{\xi}^k,\boldsymbol{v}(\tau_k)) - H_j(\tau_k,\boldsymbol{x}^p(\tau_k),\boldsymbol{\xi}^{k+1},\boldsymbol{v}(\tau_k)) \\ &\quad + \sum_{l=1}^{p}\int_{\tau_{l-1}}^{\tau_l}\left\{\frac{\partial H_j(t,\boldsymbol{x}^p(t),\boldsymbol{\xi}^l,\boldsymbol{v}(t))}{\partial\boldsymbol{x}}\frac{\partial\boldsymbol{x}^p(t)}{\partial\tau_k} + \dot{\boldsymbol{v}}(t)^\top\frac{\partial\boldsymbol{x}^p(t)}{\partial\tau_k}\right\}dt. \end{aligned}$$

Recall that $\boldsymbol{v}:[0,T]\to\mathbb{R}^n$ was chosen arbitrarily. Setting $\boldsymbol{v}=\boldsymbol{\lambda}^j(\cdot|\boldsymbol{\xi},\boldsymbol{\tau})$, and then applying equations (33)-(34), we obtain

$$\frac{\partial g_j^p(\boldsymbol{\xi},\boldsymbol{\tau})}{\partial\tau_k} = H_j(\tau_k,\boldsymbol{x}^p(\tau_k),\boldsymbol{\xi}^k,\boldsymbol{\lambda}^j(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau})) - H_j(\tau_k,\boldsymbol{x}^p(\tau_k),\boldsymbol{\xi}^{k+1},\boldsymbol{\lambda}^j(\tau_k|\boldsymbol{\xi},\boldsymbol{\tau})),$$

as required. □

Equations (32) and (35) give the partial derivatives of the canonical functions g_j^p, $j=0,\ldots,m$, with respect to the switching times. Contrary to what is widely claimed in the literature, these partial derivatives are continuous functions: since the state trajectory and the solutions of the variational and costate systems depend continuously on $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$, the derivative formulae in (32) and (35) also depend continuously on $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$. In principle, (32) and (35) can be used in conjunction with a gradient-based optimization method to optimize the switching times. However, there are several difficulties with this approach:

(i) The variational system for the switching times contains a jump condition;
(ii) The partial derivatives of the canonical functions with respect to τ_k only exist when the switching times are distinct;
(iii) It is cumbersome to integrate the state and variational/costate systems numerically when the switching times are variable, especially when two or more switching times are close together.

For these reasons, the gradient formulae in (32) and (35) are rarely implemented in practice. Instead, the *time-scaling transformation* is used to transform the problem with variable switching times into an equivalent problem with fixed switching times. This is discussed in the next section.

4.2. **The time-scaling transformation.** One way of circumventing the difficulties caused by variable switching times is to apply the *time-scaling transformation*. The time-scaling transformation works by mapping the variable switching times to fixed points in a new time horizon, thus yielding a new optimization problem in which the switching times are fixed. This new problem can then be solved readily using the gradient-based optimization techniques described in Section 3.

The time-scaling transformation—originally called the *control parameterization enhancing transform* (CPET)—was first introduced in [29] to determine optimal switching instants for time-optimal controls. It has since been applied to many other areas, such as mixed integer programming [27], impulsive systems [37], and singular optimal control [60].

To apply the time-scaling transformation, we first introduce a new time variable $s \in [0, p]$, where p is the number of control subintervals. Next, we relate the new time variable $s \in [0, p]$ to the original time variable $t \in [0, T]$ through the following boundary value problem:

$$\frac{dt(s)}{ds} = v(s), \quad s \in [0, p], \tag{36}$$

$$t(0) = 0, \tag{37}$$

$$t(p) = T, \tag{38}$$

where $v : [0, p] \to \mathbb{R}$ is a new piecewise-constant control called the *time-scaling control*. The time-scaling control is defined by

$$v(s) := \sum_{k=1}^{p} \theta_k \chi_{[k-1,k)}(s), \quad s \in [0, p],$$

where $\chi_{[k-1,k)} : \mathbb{R} \to \mathbb{R}$ is the characteristic function for the interval $[k-1, k)$ and

$$\theta_k := \tau_k - \tau_{k-1}, \quad k = 1, \ldots, p.$$

Thus, the heights of the time-scaling control are the durations of the control subintervals in the original time horizon.

It is easy to see that (36)-(38) defines t as a continuous and non-decreasing piecewise-linear function of s. In fact, integrating (36)-(38) yields

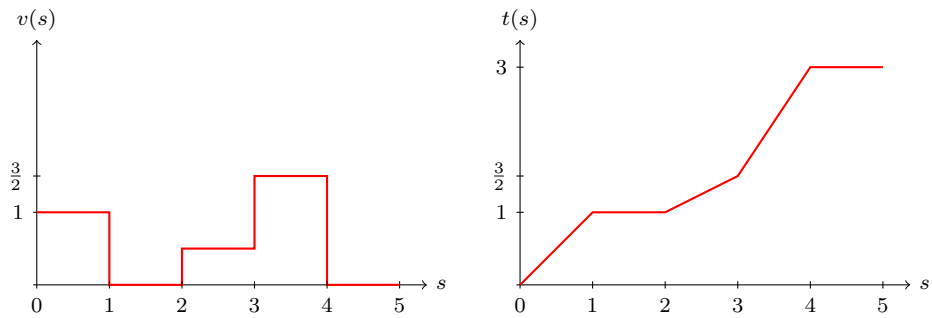
$$t(s) = \int_0^s v(\eta) d\eta = \sum_{l=1}^{\lfloor s \rfloor} \theta_l + \theta_{\lfloor s \rfloor + 1}(s - \lfloor s \rfloor), \quad s \in [0, p). \tag{39}$$

In some references (e.g. [37, 38, 45, 49]), equation (39) is used to define the time-scaling transformation directly instead of (36)-(38).

From (38) and (39), we obtain

$$t(k) = \sum_{l=1}^{k} \theta_l = \tau_k, \quad k = 0, \ldots, p.$$

This shows that the time-scaling transformation defined by (36)-(38) maps $s = k$ to the kth switching time $t = \tau_k$. An example illustrating the relationship between s and t is given in Figure 2.



(a) Time-scaling control (b) t as a function of s

FIGURE 2. The time-scaling transformation for $\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = \frac{1}{2}$, $\theta_4 = \frac{3}{2}$, and $\theta_5 = 0$.

Let

$$\boldsymbol{\theta} := [\theta_1, \ldots, \theta_p]^\top.$$

Clearly, the vector $\boldsymbol{\theta} \in \mathbb{R}^p$ must satisfy the following non-negativity constraints:

$$\theta_k \geq 0, \quad k = 1, \ldots, p. \tag{40}$$

Applying the transformation $t = t(s)$, where $t(s)$ is defined by (36)-(38), we obtain a new state vector defined as follows:

$$\tilde{\boldsymbol{x}}^p(s) := \boldsymbol{x}^p(t(s)), \quad s \in [0, p].$$

Thus, applying (36)-(38) to the dynamic system (10)-(11) yields

$$\dot{\tilde{\boldsymbol{x}}}^p(s) = \theta_k \boldsymbol{f}(t(s), \tilde{\boldsymbol{x}}^p(s), \boldsymbol{\xi}^k), \quad s \in [k-1, k), \quad k = 1, \ldots, p, \tag{41}$$

$$\tilde{\boldsymbol{x}}^p(0) = \boldsymbol{x}^0, \tag{42}$$

where the overhead dot now denotes differentiation with respect to s. Let $\tilde{\boldsymbol{x}}^p(\cdot|\boldsymbol{\xi}, \boldsymbol{\theta})$ denote the solution of (41)-(42) corresponding to $\boldsymbol{\xi} \in \mathbb{R}^{pr}$ and $\boldsymbol{\theta} \in \mathbb{R}^p$. Note that the switching times in (41)-(42) occur at the *fixed* locations $s = 1, 2, \ldots, p-1$.

The bound constraints on $\boldsymbol{\xi} \in \mathbb{R}^{pr}$ remain the same:

$$a_i \leq \xi_i^k \leq b_i, \quad k = 1, \ldots, p, \quad i = 1, \ldots, r. \tag{43}$$

However, under the time-scaling transformation (36)-(38), the canonical constraints (12) become

$$\Phi_j(\tilde{\boldsymbol{x}}^p(p|\boldsymbol{\xi}, \boldsymbol{\theta})) + \sum_{k=1}^{p} \int_{k-1}^{k} \theta_k \mathcal{L}_j(t(s), \tilde{\boldsymbol{x}}^p(s|\boldsymbol{\xi}, \boldsymbol{\theta}), \boldsymbol{\xi}^k) ds \begin{cases} = 0 \\ \geq 0 \end{cases} \quad j = 1, \ldots, m. \tag{44}$$

Similarly, the cost function (13) becomes

$$\tilde{g}_0^p(\boldsymbol{\xi}, \boldsymbol{\theta}) := \Phi_0(\tilde{\boldsymbol{x}}^p(p|\boldsymbol{\xi}, \boldsymbol{\theta})) + \sum_{k=1}^{p} \int_{k-1}^{k} \theta_k \mathcal{L}_0(t(s), \tilde{\boldsymbol{x}}^p(s|\boldsymbol{\xi}, \boldsymbol{\theta}), \boldsymbol{\xi}^k) ds. \tag{45}$$

The transformed problem is: *Given the dynamic systems* (36)-(37) *and* (41)-(42), *choose* $\boldsymbol{\xi} \in \mathbb{R}^{pr}$ *and* $\boldsymbol{\theta} \in \mathbb{R}^p$ *to minimize* (45) *subject to the constraints* (38), (40), (43), *and* (44). In this problem, the switching times τ_k, $k = 1, \ldots, p-1$, have been replaced by $\boldsymbol{\theta}$, a normal decision vector similar to $\boldsymbol{\xi}$. Furthermore, the original time variable t is now considered a state variable.

The transformed problem defined above is equivalent to the original problem. In fact, if $(\boldsymbol{\xi}^*, \boldsymbol{\theta}^*)$ is a solution of the transformed problem, then the optimal switching times for the original problem are

$$\tau_k^* = \sum_{l=1}^{k} \theta_l^*, \quad k = 1, \ldots, p-1.$$

The corresponding suboptimal control is

$$\boldsymbol{u}^p(t) = \sum_{k=1}^{p} \boldsymbol{\xi}^{k,*} \chi_{[\tau_{k-1}^*, \tau_k^*)}(t).$$

Since it only involves fixed switching times, the transformed problem can be solved readily using the techniques described in Section 3. In practice, this approach is much more effective than optimizing the switching times directly. For more details, see references [44, 49, 56, 68].

5. **Continuous inequality constraints.** Many practical optimal control problems involve *continuous inequality constraints* of the following form:

$$h(t, \boldsymbol{x}(t)) \geq 0, \quad t \in [0, T], \tag{46}$$

where $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given continuously differentiable function. Such constraints are also called *path constraints* or *functional inequality constraints* in the literature. The main difficulty with these types of constraints is that they restrict the state variables at *every* point in the time horizon—an infinite number of time points.

Continuous inequality constraints in the form of (46) arise in many applications, including robotics [18], underwater vehicles [7], and chemical engineering [82]. Since h is a continuous function, (46) can be transformed into the following canonical equality constraint:

$$\int_0^T \min\{h(t, \boldsymbol{x}(t)), 0\} dt = 0. \tag{47}$$

At first glance, this transformation looks extremely powerful: it converts the infinite number of constraints defined by (46) into just one constraint. Unfortunately, since the function $\min\{\cdot, 0\}$ is non-differentiable at the origin, (47) is a *non-smooth* constraint. Thus, gradient-based optimization methods such as sequential quadratic programming will typically struggle to handle this constraint.

An alternative integral formulation for (46) is given by

$$\int_0^T \min\{h(t, \boldsymbol{x}(t)), 0\}^2 dt = 0. \tag{48}$$

This transformation was first suggested in [63]. Although (48) is differentiable, its gradient with respect to the control values is zero at all feasible points. Thus, (48) does not satisfy the *linear independence constraint qualification*—a fundamental condition required for the convergence of nonlinear programming algorithms. Thus, standard nonlinear programming algorithms cannot handle (48) effectively.

Given the limitations of (47) and (48), these transformations are rarely used in practice. Instead, two popular approximation methods have been developed: the *constraint transcription method* and the *exact penalty method*. We now proceed to discuss these two methods.

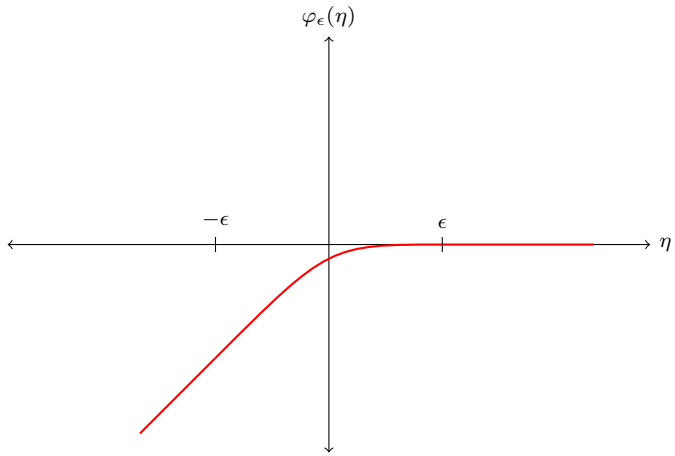


FIGURE 3. The smoothing function φ_ϵ.

5.1. **Constraint transcription: Traditional approach.** The constraint transcription method is based on the following smooth approximation of the function $\min\{\cdot, 0\}$:

$$\min\{\eta, 0\} \approx \varphi_\epsilon(\eta) := \begin{cases} \eta, & \text{if } \eta < -\epsilon, \\ -(\eta-\epsilon)^2/4\epsilon, & \text{if } -\epsilon \le \eta \le \epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$ is an adjustable parameter. The smoothing function φ_ϵ is illustrated in Figure 3. It is easy to verify that φ_ϵ is continuously differentiable and non-positive.

Substituting $\min\{\eta, 0\} \approx \varphi_\epsilon(\eta)$ into the left-hand side of (47) gives the following approximate constraint:

$$\int_0^T \varphi_\epsilon(h(t, \boldsymbol{x}(t)))dt = 0. \tag{49}$$

Like (48), this constraint does not satisfy the linear independence constraint qualification. Thus, it is not suitable for numerical computation.

Since φ_ϵ is non-positive, we can approximate (49) by

$$-\gamma \le \int_0^T \varphi_\epsilon(h(t, \boldsymbol{x}(t)))dt, \tag{50}$$

where $\gamma > 0$ is an adjustable parameter. Hence, the original continuous inequality constraint (46) is approximated by the canonical inequality constraint (50). This is the main idea of the traditional constraint transcription method.

Replacing (46) with (50) yields a standard optimal control problem that can be solved readily using control parameterization. It can be shown (see [22, 65, 66]) that for each $\epsilon > 0$, there exists a corresponding $\gamma(\epsilon) > 0$ such that whenever $0 < \gamma < \gamma(\epsilon)$, constraint (50) implies constraint (46). Furthermore, the optimal cost of the approximate problem (obtained by replacing (46) with (50)) converges to the true optimal cost as $\epsilon \to 0$. Thus, when ϵ and γ are sufficiently small, (50) is a good approximation of (46).

5.2. **Constraint transcription: Penalty approach.** An alternative constraint transcription approach is described in references [49, 54, 71]. This approach involves

appending the right-hand side of (50) to the cost function as a penalty term. This gives the modified cost function

$$\hat{g}_0(\boldsymbol{u}) := g_0(\boldsymbol{u}) - \gamma \int_0^T \varphi_\epsilon(h(t, \boldsymbol{x}(t)))dt, \tag{51}$$

where $\gamma > 0$ is a *penalty parameter* and $g_0(\boldsymbol{u})$ is the original cost function. It can be shown that for each $\epsilon > 0$, there exists a corresponding $\gamma(\epsilon) > 0$ such that for all $\gamma > \gamma(\epsilon)$, any minimizing control of (51) satisfies the original continuous inequality constraints (46). Moreover, the cost of the minimizing control of (51) converges to the true optimal cost as $\epsilon \to 0$.

Both versions of the constraint transcription method have similar convergence properties. The traditional approach, in which the continuous inequality constraints are approximated by canonical constraints, was proposed first in the early 1990s. It was originally designed to solve semi-infinite programming problems in [22], and then later applied to optimal control in [65]. The penalty approach was first proposed in [71], a few years after the introduction of the traditional approach. One advantage of the penalty approach is that unlike the traditional approach, it does not require any additional costate systems (in the traditional approach, each approximate canonical constraint requires a separate costate system). Thus, intuitively, the penalty approach should be faster than the traditional approach. However, this has yet to be demonstrated conclusively in practice.

5.3. **Exact penalty method.** For many years, constraint transcription was the method of choice for handling path-constrained optimal control problems. Recently, however, a rival method has emerged. This new method, called the *exact penalty method*, is described in [34].

In the exact penalty method, we define

$$\Delta(\boldsymbol{u}, \epsilon) := \int_0^T \max\{-h(t, \boldsymbol{x}(t)) - \epsilon\omega, 0\}^2 dt,$$

where $\epsilon > 0$ is a new decision variable and $\omega \in (0, 1)$ is a fixed constant. Note that $\Delta(\boldsymbol{u}, \epsilon) = 0$ if and only if

$$h(t, \boldsymbol{x}(t)) \geq -\epsilon\omega, \quad t \in [0, T].$$

Thus, when ϵ is small, $\Delta(\boldsymbol{u}, \epsilon)$ is an approximate measure of the violation of the continuous inequality constraint (46).

Now, consider the following *penalty function*:

$$\hat{g}_0(\boldsymbol{u}, \epsilon) := g_0(\boldsymbol{u}) + \epsilon^{-\alpha}\Delta(\boldsymbol{u}, \epsilon) + \gamma\epsilon^\beta, \tag{52}$$

where $\gamma > 0$ is a penalty parameter and α and β are fixed constants.

In the penalty function (52), the last term $\gamma\epsilon^\beta$ is designed to penalize large values of ϵ, while the middle term $\epsilon^{-\alpha}\Delta(\boldsymbol{u}, \epsilon)$ is designed to penalize violations in the continuous inequality constraint (46). When γ is large, minimizing (52) forces ϵ to be small, which in turn causes $\epsilon^{-\alpha}$ to become large, and thus constraint violations are penalized very severely. Hence, minimizing the penalty function for large values of γ will lead to feasible points satisfying constraint (46).

Under some mild technical conditions, it can be shown that the penalty function (52) is "exact" in the sense that when γ is sufficiently large, any local minimizer of the penalty function is also a local minimizer of the original optimal control problem. This is a strong convergence property that is not shared by the constraint

transcription method. Indeed, the convergence results for the constraint transcription method require global optimal solutions; there is no guarantee that a local solution of the approximate problem converges to a local solution of the original problem. Note that the approximate problems are usually non-convex, and thus finding the global solution is a difficult task in practice. Despite this, the constraint transcription method has proven to be very effective at solving optimal control problems with continuous inequality constraints.

6. **Non-standard optimal control problems.** Although the standard optimal control problem described in Section 2 is quite general, there are many practical problems that do not fit into this standard form. One of the main virtues of control parameterization is its versatility at handling such non-standard optimal control problems. The purpose of this section is to describe how several important classes of non-standard optimal control problems can be tackled using the control parameterization approach.

6.1. **System parameters.** The control function $\boldsymbol{u}$ is a *time-varying decision variable* that influences the evolution of the state variables through the governing dynamic system. In some optimal control problems, the dynamic system also involves *time-invariant decision variables*, in addition to the time-varying control function. Such time-invariant decision variables are called *system parameters*.

For example, consider the following dynamic system:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\zeta}), \quad t \in [0, T], \\ \boldsymbol{x}(0) &= \boldsymbol{x}^0(\boldsymbol{\zeta}). \end{aligned}$$

Here, $\boldsymbol{\zeta} \in \mathbb{R}^q$ is a vector of time-invariant system parameters and $\boldsymbol{x}^0 : \mathbb{R}^q \to \mathbb{R}^n$ is a given function defining the initial state in terms of $\boldsymbol{\zeta}$. Note that $\boldsymbol{\zeta}$ can be viewed as a piecewise-constant control in the form of (7) with $p = 1$ (i.e. a piecewise-constant control with only one subinterval). Thus, the control parameterization technique can be easily extended to solve optimal control problems with system parameters. See Chapter 5 of [65] for details.

6.2. **Characteristic times.** In previous sections, we considered the following canonical form for the cost and constraint functions:

$$g_j(\boldsymbol{u}) := \Phi_j(\boldsymbol{x}(T)) + \int_0^T \mathcal{L}_j(t, \boldsymbol{x}(t), \boldsymbol{u}(t))dt. \tag{53}$$

A more general canonical form is

$$g_j(\boldsymbol{u}) := \Phi_j(\boldsymbol{x}(\tau_j)) + \int_0^{\tau_j} \mathcal{L}_j(t, \boldsymbol{x}(t), \boldsymbol{u}(t))dt, \tag{54}$$

where $\tau_j \in [0, T]$ is a given constant called the *characteristic time*. This generalization is useful for modelling *interior point constraints* of the form $\boldsymbol{x}(\tau_j) = \boldsymbol{\gamma}^j$, where $\boldsymbol{\gamma}^j$ is the desired state at time $t = \tau_j$. The gradient formulae for (53) given in Section 3.2 can be easily adapted to suit (54); see Chapter 6 of [65] for details.

In some applications—for example, cancer chemotherapy [44, 51]—the constraints depend on the state at two or more discrete time points. This motivates the following generalization of (53) and (54):

$$g_j(\boldsymbol{u}) := \Phi_j(\boldsymbol{x}(\tau_1), \ldots, \boldsymbol{x}(\tau_q)) + \int_0^T \mathcal{L}_j(t, \boldsymbol{x}(t), \boldsymbol{u}(t))dt. \tag{55}$$

Here, the canonical function involves *multiple* characteristic times, not just one.

As shown in [44], the variational method can be readily extended to derive gradient formulae for (55). The costate method can also be used to derive gradient formulae for (55); see [51] for details. It is interesting to note that the costate system for (55) involves instantaneous jumps at the characteristic times. Such jumps are not present in the variational system.

References [44, 70] consider the situation when the characteristic times are decision variables instead of fixed constants. In this case, the time-scaling transformation can be used to map the characteristic times to fixed time points. The resulting transformed problem can then be solved using the methods in [51].

6.3. **Discrete-valued controls.** In a standard optimal control problem, the control variables are subject to the following *bound constraints*:

$$a_i \leq u_i(t) \leq b_i, \quad t \in [0, T], \quad i = 1, \ldots, r.$$

That is, the ith control variable u_i is allowed to assume *any* value within the convex range $[a_i, b_i]$. In many applications, however, the control cannot assume values in a convex range, but is instead restricted to a finite set of discrete values. In this case, we have the following *non-convex control constraints*:

$$\boldsymbol{u}(t) \in \{\boldsymbol{u}^1, \ldots, \boldsymbol{u}^q\}, \quad t \in [0, T], \tag{56}$$

where $\boldsymbol{u}^i \in \mathbb{R}^r$, $i = 1, \ldots, q$, are given vectors. A control function satisfying (56) is called a *discrete-valued control*.

Discrete-valued controls arise in a wide range of important applications, including diesel-electric locomotives [20], submarines [8, 55], hybrid power systems [59, 77], switching amplifiers [61], sensor scheduling [28, 76], and chromatography [9]. A discrete-valued control is completely determined by:

(i) the order in which it assumes the different values in $\{\boldsymbol{u}^1, \ldots, \boldsymbol{u}^q\}$ (the so-called *switching sequence*); and
(ii) the times at which it switches from one value to another (the so-called *switching times*).

Thus, the problem of determining an optimal discrete-valued control amounts to determining an optimal switching sequence and an optimal set of switching times. Since the switching sequence is a discrete optimization variable, this problem can be viewed as a *mixed-integer optimization problem*.

The first general method for solving optimal control problems with discrete-valued controls was proposed in [30]. This method is based on a heuristic procedure that converts the optimal control problem into a *switching time optimization problem*. The time-scaling transformation can then be applied to solve the resulting switching time optimization problem. The advantage of this approach is that it converts an optimal control problem with discrete variables into an easier problem that only involves continuous optimization variables. The disadvantage, however, is that this approach is only heuristic—the switching time optimization problem obtained via the transformation is not quite equivalent to the original optimal control problem. New transformation procedures that do preserve mathematical equivalence have recently been proposed in [79, 81]. These new approaches are based on discrete optimization techniques.

6.4. **Switched impulsive systems.** A general switched impulsive system can be described as follows:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}^i(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [t_{i-1}, t_i), \quad i = 1, \ldots, m, \tag{57}$$

with jump conditions

$$\boldsymbol{x}(t_i^+) = \Psi^i(\boldsymbol{x}(t_i^-)), \quad i = 1, \ldots, m, \tag{58}$$

and initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}^0. \tag{59}$$

Here, each function $\boldsymbol{f}^i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ describes the dynamic behaviour of a *subsystem* or *mode*. Furthermore, the function $\Psi^i : \mathbb{R}^n \to \mathbb{R}^n$ defines an *instantaneous jump* in the system state at the *switching time* $t = t_i$.

The switched impulsive system (57)-(59) operates as follows. It starts in state $\boldsymbol{x}^0$ at time $t = 0$ and runs smoothly according to equation (57) with $i = 1$ until time $t = t_1$. At $t = t_1$, the state experiences an instantaneous jump from $\boldsymbol{x}(t_1^-)$ to $\boldsymbol{x}(t_1^+)$ according to equation (58) with $i = 1$. Then, starting from $\boldsymbol{x}(t_1^+)$, the system again runs smoothly until $t = t_2$, at which time it experiences another instantaneous jump, and so on until the end of the time horizon.

Switched impulsive systems have many applications in areas such as switching power converters [48], shrimp harvesting [24, 80], robotics [6], and 1-3 propanediol production [40]. When $\boldsymbol{f}^i := \boldsymbol{f}$ for each $i = 1, \ldots, m$ (i.e. no switch in the dynamics), system (57)-(59) is called an *impulsive system*. When $\Psi^i := \boldsymbol{x}(t_i^-)$ (i.e. no jumps), system (57)-(59) is called a *switched system*.

According to equation (58), the state jump at $t = t_i$ depends entirely on the state immediately before the ith switch. In some systems, however, the magnitude of the state jump can be controlled through a set of system parameters. Thus, a more general state jump condition is

$$\boldsymbol{x}(t_i^+) = \Psi^i(\boldsymbol{x}(t_i^-), \boldsymbol{\zeta}), \quad i = 1, \ldots, m,$$

where $\boldsymbol{\zeta} \in \mathbb{R}^q$ is a vector of system parameters. In optimal control problems involving switched impulsive systems, the switching times are usually decision variables to be optimized, along with the control function and/or system parameters. To solve such problems, the control function can be approximated using the control parameterization method, and the optimal switching times can be determined using the time-scaling transformation. See [33, 39, 41, 45, 58, 78] for details.

In system (57)-(59), the sequence in which the different subsystems operate—the so-called *switching sequence*—is fixed and known:

$$\boldsymbol{f}^1 \quad \longrightarrow \quad \boldsymbol{f}^2 \quad \longrightarrow \quad \boldsymbol{f}^3 \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad \boldsymbol{f}^m.$$

In some situations, however, the switching sequence is actually a discrete optimization variable. In this case, the system dynamics can be described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}^{v(t)}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [0, T], \tag{60}$$

where $v : [0, T] \to \{1, \ldots, m\}$ is a discrete-valued control representing the current mode of the system. The technique developed in [81] for solving optimal discrete-valued control problems can be applied to determine the optimal switching sequence and the optimal switching times. This technique is based on the time-scaling transformation and the exact penalty method.

6.5. **Delay systems.** In some systems, the rate of change of the state depends not only on the current state and current control, but also on the state and control at previous times. Such systems are called *delay systems*. A general delay system can be modelled as follows:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\alpha), \boldsymbol{u}(t), \boldsymbol{u}(t-\beta)), \quad t \in [0, T],\\ \boldsymbol{x}(t) &= \boldsymbol{\gamma}(t), \quad t \le 0,\end{aligned}$$

where α is called the *state-delay*, β is called the *input-delay*, and $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n$ is a given *initial function* defining the state at negative times. Note that our previous initial condition $\boldsymbol{x}(0) = \boldsymbol{x}^0$ is not sufficient here. This is because the dynamic equations for the interval $[0, \alpha]$ depend on the state at times $t \in [-\alpha, 0]$.

Various extensions of the control parameterization method have been developed for delay systems; see references [13, 25, 32, 65, 73, 74] for details. The methods discussed in these references do not allow the control switching times to be variable. In fact, variable switching times pose major problems for delay systems because the time-scaling transformation does not work in this case. To see why, consider the following simple state-delay system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{x}(t-\alpha), \boldsymbol{u}(t)), \quad t \in [0, T],\\ \boldsymbol{x}(t) &= \boldsymbol{\gamma}(t), \quad t \le 0.\end{aligned}$$

By using the piecewise-constant approximation scheme $\boldsymbol{u}(t) \approx \boldsymbol{\xi}^k$, $t \in [\tau_{k-1}, \tau_k)$, we obtain

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{x}(t-\alpha), \boldsymbol{\xi}^k), \quad t \in [\tau_{k-1}, \tau_k), \quad k = 1, \ldots, p,\\ \boldsymbol{x}(t) &= \boldsymbol{\gamma}(t), \quad t \le 0.\end{aligned}$$

As in Section 4.2, we define the time-scaling transformation through the differential equation

$$\frac{dt(s)}{ds} = v(s), \quad t(0) = 0, \quad t(p) = T, \tag{61}$$

where $s \in [0, p]$ is the new time variable and $v : [0, p] \to [0, \infty)$ is the time-scaling control. Recall that this transformation maps $s = k$ to the kth switching time. Defining $\tilde{\boldsymbol{x}}(s) = \boldsymbol{x}(t(s))$, it follows from (61) and the chain rule of differentiation that

$$\dot{\tilde{\boldsymbol{x}}}(s) = \frac{d}{ds}\big\{\boldsymbol{x}(t(s))\big\} = v(s)\boldsymbol{f}(\boldsymbol{x}(t(s)), \boldsymbol{x}(t(s)-\alpha), \boldsymbol{\xi}^k), \quad s \in [k-1, k),$$

and thus

$$\dot{\tilde{\boldsymbol{x}}}(s) = v(s)\boldsymbol{f}(\tilde{\boldsymbol{x}}(s), \boldsymbol{x}(t(s)-\alpha), \boldsymbol{\xi}^k), \quad s \in [k-1, k).$$

It is not possible to express the delay argument $\boldsymbol{x}(t(s)-\alpha)$ in terms of $\tilde{\boldsymbol{x}}$. Thus, the new system obtained by applying the time-scaling transformation is coupled with the original system. Consequently, applying the time-scaling transformation is not beneficial for delay systems—in fact, the time-scaling transformation actually complicates the problem. One possible remedy is to use the approach proposed in [75], whereby the delay system is transformed into a large system of ordinary differential equations, and then the time-scaling transformation is applied to this new system. The disadvantage of this approach is that the new system is large and contains many boundary conditions, making it difficult to solve numerically.

In practice, the time-delays α and β are often unknown and must first be estimated before control parameterization is applied. References [10, 11, 46] consider the problem of choosing the delays to minimize the deviation between predicted

and observed system output. This problem is a time-delay optimal control problem in which the control variables are the delays themselves. Gradient formulae for the cost function with respect to the time-delays are derived in [10, 11, 46].

6.6. **Stopping constraints.** In some optimal control problems (e.g. the glider control problems in [64, 69]), the governing dynamic system terminates only when a certain *stopping constraint* is satisfied. This stopping constraint is usually of the following form:

$$\Psi(\boldsymbol{x}(T)) = 0, \tag{62}$$

where T is the terminal time and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a given function. For example, consider the time-optimal control problem in which the aim is to drive the system from the initial state $\boldsymbol{x}^0$ to a target state $\boldsymbol{x}^f$ in minimum time. In this case, the stopping constraint is

$$\left|\boldsymbol{x}(T) - \boldsymbol{x}^f\right|^2 = 0. \tag{63}$$

For a system governed by stopping constraint (62), the terminal time T is defined as the *first* time at which (62) is satisfied. Mathematically,

$$T = T(\boldsymbol{u}) = \inf\{\, t > 0 : \ \Psi(\boldsymbol{x}(t|\boldsymbol{u})) = 0 \,\},$$

where $\boldsymbol{x}(\cdot|\boldsymbol{u})$ denotes the state trajectory corresponding to the control signal $\boldsymbol{u}$. Hence, the terminal time can be viewed as an *implicit* function of the control: changing the control changes the state trajectory, which in turn changes the time at which the state trajectory satisfies the stopping constraint.

At first glance, it may appear that (62) can be formulated as a canonical equality constraint with $\Phi_j := \Psi$ and $\mathcal{L}_j := 0$. However, this is not sufficient to ensure that T is the first time at which (62) is satisfied. Consequently, special techniques are needed to handle the stopping constraint. The control parameterization method has been successfully adapted to solve optimal control problems with stopping constraints; see [23, 36, 38, 64, 69] for details.

6.7. **Penalizing control changes.** The standard optimal control problem formulated in Section 2 does not consider the cost of changing the control input. Thus, the theoretical optimal control may in fact be highly volatile, making it difficult and expensive (if not impossible) to implement in practice. Thus, it is often desirable in practical problems to impose a penalty on control changes.

To measure the volatility of the control signal, we define the *total variation* of u_i by

$$\bigvee_0^T u_i := \sup \sum_{j=1}^m \left|u_i(t_j) - u_i(t_{j-1})\right|,$$

where the supremum is taken over all finite partitions $\{t_j\}_{j=0}^m \subset [0, T]$ satisfying

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T.$$

The total variation of the vector-valued control signal $\boldsymbol{u} : [0, T] \to \mathbb{R}^r$ is defined by

$$\bigvee_0^T \boldsymbol{u} := \sum_{i=1}^r \bigvee_0^T u_i.$$

Clearly, the total variation is zero if $\boldsymbol{u}$ is constant.

Consider the following modified cost function:

$$\hat{g}_0(\boldsymbol{u}) := \Phi_0(\boldsymbol{x}(T)) + \int_0^T \mathcal{L}_0(t, \boldsymbol{x}(t), \boldsymbol{u}(t))dt + \alpha \bigvee_0^T \boldsymbol{u}, \tag{64}$$

where $\alpha \geq 0$ is a given weight. This modified cost function considers both system performance and control volatility. Theoretical conditions for minimizing cost functions in the form of (64) are given in [5, 52]. To solve such problems numerically, the control parameterization method can be adapted (see [42, 43, 67]).

7. **Conclusion.** This paper has demonstrated the power of the control parameterization approach in solving nonlinear optimal control problems. The main idea of control parameterization is simple and intuitive: it involves approximating the control by a piecewise-constant function to yield a finite-dimensional approximation of the optimal control problem. This approximate problem is essentially a nonlinear programming problem in which the cost function's gradient depends on the solution of an auxiliary system of differential equations (either the variational system or the costate system; see Section 3). Standard gradient-based optimization methods such as sequential quadratic programming can be applied to solve the approximate problem. Various convergence results are available to show that the accuracy of the approximation improves as the discretization of the time horizon is refined. As shown in Section 6, this approach is very flexible and can be applied to solve a wide range of non-standard optimal control problems.

Our hope is that this survey paper will stimulate further interest in the area of optimal control computation. There are still many possibilities for future research. For the interested reader, we give some suggestions below.

- The control parameterization method is mainly used to compute *open loop controls* in which the control is a function of time. However, often the control is required to be expressed as a function of the state—a so-called *feedback control*. Feedback controls are well-known to be more robust than open loop controls. Recent work has explored the possibility of using the control parameterization technique to determine optimal feedback controls; see [35, 47]. This research direction should be pursued in future work.
- The approximate problem obtained through control parameterization is generally solved using gradient-based optimization techniques. Such techniques only converge to a *local* optimal solution, which may or may not be globally optimal. Recently, the idea of combining gradient-based optimization methods with global optimization methods has been explored to yield convergence to a global solution. See [77, 78] for some relevant work in this area.
- In Section 6.4, we discussed two types of switched impulsive systems: one in which the switching sequence is fixed, and the other in which the switching sequence is a decision variable. There is yet another type of switched system in which the switching law is governed by a state-dependent switching criterion. Such systems arise in robotics [6] and voice-coil motors [14]. The state trajectory for these switched systems can be generated numerically using a technique based on the time-scaling transformation [31]. However, this technique is only designed to solve the system, not perform any optimization. Extending the control parameterization method to this class of switched systems is an important topic for future research.

- As mentioned in Section 6.5, delay systems pose difficulties for the time-scaling transformation. It is currently unknown whether the time-scaling transformation can be suitably modified to handle delay systems. More work is needed to provide answers to this question.
- For time-delay systems, the control parameterization method has mainly been used in the case when the delays are fixed and known. Some systems, however, involve delays that are functions of the state and/or control variables. Examples include crushing processes [57], mixing tanks with recycle loops [16], and irrigation canals [4]. Optimal control computation for such delay systems promises to be a fruitful area of research.

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Appendix A. **Proof of Theorem 4.1.** We prove the theorem in 6 steps.

Step 1: Preliminaries. For notational simplicity, we write $\boldsymbol{x}^p(\cdot)$ instead of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi},\boldsymbol{\tau})$ and $\boldsymbol{x}^{p,\epsilon}(\cdot)$ instead of $\boldsymbol{x}^p(\cdot|\boldsymbol{\xi},\boldsymbol{\tau}+\epsilon\boldsymbol{e}^k)$. This will not cause confusion because $\boldsymbol{\xi}$ and $\boldsymbol{\tau}$ are fixed throughout this proof.

Let $t \in [0,T]$ be a fixed time point. Furthermore, let $\boldsymbol{u}^p$ denote the piecewise-constant control corresponding to $(\boldsymbol{\xi},\boldsymbol{\tau})$, and let $\boldsymbol{u}^{p,\epsilon}$ denote the piecewise-constant control corresponding to $(\boldsymbol{\xi},\boldsymbol{\tau}+\epsilon\boldsymbol{e}^k)$.

Let V denote the *control restraint set* defined by

$$V := \{\, \boldsymbol{v} \in \mathbb{R}^r : \, a_i \le v_i \le b_i, \, i = 1,\ldots,r \,\}.$$

Since the control range is contained in V, $\boldsymbol{u}^{p,\epsilon}$ is uniformly bounded with respect to ϵ. That is, there exists a constant $M_1 > 0$ such that for all $\epsilon > 0$ satisfying $\tau_k + \epsilon \le \tau_{k+1}$,

$$|\boldsymbol{u}^{p,\epsilon}(s)| \le M_1, \quad s \in [0,T].$$

As is customary in the optimal control literature, we assume that the governing dynamic system satisfies the following linear growth condition: there exists a positive constant $M_2 > 0$ such that

$$|\boldsymbol{f}(s,\boldsymbol{x},\boldsymbol{u})| \le M_2(1+|\boldsymbol{x}|), \quad (s,\boldsymbol{x},\boldsymbol{u}) \in [0,T]\times\mathbb{R}^n\times V.$$

It then follows from Lemma 6.4.2 in [65] that $\boldsymbol{x}^{p,\epsilon}$ is also uniformly bounded with respect to ϵ. Hence, there exists a constant $M_3 > 0$ such that for all $\epsilon > 0$ satisfying $\tau_k + \epsilon \le \tau_{k+1}$,

$$|\boldsymbol{x}^{p,\epsilon}(s)| \le M_3, \quad s \in [0,T].$$

Step 2: Case $t \le \tau_k$. If $t \le \tau_k$, then

$$\boldsymbol{u}^{p,\epsilon}(s) = \boldsymbol{u}^p(s), \quad s \in [0,t),$$

and thus

$$\boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^p(t).$$

Consequently,

$$\lim_{\epsilon\to 0+} \frac{\boldsymbol{x}^{p,\epsilon}(t) - \boldsymbol{x}^p(t)}{\epsilon} = \boldsymbol{0}.$$

In particular, if $t = \tau_k$, then

$$\lim_{\epsilon\to 0+} \frac{\boldsymbol{x}^{p,\epsilon}(\tau_k) - \boldsymbol{x}^p(\tau_k)}{\epsilon} = \boldsymbol{0},$$

which proves equation (28). If $t < \tau_k$, then it follows from (26) that

$$\boldsymbol{\phi}^k(t) = \mathbf{0} = \lim_{\epsilon \to 0+} \frac{\boldsymbol{x}^{p,\epsilon}(t) - \boldsymbol{x}^p(t)}{\epsilon},$$

which proves equation (27) when $t < \tau_k$. It remains to prove (27) when $t > \tau_k$.

Step 3: Definition of $\boldsymbol{\varphi}^\epsilon$. Define

$$\boldsymbol{\varphi}^\epsilon(s) := \boldsymbol{x}^{p,\epsilon}(s) - \boldsymbol{x}^p(s), \quad s \in [0, T].$$

Let $\epsilon \in (0, \tau_{k+1} - \tau_k)$. Then clearly,

$$\tau_k + \epsilon < \tau_{k+1}$$

and thus

$$\boldsymbol{u}^{p,\epsilon}(s) = \boldsymbol{u}^p(s), \quad s \notin [\tau_k, \tau_k + \epsilon]. \tag{65}$$

For each $s \in [0, T]$, we have

$$\boldsymbol{\varphi}^\epsilon(s) = \boldsymbol{x}^{p,\epsilon}(s) - \boldsymbol{x}^p(s) = \int_0^s \big\{\boldsymbol{f}(\eta, \boldsymbol{x}^{p,\epsilon}(\eta), \boldsymbol{u}^{p,\epsilon}(\eta)) - \boldsymbol{f}(\eta, \boldsymbol{x}^p(\eta), \boldsymbol{u}^p(\eta))\big\} d\eta.$$

Thus, since $\boldsymbol{f}$ is continuously differentiable and $\boldsymbol{x}^{p,\epsilon}$ and $\boldsymbol{u}^{p,\epsilon}$ are uniformly bounded with respect to ϵ, there exists a Lipschitz constant $M_4 > 0$ such that

$$|\boldsymbol{\varphi}^\epsilon(s)| \le \int_0^s M_4 |\boldsymbol{\varphi}^\epsilon(\eta)| d\eta + \int_0^s M_4 |\boldsymbol{u}^{p,\epsilon}(\eta) - \boldsymbol{u}^p(\eta)| d\eta.$$

By (65),

$$\begin{aligned}
|\boldsymbol{\varphi}^\epsilon(s)| &\le \int_0^s M_4 |\boldsymbol{\varphi}^\epsilon(\eta)| d\eta + \int_{[0,s] \cap [\tau_k, \tau_k + \epsilon]} M_4 |\boldsymbol{u}^{p,\epsilon}(\eta) - \boldsymbol{u}^p(\eta)| d\eta \\
&\le \int_0^s M_4 |\boldsymbol{\varphi}^\epsilon(\eta)| d\eta + \int_{\tau_k}^{\tau_k + \epsilon} M_4 |\boldsymbol{u}^{p,\epsilon}(\eta) - \boldsymbol{u}^p(\eta)| d\eta \\
&\le 2 M_1 M_4 \epsilon + \int_0^s M_4 |\boldsymbol{\varphi}^\epsilon(\eta)| d\eta.
\end{aligned}$$

Thus, it follows from the Gronwall-Bellman lemma that

$$|\boldsymbol{\varphi}^\epsilon(s)| \le M_5 \epsilon, \tag{66}$$

where $M_5 := 2 M_1 M_4 \exp(M_4 T)$. This inequality holds for all $\epsilon \in (0, \tau_{k+1} - \tau_k)$.

Step 4: Definitions of ρ_k^ϵ and ρ_{k+1}^ϵ. Define

$$\rho_k^\epsilon := \int_{\tau_k}^{\tau_k + \epsilon} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k)\big\} ds$$

and

$$\rho_{k+1}^\epsilon := \int_{\tau_k}^{\tau_k + \epsilon} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1}) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1})\big\} ds.$$

We have

$$\begin{aligned}
|\rho_k^\epsilon| \le \int_{\tau_k}^{\tau_k + \epsilon} &\big|\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^k)\big| ds \\
&+ \int_{\tau_k}^{\tau_k + \epsilon} \big|\boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k)\big| ds.
\end{aligned}$$

Thus,

$$|\rho_k^\epsilon| \le \int_{\tau_k}^{\tau_k+\epsilon} M_4|\boldsymbol{\varphi}^\epsilon(s)|ds + \int_{\tau_k}^{\tau_k+\epsilon} M_4|s-\tau_k|ds + \int_{\tau_k}^{\tau_k+\epsilon} M_4|\boldsymbol{x}^p(s)-\boldsymbol{x}^p(\tau_k)|ds,$$

where M_4 is the Lipschitz constant defined in Step 3. Applying (66), we obtain

$$\begin{aligned}|\rho_k^\epsilon| &\le M_4M_5\epsilon^2 + \int_{\tau_k}^{\tau_k+\epsilon} M_4|s-\tau_k|ds + \int_{\tau_k}^{\tau_k+\epsilon} M_4|\boldsymbol{x}^p(s)-\boldsymbol{x}^p(\tau_k)|ds \\ &\le M_4M_5\epsilon^2 + M_4\epsilon^2 + \int_{\tau_k}^{\tau_k+\epsilon} M_4|\boldsymbol{x}^p(s)-\boldsymbol{x}^p(\tau_k)|ds, \end{aligned} \tag{67}$$

assuming that $\epsilon \in (0, \tau_{k+1}-\tau_k)$. Now, for each $s \in [\tau_k, \tau_k+\epsilon]$,

$$|\boldsymbol{x}^p(s)-\boldsymbol{x}^p(\tau_k)| \le \int_{\tau_k}^{s} |\boldsymbol{f}(\eta, \boldsymbol{x}^p(\eta), \boldsymbol{u}^p(\eta))|d\eta \le \int_{\tau_k}^{\tau_k+\epsilon} |\boldsymbol{f}(\eta, \boldsymbol{x}^p(\eta), \boldsymbol{u}^p(\eta))|d\eta.$$

Thus, since $\boldsymbol{f}$ is continuous and $\boldsymbol{x}^p$ and $\boldsymbol{u}^p$ are bounded on $[0,T]$, there exists a constant $M_6 > 0$ such that

$$|\boldsymbol{x}^p(s)-\boldsymbol{x}^p(\tau_k)| \le \int_{\tau_k}^{\tau_k+\epsilon} |\boldsymbol{f}(\eta, \boldsymbol{x}^p(\eta), \boldsymbol{u}^p(\eta))|d\eta \le \int_{\tau_k}^{\tau_k+\epsilon} M_6 d\eta = M_6\epsilon. \tag{68}$$

Substituting (68) into (67) gives

$$|\rho_k^\epsilon| \le M_4M_5\epsilon^2 + M_4\epsilon^2 + \int_{\tau_k}^{\tau_k+\epsilon} M_4M_6\epsilon ds = M_7\epsilon^2, \tag{69}$$

where $M_7 := M_4(M_5+M_6+1)$. Similarly,

$$|\rho_{k+1}^\epsilon| \le M_7\epsilon^2. \tag{70}$$

Inequalities (69) and (70) hold whenever $\epsilon \in (0, \tau_{k+1}-\tau_k)$.

Step 5: Definition of Δ_l^ϵ. Define

$$\Delta_l^\epsilon(s,\alpha) := \left\{ \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s)+\alpha\boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \right\} \boldsymbol{\varphi}^\epsilon(s).$$

Let $\gamma > 0$ be arbitrary. Then since $\partial \boldsymbol{f}/\partial \boldsymbol{x}$ is continuous, $\boldsymbol{x}^{p,\epsilon}$ is uniformly bounded with respect to ϵ, and $\boldsymbol{\varphi}^\epsilon \to \boldsymbol{0}$ uniformly as $\epsilon \to 0$, there exists an $\epsilon' \in (0, \tau_{k+1}-\tau_k)$, which depends solely on γ, such that for all $\epsilon \in (0, \epsilon')$,

$$\left| \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s)+\alpha\boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \right| < \gamma, \quad (s,\alpha) \in [0,T]\times[0,1].$$

Thus, it follows from (66) that for all $\epsilon \in (0, \epsilon')$,

$$|\Delta_l^\epsilon(s,\alpha)| \le \left| \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s)+\alpha\boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \right| \cdot |\boldsymbol{\varphi}^\epsilon(s)| \le M_5\epsilon\gamma, \tag{71}$$

uniformly with respect to $(s,\alpha) \in [0,T]\times[0,1]$.

Step 6: Case $t > \tau_k$. Let $i \in \{k+1, \ldots, p\}$ denote the unique control subinterval containing t. Then $t \in [\tau_{i-1}, \tau_i)$ if $i \leq p-1$, and $t \in [\tau_{p-1}, T]$ if $i = p$.

Let $\gamma \in (0, t-\tau_k]$ be arbitrary but fixed. Furthermore, choose ϵ so that

$$0 < \epsilon < \min\{\gamma, \epsilon'\} < \tau_{k+1} - \tau_k, \tag{72}$$

where ϵ' is as defined in Step 5. Then since $\tau_k + \epsilon < \tau_{k+1}$ and $\tau_k + \epsilon < t$,

$$\boldsymbol{x}^p(t) = \boldsymbol{x}^p(\tau_k) + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l) ds \tag{73}$$

and

$$\begin{aligned} \boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^{p,\epsilon}(\tau_k) &+ \int_{\tau_k}^{\tau_k+\epsilon} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^k) ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}+\epsilon\delta_{k+1,l}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) ds, \end{aligned} \tag{74}$$

where $\delta_{k+1,l}$ denotes the Kronecker delta function. Equation (74) can be written as

$$\begin{aligned} \boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^{p,\epsilon}(\tau_k) &+ \int_{\tau_k}^{\tau_k+\epsilon} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1})\big\} ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) ds. \end{aligned} \tag{75}$$

Since $\boldsymbol{x}^{p,\epsilon}(\tau_k) = \boldsymbol{x}^p(\tau_k)$, combining (73) and (75) gives

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = &\int_{\tau_k}^{\tau_k+\epsilon} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1})\big\} ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)\big\} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_k^\epsilon - \rho_{k+1}^\epsilon &+ \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)\big\} ds, \end{aligned} \tag{76}$$

where ρ_k^ϵ and ρ_{k+1}^ϵ are as defined in Step 4. Using the fundamental theorem of calculus, (76) can be written as

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_k^\epsilon - \rho_{k+1}^\epsilon &+ \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \int_0^1 \frac{\partial}{\partial \alpha} \Big\{\boldsymbol{f}(s, \boldsymbol{x}^p(s) + \alpha \boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)\Big\} d\alpha ds. \end{aligned}$$

Hence,

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_k^\epsilon - \rho_{k+1}^\epsilon &+ \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \int_0^1 \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s) + \alpha \boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^\epsilon(s) d\alpha ds. \end{aligned}$$

Rearranging gives

$$\begin{aligned}\boldsymbol{\varphi}^\epsilon(t) = \rho_k^\epsilon - \rho_{k+1}^\epsilon + \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l,t\}} \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^\epsilon(s) ds \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l,t\}} \int_0^1 \Delta_l^\epsilon(s,\alpha) d\alpha ds,\end{aligned} \tag{77}$$

where Δ_l^ϵ is as defined in Step 5. Now, the right-continuous solution of the variational system (24)-(26) is

$$\boldsymbol{\phi}^k(t) = \boldsymbol{\phi}^k(\tau_k^+) + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l,t\}} \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\phi}^k(s) ds,$$

where $\boldsymbol{\phi}^k(\cdot) := \boldsymbol{\phi}^k(\cdot|\boldsymbol{\xi}, \boldsymbol{\tau})$. Using the jump condition (25) gives

$$\begin{aligned}\boldsymbol{\phi}^k(t) = \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l,t\}} \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\phi}^k(s) ds.\end{aligned} \tag{78}$$

From (77) and (78), we obtain

$$\begin{aligned}\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le \epsilon^{-1}|\rho_k^\epsilon| + \epsilon^{-1}|\rho_{k+1}^\epsilon| + \int_{\tau_k}^t M_8 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l,t\}} \int_0^1 \epsilon^{-1}\left|\Delta_l^\epsilon(s,\alpha)\right| d\alpha ds,\end{aligned}$$

where $M_8 > 0$ is an upper bound for $|\partial \boldsymbol{f}/\partial \boldsymbol{x}|$ (recall that $\boldsymbol{f}$ is continuously differentiable and $\boldsymbol{x}^p$ is bounded). Recall from (72) that $\epsilon < \tau_{k+1} - \tau_k$ and $\epsilon < \epsilon'$. Thus, we can apply inequalities (69)-(71) to obtain

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le 2M_7\epsilon + M_5 T\gamma + \int_{\tau_k}^t M_8 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds.$$

Since $\epsilon < \gamma$ and $\tau_k + \gamma \le t$, this inequality becomes

$$\begin{aligned}\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le 2M_7\gamma + M_5 T\gamma + \int_{\tau_k}^{\tau_k+\gamma} M_8 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds \\ + \int_{\tau_k+\gamma}^t M_8 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds.\end{aligned} \tag{79}$$

Using (66), we have

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| \le \epsilon^{-1}|\boldsymbol{\varphi}^\epsilon(s)| + |\boldsymbol{\phi}^k(s)| \le M_5 + M_9, \quad s \in [\tau_k, \tau_k + \gamma], \tag{80}$$

where $M_9 > 0$ is an upper bound for $|\boldsymbol{\phi}^k|$ (recall that $\boldsymbol{\phi}^k$ is piecewise-continuous). Substituting (80) into (79) gives

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le M_{10}\gamma + \int_{\tau_k+\gamma}^t M_8 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds,$$

where $M_{10} := 2M_7 + M_5T + M_8(M_5 + M_9)$. This inequality holds for any $t \geq \tau_k + \gamma$ uniformly with respect to ϵ. Thus, for all ϵ satisfying (72),

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^{\epsilon}(s) - \boldsymbol{\phi}^k(s)\right| \leq M_{10}\gamma + \int_{\tau_k+\gamma}^{s} M_8\left|\epsilon^{-1}\boldsymbol{\varphi}^{\epsilon}(\eta) - \boldsymbol{\phi}^k(\eta)\right|d\eta, \quad s \in [\tau_k + \gamma, T].$$

Applying Gronwall's lemma gives

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^{\epsilon}(s) - \boldsymbol{\phi}^k(s)\right| \leq M_{10}\exp(M_8T)\gamma, \quad s \in [\tau_k + \gamma, T].$$

In particular,

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^{\epsilon}(t) - \boldsymbol{\phi}^k(t)\right| \leq M_{10}\exp(M_8T)\gamma.$$

Since $\gamma > 0$ was arbitrary, this shows that (27) holds when $t > \tau_k$.

Appendix B. **Proof of Theorem 4.2.** We prove the theorem in 3 steps.

Step 1: Preliminaries. Let $t \in [0, T]$ be a fixed time point. Furthermore, let $\boldsymbol{x}^p(\cdot)$ and $\boldsymbol{x}^{p,\epsilon}(\cdot)$ be as defined in Appendix A.

Define

$$\boldsymbol{\varphi}^{\epsilon}(s) := \boldsymbol{x}^{p,\epsilon}(s) - \boldsymbol{x}^p(s), \quad s \in [0, T].$$

In addition, define

$$\rho_k^{\epsilon} := \int_{\tau_k+\epsilon}^{\tau_k} \left\{\boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k)\right\}ds,$$

$$\rho_{k+1}^{\epsilon} := \int_{\tau_k+\epsilon}^{\tau_k} \left\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1}) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1})\right\}ds.$$

Note that these definitions of ρ_k^{ϵ} and ρ_{k+1}^{ϵ} differ from the definitions given in Appendix A: the integral limits are reversed because $\epsilon < 0$, and the first term of the integrand for ρ_k^{ϵ} depends on $\boldsymbol{x}^p$, not $\boldsymbol{x}^{p,\epsilon}$.

Using the same arguments as in Step 4 of Appendix A, it can be shown that there exists a constant $M_1 > 0$ such that for all $\epsilon \in (\tau_{k-1} - \tau_k, 0)$,

$$|\rho_k^{\epsilon}| \leq M_1\epsilon^2, \quad |\rho_{k+1}^{\epsilon}| \leq M_1\epsilon^2. \tag{81}$$

Define

$$\Delta_l^{\epsilon}(s, \alpha) := \left\{\frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s) + \alpha\boldsymbol{\varphi}^{\epsilon}(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}}\right\}\boldsymbol{\varphi}^{\epsilon}(s).$$

Let $\gamma > 0$ be arbitrary. Then it follows from Step 5 of Appendix A that there exists a corresponding $\epsilon' < 0$ such that for all $\epsilon \in (\epsilon', 0)$,

$$|\Delta_l^{\epsilon}(s, \alpha)| \leq M_2|\epsilon|\gamma, \quad (s, \alpha) \in [0, T] \times [0, 1], \tag{82}$$

where $M_2 > 0$ is a constant independent of γ and ϵ.

Step 2: Case $t < \tau_k$. If $t < \tau_k$, then for all $\epsilon < 0$ of sufficiently small magnitude,

$$\boldsymbol{u}^{p,\epsilon}(s) = \boldsymbol{u}^p(s), \quad s \in [0, t].$$

Thus,

$$\boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^p(t)$$

and

$$\lim_{\epsilon \to 0-} \frac{\boldsymbol{x}^{p,\epsilon}(t) - \boldsymbol{x}^p(t)}{\epsilon} = \boldsymbol{0} = \boldsymbol{\phi}^k(t).$$

This proves equation (29) for $t < \tau_k$.

Step 3: Case $t \geq \tau_k$. Let $i \in \{k+1, \ldots, p\}$ denote the unique control subinterval containing t. Then $t \in [\tau_{i-1}, \tau_i)$ if $i \leq p-1$, and $t \in [\tau_{p-1}, T]$ if $i = p$.

Let $\epsilon \in (\tau_{k-1} - \tau_k, 0)$. Then

$$\begin{aligned} \boldsymbol{x}^p(t) = \boldsymbol{x}^p(\tau_k + \epsilon) &+ \int_{\tau_k+\epsilon}^{\tau_k} \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^k) ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l) ds \end{aligned} \tag{83}$$

and

$$\boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^{p,\epsilon}(\tau_k + \epsilon) + \sum_{l=k+1}^{i} \int_{\tau_{l-1}+\epsilon\delta_{k+1,l}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) ds, \tag{84}$$

where $\delta_{k+1,l}$ denotes the Kronecker delta function. Equation (84) can be written as

$$\begin{aligned} \boldsymbol{x}^{p,\epsilon}(t) = \boldsymbol{x}^{p,\epsilon}(\tau_k + \epsilon) &+ \int_{\tau_k+\epsilon}^{\tau_k} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1}) ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) ds. \end{aligned} \tag{85}$$

Since $\boldsymbol{x}^{p,\epsilon}(\tau_k + \epsilon) = \boldsymbol{x}^p(\tau_k + \epsilon)$ for all $\epsilon < 0$, combining (83) and (85) gives

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = &\int_{\tau_k+\epsilon}^{\tau_k} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^{k+1}) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^k)\big\} ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)\big\} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_{k+1}^\epsilon - \rho_k^\epsilon &- \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) + \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \big\{\boldsymbol{f}(s, \boldsymbol{x}^{p,\epsilon}(s), \boldsymbol{\xi}^l) - \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)\big\} ds, \end{aligned} \tag{86}$$

where ρ_k^ϵ and ρ_{k+1}^ϵ are as defined in Step 1. Using the fundamental theorem of calculus, (86) can be written as

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_{k+1}^\epsilon - \rho_k^\epsilon &- \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) + \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \int_0^1 \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s) + \alpha \boldsymbol{\varphi}^\epsilon(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^\epsilon(s) d\alpha ds. \end{aligned}$$

Rearranging gives

$$\begin{aligned} \boldsymbol{\varphi}^\epsilon(t) = \rho_{k+1}^\epsilon - \rho_k^\epsilon &- \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) + \epsilon \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\varphi}^\epsilon(s) ds \\ &+ \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \int_0^1 \Delta_l^\epsilon(s, \alpha) d\alpha ds, \end{aligned} \tag{87}$$

where Δ_l^ϵ is as defined in Step 1. Now, the right-continuous solution of the variational system (24)-(26) is

$$\begin{aligned}\boldsymbol{\phi}^k(t) = \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^k) - \boldsymbol{f}(\tau_k, \boldsymbol{x}^p(\tau_k), \boldsymbol{\xi}^{k+1}) \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}^p(s), \boldsymbol{\xi}^l)}{\partial \boldsymbol{x}} \boldsymbol{\phi}^k(s) ds.\end{aligned} \tag{88}$$

From (87) and (88), we obtain

$$\begin{aligned}\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le |\epsilon|^{-1}|\rho_k^\epsilon| + |\epsilon|^{-1}|\rho_{k+1}^\epsilon| + \int_{\tau_k}^{t} M_3 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds \\ + \sum_{l=k+1}^{i} \int_{\tau_{l-1}}^{\min\{\tau_l, t\}} \int_0^1 |\epsilon|^{-1} \left|\Delta_l^\epsilon(s, \alpha)\right| d\alpha ds,\end{aligned}$$

where M_3 is an upper bound for $|\partial \boldsymbol{f}/\partial \boldsymbol{x}|$. Let $\gamma > 0$ be arbitrary and choose ϵ so that

$$\max\{-\gamma, \epsilon', \tau_{k-1} - \tau_k\} < \epsilon < 0.$$

Then using inequalities (81) and (82), we see that

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le 2M_1\gamma + M_2 T\gamma + \int_{\tau_k}^{t} M_3 \left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(s) - \boldsymbol{\phi}^k(s)\right| ds.$$

This inequality holds for any $t \ge \tau_k$ uniformly with respect to ϵ. Thus, from Gronwall's lemma,

$$\left|\epsilon^{-1}\boldsymbol{\varphi}^\epsilon(t) - \boldsymbol{\phi}^k(t)\right| \le (2M_1\gamma + M_2 T\gamma)\exp(M_3 T).$$

Since $\gamma > 0$ was arbitrary, this shows that (29) holds when $t \ge \tau_k$.

REFERENCES

[1] B. Açikmeşe and L. Blackmore, *Lossless convexification of a class of optimal control problems with non-convex control constraints*, Automatica J. IFAC, **47** (2011), 341–347.

[2] N. U. Ahmed, "Elements of Finite-Dimensional Systems and Control Theory," Longman Scientific and Technical, Essex, 1988.

[3] N. U. Ahmed, "Dynamic Systems and Control with Applications," World Scientific, Singapore, 2006.

[4] Z. Benayache, G. Besançon and D. Georges, *A new nonlinear control methodology for irrigation canals based on a delayed input model*, in "Proceedings of the 17th World Congress of the International Federation of Automatic Control," 2008.

[5] J. M. Blatt, *Optimal control with a cost of switching control*, Journal of the Australian Mathematical Society – Series B: Applied Mathematics, **19** (1976), 316–332.

[6] M. Boccadoro, Y. Wardi, M. Egerstedt and E. Verriest, *Optimal control of switching surfaces in hybrid dynamical systems*, Discrete Event Dynamic Systems: Theory and Applications, **15** (2005), 433–448.

[7] C. Büskens and H. Maurer, *SQP-methods for solving optimal control problems with control and state constraints: Adjoint variables, sensitivity analysis and real-time control*, Journal of Computational and Applied Mathematics, **120** (2000), 85–108.

[8] L. Caccetta, I. Loosen and V. Rehbock, *Computational aspects of the optimal transit path problem*, Journal of Industrial and Management Optimization, **4** (2008), 95–105.

[9] Q. Chai, R. Loxton, K. L. Teo and C. Yang, *A max-min control problem arising in gradient elution chromatography*, Industrial and Engineering Chemistry Research, **51** (2012), 6137–6144.

[10] Q. Chai, R. Loxton, K. L. Teo and C. Yang, *A unified parameter identification method for nonlinear time-delay systems*, Journal of Industrial and Management Optimization, **9** (2013), 471–486.

[11] Q. Chai, R. Loxton, K. L. Teo and C. Yang, *A class of optimal state-delay control problems*, Nonlinear Analysis: Real World Applications, **14** (2013), 1536–1550.

[12] Q. Chai, R. Loxton, K. L. Teo and C. Yang, *Time-delay estimation for nonlinear systems with piecewise-constant input*, Applied Mathematics and Computation, **219** (2013), 9543–9560.

[13] Q. Q. Chai, C. H. Yang, K. L. Teo and W. H. Gui, *Optimal control of an industrial-scale evaporation process: Sodium aluminate solution*, Control Engineering Practice, **20** (2012), 618–628.

[14] B. Christiansen, H. Maurer and O. Zirn, *Optimal control of a voice-coil-motor with Coulombic friction*, in "Proceedings of the 47th IEEE Conference on Decision and Control," 2008.

[15] M. Chyba, T. Haberkorn, R. N. Smith and S. K. Choi, *Design and implementation of time efficient trajectories for autonomous underwater vehicles*, Ocean Engineering, **35** (2008), 63–76.

[16] J. Y. Dieulot and J. P. Richard, *Tracking control of a nonlinear system with input-dependent delay*, in "Proceedings of the 40th IEEE Conference on Decision and Control," 2001.

[17] B. Farhadinia, K. L. Teo and R. Loxton, *A computational method for a class of non-standard time optimal control problems involving multiple time horizons*, Mathematical and Computer Modelling, **49** (2009), 1682–1691.

[18] M. Gerdts and M. Kunkel, *A nonsmooth Newton's method for discretized optimal control problems with state and control constraints*, Journal of Industrial and Management Optimization, **4** (2008), 247–270.

[19] C. J. Goh and K. L. Teo, *Control parametrization: A unified approach to optimal control problems with general constraints*, Automatica J. IFAC, **24** (1988), 3–18.

[20] P. G. Howlett, P. J. Pudney and X. Vu, *Local energy minimization in optimal train control*, Automatica J. IFAC, **45** (2009), 2692–2698.

[21] L. S. Jennings, M. E. Fisher, K. L. Teo and C. J. Goh, "MISER3 Optimal Control Software: Theory and User Manual," University of Western Australia, Perth, 2004.

[22] L. S. Jennings and K. L. Teo, *A computational algorithm for functional inequality constrained optimization problems*, Automatica J. IFAC, **26** (1990), 371–375.

[23] C. Jiang, Q. Lin, C. Yu, K. L. Teo and G. R. Duan, *An exact penalty method for free terminal time optimal control problem with continuous inequality constraints*, Journal of Optimization Theory and Applications, **154** (2012), 30–53.

[24] C. Jiang, K. L. Teo, R. Loxton and G. R. Duan, *A neighboring extremal solution for an optimal switched impulsive control problem*, Journal of Industrial and Management Optimization, **8** (2012), 591–609.

[25] K. Kaji and K. H. Wong, *Nonlinearly constrained time-delayed optimal control problems*, Journal of Optimization Theory and Applications, **82** (1994), 295–313.

[26] C. Y. Kaya and J. L. Noakes, *Computational method for time-optimal switching control*, Journal of Optimization Theory and Applications, **117** (2003), 69–92.

[27] H. W. J. Lee, K. L. Teo and X. Q. Cai, *An optimal control approach to nonlinear mixed integer programming problems*, Computers and Mathematics with Applications, **36** (1998), 87–105.

[28] H. W. J. Lee, K. L. Teo and A. E. B. Lim, *Sensor scheduling in continuous time*, Automatica J. IFAC, **37** (2001), 2017–2023.

[29] H. W. J. Lee, K. L. Teo, V. Rehbock and L. S. Jennings, *Control parametrization enhancing technique for time optimal control problems*, Dynamic Systems and Applications, **6** (1997), 243–261.

[30] H. W. J. Lee, K. L. Teo, V. Rehbock and L. S. Jennings, *Control parametrization enhancing technique for optimal discrete-valued control problems*, Automatica J. IFAC, **35** (1999), 1401–1407.

[31] H. W. J. Lee and K. L. Teo, *Control parametrization enhancing technique for solving a special ODE class with state dependent switch*, Journal of Optimization Theory and Applications, **118** (2003), 55–66.

[32] H. W. J. Lee and K. H. Wong, *Semi-infinite programming approach to nonlinear time-delayed optimal control problems with linear continuous constraints*, Optimization Methods and Software, **21** (2006), 679–691.

[33] R. Li, K. L. Teo, K. H. Wong and G. R. Duan, *Control parameterization enhancing transform for optimal control of switched systems*, Mathematical and Computer Modelling, **43** (2006), 1393-1403.

[34] B. Li, C. J. Yu, K. L. Teo and G. R. Duan, *An exact penalty function method for continuous inequality constrained optimal control problem*, Journal of Optimization Theory and Applications, **151** (2011), 260–291.

[35] B. Li, K. L. Teo, C. C. Lim and G. R. Duan, *An optimal PID controller design for nonlinear constrained optimal control problems*, Discrete and Continuous Dynamical Systems: Series B, **16** (2011), 1101–1117.

[36] Q. Lin, R. Loxton, K. L. Teo and Y. H. Wu, *A new computational method for a class of free terminal time optimal control problems*, Pacific Journal of Optimization, **7** (2011), 63–81.

[37] Q. Lin, R. Loxton, K. L. Teo and Y. H. Wu, *A new computational method for optimizing nonlinear impulsive systems*, Dynamics of Continuous, Discrete and Impulsive Systems – Series B: Applications and Algorithms, **18** (2011), 59–76.

[38] Q. Lin, R. Loxton, K. L. Teo and Y. H. Wu, *Optimal control computation for nonlinear systems with state-dependent stopping criteria*, Automatica J. IFAC, **48** (2012), 2116–2129.

[39] Y. Liu, A. Eberhard and K. L. Teo, *A numerical method for a class of mixed switching and impulsive optimal control problems*, Computers and Mathematics with Applications, **52** (2006), 625–636.

[40] C. Liu, Z. Gong, E. Feng and H. Yin, *Modelling and optimal control for nonlinear multistage dynamical system of microbial fed-batch culture*, Journal of Industrial and Management Optimization, **5** (2009), 835–850.

[41] Y. Liu, K. L. Teo, L. S. Jennings and S. Wang, *On a class of optimal control problems with state jumps*, Journal of Optimization Theory and Applications, **98** (1998), 65–82.

[42] R. Loxton, Q. Lin and K. L. Teo, *Minimizing control variation in nonlinear optimal control*, Automatica J. IFAC, **49** (2013), 2652–2664.

[43] R. Loxton, Q. Lin, V. Rehbock and K. L. Teo, *Control parameterization for optimal control problems with continuous inequality constraints: New convergence results*, Numerical Algebra, Control and Optimization, **2** (2012), 571–599.

[44] R. Loxton, K. L. Teo and V. Rehbock, *Optimal control problems with multiple characteristic time points in the objective and constraints*, Automatica J. IFAC, **44** (2008), 2923–2929.

[45] R. Loxton, K. L. Teo and V. Rehbock, *Computational method for a class of switched system optimal control problems*, IEEE Transactions on Automatic Control, **54** (2009), 2455–2460.

[46] R. Loxton, K. L. Teo and V. Rehbock, *An optimization approach to state-delay identification*, IEEE Transactions on Automatic Control, **55** (2010), 2113–2119.

[47] R. Loxton, K. L. Teo and V. Rehbock, *Robust suboptimal control of nonlinear systems*, Applied Mathematics and Computation, **217** (2011), 6566–6576.

[48] R. Loxton, K. L. Teo, V. Rehbock and W. K. Ling, *Optimal switching instants for a switched-capacitor DC/DC power converter*, Automatica J. IFAC, **45** (2009), 973–980.

[49] R. Loxton, K. L. Teo, V. Rehbock and K. F. C. Yiu, *Optimal control problems with a continuous inequality constraint on the state and the control*, Automatica J. IFAC, **45** (2009), 2250–2257.

[50] D. G. Luenberger and Y. Ye, "Linear and Nonlinear Programming," 3rd Edition, Springer, New York, 2008.

[51] R. B. Martin, *Optimal control drug scheduling of cancer chemotherapy*, Automatica J. IFAC, **28** (1992), 1113–1123.

[52] J. Matula, *On an extremum problem*, Journal of the Australian Mathematical Society – Series B: Applied Mathematics, **28** (1987), 376–392.

[53] J. Nocedal and S. J. Wright, "Numerical Optimization," 2nd Edition, Springer, New York, 2006.

[54] V. Rehbock, "Tracking Control and Optimal Control," PhD thesis, University of Western Australia, Perth, 1994.

[55] V. Rehbock and L. Caccetta, *Two defence applications involving discrete valued optimal control*, ANZIAM Journal, **44** (2002), E33-E54.

[56] V. Rehbock, K. L. Teo, L. S. Jennings and H. W. J. Lee, *A survey of the control parametrization and control parametrization enhancing methods for constrained optimal control problems*, in "Progress in Optimization: Contributions from Australasia," Kluwer Academic Publishers, Dordrecht, 1999.

[57] J. P. Richard, *Time-delay systems: An overview of some recent advances and open problems*, Automatica J. IFAC, **39** (2003), 1667–1694.

[58] T. Ruby and V. Rehbock, *Numerical solutions of optimal switching control problems*, in "Optimization and Control with Applications," Springer, New York, 2005.

[59] T. Ruby, V. Rehbock and W. B. Lawrance, *Optimal control of hybrid power systems*, Dynamics of Continuous, Discrete and Impulsive Systems – Series B: Applications and Algorithms, **10** (2003), 429–439.

[60] A. Siburian and V. Rehbock, *Numerical procedure for solving a class of singular optimal control problems*, Optimization Methods and Software, **19** (2004), 413–426.

[61] D. E. Stewart, *A numerical algorithm for optimal control problems with switching costs*, Journal of the Australian Mathematical Society – Series B: Applied Mathematics, **34** (1992), 212–228.

[62] K. L. Teo, *Control parametrization enhancing transform to optimal control problems*, Nonlinear Analysis: Theory, Methods and Applications, **63** (2005), e2223–e2236.

[63] K. L. Teo and C. J. Goh, *A simple computational procedure for optimization problems with functional inequality constraints*, IEEE Transactions on Automatic Control, **32** (1987), 940–941.

[64] K. L. Teo, C. J. Goh and C. C. Lim, *A computational method for a class of dynamical optimization problems in which the terminal time is conditionally free*, IMA Journal of Mathematical Control and Information, **6** (1989), 81–95.

[65] K. L. Teo, C. J. Goh and K. H. Wong, "A Unified Computational Approach to Optimal Control Problems," Longman Scientific and Technical, Essex, 1991.

[66] K. L. Teo and L. S. Jennings, *Nonlinear optimal control problems with continuous state inequality constraints*, Journal of Optimization Theory and Applications, **63** (1989), 1–22.

[67] K. L. Teo and L. S. Jennings, *Optimal control with a cost on changing control*, Journal of Optimization Theory and Applications, **68** (1991), 335–357.

[68] K. L. Teo, L. S. Jennings, H. W. J. Lee and V. Rehbock, *The control parameterization enhancing transform for constrained optimal control problems*, Journal of the Australian Mathematical Society – Series B: Applied Mathematics, **40** (1999), 314–335.

[69] K. L. Teo, G. Jepps, E. J. Moore and S. Hayes, *A computational method for free time optimal control problems, with application to maximizing the range of an aircraft-like projectile*, Journal of the Australian Mathematical Society – Series B: Applied Mathematics, **28** (1987), 393–413.

[70] K. L. Teo, W. R. Lee, L. S. Jennings, S. Wang and Y. Liu, *Numerical solution of an optimal control problem with variable time points in the objective function*, ANZIAM Journal, **43** (2002), 463–478.

[71] K. L. Teo, V. Rehbock and L. S. Jennings, *A new computational algorithm for functional inequality constrained optimization problems*, Automatica J. IFAC, **29** (1993), 789–792.

[72] T. L. Vincent and W. J. Grantham, "Optimality in Parametric Systems," John Wiley, New York, 1981.

[73] L. Y. Wang, W. H. Gui, K. L. Teo, R. Loxton and C. H. Yang, *Time delayed optimal control problems with multiple characteristic time points: Computation and industrial applications*, Journal of Industrial and Management Optimization, **5** (2009), 705–718.

[74] L. Y. Wang, W. H. Gui, K. L. Teo, R. Loxton and C. H. Yang, *Optimal control problems arising in the zinc sulphate electrolyte purification process*, Journal of Global Optimization, **54** (2012), 307–323.

[75] K. H. Wong, L. S. Jennings and F. Benyah, *The control parametrization enhancing transform for constrained time-delayed optimal control problems*, ANZIAM Journal, **43** (2002), E154–E185.

[76] S. F. Woon, V. Rehbock and R. Loxton, *Global optimization method for continuous-time sensor scheduling*, Nonlinear Dynamics and Systems Theory, **10** (2010), 175–188.

[77] S. F. Woon, V. Rehbock and R. Loxton, *Towards global solutions of optimal discrete-valued control problems*, Optimal Control Applications and Methods, **33** (2012), 576–594.

[78] C. Z. Wu and K. L. Teo, *Global impulsive optimal control computation*, Journal of Industrial and Management Optimization, **2** (2006), 435–450.

[79] C. Z. Wu, K. L. Teo and V. Rehbock, *A filled function method for optimal discrete-valued control problems*, Journal of Global Optimization, **44** (2009), 213–225.

[80] R. Yu and P. Leung, *Optimal partial harvesting schedule for aquaculture operations*, Marine Resource Economics, **21** (2006), 301–315.

[81] C. Yu, B. Li, R. Loxton and K. L. Teo, *Optimal discrete-valued control computation*, Journal of Global Optimization, **56** (2013), 503–518.

[82] Y. Zhao and M. A. Stadtherr, *Rigorous global optimization for dynamic systems subject to inequality path constraints*, Industrial and Engineering Chemistry Research, **50** (2011), 12678–12693.

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E-mail address: q.lin@curtin.edu.au
E-mail address: r.loxton@curtin.edu.au
E-mail address: k.l.teo@curtin.edu.au