Horizontal well's path planning: An optimal switching control approach

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Abstract

In this paper, we consider a three-dimensional horizontal well's path planning problem, where the well's path evolves as a combination of several constant-curvature smooth turn segments. The problem is formulated as an optimal switching control problem subject to continuous state inequality constraints. By applying the time-scaling transformation and constraint transcription in conjunction with local smooth approximation technique, the optimal switching control problem is approximated by a sequence of optimal parameter selection problems with only box constraints, each of which is solvable by gradient-based optimization techniques. The optimal path planning problems of the wells Ci-16-Cp146 and Jin27 in Liaohe oil field are solved to demonstrate the applicability of the approach proposed.

Key words: Horizontal well's path planning; Optimal switching control; Continuous state inequality constraint; Time-scaling transformation; Computational method

1. Introduction

In practice, a well path is a three-dimensional (3-D) curve, rather than lying on a plane, where the well path is described as a combination of turn sections and straight sections. The goal of planning a horizontal well's path is to construct a trajectory that reaches a given target at a specified inclination and azimuth from a given starting location, subject to various constraints arising from engineering specifications. Several well planning softwares are available commercially for finding a horizontal well's path. However, most of them use a trial-and-error procedure to obtain a solution. In addition to being time-consuming and depending on user-experience, these techniques are limited to simple well's paths, and may not generate an optimal path, as the trial-and-error search is usually user-driven.

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Mathematical optimization theory provides a much more sophisticated, rigorous, and efficient approach to the well's path planning [1]. An optimal control approach is proposed to determine a least-length trajectory in [2]. A sequential unconstrained minimization technique is presented to determine a minimum-length path for a two-dimensional S-shaped well, subject to build and drop-rate restrictions [3]. A procedure using nonlinear optimization theory is developed for 3-D well paths and path corrections in [4]. In [5, 6, 7, 8], it is assumed that the 3-D well's path is expressed as a combination of several constant-curvature smooth turn segments. A nonlinear multistage dynamical system is thus proposed to describe the 3-D well's path. Taking the weighted sum of the target error and the length of the well's path as the cost function, a multistage optimal control problem subject to continuous state inequality constraints is investigated. For this multistage optimal control problem, the optimality conditions are derived via nonsmooth optimization theory in [9] and maximum principle in [10]. However, the solution methods used in [5, 6, 7, 8, 9, 10] are based on either heuristic or direction search methods. They are computationally expensive with poor convergence properties. In addition, the continuous state inequality constraints arising from engineering specifications are ignored in [9, 10]. Consequently, some so-called optimal solutions obtained actually fail to satisfy the continuous state inequality constraints at some points along the well's path. This is clearly undesirable in practice.

In this paper, we formulate the 3-D horizontal well's path planning problem as an optimal switching control problem subject to continuous state inequality constraints. We develop a new solution method. First, by the time-scaling transformation [11], the optimal switching control problem is transformed equivalently into an optimal control problem with fixed switching instants. Then the constraint transcription is used in conjunction with local smoothing approximation technique [12] to approximate the constrained optimal control problem as a sequence of optimal parameter selection problems with only box constraints. Each of these optimal parameter selection problems is solvable by gradient-based optimization methods. For illustration, two practical optimal path planning problems of the wells Ci-16-Cp146 and Jin27 in Liaohe oil field in China are solved by using the proposed approach. Our results show that for each case, the length of the whole path is reduced by around 3%-4% and the precision of reaching the target is higher compared with the best results obtained previously in the literature. Moreover, the continuous state inequality constraints arising from the engineering specifications are fulfilled everywhere.

2. Problem formulation

Consider a 3-D horizontal well's path planning as shown in Figure 1. The well's path, which is required to reach a given target from the kick-off point (i.e., the location of the point where the curve begins to deviate from vertical) at a specified inclination and azimuth, is described in the Cartesian coordinate system, with $\bar{x}$-axis representing North/South (positive $\bar{x}$ being North), $\bar{y}$-axis

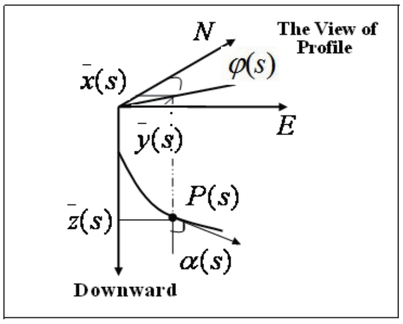

Figure 1: Coordinate system for well's path

representing East/West (positive $\bar{y}$ being East), and the $\bar{z}$-axis ($\bar{z}$ positive downwards) representing the true vertical depth (TVD). The arc length from the kick-off point is denoted by s, and any point $P(s)$ on the curve is described by its inclination $\alpha(s)$ and azimuth $\varphi(s)$, and coordinate $(\bar{x}(s), \bar{y}(s), \bar{z}(s))$. The tool-face angle ω and the curvature K are the decision parameters. In line with practical situation, we assume that

(H_1). The well's path is a combination of n smooth turn segments.

(H_2). The curvature K_i and the tool-face angle ω_i are constants in each ith turn segment, $i = 1, \ldots, n$.

(H_3). The design of the well trajectory is only for non-straight horizontal well. That is to say, the inclination $\alpha(s) \notin (k\pi - \varepsilon_0, k\pi + \varepsilon_0)$, $k = 0, 1, \cdots$, where $\varepsilon_0 > 0$ is a given real number. Therefore, we may assume, without loss of generality, that $\alpha(s) = k\pi + \varepsilon_0$ whenever $|\alpha(s) - k\pi| \leq \varepsilon_0$, $k = 0, 1, \cdots$.

Under Assumptions (H_1)-(H_3), the change rates of $\alpha(s)$ and $\varphi(s)$ obey, respectively, the following rules [6]:

$$\dot{\alpha}(s) = K \cos\omega, \quad \dot{\varphi}(s) = \frac{K \sin\omega}{\sin\alpha(s)}. \tag{1}$$

Furthermore, the change rates of $\bar{x}(s), \bar{y}(s)$ and $\bar{z}(s)$ with respect to arc length s are given as follows [6]:

$$\dot{\bar{x}}(s) = \sin\alpha(s)\cos\varphi(s), \quad \dot{\bar{y}}(s) = \sin\alpha(s)\sin\varphi(s), \quad \dot{\bar{z}}(s) = \cos\alpha(s). \tag{2}$$

Now, let $x(s) = (\alpha(s), \varphi(s), \bar{x}(s), \bar{y}(s), \bar{z}(s))^\top$, $\xi^i = (K_i, \omega_i)$, $i = 1, \ldots, n$, and

$$f(x(s), \xi^i) = \begin{pmatrix} \xi_1^i \cos \xi_2^i \\ \frac{\xi_1^i \sin \xi_2^i}{\sin x_1(s)} \\ \sin x_1(s) \cos x_2(s) \\ \sin x_1(s) \sin x_2(s) \\ \cos x_1(s) \end{pmatrix}. \tag{3}$$

Then, we obtain the nonlinear dynamic system to describe the 3-D horizontal well's path planning given below:

$$\dot{x}(s) = f(x(s), \xi^i), \quad s \in [s_{i-1}, s_i), \quad i = 1, \ldots, n, \tag{4}$$

with the initial and intermediate conditions

$$x(s_i) = x(s_i-), \quad i = 1, \ldots, n-1, \tag{5}$$

$$x(0) = x^0, \tag{6}$$

where $x(s_i-)$ denotes the limit of state from the left at s_i, x^0 is a given initial state (i.e., the kick-off point), $s_0 = 0$ is the initial arc length, s_i, $i = 1, \ldots, n-1$, are the switching instants, and s_n denotes the well's whole path length. Note that $s_i, i = 1, \ldots, n$, are decision variables.

Due to the operational and geological limitations, the following bound constraints are imposed:

$$\xi^i \in [a_1, b_1] \times [a_2, b_2] \subset R^2, \quad i = 1, \ldots, n,$$

and

$$s_i - s_{i-1} \in [a_3, b_3], \quad i = 1, \ldots, n,$$

where a_l and $b_l, l = 1, 2, 3$, are given real numbers such that $a_l < b_l$. Define

$$\Gamma = \{\tau = (s_1, \ldots, s_n)^\top \in R^n : s_i - s_{i-1} \in [a_3, b_3], i = 1, \ldots, n\},$$

and

$$\Xi = \{\xi = (\xi^1, \ldots, \xi^n) \in R^{2n} : \xi^i \in [a_1, b_1] \times [a_2, b_2], i = 1, \ldots, n\}.$$

Any $(\tau, \xi) \in \Gamma \times \Xi$ is called an admissible pair.

Following a similar argument as that given for Property 3.1 in [9], the following property can be established.

Property 1. *Under Assumptions (H_1)-(H_3), the function $f(\cdot, \cdot)$ defined by (3) satisfies the following conditions:*

(a). $f(\cdot, \cdot) : R^5 \times \big([a_1, b_1] \times [a_2, b_2]\big) \to R^5$ is continuously differentiable;

(b). There exists a real number $L > 0$ such that

$$\|f(x, \xi^i)\| \leq L(1 + \|x\|), \quad \forall (x, \xi^i) \in R^5 \times \big([a_1, b_1] \times [a_2, b_2]\big),$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

Based on the Property 1, system (4)-(5) has a unique solution, $x(s|\tau,\xi)$, for each $s \in [s_{i-1}, s_i), i = 1,\ldots,n$. Due to engineering specifications, $x(s|\tau,\xi)$ is required to satisfy the following constraints:

$$x_1(s|\tau,\xi) \in [\varepsilon_0, \frac{\pi}{2}], \qquad \forall s \in [0, s_n], \tag{7a}$$

$$x_2(s|\tau,\xi) \in [0, 2\pi], \qquad \forall s \in [0, s_n], \tag{7b}$$

where $\varepsilon_0 > 0$ is a given real number.

Our goal is to construct a well's path to reach the target $x^f = (x_1^f, \ldots, x_5^f)^\top$ from the kick-off point, x^0, by adjusting the decision vectors τ and ξ, such that the weighted sum of the path length and the error of reaching the target, i.e., the following cost function

$$J(\tau,\xi) = s_n + \rho \sum_{j=1}^{5} (x_j(s_n|\tau,\xi) - x_j^f)^2, \tag{8}$$

where ρ is a given weight coefficient, is minimized.

Now we may state the optimal switching control problem as follows.

Problem (P1). *Given the dynamic system (4)-(5), choose an admissible pair $(\tau,\xi) \in \Gamma \times \Xi$ such that the cost function (8) is minimized subject to the continuous state inequality constraints (7).*

3. Problem transformation

Problem (P1) can, in principle, be solved by using nonlinear optimization techniques. However, as pointed out in [13, 14, 15], there are three deficiencies associated with Problem (P1), if it is to be solved directly as such.

1). It is cumbersome to integrate the state and costate systems numerically when the switching instants are variables, especially when two or more switching instants are closed.
2). The gradients of the cost function with respect to switching instants only exist when the switching instants are distinct.
3). Each of the continuous state inequality constraints (7) represents an infinite number of inequality constraints.

3.1. Time-scaling transformation

To overcome the first and second deficiencies, we shall apply a time-scaling transformation [11] to map the variable switching and terminal instants into a set of fixed points in a new horizon. First, we introduce a new variable $t \in [0, n]$ and relate the new variable $t \in [0, n]$ to the original arc length variable $s \in [0, s_n]$ through the following differential equation:

$$\dot{s}(t) = v(t), \qquad t \in [0, n], \tag{9}$$

with the initial condition

$$s(0) = 0, \tag{10}$$

where $v : [0, n] \to R$ is a new piecewise-constant control defined by

$$v(t) = \sum_{i=1}^{n} \theta_i \chi_{[i-1,i)}(t), \qquad t \in [0, n]. \tag{11}$$

Here, $\chi_{[i-1,i)}(\cdot)$ is the characteristic function of the interval $[i-1, i)$, and

$$0 < a_3 \leq \theta_i = s_i - s_{i-1} \leq b_3, \quad i = 1, \dots, n. \tag{12}$$

Thus, θ_i is the duration of the ith turn segment in the original arc length horizon. Let $\Theta = \{\theta = (\theta_1, \dots, \theta_n)^\top \in R^n : 0 < a_3 \leq \theta_i = s_i - s_{i-1} \leq b_3, i = 1, \dots, n\}$. Taking integration of (9)-(10) yields

$$s(t|\theta) = \sum_{j=1}^{\llcorner t \lrcorner} \theta_j + \theta_{\llcorner t \lrcorner + 1}(t - \llcorner t \lrcorner), \tag{13}$$

where $\llcorner \cdot \lrcorner$ denotes the floor function. It can be verified that $s(t|\theta)$ is continuous and monotonically increasing on $[0, n]$. From (13), the new switching and terminal instants are at

$$t = 1, \dots, n. \tag{14}$$

Note that $v(\cdot)$ depends on the choice of $\theta \in \Theta$, so we denote it by $v(\cdot|\theta)$. Let $y(t) = (x(s(t))^\top, s(t))^\top$, $h(y(t), \xi^i, \theta) = ((v(t|\theta)f(y(t)))^\top, \xi^i), v(t|\theta))^\top$. Then, from (4)-(5) and (9)-(13), we have

$$\dot{y}(t) = h(y(t), \xi^i, \theta), \quad t \in [i-1, i), \quad i = 1, \dots, n, \tag{15}$$

where

$$y(i) = \begin{cases} ((x^0)^\top, 0)^\top, & \text{if } i = 0, \\ y(i-), & \text{if } i = 1, \dots, n. \end{cases} \tag{16}$$

Let $y(\cdot|\theta, \xi)$ be the unique solution of system (15)-(16) corresponding to each $(\theta, \xi) \in \Theta \times \Xi$. Now, the continuous state inequality constraints (7) and the cost function (8) become, respectively,

$$y_1(t|\theta, \xi) \in [\varepsilon_0, \frac{\pi}{2}], \qquad \forall t \in [0, n], \tag{17a}$$

$$y_2(t|\theta, \xi) \in [0, 2\pi], \qquad \forall t \in [0, n], \tag{17b}$$

and

$$\tilde{J}(\theta, \xi) = y_6(n|\theta) + \rho \sum_{j=1}^{5} (y_j(n|\theta, \xi) - x_j^f)^2. \tag{18}$$

Problem (P1) may now be stated equivalently as the following optimal switching control problem with fixed switching instants:

Problem (P2). *Given the dynamic system (15)-(16), choose a pair $(\theta, \xi) \in \Theta \times \Xi$ such that the cost function (18) is minimized subject to the continuous state inequality constraints (17).*

3.2. Constraint approximation

In Problem (P2), the third deficiency mentioned in Section 3 remains. Namely, each continuous state inequality constraint represents an infinite number of inequality constraints to be satisfied. Thus, Problem (P2) still cannot be solved directly. We shall apply the constraint transcription in conjunction with a local smoothing technique [11] to approximate each of these continuous state inequality constraints.

To begin, the continuous state inequality constraints (17) are written as

$$g_j(y(t|\theta,\xi)) \le 0, \quad \forall t \in [0,n], \; j = 1,2,3,4, \tag{19}$$

where

$$g_1(y(t|\theta,\xi)) = -y_1(t|\theta,\xi) + \varepsilon_0, \tag{20a}$$
$$g_2(y(t|\theta,\xi)) = y_1(t|\theta,\xi) - \pi/2, \tag{20b}$$
$$g_3(y(t|\theta,\xi)) = -y_2(t|\theta,\xi), \tag{20c}$$
$$g_4(y(t|\theta,\xi)) = y_2(t|\theta,\xi) - 2\pi. \tag{20d}$$

Clearly, the constraints (19) hold if and only if

$$\sum_{j=1}^{4} \int_0^n \max\{g_j(y(t|\theta,\xi)), 0\} dt = 0. \tag{21}$$

However, $\max\{g_j(y(t|\theta,\xi)),0\}, j = 1,2,3,4$, are non-smooth functions in θ and ξ. Thus, gradient-based optimization routines would have difficulties in dealing with this type of equality constraints. To overcome this difficult, a local smoothing technique is applied to approximate $\max\{g_j(y(t|\theta,\xi)),0\}$ by

$$\varphi_{j,\epsilon}(y(t|\theta,\xi)) = \begin{cases} 0, & \text{if } g_j(y(t|\theta,\xi)) < -\epsilon, \\ \dfrac{(\eta+\epsilon)^2}{4\epsilon}, & \text{if } -\epsilon \le g_j(y(t|\theta,\xi)) \le \epsilon, \\ \eta, & \text{if } g_j(y(t|\theta,\xi)) > \epsilon, \end{cases} \tag{22}$$

where $\epsilon > 0$ is an adjustable parameter.

We now consider the following approximate optimal switching control problem with fixed switching instants.

Problem ($\mathbf{P}_{\epsilon,\gamma}$). *Given the dynamic system (15)-(16), find a pair $(\theta,\xi) \in \Theta\times\Xi$ such that the cost function*

$$\tilde{J}_{\epsilon,\gamma}(\theta,\xi) = \tilde{J}(\theta,\xi) + \gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta,\xi)) dt \tag{23}$$

is minimized, where γ is a penalty parameter.

The relationship between Problem ($P_{\epsilon,\gamma}$) and Problem (P2) is furnished in two theorems. Note that the following theorem ensures that the optimal solution of Problem ($P_{\epsilon,\gamma}$) is feasible for Problem (P2).

Theorem 1. *For each $\epsilon > 0$, there exists a $\gamma(\epsilon) > 0$ such that for each $\gamma > \gamma(\epsilon)$, the optimal solution of Problem $(P_{\epsilon,\gamma})$ is feasible for Problem (P2).*

Proof. To prove this theorem, we define

$$\mathcal{F} = \{(\theta, \xi) \in \Theta \times \Xi : g_j(y(t|\theta, \xi)) \leq 0, j = 1, 2, 3, 4, t \in [0, n]\}, \tag{24}$$

and

$$\mathcal{F}_\epsilon = \{(\theta, \xi) \in \Theta \times \Xi : g_j(y(t|\theta, \xi)) \leq -\epsilon, j = 1, 2, 3, 4, t \in [0, n]\}. \tag{25}$$

Clearly, $\mathcal{F}_\epsilon \subset \mathcal{F}$ for each $\epsilon > 0$. Let $(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma})$ be the optimal solution of Problem $(P_{\epsilon,\gamma})$. Then, for all $(\theta, \xi) \in \Theta \times \Xi$, we have

$$\begin{aligned} \tilde{J}(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}) + \gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}))dt \\ \leq \tilde{J}(\theta, \xi) + \gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta, \xi))dt. \end{aligned} \tag{26}$$

Let $(\theta_\epsilon, \xi_\epsilon) \in \mathcal{F}_\epsilon$ be fixed. Then

$$\int_0^n \varphi_{j,\epsilon}(y(t|\theta_\epsilon, \xi_\epsilon))dt = 0, \quad j = 1, 2, 3, 4. \tag{27}$$

Since Θ and Ξ are compact and $\tilde{J}$ is continuous, there exists a $(\bar{\theta}, \bar{\xi}) \in \Theta \times \Xi$ such that

$$\tilde{J}(\bar{\theta}, \bar{\xi}) \leq \tilde{J}(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}). \tag{28}$$

Hence,

$$\begin{aligned} \tilde{J}(\bar{\theta}, \bar{\xi}) + \gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}))dt &\leq \tilde{J}(\theta, \xi) + \gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta, \xi))dt \\ &\leq \tilde{J}(\theta_\epsilon, \xi_\epsilon). \end{aligned} \tag{29}$$

Rearranging (29), we obtain

$$\gamma \sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}))dt \leq \tilde{J}(\theta_\epsilon, \xi_\epsilon) - \tilde{J}(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}). \tag{30}$$

Set $\sigma = \tilde{J}(\theta_\epsilon, \xi_\epsilon) - \tilde{J}(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma})$, then we have

$$\sum_{j=1}^{4} \int_0^n \varphi_{j,\epsilon}(y(t|\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}))dt \leq \frac{\sigma}{\gamma}. \tag{31}$$

Let $0 < \kappa < \kappa(\epsilon)$ and $\gamma(\epsilon) \geq \frac{\sigma}{\kappa(\epsilon)}$, it follows that for all $\gamma > \gamma(\epsilon)$,

$$\sum_{j=1}^{4} \int_{0}^{n} \varphi_{j,\epsilon}(y(t|\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}))dt < \kappa. \tag{32}$$

On the basis of Theorem 2.3 [16], we have $(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}) \in \mathcal{F}$. This completes the proof. □

The next theorem guarantees the convergence of the approximation procedure.

Theorem 2. *Suppose that Problem (P2) has an optimal solution* $(\theta^*, \xi^*) \in \Theta \times \Xi$. *For each* $\epsilon > 0$, *let* $(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma})$ *denote the optimal solution of Problem* $(P_{\epsilon,\gamma})$, *where* $\gamma > 0$ *is chosen such that* $(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma})$ *is feasible for Problem (P2). Then*

$$\lim_{\epsilon \to 0} \tilde{J}_{\epsilon,\gamma}(\theta^*_{\epsilon,\gamma}, \xi^*_{\epsilon,\gamma}) = \tilde{J}(\theta^*, \xi^*), \tag{33}$$

where (θ^*, ξ^*) *is the optimal solution of Problem (P2).*

Proof. The proof is similar to that given for Theorem 2.2 in [12]. □

To solve Problem (P2), we need to solve a sequence of Problems $\{(P_{\epsilon,\gamma})\}$. Each Problem $(P_{\epsilon,\gamma})$ can be regarded as a smooth nonlinear mathematical programming problem with only box constraints, and it can be solved using existing gradient-based optimization techniques [17]. For this, we need the gradients of the cost function (23) with respect to the decision variables. These gradients are given in the following theorem.

Theorem 3. *For each* $\epsilon > 0$ *and* $\gamma > 0$, *the gradients of the cost function* $\tilde{J}_{\epsilon,\gamma}(\theta, \xi)$ *with respect to* θ *and* ξ *are, respectively,*

$$\frac{\partial \tilde{J}_{\epsilon,\gamma}(\theta, \xi)}{\partial \theta_i} = \int_{i-1}^{i} \frac{\partial H(y(t|\theta, \xi), \theta, \xi, \lambda(t))}{\partial \theta_i} dt, \quad i = 1, \ldots, n, \tag{34}$$

and

$$\frac{\partial \tilde{J}_{\epsilon,\gamma}(\theta, \xi)}{\partial \xi^i} = \int_{i-1}^{i} \frac{\partial H(y(t|\theta, \xi), \theta, \xi, \lambda(t))}{\partial \xi^i} dt, \quad i = 1, \ldots, n, \tag{35}$$

where

$$H(y(t|\theta, \xi), \theta, \xi, \lambda(t)) = \gamma \sum_{j=1}^{4} \varphi_{j,\epsilon}(y(t|\theta, \xi)) + \lambda^\top(t) h(y(t|\theta, \xi), \xi^i, \theta), \tag{36}$$

and $\lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_6(t))^\top$ *is the solution of the costate system*

$$\dot{\lambda}(t) = -\left(\frac{\partial H(y(t|\theta, \xi), \theta, \xi, \lambda(t))}{\partial y}\right)^\top, \quad t \in [0, n], \tag{37}$$

with the condition

$$\lambda(n) = (2\rho(y_1(n|\theta,\xi) - x_1^f), \ldots, 2\rho(y_5(n|\theta,\xi) - x_5^f), 1)^\top. \tag{38}$$

Proof. The derivation of the gradients of the cost function $\tilde{J}_{\epsilon,\gamma}(\theta,\xi)$ with respect to θ and ξ are similar. Thus, only the derivation of the gradient of the cost function $\tilde{J}_{\epsilon,\gamma}(\theta,\xi)$ with respect to θ is given below.

Let $\nu : [0,n] \to R^6$ be an arbitrary function that is continuously differentiable. Then the cost function can be expressed as follows.

$$\begin{aligned}\tilde{J}_{\epsilon,\gamma}(\theta,\xi) \quad &= \phi(y(n)) + \gamma\sum_{j=1}^{4}\int_0^n \varphi_{j,\epsilon}(y(t))dt \\ &+ \sum_{i=1}^{n}\int_{i-1}^{i} \nu^\top(t)(h(y(t),\xi^i,\theta) - \dot{y}(t))dt,\end{aligned} \tag{39}$$

where we have omitted the arguments θ and ξ in $y(\cdot|\theta,\xi)$ for brevity, and

$$\phi(y(n)) = y_6(n) + \rho\sum_{j=1}^{5}(y_j(n) - x_j^f)^2.$$

Applying integration by parts to equation (39) gives

$$\begin{aligned}\tilde{J}_{\epsilon,\gamma}(\theta,\xi) \quad = \quad &\phi(y(n)) + \gamma\sum_{j=1}^{4}\int_0^n \varphi_{j,\epsilon}(y(t))dt - \nu^\top(n)y(n) + \nu^\top(0)y(0) \\ &+ \sum_{i=1}^{n}\int_{i-1}^{i} \dot{\nu}^\top(t)y(t)dt + \sum_{i=1}^{n}\int_{i-1}^{i} \nu^\top(t)h(y(t),\xi^i,\theta)dt.\end{aligned} \tag{40}$$

Differentiating (40) with respect to θ_i yields

$$\begin{aligned}\frac{\partial\tilde{J}_{\epsilon,\gamma}(\theta,\xi)}{\partial\theta_i} \quad = \quad &\left(\frac{\partial\phi(y(n))}{\partial y} - \nu^\top(t)\right)\frac{\partial y(n)}{\partial\theta_i} + \sum_{i=1}^{n}\int_{i-1}^{i} \nu^\top(t)\frac{\partial h(y(t),\xi^i,\theta)}{\partial\theta_i}dt \\ &+ \sum_{i=1}^{n}\int_{i-1}^{i}\left(\dot{\nu}(t) + \gamma\sum_{j=1}^{4}\frac{\partial\varphi_{j,\epsilon}(y(t))}{\partial y} + \frac{\partial h(y(t),\xi^i,\theta)}{\partial y}\right)\frac{\partial y(t)}{\partial\theta_i}dt.\end{aligned}$$

Choosing $\nu(\cdot) = \lambda(\cdot)$ and substituting (36)-(38) into the equation above gives equation (34). Similarly, we can obtain equation (35). The proof is completed. □

Based on Theorems 2 and 3, we can present the following algorithm to solve Problem (P2).

Algorithm 1.

Step 1. Choose initial values of ϵ^0, γ^0 and $(\theta_{\epsilon^0,\gamma^0}, \xi_{\epsilon^0,\gamma^0}) \in \Theta \times \Xi$, set weight coefficient $\rho > 0$, parameters $\alpha > 1, 0 < \beta < 1, \bar{\gamma} > 0, \bar{\epsilon} > 0$, and set iteration numbers $k = 0$ and $l = 0$.

Step 2. Solve Problem $(P_{\epsilon^k,\gamma^l})$ by using a gradient-based optimization technique (the gradients are given in Theorem 3) to give $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$.

Step 3. Check the feasibility of $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$. If $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$ is feasible, go to Step 4. Otherwise, set $\gamma^{l+1} := \alpha\gamma^l$ and $l = l + 1$. If $\gamma^l > \bar{\gamma}$, we have an abnormal exit. Otherwise, set $(\theta_{\epsilon^k,\gamma^l}, \xi_{\epsilon^k,\gamma^l}) := (\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^{l-1}})$, and go to Step 2.

Step 4. Set $\epsilon^{k+1} := \beta\epsilon^k$. If $\epsilon^k < \bar{\epsilon}$, output $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$ and stop. Otherwise, set $k := k + 1$, $(\theta_{\epsilon^k,\gamma^l}, \xi_{\epsilon^k,\gamma^l}) := (\theta^*_{\epsilon^{k-1},\gamma^l}, \xi^*_{\epsilon^{k-1},\gamma^l})$ and go to Step 2.

Remark 1. *From Theorem 1, it is easy to observe that the loop between Step 2 and Step 3 is finite, i.e., the number of iterations within the loop is finite.*

Remark 2. $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$ *is an optimal solution of Problem* $(P_{\epsilon^k,\gamma^l})$. *The corresponding suboptimal solution of Problem (P1) can be readily constructed from* $(\theta^*_{\epsilon^k,\gamma^l}, \xi^*_{\epsilon^k,\gamma^l})$.

4. Numerical results

Algorithm 1 is implemented within the optimal control software MISER 3 [18] to design the optimal paths of the wells Ci-16-Cp146 and Jin27 in an oil field of Liaohe, China. The basic data and the bounds for the control variables of the wells Ci-16-Cp146 and Jin27 are listed in Tables 1-4, respectively.

Table 1: Basic data of the Well Ci-16-Cp146

	Inclination (∘)	Azimuth (∘)	Coordinate x(m)	Coordinate y(m)	Vertical depth z(m)
Bottom	10.4	228.18	102.69	−156.39	1673.15
Target	89.5	205.5	62.5	192.9	1718.0

Table 2: Bounds for the control variables of the Well Ci-16-Cp146

	tool-face angle(∘)	radius(m)	Curve length(m)
1st seg.	[-50,50]	[40,60]	[10,100]
2nd seg.	[-50,50]	[40,60]	[10,100]
3rd seg.	[-50,50]	[40,60]	[10,100]

In the computation, the initial values of ϵ^0, γ^0, the weight coefficient ρ, and the parameters $\alpha, \beta, \bar{\gamma}, \bar{\epsilon}$ are chosen as $10^{-2}, 0.75, 10, 2, 0.1, 10^6, 10^{-7}$, respectively. By running MISER 3, the optimal controls, the error of reaching the target and the total curve length are listed in Table 5. Here, the error of reaching the

Table 3: Basic data of the Well Jin27

	Inclination (◦)	Azimuth (◦)	Coordinate x(m)	Coordinate y(m)	Vertical depth z(m)
Bottom	10.0	265.0	0	0	925.0
Target	90.0	270	3.8362	-285.3	1204.0

Table 4: Bounds for the control variables of the Well Jin27

	tool-face angle(◦)	radius(m)	Curve length(m)
1st seg.	[-30,30]	[260,360]	[30,400]
2nd seg.	[-30,30]	[260,360]	[30,400]
3rd seg.	[0,20]	[260,360]	[30,400]

target is computed according to $||x(3|\theta^*, \xi^*) - x^f||^2$. For comparison, the results obtained previously in [6] and [7], which are the best results amongst those available in previous studies, are also listed in Table 5. Furthermore, under the optimal results obtained in this work, the 3-D views of the optimal paths for the wells Ci-16-Cp146 and Jin27 are plotted in Figures 2 and 3, respectively. The variations of the inclination and the azimuth with respect to the path length of the wells Ci-16-Cp146 and Jin47 are shown in Figures 4-7, respectively. From Table 5 and Figures 4-7, we conclude that the results obtained in this work are clearly superior to the best results in the existing literature:

1). The length of the path of each of the two wells, Ci-16-Cp146 and Jin27, is decreased by around 3%-4%, resulting in the cost reduction of the drilling the horizontal well for each of the two cases.
2). The precision of reaching the target is higher, meaning that the optimal 3-D well's path reaches the target at the desired inclination, azimuth, and coordinate much more accurately.
3). The continuous state inequality constraints arising from engineering specifications are fulfilled for the whole arc length $[0, s_n]$.

5. Conclusions

We proposed a realistic optimal switching control model for the 3-D horizontal well's path planning. We developed a gradient-based solution method to solve this problem. This approach produces superior results when compared with those obtained in previous studies. It appears that the exact penalty method proposed in [19] and [20] can also be applied to handle the continuous state inequality constraints. This is an interesting research topic for future study.

Table 5. The results obtained for the Well Ci-16-Cp146 and Well Jin27.

Segment	Decision parameters	Well Ci-16-Cp146		Well Jin27	
		this work	[6]	this work	[7]
1st	Radius of curvature (m)	40.0	50.0	273.2	260.5
	Tool-face angle (○)	-6.3	-2.5	14.5	1.4
	Curvature length (m)	24.5	31.5	113.5	119.3
2nd	Radius of curvature(m)	59.9	59.8	359.7	292.6
	Tool-face angle (○)	49.9	47.8	-29.9	-2.5
	Curvature length (m)	15.9	10.1	170.5	175.9
3rd	Radius of curvature (m)	59.9	44.4	359.7	306.1
	Tool-face angle (○)	-43.7	-35.3	19.9	17.0
	Curvature length (m)	33.9	34.7	137.6	143.2
	Target error (m)	0.21	0.62	0.21	0.24
	Total curve length (m)	74.4	76.3	421.7	438.5

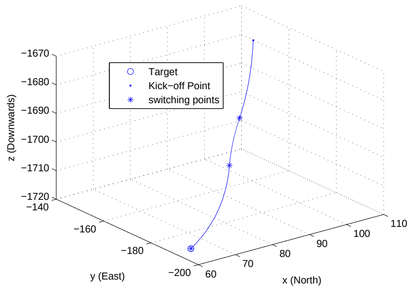

Figure 2: 3-D view of the optimal path of the Well Ci-16-Cp146.

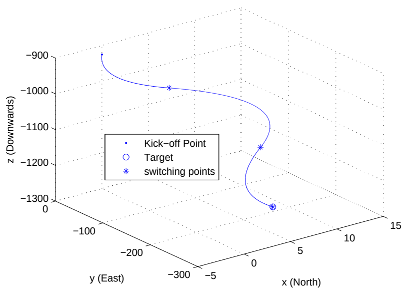

Figure 3: 3-D view of the optimal path of the Well Jin27.

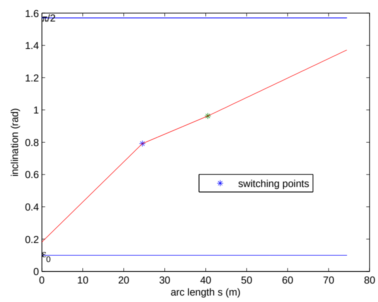

Figure 4: Variation of the inclination with the path length of the Well Ci-16-Cp146.

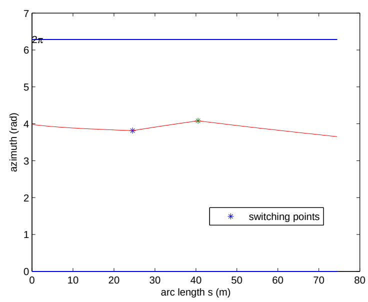

Figure 5: Variation of the azimuth with the path length of the Well Ci-16-Cp146.

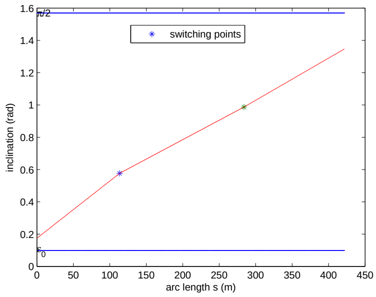

Figure 6: Variation of the inclination with the path length of the Well Jin27.

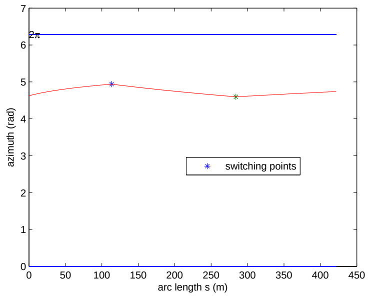

Figure 7: Variation of the azimuth with the path length of the Well Jin27.

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