MATERIAL SYMMETRIES OF ELASTICITY TENSORS

ANDREJ BÓNA, IOAN BUCATARU, AND MICHAEL A. SLAWINSKI

Abstract. We prove that there are eight subgroups of the orthogonal group $O(3)$ that determine all symmetry classes of an elasticity tensor. Then, we provide the necessary and sufficient conditions that allow us to determine the symmetry class to which a given elasticity tensor belongs. We also give a method to determine the natural coordinate system for each symmetry class.

1. Introduction

Any linearly elastic continuum is defined at a given point in terms of the corresponding elasticity tensor, which possesses intrinsic symmetries that result from the fundamentals of elasticity theory. Also, an elasticity tensor might exhibit additional symmetries that depend on the properties of a given elastic continuum. The latter symmetries, which are the subject of this paper, are referred to as material symmetries.

This paper consists of two main sections. In the first one, we discuss the classification of elasticity tensors according to their material symmetries. We determine all possible symmetry classes for an elasticity tensor; for each of these classes the corresponding symmetry group is explicitly given. These symmetry groups are determined using simpler symmetries of associated second-rank tensors. The symmetry groups are presented in an invariant form and, when a basis is fixed, we represent them using their matrix representation. Accordingly, our study does not depend on a particular choice of a basis for the three-dimensional Euclidean space $\mathbb{E}^3$. This independence is especially important in discussion of trigonal and tetragonal cases, when particular basis and particular representations of the symmetry groups help us to determine the symmetry groups. In the second section, we show how to recognize to which symmetry class a given elasticity tensor expressed in an arbitrary basis belongs. Using the eigenvectors of the associated second-rank tensors, we determine for each symmetry class a natural coordinate system. To make the paper self-contained, we provide a complete and rigorous, albeit laborious, formulation of these results.

Symmetries of elastic media are discussed by numerous authors, notably, by Love [1] and Voigt [2]. There are several papers in which the authors prove that the number of symmetry classes of an elasticity tensor is eight; notably, those of Chadwick et al [3], Forte and Vianello [4] and Ting [5]. In recent literature, there are several discussions about symmetries of elasticity tensors that are pertinent to our study; in particular, those by Backus [6], Baerheim [7, 8], Cowin and Mehrabadi [9, 10, 11], Forte and Vianello [12], Helbig [13], as well as Huo and del Piero [14].

Our paper differs from previous publications in several ways. Unlike Love [1] and Voigt [2], we do not use the strain-energy function, which contains components of the elasticity tensor, to discuss material symmetries. Instead, we deal directly with the tensor itself. Unlike Baerheim [7], Chadwick et al [3], and Ting [5], our discussion of hierarchy of symmetries is based on symmetries groups of associated second-rank tensors and their intersections. In particular, the symmetry group associated with each symmetry class of an elasticity tensor allows us to discuss the possible routes of increasing symmetries. A diagram that shows the relations between these classes of symmetry of an elasticity tensor is presented in Section 4. Our study is based on eigenvectors and eigenspaces of associated second-rank tensors and not on eigentensors of the elasticity tensor as done by Cowin and Mehrabadi [10, 11].

As shown by Forte and Vianello [12, 4], there are two different ways to define symmetries for an elasticity tensor. In this paper we follow the definition used by Forte and Vianello [4] and Chadwick et al [3]. A different way of defining the symmetry classes of an elasticity tensor has been introduced by Huo and del Piero [14]. According to their definition, there are ten symmetry classes of an elasticity tensor.

We begin our investigation with the elasticity tensor as a four-linear map that possesses all the intrinsic symmetries, and not as a linear map between two spaces of symmetric bi-linear forms, as Huo and del Piero [14] and Forte and Vianello [12, 4] do. Using this four-linear map, we discuss the material symmetries of the elasticity tensor, which is a fourth-rank tensor, by considering two associated second-rank tensors. We prove that the symmetry group of this fourth-rank tensor must be a subgroup of both symmetry groups of the two associated second-rank tensors. Studying all possible intersections of eigenspaces and the corresponding symmetry groups for the second-rank tensors, we obtain eight subgroups of $O(3)$ that correspond to all symmetry classes of an elasticity tensor. Moreover, we determine symmetry planes using the eigenspaces of the two second-rank tensors and, depending on these spaces, we identify the natural coordinate system. Hence, it is the material symmetry that determines the coordinate system, rather than its being determined by some a priori considerations. This allows us to identify the symmetry class to which a given elasticity tensor belongs by starting with an arbitrary coordinate system.

We wish to emphasize that, in spite of similarities to crystallographic classifications, such as Bravais lattices and Fedorov groups, the classification in this paper is based solely on the concept of continua. Also, we note that we do not use for our discussions the 6×6 elasticity matrix; we deal only with tensors.

2. Classification of elasticity tensors

2.1. **Elasticity tensors.** To study elasticity tensors, we begin by stating several properties of tensors that are pertinent to our present work. For this purpose, consider $\mathbb{E}^3$ – the Euclidean three-dimensional space.

An n-th rank tensor in $\mathbb{E}^3$ is an n-linear map

$$T : \underbrace{\mathbb{E}^3 \times \cdots \times \mathbb{E}^3}_{n\text{-times}} \longrightarrow \mathbb{R}.$$

The Euclidean scalar product, $\langle \cdot, \cdot \rangle$, in $\mathbb{E}^3$ determines the canonical isomorphism between $\mathbb{E}^3$ and its dual space $(\mathbb{E}^3)^* = \{\theta : \mathbb{E}^3 \longrightarrow \mathbb{R}$, where θ is linear$\}$. Due to this isomorphism, we shall identify the two spaces – $\mathbb{E}^3$ and its dual, $(\mathbb{E}^3)^*$ – and, consequently, we shall make no distinction between the elements of these two spaces. Also, this identification allows us to view an n-th rank tensor as a linear map $T : \mathbb{E}^3 \times \cdots \times \mathbb{E}^3 ((n-k)\text{-times}) \longrightarrow \mathbb{E}^3 \times \cdots \times \mathbb{E}^3$ (k-times), for any $k = 0, \ldots, n-1$. For example, a second-rank tensor, g, can be viewed either as the bi-linear map given by $g : \mathbb{E}^3 \times \mathbb{E}^3 \longrightarrow \mathbb{R}$ or as the linear map given by $\widetilde{g} : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$, where the connection between these two maps is given via the Euclidean scalar product as

$$g(u, v) = \langle \widetilde{g}(u), v \rangle \text{ , for } u, v \in \mathbb{E}^3.$$

If the second-rank tensor, g, is symmetric, namely $g\,(u, v) = g\,(v, u)$, then the symmetry of the induced tensor $\widetilde{g}$ is expressed by $\langle u, \widetilde{g}(v) \rangle = \langle \widetilde{g}(u), v \rangle$.

Now, we turn our attention to c. The *elasticity tensor*, c, is a fourth-rank tensor, namely a four-linear map $c : \mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3 \longrightarrow \mathbb{R}$ that satisfies the following conditions:

$$c(u, v, z, w) = c(v, u, z, w) = c(z, w, u, v), \tag{1}$$

for all $u, v, z, w \in \mathbb{E}^3$, and

$$c(u, v, u, v) \geq 0, \tag{2}$$

for all $u, v \in \mathbb{E}^3$, where

$$c(u, v, u, v) = 0, \tag{3}$$

if and only if $u = 0$ or $v = 0$. Note that the positive-definiteness of c is not pertinent to the discussion of the material symmetries.

A basis $\{e_1, e_2, e_3\}$ in $\mathbb{E}^3$ allows us to express the components of c with respect to this basis as

$$c_{ijkl} = c(e_i, e_j, e_k, e_l), \quad i, j, k, l \in \{1, 2, 3\}.$$

Thus, c has $3^4 = 81$ components, c_{ijkl}. With respect to these components, we can write tensor c as

$$c(u, v, z, w) = c_{ijkl} u^i v^j z^k w^l,$$

where $u = u^i e_i$, $v = v^i e_i$, $z = z^i e_i$ and $w = w^i e_i$. [1] If we denote the coordinate functions that correspond to the given basis as $\{x^1, x^2, x^3\}$, then we can write the elasticity tensor as

$$c = c_{ijkl} dx^i dx^j dx^k dx^l.$$

Using this coordinate expression, condition (1) can also be written as

$$c_{ijkl} = c_{jikl} = c_{klij}, \tag{4}$$

for all $i, j, k, l \in \{1, 2, 3\}$.

Due to the intrinsic symmetries stated by condition (4), we conclude that c has only twenty-one independent components. We choose to represent all these components by

$$\begin{gathered} 1c_{1111},\ 1c_{2222},\ 1c_{3333}, \\ 4c_{2323},\ 2c_{2233},\ 4c_{1212},\ 2c_{1122},\ 4c_{1313},\ 2c_{1133}, \\ 4c_{1123},\ 8c_{1213},\ 4c_{1233},\ 8c_{1323},\ 4c_{2213},\ 8c_{1223}, \\ 4c_{1222},\ 4c_{1112},\ 4c_{2223},\ 4c_{1113},\ 4c_{2333},\ 4c_{1333}. \end{gathered}$$

In this list, the number in front of each component corresponds to the number of components of c that the particular component represents. We will use this representation to describe material symmetries of the elasticity tensor.

2.2. **Symmetries of second-rank tensor.**

Any symmetry of a fourth-rank tensor translates to symmetries of associated second-rank tensors. Since it is more convenient to study symmetries of second-rank tensors, we turn our attention to these tensors.

To study symmetries of a tensor, we consider linear isomorphisms $A : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$; in other words, A and its inverse, A^{-1}, are linear maps. The isomorphism, A, is said to be an orthogonal transformation with respect to the Euclidean scalar product in $\mathbb{E}^3$ if the scalar product is preserved, namely,

$$\langle A(u), A(v) \rangle = \langle u, v \rangle, \quad \forall u, v \in \mathbb{E}^3.$$

The set of all orthogonal transformations of $\mathbb{E}^3$ with respect to the composition of maps is called the orthogonal group of $\mathbb{E}^3$ and denoted by $O(3)$.

Orthogonal group $O(3)$ acts on the space of nth-rank tensors through

$$(A, T(\cdot, \cdot, \ldots, \cdot)) \mapsto A * T(\cdot, \cdot, \ldots, \cdot) := T(A\cdot, A\cdot, \ldots, A\cdot). \tag{5}$$

Orthogonal transformation A of $\mathbb{E}^3$ is said to be a *symmetry* of an nth-rank tensor, T, if $A * T = T$, i.e.

$$T(Au_1, \cdots, Au_n) = T(u_1, \cdots, u_n), \quad \forall u_1, u_2, \cdots, u_n \in \mathbb{E}^3.$$

If orthogonal transformation A is a symmetry of tensor T, then we can say that this tensor is invariant under orthogonal transformation A.

The set of all symmetries for a given nth-rank tensor, T, is a subgroup G_T of orthogonal group $O(3)$. For an n-th rank tensor T and an orthogonal

[1] Unless stated otherwise, we use throughout the paper the summation convention of repeated indices.

transformation A, the symmetry groups of T and $A*T$ are *orthogonally conjugate*, i.e.

$$G_T = A^{-1} G_{A*T} A.$$

It is important to note that the symmetry group of any even-rank tensor contains the point symmetry, namely, $-I$. Thus, all tensors considered in this paper intrinsically possess the point symmetry.

To establish the symmetries of a given tensor, it is useful to find the basis with respect to which this tensor has the simplest possible form. For symmetric second-rank tensor $g : \mathbb{E}^3 \times \mathbb{E}^3 \longrightarrow \mathbb{R}$, there is an orthonormal basis $\{e_1, e_2, e_3\}$ with respect to which the components of g are $g_{ij} := g(e_i, e_j) = \lambda_i \delta_{ij}$ (no summation here). This is equivalent to saying that the induced linear map $\widetilde{g}$ has eigenvectors $\{e_1, e_2, e_3\}$; in other words, $\widetilde{g}(e_i) = \lambda_i e_i$. Consequently, if A is a symmetry of g, namely $g\left(Ae_i, Ae_j\right) = g\left(e_i, e_j\right)$, then $\widetilde{g}\left(Ae_i\right) = \lambda_i \left(Ae_i\right)$, since $g(e_i, e_j) = \lambda_i \delta_{ij}$. This can be stated as the following lemma.

Lemma 1. *An orthogonal transformation A is a symmetry of a symmetric second-rank tensor g, if and only if it preserves the eigenspaces that correspond to the eigenvalues of $\widetilde{g}$.*

Since any symmetry preserves the eigenspaces of a second-rank tensor, we can obtain useful information about the tensor symmetries by studying the eigenspaces of such a tensor. A symmetric second-rank tensor in $\mathbb{E}^3$ has three real eigenvalues, λ_i, where $i \in \{1, 2, 3\}$. There are, in general, three possibilities, namely, all three eigenvalues are distinct, two among the three eigenvalues are equal or all three eigenvalues are equal.

The largest possible symmetry groups of the second-rank tensor, g, corresponding to the three possibilities for eigenvalues of $\tilde{g}$ are stated as the following three types.

1): If all eigenvalues of $\tilde{g}$ are distinct; that is $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then $\tilde{g}$ has three one-dimensional eigenspaces. According to Lemma 1, the symmetry group of g is $G_g = \{A \in O\left(3\right), A\left(e_i\right) = \pm e_i, i = 1, 2, 3\}$. Consequently, $G_g = \{\pm I, \pm R_{e_i}, i \in \{1, 2, 3\}\}$, where R_{e_i} is the reflection about the plane that is orthogonal to vector e_i, $i \in \{1, 2, 3\}$.

2): If two eigenvalues of $\tilde{g}$ are equal to one another, say, $\lambda_1 = \lambda_2$, then $\tilde{g}$ has two eigenspaces, a two-dimensional space generated by e_1 and e_2, and a one-dimensional space generated by e_3. According to Lemma 1, the symmetry group of g is

$$G_g = \left\{A \in O\left(3\right), A\left(e_3\right) = \pm e_3, Ae_i = \alpha_i e_1 + \beta_i e_2, \alpha_i, \beta_i \in \mathbb{E}, \forall i \in \{1, 2\}\right\}.$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, one can express the elements of the symmetry group as follows.

(6)
$$G_g = \left\{ \pm \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \theta \in (-\pi, \pi] \right\}.$$

The first elements of the symmetry group are $\pm R_{\theta,e_3}$, which are the rotations by angle θ about vector e_3. The last elements are $\pm R_{u(\theta)}$, which are the reflections about the plane that is orthogonal to $u(\theta) = \sin(\theta/2)e_1 - \cos(\theta/2)e_2$.
Consequently, the symmetry group contains all rotations around e_3 and all reflections about planes that contain e_3. It coincides with $O(2)$, as a subgroup of $O(3)$.

3): If all eigenvalues of $\tilde{g}$ are equal to each other, that is, $\lambda_1 = \lambda_2 = \lambda_3$, then $\tilde{g}$ has one three-dimensional eigenspace. In other words, the whole space $\mathbb{E}^3$ is the eigenspace of $\tilde{g}$. Consequently, $G_g = O(3)$.

To distinguish between possible cases within the second and the third possibility, following Herman [15], we state another important property of tensor symmetries in the following theorem .

Theorem 2. *If an nth-rank tensor is invariant under an* $(n+k)$*-fold rotation* $(k \geq 1)$ *about a given axis, then it is invariant under any rotation about this axis.*

We shall use this theorem for second-rank and fourth-rank tensors. Specifically, for these tensors, we can state the following two corollaries.

Corollary 3. *If a second-rank tensor is invariant under rotation about a given axis by an angle smaller than* π*, then it is invariant under any rotation about this axis.*

Corollary 4. *If a fourth-rank tensor is invariant under rotation about a given axis by an angle smaller than* $\pi/2$*, then it is invariant under any rotation about this axis.*

To use the discussed properties of second-rank tensors for studying the symmetries of elasticity tensors, we choose to associate this fourth-rank tensor with particular second-rank tensors.

2.3. **Second-rank tensors associated with elasticity tensors.** To associate second-rank tensors with elasticity tensors c, we begin by defining two bi-linear maps, $\Gamma(u) : \mathbb{E}^3 \times \mathbb{E}^3 \longrightarrow \mathbb{R}$ and $\Delta(u) : \mathbb{E}^3 \times \mathbb{E}^3 \longrightarrow \mathbb{R}$ for any fixed $u \in \mathbb{E}^3$, by

$$\Gamma(u)(v,z) = c(u,v,u,z) \quad \text{and} \quad \Delta(u)(v,z) = c(u,u,v,z),$$

respectively. For each $u \in \mathbb{E}^3$, these two bi-linear maps are symmetric. For fixed basis $\{e_1, e_2, e_3\}$, the components of these two maps are given by

$$\Gamma_{ij}(u) = c(u,e_i,u,e_j) = c_{kilj}u^k u^l \quad \text{and} \quad \Delta_{ij}(u) = c(u,u,e_i,e_j) = c_{klij}u^k u^l.$$

Map $\Gamma(u)$ has also been used by Chadwick et al [3] and Ting [5] to study symmetries of elasticity tensors. We can consider bi-linear maps $\Gamma(u)$ and $\Delta(u)$ also as linear maps that map $\mathbb{E}^3$ to $\mathbb{E}^3$. The traces of these two linear maps define two maps $\mathcal{V}, \mathcal{D} : \mathbb{E}^3 \longrightarrow \mathbb{R}$, namely,

$$\mathcal{V}(u) = \operatorname{Tr}\{\Gamma(u)\} \quad \text{and} \quad \mathcal{D}(u) = \operatorname{Tr}\{\Delta(u)\}.$$

With respect to orthonormal basis $\{e_1, e_2, e_3\}$, we see that

$$\mathcal{V}(u) = \Gamma_{ii}(u) = c_{kili} u^k u^l \quad \text{and} \quad \mathcal{D}(u) = \Delta_{ii}(u) = c_{klii} u^k u^l.$$

$\mathcal{V}(u)$ and $\mathcal{D}(u)$ are quadratic forms. Consequently, to each of them corresponds a symmetric second-rank tensor. The components of these two tensors, with respect to orthonormal basis $\{e_1, e_2, e_3\}$, are given by

$$\mathcal{V}_{ij} = c_{i1j1} + c_{i2j2} + c_{i3j3} \quad \text{and} \quad \mathcal{D}_{ij} = c_{ij11} + c_{ij22} + c_{ij33}.$$

Tensors $\mathcal{V}$ and $\mathcal{D}$ and their eigenvectors have also been used by Chadwick et al [3], Cowin and Mehrabadi [11] to study symmetry classes of an elasticity tensor. The symmetry group of $\mathcal{V}$ – namely, $\{A \in O(3) | \mathcal{V}(Au) = \mathcal{V}(u), \forall u \in \mathbb{E}^3\}$ – is denoted by $G_{\mathcal{V}}$, while the symmetry group of $\mathcal{D}$ – namely, $\{A \in O(3) | \mathcal{D}(Au) = \mathcal{D}(u), \forall u \in \mathbb{E}^3\}$ – is denoted by $G_{\mathcal{D}}$.

In the next section, we shall classify all the symmetry groups of elasticity tensors using the symmetry groups of tensors $\mathcal{V}$ and $\mathcal{D}$, as well as using the symmetries of the two maps, $\Gamma(u)$ and $\Delta(u)$.

2.4. **Possible material symmetries of elasticity tensors.** In this section, we show that an elasticity tensor can belong to one among eight distinct symmetry classes.

Orthogonal transformation A is said to be a *symmetry of an elasticity tensor*, c, if

$$c(Au, Av, Aw, Az) = c(u, v, w, z), \qquad \forall u, v, w, z \in \mathbb{E}^3.$$

The set of all symmetries of tensor c is a subgroup of the orthogonal group, $O(3)$. We denote it by G_c and we refer to it as the symmetry group of the elasticity tensor.

Two elasticity tensors, c_1 and c_2, belong to the same *symmetry class* if their symmetry groups, G_{c_1} and G_{c_2}, are orthogonally conjugate, i.e. there is an orthogonal transformation A such that $G_{c_1} = A^{-1} G_{c_2} A$ or $G_{c_1} = G_{A * c_2}$.

Since the elasticity tensor is an even-rank tensor, its symmetry group always contains the point symmetry of $\mathbb{E}^3$. This means that $-I \in G_c$. Consequently, the symmetry class of an elasticity tensor is determined by only a subgroup of the rotation group $SO(3)$, rather than the entire $O(3)$.

Since we can express tensor c using map Δ as

$$c(u, v, w, z) = \Delta\left(\frac{u+v}{2}\right)(w, z) - \Delta\left(\frac{u-v}{2}\right)(w, z),$$

orthogonal transformation A is a symmetry of c, if and only if

$$\Delta(Au)(Av, Aw) = \Delta(u)(v, w), \quad \forall u, v, w \in \mathbb{E}^3.$$

For map $\Gamma(u)$, we can only state that if A is a symmetry of c, then:

$$\Gamma(Au)(Av, Aw) = \Gamma(u)(v, w), \quad \forall u, v, w \in \mathbb{E}^3.$$

These statements also show that if $Au = \pm u$ and A is a symmetry of c, then A is a symmetry of the two bi-linear transformations, $\Gamma(u)$ and $\Delta(u)$, namely,

$$\Gamma(u)(Av, Aw) = \Gamma(u)(v, w) \text{ and } \quad \Delta(u)(Av, Aw) = \Delta(u)(v, w), \quad \forall v, w \in \mathbb{E}^3.$$

We can reconstruct c from Δ. However, c cannot be reconstructed from $\mathcal{V}$ and $\mathcal{D}$. This implies that any symmetry of c is a symmetry of both $\mathcal{V}$ and $\mathcal{D}$, whereas a symmetry of $\mathcal{V}$ or $\mathcal{D}$ is not necessarily a symmetry of c. Hence, we can state the following result, that has been proved also by Forte and Vianello in [4].

Lemma 5. *Symmetry group G_c of an elasticity tensor c is a subgroup of the group $G_{\mathcal{V}} \cap G_{\mathcal{D}}$.*

In other words, G_c can be, at most, $G_{\mathcal{V}} \cap G_{\mathcal{D}}$. We note that throughout this paper we use the expression "at most" to state the fact that an elasticity tensor c cannot have more symmetries than both tensors $\mathcal{V}$ and $\mathcal{D}$ possess.

The fact that G_c can be, at most, $G_{\mathcal{V}} \cap G_{\mathcal{D}}$ is the reason to study all possible intersections of the two groups, $G_{\mathcal{V}}$ and $G_{\mathcal{D}}$. Since tensors $\mathcal{V}$ and $\mathcal{D}$ are second-rank symmetric tensors, according to Section 2.2, there are three possible types of symmetry groups for each of these two tensors. To study these groups, in view of Lemma 1, we turn our attention to the eigenspaces of $\mathcal{V}$ or $\mathcal{D}$.

Let us denote the eigenvalues of tensors $\mathcal{V}$ and $\mathcal{D}$ by $\{\lambda_i^{\mathcal{V}}\}_{i=1,2,3}$ and $\{\lambda_i^{\mathcal{D}}\}_{i=1,2,3}$, respectively. We also consider the unit eigenvectors, $\{e_i^{\mathcal{V}}\}_{i=1,2,3}$ and $\{e_i^{\mathcal{D}}\}_{i=1,2,3}$, and the eigenspaces, $L(e_i^{\mathcal{V}})$ and $L(e_i^{\mathcal{D}})$, that correspond to these two tensors. Now, let us consider the four possible possibilities of intersections of eigenspaces of $\mathcal{V}$ and $\mathcal{D}$, depending on the dimension of the intersection.

(1) **The intersections of any eigenspace of $\mathcal{D}$ with any eigenspace of $\mathcal{V}$ are zero-dimensional**. In other words, $L(e_i^{\mathcal{V}}) \cap L(e_j^{\mathcal{D}}) = \{0\}$, $\forall i, j \in \{1, 2, 3\}$, and hence $G_{\mathcal{V}} \cap G_{\mathcal{D}} = \{\pm I\}$. Since $\{\pm I\} \subset G_c \subset G_{\mathcal{V}} \cap G_{\mathcal{D}} = \{\pm I\}$, we conclude that $G_c = \{\pm I\}$. Hence, c is a *generally-anisotropic* tensor.

(2) **The intersection of any eigenspace of $\mathcal{D}$ with any eigenspace of $\mathcal{V}$ is at most one-dimensional.** Thus the symmetry group of c is a group that preserves the one-dimensional eigenspace(s) and may contain one or three reflections about orthogonal planes. Within this possibility, we can distinguish the following two cases.
   (a) If the eigenspaces of $\mathcal{V}$ and the eigenspaces of $\mathcal{D}$ have in common only a single one-dimensional space, say spanned by e_3, then $G_c \subset G_{\mathcal{V}} \cap G_{\mathcal{D}} = \{\pm I, \pm R_{e_3}\}$. Hence, in view of Lemma 5, c is, at most, a *monoclinic* tensor.
   (b) If the intersection of the eigenspaces of $\mathcal{V}$ and the eigenspaces of $\mathcal{D}$ are three one-dimensional spaces, spanned by e_1, e_2 and e_3, then $G_c \subset G_{\mathcal{V}} \cap G_{\mathcal{D}} = \{\pm I, \pm R_{e_1}, \pm R_{e_2}, \pm R_{e_3}\}$. Hence, c is, at most, an *orthotropic* tensor.

Note that the existence of two reflection planes together with the point symmetry implies the existence of the third reflection plane. Hence, there is no intermediate case between the monoclinic and orthotropic cases.

(3) **The intersection of some eigenspace of $\mathcal{D}$ with some eigenspace of $\mathcal{V}$ is two-dimensional.** We can choose the coordinate system such that the common one-dimensional eigenspace is spanned by e_3 and the two-dimensional eigenspace is spanned by $\{e_1, e_2\}$. Then, following expression (6), we see that $G_{\mathcal{V}} \cap G_{\mathcal{D}} = O(2)$. In view of Lemma 5, symmetry group G_c is a subgroup of $O(2)$ and may contain rotations about e_3 as well as reflections about planes that contain e_3. Then, using Corollary 4, there are the following three possible cases for the symmetries of c.

(a) If $G_c \subset \left\{\pm I, \pm R_{u_\alpha}, \pm R_{\pm 2\pi/3, e_3}, \alpha \in \{1,2,3\}\right\} \subset G_{\mathcal{V}} \cap G_{\mathcal{D}}$, which means that G_c may contain a rotation by $\theta = 2\pi/3$ about e_3 and reflections about three planes that contain the axis of rotation, then c is, at most, a *trigonal* tensor. The angle between two reflection planes is $2\pi/3$. One can choose e_2 to be orthogonal to one plane of reflection.
For a trigonal tensor the symmetry group does not contain R_{e_3} and $-R_{e_3}$. We can see this from the fact that rotation $R_{2\pi/3, e_3}$ combined with reflection R_{e_3} would result in rotation by $\pi/3$ about e_3 and, hence – by Corollary 4 – such a tensor would be transversely isotropic, which is a class discussed below.
We shall see in Section 3 that if an elasticity tensor c is invariant under $R_{\pm 2\pi/3}$, then it is also invariant under reflections about three planes that contain the axis of rotation.

(b) If $G_c \subset \left\{\pm I, \pm R_{\pm\pi/2, e_3}, \pm R_{\pi, e_3}, \pm R_{u_\alpha}, \alpha \in \{1,2,3,4\}\right\} \subset G_{\mathcal{V}} \cap G_{\mathcal{D}}$, which means that G_c may contain a rotation by $\theta = \pi/2$ about e_3 and reflections about four planes that contain the axis of rotation, then c is, at most, a *tetragonal* tensor. The angle between consecutive reflection planes is $\pi/4$. One can choose e_1, e_2 to be orthogonal to two planes of reflection.
Note that the symmetry group of a tetragonal tensor contains R_{e_3} and $-R_{e_3}$. We can see this from the fact that $R_{e_3} = -R_{\pi, e_3}$ and $-R_{e_3} = R_{\pi, e_3}$.
We shall see in Section 3 that if an elasticity tensor c is invariant under $R_{\pm\pi/2, e_3}$, then it is invariant also under four reflections $R_{u_\alpha}, \alpha \in \{1,2,3,4\}$ about four planes that contain the axis of rotation.

(c) If $G_c \subset G_{\mathcal{V}} \cap G_{\mathcal{D}} = O(2)$, which means that G_c may contain any other rotations by $\theta \in (-\pi, \pi]$ about e_3 and reflections about planes that contain e_3, then c is, at most, a *transversely-isotropic* tensor.

(4) **The intersection of the eigenspace of $\mathcal{D}$ with the eigenspace of $\mathcal{V}$ is three-dimensional.** In this possibility, the symmetry groups of tensors $\mathcal{V}$ and $\mathcal{D}$ coincide with the orthogonal group $O(3)$ and $G_{\mathcal{V}} \cap G_{\mathcal{D}} = O(3)$. Hence, the symmetry group, G_c, of an elasticity tensor c is a subgroup of $O(3)$ that may contain rotations about different axes. In such a case, as we shall see in Section 3, there are only two possible subgroups of $O(3)$, which are symmetry groups for the elasticity tensor c. These two groups correspond to the cubic and the isotropic cases. They are as follows.
   (a) If $G_c \subset \{A \in O(3), A(e_i) = \pm e_j, i, j \in \{1,2,3\}\}$, then c is, at most, a *cubic* tensor.
   (b) If $G_c \subset O\,(3)$, then c is, at most, an *isotropic* tensor.

This completes the classification of all the possible symmetries of the elasticity tensor. All classes are given by 1, 2(a), 2(b), 3(a), 3(b), 3(c), 4(a) and 4(b). These consist of both discrete and continuous symmetry classes. While the discrete classes have their analogies in crystallography, the continuous ones, namely transverse isotropy and isotropy, are specific to continuum mechanics.

3. Symmetry recognition of given elasticity tensor

In order to recognize to which symmetry class any given elasticity tensor belongs, we use a method that results in expressing c in the natural basis where it has the simplest possible form.

To distinguish between the cases within the four possibilities of possible intersections of eigenspaces, discussed in the previous section, we have to study how $c\,(e_i, e_j, e_k, e_l)$ transforms under the given symmetry. Since there are only three basis vectors $\{e_1, e_2, e_3\}$ and c is a fourth-rank tensor, instead of studying transformations of c, we can equivalently study transformations of both $\Gamma\,(e_i)\,(e_j, e_k)$ and $\Delta\,(e_i)\,(e_j, e_k)$. Note that under transformations that preserve u, bilinear maps $\Gamma(u)$ and $\Delta(u)$ behave as second-rank tensors and, hence, we can use the methods developed above to study their symmetries that preserve a fixed u. Namely, in view of Lemma 1, we are going to study the eigenspaces of these maps, which must be preserved under such symmetries. Following the same order as in Section 2.4, we will discuss each of the four possibilities in detail.

(1) **The intersection of any eigenspace of $\mathcal{D}$ with any eigenspace of $\mathcal{V}$ is zero-dimensional.** This is the case of the *generally-anisotropic tensor*. In this case, there is nothing left to decide, since the symmetry group of the elasticity tensor is $G_c = \{\pm I\}$ – the elasticity tensor has twenty-one nonzero independent components and it is a generally-anisotropic tensor.
(2) **The intersection of any eigenspace of $\mathcal{D}$ with any eigenspace of $\mathcal{V}$ is at most one-dimensional.** Within this possibility, we can distinguish the following cases.

(a) *Monoclinic tensor*: Let $\{e_1, e_2, e_3\}$ be an arbitrary orthonormal basis such that e_3 is a common eigenvector of $\mathcal{V}$ and $\mathcal{D}$. The condition that $\mathcal{V}$ and $\mathcal{D}$ have the same eigenvector e_3 can be expressed as

$$\begin{aligned} \mathcal{V}_{13} &= c_{1113} + c_{1223} + c_{1333} = 0, \\ \mathcal{V}_{23} &= c_{1213} + c_{2223} + c_{2333} = 0, \end{aligned} \tag{7}$$

and as

$$\begin{aligned} \mathcal{D}_{13} = c_{1113} + c_{1322} + c_{1333} &= 0, \\ \mathcal{D}_{23} = c_{1123} + c_{2223} + c_{2333} &= 0. \end{aligned} \tag{8}$$

We remark here that equations (7) are equivalent to $G_{\mathcal{V}} = \{\pm I, \pm R_{e_3}\}$, while equations (8) are equivalent to $G_{\mathcal{D}} = \{\pm I, \pm R_{e_3}\}$. If the symmetries of c contain R_{e_3}, then — since any symmetry of a second-rank tensor preserves its eigenspaces and $R_{e_3} e_3 = -e_3$, e_3 defines an eigenspace of $\Gamma(e_3)$, and according to Lemma 1 — we see that $G_{\Gamma(e_3)} = \{\pm I, \pm R_{e_3}\}$. This implies that

$$c_{1333} = c_{2333} = 0. \tag{9}$$

Note that following Lemma 1, these equalities are true also for any symmetries that preserve $\pm e_3$.
We remark that we also get equations (9) if we require e_3 to be an eigenvector for $\Delta(e_3)$. Similarly, if R_{e_3} is a symmetry of c, then R_{e_3} also preserves the vectors $\{e_1, e_2\}$; consequently e_3 is an eigenvalue of $\Gamma(e_1)$. Note that this is equivalent to saying that e_3 is also an eigenvalue of $\Gamma(e_2)$ or $\Delta(e_1)$ or $\Gamma(e_2)$. Any of these conditions can be written as

$$c_{1113} = c_{1213} = 0. \tag{10}$$

Equations (7) through (10) imply that

$$c_{1113} = c_{1123} = c_{1322} = c_{2223} = c_{1333} = c_{2333} = c_{1213} = c_{1223} = 0.$$

The last equations are equivalent to

$$G_c = G_{\mathcal{V}} = G_{\mathcal{D}} = G_{\Gamma(e_i)} = G_{\Delta(e_i)} = \{\pm I, \pm R_{e_3}\}, \forall i = 1, 2, 3.$$

Consequently, a monoclinic tensor has thirteen nonzero independent components. With respect to an orthonormal basis $\{e_1, e_2, e_3\}$, where e_3 is the common eigenvector of $\mathcal{V}$ and $\mathcal{D}$ and e_1, e_2 are arbitrary, the components of a monoclinic tensor are given by

$$\begin{array}{llll} c_{1111}, & c_{2222}, & c_{3333}, & \\ c_{1122}, & c_{1133}, & c_{2233}, & \\ c_{1212}, & c_{1313}, & c_{2323}, & \\ c_{1112}, & c_{1222}, & c_{1233}, & c_{1323}. \end{array} \tag{11}$$

One can rotate $\{e_1, e_2, e_3\}$ by an angle θ around e_3 such that

$$\tan(2\theta) = \frac{2c_{1323}}{c_{2323} - c_{1313}}.$$

With respect to the new basis, c has twelve components, since c_{1323} vanishes. This simpler form of c does not bring any new information about the tensor, as we shall also see later for trigonal and tetragonal cases.

(b) *Orthotropic tensor*: Let e_1, e_2 and e_3 be the orthonormal basis of common eigenvectors of tensors $\mathcal{V}$ and $\mathcal{D}$. The fact that vectors e_1, e_2 and e_3 are eigenvectors for both tensors $\mathcal{V}$ and $\mathcal{D}$ results in

$$\begin{array}{rcl} \mathcal{V}_{13} = c_{1113} + c_{1223} + c_{1333} & = & 0, \\ \mathcal{V}_{23} = c_{1213} + c_{2223} + c_{2333} & = & 0, \\ \mathcal{V}_{12} = c_{1112} + c_{1222} + c_{1323} & = & 0 \end{array} \tag{12}$$

and

$$\begin{array}{rcl} \mathcal{D}_{13} = c_{1113} + c_{1322} + c_{1333} & = & 0, \\ \mathcal{D}_{23} = c_{1123} + c_{2223} + c_{2333} & = & 0, \\ \mathcal{D}_{12} = c_{1112} + c_{1222} + c_{1233} & = & 0. \end{array} \tag{13}$$

We remark here that equations (12) are equivalent to $G_{\mathcal{V}} = \{\pm I, \pm R_{e_1}, \pm R_{e_2}, \pm R_{e_3}\}$, while equations (13) are equivalent to $G_{\mathcal{D}} = \{\pm I, \pm R_{e_1}, \pm R_{e_2}, \pm R_{e_3}\}$. If the symmetry group of c contains all reflections, R_{e_1}, R_{e_2} and R_{e_3}, then, by a similar argument as in case 2(a), e_1, e_2 and e_3 are eigenvectors of $\Gamma(e_i)$ for all $i = 1, 2, 3$. Equivalently,

$$c_{1112} = c_{1113} = c_{1222} = c_{2223} = c_{1333} = c_{2333} = 0. \tag{14}$$

Note that the requirement for e_1, e_2 and e_3 to be eigenvectors of $\Delta(e_i)$ for all $i \in \{1, 2, 3\}$ gives the same result. Then equations (14) are equivalent to

$$G_{\Gamma(e_i)} = G_{\Delta(e_i)} = \{\pm I, \pm R_{e_1}, \pm R_{e_2}, \pm R_{e_3}\},$$

for all $i = 1, 2, 3$.

In this case, six equations given by expressions (12) and (13) as well as equalities (14) imply that the elasticity tensor has only nine nonzero independent components with respect to basis $\{e_1, e_2, e_3\}$ of common eigenvector of $\mathcal{V}$ and $\mathcal{D}$. Such an elasticity tensor is an orthotropic tensor. In such a case, the symmetry group is given by:

$$G_c = G_{\mathcal{V}} = G_{\mathcal{D}} = G_{\Gamma(e_i)} = G_{\Delta(e_i)} = \{\pm I, \pm R_{e_1}, \pm R_{e_2}, \pm R_{e_3}\}.$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, where e_1, e_2, e_3 are common eigenvectors of $\mathcal{V}$ and $\mathcal{D}$, the components of an orthotropic tensor are given by:

$$\begin{array}{lll} c_{1111}, & c_{2222}, & c_{3333}, \\ c_{1122}, & c_{1133}, & c_{2233}, \\ c_{1212}, & c_{1313}, & c_{2323}. \end{array} \tag{15}$$

(3) **The intersection of some eigenspace of $\mathcal{D}$ with some eigenspace of $\mathcal{V}$ is two-dimensional.** We can choose the coordinate system such that the common one-dimensional eigenspace is spanned by e_3 and the two-dimensional eigenspace is spanned by two arbitrarily orthonormal vectors $\{e_1, e_2\}$. Consequently, with respect to orthonormal basis $\{e_1, e_2, e_3\}$, both tensors $\mathcal{V}$ and $\mathcal{D}$ have a diagonal form. This translates to the fact that equations (12) and (13) are satisfied.

As the two tensors have in common a two-dimensional eigenspace, the eigenvalues of $\mathcal{V}$ that correspond to eigenvectors e_1 and e_2 are equal to one another. The same is true for $\mathcal{D}$. This imposes

$$c_{1111} + c_{1212} + c_{1313} = \mathcal{V}(e_1, e_1) = \mathcal{V}(e_2, e_2) = c_{1212} + c_{2222} + c_{2323}. \tag{16}$$

Similarly, for $\mathcal{D}$, we can write

$$c_{1111} + c_{1122} + c_{1133} = \mathcal{D}(e_1, e_1) = \mathcal{D}(e_2, e_2) = c_{1122} + c_{2222} + c_{3322}. \tag{17}$$

Equations (12) and (16) imply that $G_{\mathcal{V}} = O(2)$, while equations (13) and (17) imply that $G_{\mathcal{D}} = O(2)$. In this case, the only possible symmetries leaving c invariant are rotations about e_3. The invariance of e_3 under these symmetries implies that e_3 is an eigenvector of $\Gamma(e_3)$ and $\Delta(e_3)$. This means that equations (9) are satisfied.

Let us examine how rotations about e_3 act on c. For vectors e_1 and e_2, we have

$$\begin{aligned} R_{\theta, e_3} e_1 &= \cos\theta e_1 + \sin\theta e_2 \\ R_{\theta, e_3} e_2 &= \sin\theta e_1 - \cos\theta e_2. \end{aligned} \tag{18}$$

Vector e_3 is an eigenvector for rotations (18). For a trigonal, tetragonal and transversely isotropic tensor, we need rotations (18) to be symmetries for bilinear maps $\Gamma(e_3)$ and $\Delta(e_3)$. This means that the symmetry groups of these maps are given by $G_{\Gamma(e_3)} = G_{\Delta(e_3)} = O(3)$. First, let us note that the invariance of each among $\Gamma(e_3)(e_1, e_3)$, $\Gamma(e_3)(e_2, e_3)$, $\Delta(e_3)(e_1, e_3)$ and $\Delta(e_3)(e_2, e_3)$ results in equation (9). Next, let us study the invariance of the remaining entities associated with $\Gamma(e_3)$ and $\Delta(e_3)$.

The invariance of $\Gamma(e_3)(e_1, e_1)$ under rotations (18) imposes

$$c_{1323} = 0 \text{ and } c_{1313} = c_{2323}. \tag{19}$$

Analogous computations for $\Delta(e_3)(e_1, e_1)$ give us

$$c_{1233} = 0 \text{ and } c_{1133} = c_{2233}. \tag{20}$$

At this moment, we have six equations given by expressions (12) and (13), four equations given by expressions (16), (17), (9) and four equations given by expressions (19) and (20). Hence, there are thirteen equations. However, only twelve among them are independent. All these equations are true for the trigonal, tetragonal and transversely isotropic cases. Note that the symmetry group of an elasticity tensor that satisfies the above-mentioned twelve independent equations satisfies

$$G_c \subset G_{\mathcal{V}} = G_{\mathcal{D}} = G_{\Gamma(e_3)} = G_{\Delta(e_3)} = O(2).$$

To distinguish between these cases, we need to express the following invariances.

(a) *Trigonal tensor*: For this case, $\Gamma(e_1)$ has to be invariant under rotations (18) for $\theta \in \{2\pi/3, 4\pi/3\}$.
The invariance of $\Gamma(e_1)(e_2, e_2)$ under rotations (18) with $2\pi/3$ and $4\pi/3$ for θ, implies

$$c_{1112} = 0 \text{ and } 2c_{1212} = c_{1111} - c_{1122}. \tag{21}$$

Equations (9), (12), (13), (21) and equations (16) through (20) imply that c has – with respect to orthonormal basis $\{e_1, e_2, e_3\}$ – seven nonzero independent components. These components are given by

$$\begin{gathered} c_{1111} = c_{2222},\ c_{3333}, \\ c_{1122},\ c_{1133} = c_{2233}, \\ c_{1212} = \tfrac{1}{2}(c_{1111} - c_{1122}),\ c_{1313} = c_{2323}, \\ c_{1123} = -c_{2223} = c_{1213},\ c_{1113} = -c_{1322} = -c_{1223}. \end{gathered} \tag{22}$$

One can rotate the orthonormal basis $\{e_1, e_2, e_3\}$ by angle θ around e_3, such that

$$\tan(3\theta) = \frac{c_{1123}}{c_{1113}}. \tag{23}$$

With respect to the new basis, c_{1123} vanishes. The six nonzero independent components are given by expression (22). One can check by direct calculation that an elasticity tensor that has the components given by expression (22) with $c_{1123} = 0$ is invariant under reflections R_u, namely,

$$\begin{aligned} R_u(e_1) &= \cos\theta e_1 + \sin\theta e_2 \\ R_u(e_2) &= \sin\theta e_1 - \cos\theta e_2, \end{aligned} \tag{24}$$

where $u = \sin(\theta/2)e_1 - \cos(\theta/2)e_2$, if and only if $\theta \in \{0, \pm 2\pi/3\}$. Since the invariance of the elasticity tensor under orthogonal transformation is independent of the coordinate system choice, we conclude that for the trigonal case the symmetry group contains also three reflections.

An elasticity tensor that has the components given by conditions (22) with $c_{1123} = 0$ has the symmetry group given by

$$G_c = \left\{ \pm I, \pm R_{u_\alpha}, \pm R_{\pm 2\pi/3, e_3}, \alpha \in \{1,2,3\} \right\},$$

where the three vectors $u_\alpha, \alpha \in \{1,2,3\}$ are orthogonal to e_3 and the angle between any two of them is $2\pi/3$.

We remark here that we could choose the angle of rotation of the basis $\{e_1, e_2, e_3\}$ to be such that

$$\tan(3\theta) = \frac{c_{1113}}{c_{1123}}.$$

In this case with respect to the new basis, the components of the elasticity tensor c are given by expression (22) with $c_{1113} = 0$. In this case we can draw analogous conclusions as in the case when θ is given by expression (23).

(b) *Tetragonal tensor*: For this case, $\Gamma(e_1)$ and $\Delta(e_1)$ have to be invariant under rotation (18) for $\theta = \pi/2$.

The invariance of $\Delta(e_1)(e_2, e_3)$ under rotations (18) for $\theta = \pi/2$, implies

$$c_{1213} = c_{1223} \text{ and } c_{1123} = c_{1322}. \tag{25}$$

Equations (9), (12), (13), equations (16) through (20), as well as equation (25) imply that c is a tetragonal tensor. With respect to the orthonormal basis, tensor c has seven nonzero independent components given by

$$\begin{gathered} c_{1111} = c_{2222},\ c_{3333}, \\ c_{1122},\ c_{1133} = c_{2233}, \\ c_{1212},\ c_{1313} = c_{2323}, \\ c_{1112} = -c_{1222}. \end{gathered} \tag{26}$$

One can rotate the orthonormal basis $\{e_1, e_2, e_3\}$ by angle θ around e_3, such that

$$\tan(4\theta) = \frac{-4c_{1112}}{2c_{1212} - c_{1111} + c_{1122}}. \tag{27}$$

With respect to the new basis, c_{1112} vanishes and the six nonzero independent components are given by the first three lines of expression (26). One can check by direct calculation that an elasticity tensor that has the components (26) with $c_{1112} = 0$ is invariant under reflections (24), if and only if $\theta \in \{0, \pm\pi/2, \pi\}$. Since the invariance of c under orthogonal transformation is independent of the coordinate system choice, we conclude that

for the trigonal case, the symmetry group contains also four reflections about planes that contain e_3, given by expression (24), where $\theta \in \{0, \pm\pi/2, \pi\}$.
With respect to the new basis, $e_1$and e_2 are orthogonal to two reflection planes. This implies that $R_{e_1}, R_{e_2} \in G_c$. Both of these imply that $c_{1112} = -c_{1222} = 0$. A tetragonal elasticity tensor is also invariant under $R_{e_3} = -R_{\pi,e_3}$.
An elasticity tensor whose components with respect to the orthonormal basis are given by the first three lines of (26) has the symmetry group given by

$$G_c = \left\{\pm I, \pm R_{\pm\pi/2,e_3}, \pm R_{\pi,e_3}, \pm R_{u_\alpha}, \alpha \in \{1,2,3,4\}\right\}.$$

(c) *Transversely isotropic tensor*: For this case, all the above equations have to be satisfied. Equations (9), (12), (13), equations (16) through (20), and equations (21) and (25) imply that c describes a transversely isotropic tensor. With respect to the orthonormal basis, tensor c has five nonzero independent components given by

$$\begin{gathered} c_{1111} = c_{2222},\ c_{3333}, \\ c_{1122},\ c_{1133} = c_{2233}, \\ c_{1212} = \tfrac{1}{2}(c_{1111} - c_{1122}),\ c_{1313} = c_{2323}. \end{gathered} \tag{28}$$

An elasticity tensor whose components with respect to the orthonormal basis are given by conditions (28) has the symmetry group given by

$$G_c = G_{\mathcal{V}} \cap G_{\mathcal{D}} \cap G_{\Gamma(e_3)} \cap G_{\Delta(e_3)} = O(2).$$

(4) **The intersection of the eigenspace of $\mathcal{D}$ with the eigenspace of $\mathcal{V}$ is three-dimensional.** This means that, with respect to any orthonormal basis, $\{e_1, e_2, e_3\}$, these tensors are diagonal. In this case, equations (12)) and (13) are satisfied. This also implies that $\mathcal{V}_{11} = \mathcal{V}_{22} = \mathcal{V}_{33}$; in other words,

$$\begin{gathered} c_{1111} + c_{1313} = c_{2222} + c_{2323}, \\ c_{1212} + c_{2222} = c_{1313} + c_{3333}. \end{gathered} \tag{29}$$

Similar equations apply to tensor $\mathcal{D}$, which means that $\mathcal{D}_{11} = \mathcal{D}_{22} = \mathcal{D}_{33}$, and which can be written as

$$\begin{gathered} c_{1111} + c_{1133} = c_{2222} + c_{2233}, \\ c_{1122} + c_{2222} = c_{1133} + c_{3333}. \end{gathered} \tag{30}$$

Also, each vector e_i from the orthonormal basis is an eigenvector for $\Gamma(e_i)$ and $\Delta(e_i)$. This implies that equations (14) are also satisfied. Equations (12) and (29) imply that $G_{\mathcal{V}} = O(3)$, while equations (13) and (30) imply that $G_{\mathcal{D}} = O(3)$.
At this moment we have sixteen equations (12), (13), (14), (29) and (30). If an elasticity tensor c is invariant under a rotation by any

angle except $\pi/2$, then – in view of the above equations – we see that c has isotropic symmetry and its components are given by expressions (34), below.

(a) *Cubic tensor*: Since cubic tensor is invariant under tetragonal symmetries, we can consider only the nonzero coefficients given by expressions (26). Considering rotations about the e_1 axis by an arbitrary angle, we can use equations equivalent to (19) and (20), from which we obtain

$$c_{1313} = c_{1212} \text{ and } c_{1122} = c_{1133}. \tag{31}$$

If we use equations (29) or (30), as well as equations (26) and (31), we also see that $c_{1111} = c_{2222} = c_{3333}$. Thus, with respect to any orthonormal basis $\{e_1, e_2, e_3\}$, a cubic tensor c has three nonzero independent components, which are given by

$$\begin{aligned} c_{1111} &= c_{2222} = c_{3333}, \\ c_{1122} &= c_{1133} = c_{2233}, \\ c_{1212} &= c_{1313} = c_{2323}. \end{aligned} \tag{32}$$

The symmetry group of a cubic tensor is given by:

$$G_c = \{A \in O(3), A(e_i) = \pm e_j, i, j \in \{1, 2, 3\}\}.$$

Note that the cubic-symmetry group has forty-eight elements which is the full symmetry group of a cube.

(b) *Isotropic tensor*: In this case, all the equations considered above must be satisfied. The only remaining condition to consider is equation (21), which, in this case, reduces to

$$2c_{1212} = c_{1111} - c_{1122}. \tag{33}$$

In this case, with respect to any orthonormal basis $\{e_1, e_2, e_3\}$, an isotropic tensor c has only two nonzero independent components given by

$$\begin{aligned} c_{1111} = c_{2222} = c_{3333} &= 2c_{1212} + c_{1122}, \\ c_{1122} &= c_{1133} = c_{2233}, \\ c_{1212} &= c_{1313} = c_{2323}. \end{aligned} \tag{34}$$

The symmetry group of an isotropic tensor is given by

$$G_c = O(3).$$

One can check directly that if we add any other symmetry to the symmetry group of cubic tensor then equation (33) is satisfied. Consequently there is no intermediate case between cubic symmetry and isotropic symmetry.

This completes the process by which any given elasticity tensor is recognized as belonging to a particular symmetry class. All symmetry classes are described in 1, 2(a), 2(b), 3(a), 3(b), 3(c), 4(a) and 4(b).

4. Discussion

We note that there are several distinct routes of increasing symmetries of an elasticity tensor. Yet, each of these routes commences with general anisotropy and finishes with isotropy. We recall that if an elasticity tensor is, at most, of a particular symmetry class, then this tensor possesses all the symmetries that are connected to its class from below, as illustrated in Figure 1.

The symmetry classes of an elasticity tensor are presented in this figure on four levels that correspond to the four possibilities discussed in Section 3.

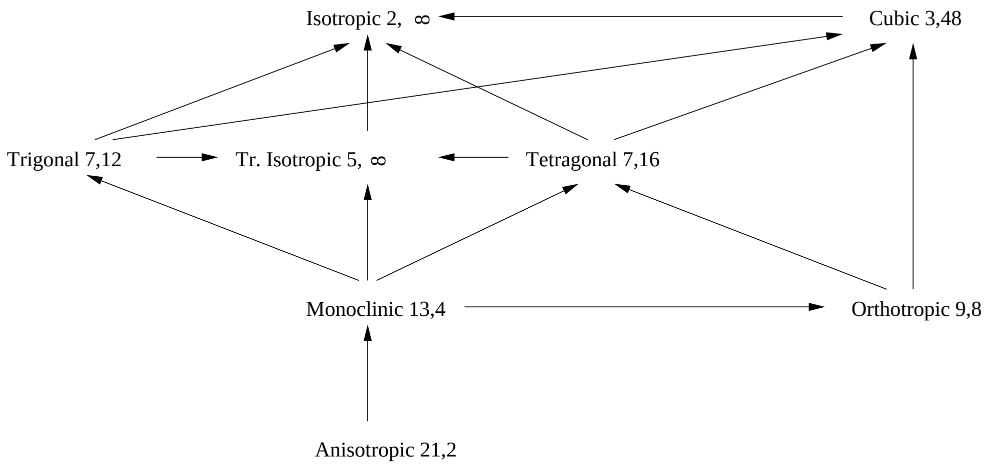

Figure 1. Possible symmetry classes of the elasticity tensor.

In this figure the first number after the name of a symmetry class refers to the number of independent coefficients the given tensor possesses, whereas the second number is the number of elements of the corresponding symmetry group.

It is interesting to note that the tetragonal and trigonal tensors possess the same number of independent components, but refer to two distinct symmetry classes with distinct number of symmetries.

Also note, that there are subgroups of the trigonal and tetragonal symmetry groups that do not contain reflections. These subgroups are sometimes considered as different symmetries for an elasticity tensor. However, an elasticity tensor that is symmetric under these subgroups is identical to an elasticity tensor that is symmetric under the bigger groups. Hence, one cannot distinguish between these groups from the point of view of the elasticity tensor.

5. Acknowledgements

We would like to acknowledge fruitful discussions with Peter Gibson, Klaus Helbig and Misha Kotchetov and the valuable editorial work of Cathy Beveridge. Also, helpful comments of the referees as well as the guidance of Paul Martin, the executive editor, are gratefully acknowledged.

This study has been done in the context of The Geomechanics Project. Hence, the authors wish to acknowledge the support of EnCana Energy and Husky Energy.

References

[1] A.E.H. Love, *A treatise on the mathematical theory of elasticity*, Dover, 1892.

[2] W. Voigt, *Lehrbuch der kristallphysics*, Teubner, 1928.

[3] P. Chadwick, S. C. Cowin, and M. Vianello, A new proof that the number of linear elastic symmetries is eight, *J. Mech. Phys. Solids*, **49**(2001), 2471-2492.

[4] S.Forte and M. Vianello, Symmetry classes and harmonic decomposition for photoelasticity tensors, *Internat. J. Engrg. Sci.*, **35**(1997), no 14, 1317-1326.

[5] T.C.T. Ting, *Generalized Cowin-Mehrabadi theorems and a direct proof that the number of linear elastic symmetries is eight*, International Journal of Solids and Structures, **40**(2003), 7129-7142.

[6] G.Backus, A Geometrical Picture of Anisotropic Elastic Tensors, *Rev. Geophys.* **8**(1970), no.3, 633-671.

[7] R. Baerheim, Classification of symmetry by means of Maxwell multipoles, *Quart. J. Mech. Appl. Math.*, **51**(1998), 73-103.

[8] R. Baerheim, Harmonic decomposition of the anisotropic elasticity tensor, *Quart. J. Mech. Appl. Math.*, **46**(1993), no.3, 391-418.

[9] S.C. Cowin and M.M. Mehrabadi, Anisotropic symmetries of linear elasticity, *Appl.Mech. Rev.*, **48**(1995), 247-285.

[10] S.C. Cowin and M.M. Mehrabadi, The structure of the linear anisotropic elastic symmetries, *J. Mech. Phys. Solids*, **40**(1992), no 7, 1459-1471.

[11] S.C. Cowin and M.M. Mehrabadi, On the identification of material symmetry for anisotropic elastic materials, *Quart. J. Mech. Appl. Math.*, **40**(1987), 451-476.

[12] S. Forte and M. Vianello, Symmetry classes for elasticity tensors, *J. Elasticity*, **43**(1996), no 2, 81-108.

[13] K. Helbig, *Foundation of anisotropy for exploration seismics*, Pergamon, 1994.

[14] Y.Z. Huo and G.del Piero, On the completeness of the crystallographic symmetries in the description of the symmetries of the elastic tensor, *Journal of Elasticity*, **25**(1991), 203-246.

[15] B. Herman, Some theorems of the theory of anisotropic media, *Comptes Rendus (Doklady) de l'Académy des Sciences de l'URSS*, **48**(1945), no. 2, 89-92.

Dept. of Earth Sciences, Memorial University, St. John's, Newfoundland, A1B 3X7, Canada