

Asymptotic stability of impulsive high-order Hopfield type neural networks

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Abstract

In this paper, we discuss impulsive high-order Hopfield type neural networks. Investigating their global asymptotic stability, by using Lyapunov function method, sufficient conditions that guarantee global asymptotic stability of networks are given. These criteria can be used to analysis the dynamics of biological neural systems or to design globally stable artificial neural networks. Two numerical example are given to illustrate the effectiveness of the proposed method.

Key words: Impulsive high-order Hopfield type neural networks, asymptotic stability, Lyapunov function

1 Introduction

Hopfield neural networks have been extensively studied and developed in recent years, and found many applications in different areas(See,e.g.,[1-3]). Artificial neural networks with impulses have been given in [4-9], and the stability, existence of the equilibrium of such networks have been investigated. In the present paper, we introduce impulsive high-order Hopfield type neural networks, and Lyapunov method is employed to investigate the sufficient conditions for the global asymptotic stability. This paper is organized as follows. In Section 2, impulsive high-order Hopfield type neural networks model is

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described and some preliminaries are given. Based on the Lyapunov stability theory, some global asymptotic stability criteria for neural networks are derived in Section 3. Two examples are given in Section 4 to illustrate the applicability of our results, and conclusions follow in Section 5.

2 Model description and preliminaries

We consider the impulsive high-order Hopfield type neural networks described by

$$\begin{cases} C_i \dot{u}_i = -\frac{u_i}{R_i} + \sum_{j=1}^n T_{ij} g_j(u_j) + \sum_{j=1}^n \sum_{l=1}^n T_{ijl} g_j(u_j) g_l(u_l) + I_i, t \neq t_k \\ \Delta u_i = d_i u_i + \sum_{j=1}^n W_{ij} h_j(u_j) + \sum_{j=1}^n \sum_{l=1}^n W_{ijl} h_j(u_j) h_l(u_l), t = t_k \\ i = 1, 2, \dots, n \end{cases} \quad (1)$$

where $\Delta u_i(t_k) = u_i(t_k^+) - u_i(t_k)$, $u_i(t_k^+) = \lim_{t \rightarrow t_k^+} u_i(t)$, $k \in Z = \{1, 2, \dots\}$, the time sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$, $\lim_{k \rightarrow \infty} t_k = \infty$, $C_i > 0$, $R_i > 0$, and I_i are the capacitance, the resistance, and the external input of the i th neuron respectively, T_{ij} , W_{ij} and T_{ijl} , W_{ijl} are the first and second order synaptic weights of the neural networks, respectively.

Throughout this paper, we assume that the neuron activation functions in (1), $g_i(u)$, $h_i(u)$, $i = 1, 2, \dots, n$, are continuous and satisfies the following condition:

$$(H) \quad \begin{cases} |g_i(u_i)| \leq M_i, 0 \leq \dot{g}_i(u_i) \leq K_i, \forall u_i \in \mathfrak{R}, \\ |h_i(u_i)| \leq N_i, 0 \leq \dot{h}_i(u_i) \leq L_i, \forall u_i \in \mathfrak{R}, \end{cases} \quad i = 1, 2, \dots, n \quad (2)$$

or

$$(H^*) \quad \begin{cases} |g_i(u_i)| \leq M_i, 0 \leq \frac{g_i(u_i) - g_i(v_i)}{u_i - v_i} \leq K_i, \forall u_i \neq v_i, u_i, v_i \in \mathfrak{R}, \\ |h_i(u_i)| \leq N_i, 0 \leq \frac{h_i(u_i) - h_i(v_i)}{u_i - v_i} \leq L_i, \forall u_i \neq v_i, u_i, v_i \in \mathfrak{R}, \\ i = 1, 2, \dots, n \end{cases} \quad (3)$$

Let $u = u^*$, i.e., $u_i = u_i^*$, $i = 1, 2, \dots, n$ be an equilibrium point of system (1), and set $x = u - u^* = (x_1, x_2, \dots, x_n)^T$,
 $d_i u_i^* + \sum_{j=1}^n W_{ij} h_j(u_j^*) + \sum_{j=1}^n \sum_{l=1}^n W_{ijl} h_j(u_j^*) h_l(u_l^*) = 0$,
 $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$, and $\varphi_i(x_i) = h_i(x_i + u_i^*) - h_i(u_i^*)$, $i = 1, 2, \dots, n$.

Then we see that

$$\begin{cases} |f_i(z)| \leq K_i|z|, z f_i(z) \geq 0, \forall z \in \mathfrak{R}, \\ |\varphi_i(z)| \leq L_i|z|, z \varphi_i(z) \geq 0, \forall z \in \mathfrak{R}, \end{cases} \quad i = 1, 2, \dots, n, \quad (4)$$

and it is easy to transform system (1) into the following

$$\begin{cases} C_i \dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^n T_{ij} f_j(x_j) \\ \quad + \sum_{j=1}^n \sum_{l=1}^n T_{ijl} \left(f_j(x_j) f_l(x_l) + f_j(x_j) g_l(u_l^*) \right. \\ \quad \quad \quad \left. + f_l(x_l) g_j(u_j^*) \right), t \neq t_k, \\ \Delta x_i = d_i u_i + \sum_{j=1}^n W_{ij} \varphi_j(u_j) \\ \quad + \sum_{j=1}^n \sum_{l=1}^n W_{ijl} \left(\varphi_j(x_j) \varphi_l(x_l) + \varphi_j(x_j) h_l(u_l^*) \right. \\ \quad \quad \quad \left. + \varphi_l(x_l) h_j(u_j^*) \right), t = t_k, \\ i = 1, 2, \dots, n, \end{cases} \quad (5)$$

By a simple transformation, system (5) may be written as follows.

$$\begin{cases} C_i \dot{x}_i = -\frac{x_i}{R_i} + \sum_{j=1}^n T_{ij} f_j(x_j) + \sum_{j=1}^n \sum_{l=1}^n (T_{ijl} + T_{ilj}) \zeta_l f_j(x_j), t \neq t_k, \\ \Delta x_i = d_i x_i + \sum_{j=1}^n W_{ij} \varphi_j(x_j) + \sum_{j=1}^n \sum_{l=1}^n (W_{ijl} + W_{ilj}) \xi_l \varphi_j(x_j), t = t_k, \\ i = 1, 2, \dots, n, \end{cases} \quad (6)$$

where ζ_l is between $g_l(u_l)$ and $g_l(u_l^*)$, while ξ_l is between $h_l(u_l)$ and $h_l(u_l^*)$.

Let $C = \text{diag}(C_1, C_2, \dots, C_n)$, $R = \text{diag}(R_1, R_2, \dots, R_n)$, $T = (T_{ij})_{n \times n}$,
 $D = \text{diag}(d_1, d_2, \dots, d_n)$, $W = (W_{ij})_{n \times n}$, $W_i = (W_{ijl})_{n \times n}$, $T_i = (T_{ijl})_{n \times n}$,
 $(i = 1, 2, \dots, n)$, $T_H = (T_1 + T_1^T, T_2 + T_2^T, \dots, T_n + T_n^T)^T$, $\Xi = (W_1 + W_1^T, W_2 + W_2^T, \dots, W_n + W_n^T)^T$, $\varphi(x) = (\varphi_1(x_1), \varphi_2(x_2), \dots, \varphi_n(x_n))^T$,
 $\Phi(x) = \text{diag}(\varphi(x), \varphi(x), \dots, \varphi(x))$, $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^T$,
 $F(x) = \text{diag}(f(x), f(x), \dots, f(x))$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^T$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$,
 $M = (M_1, M_2, \dots, M_n)^T$, $N = (N_1, N_2, \dots, N_n)^T$, $L = \text{diag}(L_1, L_2, \dots, L_n)$,
 $K = \text{diag}(K_1, K_2, \dots, K_n)$, $x = (x_1, x_2, \dots, x_n)^T$, $\Delta x = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)^T$.

Then system (6) may be written in the following equivalent form

$$\begin{cases} C\dot{x} = -R^{-1}x + Tf(x) + F^T(x)T_H\zeta, t \neq t_k, \\ \Delta x = Dx + W\varphi(x) + \Phi^T(x)\Xi\xi, t = t_k, \end{cases} \quad (7)$$

The following notations will be used throughout the paper: \mathfrak{R}^+ the set of nonnegative real numbers, and \mathfrak{R}^n the n -dimensional Euclidean space. The notation $P > 0$, (respectively, $P < 0$) means that P is symmetric and positive definite (respectively, negative definite) matrix. We use P^{-1} , $\lambda_{\min}(P)$, $\lambda_{\max}(P)$, to denote, respectively, the inverse of, the smallest and the largest eigenvalues of a square matrix P . The norms $\|\cdot\|$ is either the Euclidean vector norm or the induced matrix norm.

The following lemmas will be used in the proof of our main results.

Consider an impulsive differential system

$$\begin{cases} \dot{x}(t) = f(t, x), t \neq t_k \\ \Delta x = I_k(x), t = t_k \\ x(t_0^+) = x_0, k = 1, 2, \dots \end{cases} \quad (8)$$

where $\Delta x(t_k) = x_k^+ - x(t_k)$.

We denote by \mathcal{L} the class of maps $h : \mathfrak{R}^+ \times \mathfrak{R}^n \mapsto \mathfrak{R}^+$, which are continuous and $\inf h(t, x) = 0$.

Definition 1. Let $h, h_0 \in \mathcal{L}$. Then the impulsive differential system (8) is called

(1) (h_0, h) -stable if for any $\varepsilon > 0$ and $t_0 \in \mathfrak{R}^+$ given, there exist a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$h_0(t_0, x_0) < \delta \quad \text{implies} \quad h(t, x(t)) < \varepsilon, \quad t \geq t_0$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (8);

(2) (h_0, h) -attractive if for $t_0 \in \mathfrak{R}^+$, there exist a $\sigma = \sigma(t_0) > 0$ such that

$$h_0(t_0, x_0) < \sigma \quad \text{implies} \quad \lim_{t \rightarrow \infty} h(t, x(t)) = 0$$

(3) (h_0, h) -asymptotically stable if (1) and (2) hold together.

Let ϑ_0 denote the class of functions $V : \mathfrak{R}^+ \times \mathfrak{R}^n \mapsto \mathfrak{R}^+$, where V is locally Lipschitz in x , continuous everywhere except t_k 's at which V is left continuous and the right limit $V(t_k^+, x)$ exists for any $x \in \mathfrak{R}^n$. For $V \in \vartheta_0$, $(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n$ and $t \neq t_k$, we define $D^+V(t, x)$ by

$$D^+V(t, x) = \lim_{\theta \rightarrow 0^+} \sup \frac{1}{\theta} [V(t + \theta, x + \theta f(t, x)) - V(t, x)]$$

We denote by \aleph the class of functions $\phi : \mathfrak{R}^+ \mapsto \mathfrak{R}^+$ which are continuous, strictly increasing and $\phi(0) = 0$, \aleph_0 the class of continuous functions $\psi : \mathfrak{R}^+ \mapsto \mathfrak{R}^+$ such that $\psi(0) = 0$ if and only if $s = 0$, and PC the class of functions $\lambda : \mathfrak{R}^+ \mapsto \mathfrak{R}^+$, where λ is continuous everywhere except t_k 's at which λ is left continuous and right limit $\lambda(t_k^+)$ exists.

Definition 2. Let $V \in \vartheta_0$ and $h \in \mathcal{L}$. Then V is said to be

(1) h -positive definite if there exist a constant $\rho > 0$ and a function $b \in \aleph$ such that

$$b(h(t, x)) \leq V(t, x) \quad \text{if} \quad h(t, x) < \rho$$

(2) h -decrecent if there exist a constant $\delta > 0$ and a function $a \in \aleph$ such that

$$V(t, x) \leq a(h(t, x)) \quad \text{whenever} \quad h(t, x) \leq \rho$$

Definition 3. Let $h, h_0 \in \mathcal{L}$. Then we say that h_0 is finer than h if h is h_0 -decrecent.

Let $s(h, \rho) = \{(t, x) \in \mathfrak{R}^+ \times \mathfrak{R}^n; h(t, x) < \rho\}$.

Lemma 1^[10]. Assume that

(1) $h, h_0 \in \mathcal{L}$, h_0 is finer than h , and there exist constants, ρ, ρ_0 , with $0 < \rho_0 < \rho$ such that $(t_k, x) \in s(h, \rho_0)$ implies $(t_k, x + I_k(x)) \in s(h, \rho)$ for all $k = 1, 2, \dots$;

(2) $V \in \vartheta_0$, $v(t, x)$ is h -positive definite, h_0 -decrecent and there exists $\psi_k \in \aleph_0$ such that

$$V(t_k^+, x + I_k(x)) \leq \psi_k(V(t_k, x)), \quad k = 1, 2, \dots$$

(3) there exist $c \in \aleph$ and $p \in PC$ such that

$$D^+V(t, x) \leq p(t)c(V(t, x)), \quad (t, x) \in s(h, \rho), \quad t \neq t_k$$

(4) there exists a constant $\sigma > 0$ such that for all $z \in (0, \sigma)$

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \leq -\gamma_k$$

for some constant γ_k and $k = 1, 2, \dots$.

Then the system (8) is (h_0, h) -asymptotically stable if $\gamma_k \geq 0$ for all $k = 1, 2, \dots$, and $\sum_{k=1}^{\infty} \gamma_k = \infty$.

Lemma 2^[10]. Assume that conditions (1) and (2) of Lemma 1 hold. Suppose further that

(3*) there exist functions $c \in \mathbb{N}$ and $\lambda \in PC$ such that

$$D^+V(t, x) \leq -\lambda(t)c(V(t, x)), \quad (t, x) \in s(h, \rho), \quad t \neq t_k$$

(4*) there exists a constant $\sigma > 0$ such that for all $z \in (0, \sigma)$

$$-\int_{t_{k-1}}^{t_k} \lambda(s)ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \leq -\gamma_k$$

for some constant γ_k and $k = 1, 2, \dots$.

Then the system (8) is (h_0, h) -asymptotically stable if $\gamma_k \geq 0$ for all $k = 1, 2, \dots$, and $\sum_{k=1}^{\infty} \gamma_k = \infty$.

Lemma 3. System (1) has at least one equilibrium point.

The proof of Lemma 3 is similar to that given in [11, Theorem 1]. An additional difference is the consideration of the impulse effect.

3 Global asymptotic stability

In this section, we shall establish some sufficient conditions for global asymptotic stability of the impulsive high-order Hopfield type neural networks.

If $u^* = (u_1^*, u_2^*, \dots, u_n^*)^T$ is an equilibrium point of system (1), then $x^* = (0, 0, \dots, 0)^T$ is an equilibrium point of system (5), (6) and (7). To prove the global asymptotic stability of the equilibrium point u^* of system (1), it is

sufficient to prove the global asymptotic stability of the trivial solution of one of system (5), (6) and (7).

Theorem 1. The equilibrium point u^* of system (1) is globally asymptotically stable if the condition **(H)** and the following conditions hold:

(i) $A = (a_{ij})_{n \times n}$ is an M -matrix, where

$$a_{ij} = \begin{cases} \frac{1}{R_j} - \left(T_{jj} + \sum_{k=1}^n |T_{jjk} + T_{jkj}| M_k \right) K_j, & j = i \\ - \left(|T_{ij}| + \sum_{k=1}^n |T_{ijk} + T_{ikj}| M_k \right) K_j, & j \neq i \end{cases}$$

$$(ii) - \frac{\min_{1 \leq j \leq n} \left\{ \sum_{i=1}^n p_i a_{ij} \right\}}{\max_{1 \leq i \leq n} \{p_i C_i\}} (t_k - t_{k-1}) + \ln \frac{\max_{1 \leq i \leq n} \{p_i C_i\}}{\min_{1 \leq i \leq n} \{p_i C_i\}} + \ln(\sqrt{n} \varrho) \leq -\gamma_k,$$

where $\varrho = \|I + D\| + \max_{1 \leq i \leq n} \{L_i\} (\|W\| + \|\Xi\| \|N\|)$, $\gamma_k \geq 0$ for all $k = 1, 2, \dots$,

and $\sum_{k=1}^{\infty} \gamma_k = \infty$, and $p_i > 0$ for all $i = 1, 2, \dots, n$, such that $\sum_{i=1}^n p_i a_{ij} > 0$, ($j = 1, 2, \dots, n$).

Proof. Since $A = (a_{ij})_{n \times n}$ is an M -matrix, there exist a set of constant $p_i > 0$, ($i = 1, 2, \dots, n$), such that $q_j = \sum_{i=1}^n p_i a_{ij} > 0$ for all $j = 1, 2, \dots, n$.

Construct a radially unbounded Lyapunov function $V(t)$ by

$$V(t, x) = \sum_{i=1}^n p_i C_i |x_i|,$$

it is easy to see that

$$\min_{1 \leq i \leq n} \{p_i C_i\} \|x\| \leq V(t, x) \leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \|x\| .$$

When $t \neq t_k$, computing the upper right derivative of $V(t, x)$ along the trajectories of system (5), we obtain

$$\begin{aligned} D^+ V(t, x)|_{(5)} &= - \sum_{j=1}^n \frac{p_j}{R_j} |x_j| + \sum_{j=1}^n \left(p_j T_{jj} |f_j(x_j)| + \sum_{\substack{i=1 \\ i \neq j}}^n p_i T_{ij} f_j(x_j) \operatorname{sgn}(x_i) \right) + \omega \\ &\leq - \sum_{j=1}^n \frac{p_j}{R_j} |x_j| + \sum_{j=1}^n \left(p_j T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^n p_i |T_{ij}| \right) |f_j(x_j)| + \omega \end{aligned}$$

where $\omega = \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n p_i T_{ijl} \left(f_j(x_j) f_l(x_l) + f_j(x_j) g_l(u_l^*) + f_l(x_l) g_j(u_j^*) \right) \text{sgn}(x_i)$.

By the Taylor's mean value theorem, we obtain

$$\begin{aligned} & \sum_{j=1}^n \sum_{l=1}^n T_{ijl} \left(f_j(x_j) f_l(x_l) + f_j(x_j) g_l(u_l^*) + f_l(x_l) g_j(u_j^*) \right) \\ &= \sum_{j=1}^n \sum_{l=1}^n (T_{ijl} + T_{ilj}) \dot{g}_j(\xi_j) g_l(\xi_l) x_j \end{aligned} \quad (9)$$

where ξ_j is between u_j and u_j^* .

By (9) and taking to account the condition **(H)** and (4), we obtain

$$\begin{aligned} D^+V(t, x)|_{(5)} &\leq - \sum_{j=1}^n \frac{p_j}{R_j} |x_j| + \sum_{j=1}^n \left(p_j T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^n p_i |T_{ij}| \right) K_j |x_j| \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n p_i |T_{ijl} + T_{ilj}| M_l K_j |x_j| \\ &= - \sum_{j=1}^n \left[\frac{p_j}{R_j} - \left(p_j T_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^n p_i |T_{ij}| \right) K_j \right. \\ &\quad \left. - \sum_{i=1}^n \sum_{l=1}^n p_i |T_{ijl} + T_{ilj}| M_l K_j \right] |x_j| \\ &= - \sum_{j=1}^n q_j |x_j| \leq - \min_{1 \leq i \leq n} \{q_i\} \sum_{j=1}^n |x_j| \\ &\leq - \frac{\min_{1 \leq i \leq n} \{q_i\}}{\max_{1 \leq i \leq n} \{p_i C_i\}} V(t, x), \quad t \neq t_k. \end{aligned}$$

By the Taylor's mean value theorem, we obtain

$$\begin{aligned} & \sum_{j=1}^n \sum_{l=1}^n W_{ijl} \left(\varphi_j(x_j) \varphi_l(x_l) + \varphi_j(x_j) h_l(u_l^*) + \varphi_l(x_l) h_j(u_j^*) \right) \\ &= \sum_{j=1}^n \sum_{l=1}^n (W_{ijl} + W_{ilj}) \dot{h}_j(\theta_j) h_l(\theta_l) x_j \end{aligned}$$

where θ_j is between u_j and u_j^* .

Let $\sigma(x) = (\dot{h}_1(\theta_1)x_1, \dot{h}_2(\theta_2)x_2, \dots, \dot{h}_n(\theta_n)x_n)^T$,
 $Y(x) = \text{diag}(\sigma(x), \sigma(x), \dots, \sigma(x))$,

$h(\theta) = (h_1(\theta_1), h_2(\theta_2), \dots, h_n(\theta_n))^T$, from (5) we get

$$\Delta x = Dx + W\varphi(x) + Y^T(x)\Xi h(\theta), t = t_k$$

Hence, by condition **(H)** and (4) we obtain

$$\begin{aligned} V(t_k^+, x + \Delta x) &\leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \|x(t_k) + \Delta x(t_k)\| \\ &= \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \|(I + D)x(t_k) + W\varphi(x(t_k)) \\ &\quad + Y^T(x(t_k))\Xi h(\theta)\| \\ &\leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} (\|I + D\| \|x(t_k)\| + \|W\| \|\varphi(x(t_k))\| \\ &\quad + \|Y^T(x(t_k))\| \|\Xi\| \|h(\theta)\|) \\ &\leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \left(\|I + D\| \|x(t_k)\| \right. \\ &\quad \left. + \max_{1 \leq i \leq n} \{L_i\} \|W\| \|x(t_k)\| + \|\sigma(x(t_k))\| \|\Xi\| \|N\| \right) \\ &\leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \varrho \|x(t_k)\| \\ &\leq \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \varrho \frac{V(t_k, x)}{\min_{1 \leq i \leq n} \{p_i C_i\}}, \quad k \in Z. \end{aligned}$$

Let $\psi_k(s) = \max_{1 \leq i \leq n} \{p_i C_i\} \sqrt{n} \varrho \frac{s}{\min_{1 \leq i \leq n} \{p_i C_i\}}$,

by condition (ii) we get

$$- \int_{t_{k-1}}^{t_k} \frac{\min_{1 \leq i \leq n} \{q_i\}}{\max_{1 \leq i \leq n} \{p_i C_i\}} ds + \int_z^{\psi_k(z)} \frac{ds}{s} \leq -\gamma_k.$$

Thus by Lemma 2 we see that the trivial solution of system (5) is globally asymptotically stable. This completes the proof.

Theorem 2. The equilibrium point u^* of system (1) is globally asymptotically stable if the condition **(H)** and the following conditions hold:

(i) $\Gamma = (\gamma_{ij})_{n \times n}$ is Lyapunov diagonal stable^[12], where

$$\gamma_{ij} = \begin{cases} T_{ii}^+ K_i - \frac{1}{R_i} + \sum_{l=1}^n |T_{il} + T_{li}| M_l K_i, & i = j, \\ \left(|T_{ij}| + \sum_{l=1}^n |T_{ijl} + T_{ilj}| M_l \right) K_j, & i \neq j, \end{cases} \quad T_{ii}^+ = \max \{T_{ii}, 0\};$$

$$(ii) \frac{2\lambda_{\max}\left(\frac{P\Gamma + \Gamma^T P}{2}\right)}{\max_{1 \leq i \leq n} \{p_i C_i\}}(t_k - t_{k-1}) + \ln \frac{\max_{1 \leq i \leq n} \{p_i C_i\}}{\min_{1 \leq i \leq n} \{p_i C_i\}} + 2\ln \varrho \leq -\gamma_k,$$

where $\varrho = \|I + D\| + \max_{1 \leq i \leq n} \{L_i\}(\|W\| + \|\Xi\|\|N\|)$, $\gamma_k \geq 0$ for all $k = 1, 2, \dots$,

and $\sum_{k=1}^{\infty} \gamma_k = \infty$, and $P = \text{diag}(p_1, p_2, \dots, p_n) > 0$, such that $P\Gamma + \Gamma^T P < 0$.

Proof. Since $\Gamma = (\gamma_{ij})_{n \times n}$ is Lyapunov diagonal stable, there exists an matrix $P = \text{diag}(p_1, p_2, \dots, p_n) > 0$, such that $P\Gamma + \Gamma^T P < 0$, hence

$$\lambda_{\max}\left(\frac{P\Gamma + \Gamma^T P}{2}\right) < 0$$

Construct a Lyapunov function

$$V(t, x) = \frac{1}{2} \sum_{i=1}^n p_i C_i x_i^2,$$

it is easy to verify that

$$\frac{1}{2} \min_{1 \leq i \leq n} \{p_i C_i\} \|x\|^2 \leq V(t, x) \leq \frac{1}{2} \max_{1 \leq i \leq n} \{p_i C_i\} \|x\|^2.$$

When $t \neq t_k$, computing the derivative of $V(t, x)$ along the trajectories of system (5), by (9) and taking to account the condition **(H)** and (4), we obtain

$$\begin{aligned} \dot{V}(t, x)|_{(5)} &= \sum_{i=1}^n p_i x_i \left(-\frac{x_i}{R_i} + \sum_{j=1}^n T_{ij} f_j(x_j) \right) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n p_i T_{ijl} x_i \left(f_j(x_j) f_l(x_l) + f_j(x_j) g_l(u_l^*) + f_l(x_l) g_j(u_j^*) \right) \\ &\leq -\sum_{i=1}^n \frac{p_i}{R_i} x_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n p_i T_{ij} x_i f_j(x_j) + \sum_{i=1}^n p_i T_{ii} x_i f_i(x_i) \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n |T_{ijl} + T_{ilj}| p_i K_j M_l |x_i| |x_j| \\ &\leq -\sum_{i=1}^n \frac{p_i}{R_i} x_i^2 + \sum_{i=1}^n p_i T_{ii}^+ x_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n p_i K_j |T_{ij}| |x_i| |x_j| \\ &\quad + \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n |T_{ijl} + T_{ilj}| p_i K_j M_l |x_i| |x_j| \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n p_i \left(T_{ii}^+ K_i - \frac{1}{R_i} + \sum_{l=1}^n |T_{iil} + T_{ili}| M_l K_i \right) x_i^2 \\
&\quad + \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n p_i \left(|T_{ij}| + \sum_{l=1}^n |T_{ijl} + T_{ilj}| M_l \right) K_j |x_i| |x_j| \\
&= |x|^T P \Gamma |x| \leq \lambda_{\max} \left(\frac{P \Gamma + \Gamma^T P}{2} \right) \|x\|^2 \\
&\leq \frac{2 \lambda_{\max} \left(\frac{P \Gamma + \Gamma^T P}{2} \right)}{\max_{1 \leq i \leq n} \{p_i C_i\}} V(t, x),
\end{aligned}$$

where $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$.

By the arguments similar to that used in the proof of Theorem 1, we show that

$$V(t_k^+, x + \Delta x) \leq \max_{1 \leq i \leq n} \{p_i C_i\} \varrho^2 \frac{V(t_k, x)}{\min_{1 \leq i \leq n} \{p_i C_i\}}, \quad k \in Z.$$

Let $\psi_k(s) = \max_{1 \leq i \leq n} \{p_i C_i\} \varrho^2 \frac{s}{\min_{1 \leq i \leq n} \{p_i C_i\}}$, then

$$\int_z^{\psi_k(z)} \frac{ds}{s} = \ln \frac{\psi_k(z)}{z} = \ln \frac{\max_{1 \leq i \leq n} \{p_i C_i\}}{\min_{1 \leq i \leq n} \{p_i C_i\}} + 2 \ln \varrho.$$

From condition (ii) we get

$$\int_{t_{k-1}}^{t_k} \frac{2 \lambda_{\max} \left(\frac{P \Gamma + \Gamma^T P}{2} \right)}{\max_{1 \leq i \leq n} \{p_i C_i\}} ds + \int_z^{\psi_k(z)} \frac{ds}{s} \leq -\gamma_k.$$

Thus by Lemma 2 we see that the trivial solution of system (5) is globally asymptotically stable. This completes the proof.

Theorem 3. The equilibrium point u^* of system (1) is globally asymptotically stable if the condition (\mathbf{H}^*) hold and there exists an matrix $P > 0$ such that the following conditions hold:

$$\begin{aligned}
\text{(i)} \quad \alpha &= 2(\|PC^{-1}T\| + \|PC^{-1}\| \|T_H\| \|M\|) \max_{1 \leq i \leq n} \{K_i\} \\
&\quad - \lambda_{\min}(R^{-1}C^{-1}P + PC^{-1}R^{-1}) > 0;
\end{aligned}$$

$$(ii) \frac{\alpha}{\lambda_{\min}(P)}(t_{k+1} - t_k) + \ln \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + 2\ln \varrho \leq -\gamma_k,$$

where $\|I + D\| + \max_{1 \leq i \leq n} \{L_i\}(\|W\| + \|\Xi\|\|N\|)$, $\gamma_k \geq 0$ for all $k = 1, 2, \dots$, and

$$\sum_{k=1}^{\infty} \gamma_k = \infty.$$

Proof. Consider a Lyapunov function $V(t, x) = x^T P x$, it is easy to verify that

$$\lambda_{\min}(P)\|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P)\|x\|^2.$$

When $t \neq t_k$, computing the derivative of $V(t, x)$ along the trajectories of system (7), we obtain

$$\begin{aligned} \dot{V}(t, x)|_{(7)} &= -x^T(R^{-1}C^{-1}P + PC^{-1}R^{-1})x \\ &\quad + 2x^T PC^{-1}Tf(x) + 2x^T PC^{-1}F^T(x)T_H\zeta \\ &\leq -\lambda_{\min}(R^{-1}C^{-1}P + PC^{-1}R^{-1})\|x\|^2 \\ &\quad + 2\|PC^{-1}T\|\|x\|\|f(x)\| + 2\|PC^{-1}\|\|T_H\|\|x\|\|F^T(x)\|\|\zeta\|. \end{aligned}$$

From condition (\mathbf{H}^*) and (4) we get

$$\|F^T(x)\| = \|f(x)\| \leq \max_{1 \leq i \leq n} \{K_i\}\|x\| \quad \text{and} \quad \|\zeta\| \leq \|M\|$$

Hence

$$\begin{aligned} \dot{V}(t, x)|_{(7)} &\leq 2(\|PC^{-1}T\| + \|PC^{-1}\|\|M\|\|T_H\|) \max_{1 \leq i \leq n} \{K_i\}\|x\|^2 \\ &\quad - \lambda_{\min}(R^{-1}C^{-1}P + PC^{-1}R^{-1})\|x\|^2 \\ &= \alpha\|x\|^2 \leq \frac{\alpha}{\lambda_{\min}(P)}V(t, x), t \neq t_k. \end{aligned}$$

From condition (\mathbf{H}^*) and (4), we get

$$\|\Phi^T(x(t_k))\| = \|\varphi(x(t_k))\| \leq \max_{1 \leq i \leq n} \{L_i\}\|x(t_k)\| \quad \text{and} \quad \|\xi\| \leq \|N\|.$$

By system (7) we obtain $\|x(t_k) + \Delta x(t_k)\| \leq \varrho\|x(t_k)\|$.

Hence

$$V(t_k^+, x + \Delta x) \leq \varrho^2 \lambda_{\max}(P) \frac{V(t_k, x)}{\lambda_{\min}(P)}, \quad k \in Z.$$

Let $\psi_k(s) = \varrho^2 \lambda_{\max}(P) \frac{s}{\lambda_{\min}(P)}$, then from condition (ii) we obtain

$$\int_{t_k}^{t_{k+1}} \frac{\alpha}{\lambda_{\min}(P)} ds + \int_z^{\psi_k(z)} \frac{ds}{s} \leq -\gamma_k.$$

Thus by Lemma 1 we see that the trivial solution of system (7) is globally asymptotically stable. This completes the proof.

Theorem 4. The equilibrium point u^* of system (1) is globally asymptotically stable if the condition (\mathbf{H}^*) hold and there exists an matrix $P > 0$ such that the following conditions hold:

$$(i) \alpha = 2(\|PC^{-1}T\| + \|PC^{-1}\| \|T_H\| \|M\|) \max_{1 \leq i \leq n} \{K_i\} - \lambda_{\min}(R^{-1}C^{-1}P + PC^{-1}R^{-1}) < 0;$$

$$(ii) \frac{\alpha}{\lambda_{\max}(P)}(t_k - t_{k-1}) + \ln \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + 2 \ln \varrho \leq -\gamma_k,$$

where $\varrho = \|I + D\| + \max_{1 \leq i \leq n} \{L_i\}(\|W\| + \|\Xi\| \|N\|)$, $\gamma_k \geq 0$ for all $k = 1, 2, \dots$,

and $\sum_{k=1}^{\infty} \gamma_k = \infty$.

Proof. Consider the Lyapunov function $V(t, x) = x^T P x$, by the arguments similar to that used in the proof of theorem 3, we show that

$$\dot{V}(t, x)|_{(\tau)} \leq \frac{\alpha}{\lambda_{\max}(P)} V(t, x), t \neq t_k$$

Let $\psi_k(s)$ be same as in the proof of theorem 3, we get

$$V(t_k^+, x + \Delta x) \leq \psi_k(V(t_k, x)), k \in Z.$$

Form condition (ii) we obtain

$$\int_{t_{k-1}}^{t_k} \frac{\alpha}{\lambda_{\max}(P)} ds + \int_z^{\psi_k(z)} \frac{ds}{s} \leq -\gamma_k.$$

Thus, by Lemma 2 we see that the trivial solution of system (7) is globally asymptotically stable. This completes the proof.

4 Examples

In this section, we will give two examples to illustrate the main results of this paper.

Example 1. Consider the neural network

$$\begin{cases} C_i \dot{u}_i = -\frac{u_i}{R_i} + \sum_{j=1}^3 T_{ij} g_j(u_j) + \sum_{j=1}^3 \sum_{l=1}^3 T_{ijl} g_j(u_j) g_l(u_l), t \neq t_k, \\ \Delta u_i = d_i u_i + \sum_{j=1}^3 W_{ij} h_j(u_j) + \sum_{j=1}^3 \sum_{l=1}^3 W_{ijl} h_j(u_j) h_l(u_l), t = t_k, \\ i = 1, 2, 3, \end{cases} \quad (10)$$

where $g_1(u_1) = \tanh(0.16u_1)$, $g_2(u_2) = \tanh(0.14u_2)$, $g_3(u_3) = \tanh(0.07u_3)$,
 $h_1(u_1) = 0.9 \tanh(0.1u_1)$, $h_2(u_2) = 0.9 \tanh(0.02u_2)$, $h_3(u_3) = 0.9 \tanh(0.19u_3)$,
 $C = \text{diag}(C_1, C_2, C_3) = \text{diag}(2.6, 6.8, 8.3)$,
 $R = \text{diag}(R_1, R_2, R_3) = \text{diag}(0.4, 2.86, 0.5)$,
 $D = \text{diag}(d_1, d_2, d_3) = \text{diag}(-0.95, -0.84, -0.99)$,

$$T = (T_{ij})_{3 \times 3} = \begin{bmatrix} 0.13 & 0.11 & 0.02 \\ 0.44 & -0.57 & 0.14 \\ 0.11 & 0.16 & -0.37 \end{bmatrix}, T_1 = (T_{1ij})_{3 \times 3} = \begin{bmatrix} 0.01 & 0.01 & 0.05 \\ 0.07 & 0.11 & 0.02 \\ 0 & -0.02 & -0.11 \end{bmatrix},$$

$$T_2 = (T_{2ij})_{3 \times 3} = \begin{bmatrix} 0.12 & 0 & -0.01 \\ -0.01 & 0.02 & 0.04 \\ -0.02 & 0.07 & 0.01 \end{bmatrix},$$

$$T_3 = (T_{3ij})_{3 \times 3} = \begin{bmatrix} 0.04 & -0.01 & -0.13 \\ -0.01 & -0.08 & -0.02 \\ -0.04 & -0.16 & -0.04 \end{bmatrix},$$

$$W = (W_{ij})_{3 \times 3} = \begin{bmatrix} -0.04 & -0.05 & 0.16 \\ 0.19 & -0.17 & -0.02 \\ 0.03 & 0.13 & 0.04 \end{bmatrix},$$

$$W_1 = (W_{1ij})_{3 \times 3} = \begin{bmatrix} -0.01 & 0.01 & -0.03 \\ 0.08 & -0.09 & 0.07 \\ 0.08 & -0.01 & 0.01 \end{bmatrix},$$

$$W_2 = (W_{2ij})_{3 \times 3} = \begin{bmatrix} 0.06 & 0 & 0.04 \\ 0.04 & -0.07 & 0.07 \\ -0.02 & -0.06 & 0.05 \end{bmatrix},$$

$$W_3 = (W_{3ij})_{3 \times 3} = \begin{bmatrix} 0.04 & -0.04 & 0.01 \\ 0.02 & 0.05 & -0.05 \\ -0.02 & 0.03 & -0.02 \end{bmatrix}.$$

$$\text{In this case } M = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, N = \begin{bmatrix} 0.9 \\ 0.9 \\ 0.9 \end{bmatrix}, K = \text{diag}(0.16, 0.14, 0.07),$$

$L = \text{diag}(0.09, 0.02, 0.17)$. $u^* = (0, 0, \dots, 0)^T$ is an equilibrium point of system (10).

By direct computation, it follows that the matrix

$$A = \begin{bmatrix} 2.4552 & -0.0574 & -0.0203 \\ -0.1152 & 0.4071 & -0.0210 \\ -0.0608 & -0.0728 & 2.0014 \end{bmatrix}$$

in Theorem 1 is an M -matrix, and there exist constants $p_1 = 0.04, p_2 = 0.02, p_3 = 0.02$, such that $q_j = \sum_{i=1}^3 p_i a_{ij} > 0, j = 1, 2, 3$.

Let $\gamma_k \leq 0.2471, k \in Z$. Then by Theorem 1 we see that the equilibrium point u^* of system (10) is globally asymptotically stable for $t_k - t_{k-1} \geq 0, k \in Z$.

By Matlab Toolbox, we see that the matrix $P = \text{diag}(9.5603, 45.1883, 12.1201)$ such that $P\Gamma + \Gamma^T P < 0$. Thus the matrix

$$\Gamma = \begin{bmatrix} -2.4552 & 0.0574 & 0.0203 \\ 0.1152 & -0.3273 & 0.0210 \\ 0.0608 & 0.0728 & -1.9755 \end{bmatrix}$$

in Theorem 2 is Lyapunov diagonal stable.

Let $\gamma_k \leq 0.0134, k \in Z$. Then by Theorem 2 we see that, the equilibrium point u^* of system (10) is globally asymptotically stable for $t_k - t_{k-1} \geq 0, k \in Z$.

Example 2. Consider the neural network (10) with $g_1(u_1) = \tanh(0.7u_1)$, $g_2(u_2) = \tanh(0.5u_2)$, $g_3(u_3) = \tanh(0.8u_3)$, $h_1(u_1) = 0.9 \tanh(0.09u_1)$, $h_2(u_2) = 0.9 \tanh(0.08u_2)$, $h_3(u_3) = 0.9 \tanh(0.07u_3)$, $C = \text{diag}(C_1, C_2, C_3) = \text{diag}(1.4, 1.5, 1.3)$, $R = \text{diag}(R_1, R_2, R_3) = \text{diag}(1.9, 2.1, 0.8)$, $D = \text{diag}(d_1, d_2, d_3) = \text{diag}(-0.57, -0.77, -0.71)$,

$$T = (T_{ij})_{3 \times 3} = \begin{bmatrix} 0.02 & -0.04 & 0.01 \\ -0.14 & 0.65 & 0.74 \\ -1.2 & 0.01 & 0.25 \end{bmatrix}, T_1 = (T_{1ij})_{3 \times 3} = \begin{bmatrix} 0.02 & 0.07 & -0.07 \\ 0.01 & 0.08 & 0.04 \\ 0.04 & 0.07 & 0.02 \end{bmatrix},$$

$$T_2 = (T_{2ij})_{3 \times 3} = \begin{bmatrix} 0.03 & 0.08 & -0.01 \\ 0.01 & 0.13 & 0.07 \\ -0.04 & -0.01 & 0.06 \end{bmatrix},$$

$$T_3 = (T_{3ij})_{3 \times 3} = \begin{bmatrix} 0.08 & 0.02 & 0.01 \\ 0.01 & 0.08 & 0.01 \\ -0.05 & -0.06 & 0.03 \end{bmatrix},$$

$$W = (W_{ij})_{3 \times 3} = \begin{bmatrix} 1.36 & 0.6 & 0.64 \\ 1.33 & -0.15 & 0.2 \\ 0.38 & 0.23 & -0.62 \end{bmatrix},$$

$$W_1 = (W_{1ij})_{3 \times 3} = \begin{bmatrix} 0.03 & -0.04 & -0.01 \\ 0.01 & -0.03 & -0.04 \\ 0.02 & 0.05 & -0.04 \end{bmatrix},$$

$$W_2 = (W_{2ij})_{3 \times 3} = \begin{bmatrix} -0.08 & 0.01 & -0.03 \\ 0.07 & 0.02 & 0.01 \\ -0.03 & -0.03 & -0.01 \end{bmatrix},$$

$$W_3 = (W_{3ij})_{3 \times 3} = \begin{bmatrix} -0.03 & 0.05 & -0.05 \\ 0.03 & 0.02 & 0.04 \\ 0.05 & -0.06 & -0.06 \end{bmatrix}.$$

$$\text{In this case } M = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, K = \text{diag}(0.7, 0.5, 0.8), N = \begin{bmatrix} 0.9 \\ 0.9 \\ 0.9 \end{bmatrix},$$

$L = \text{diag}(0.08, 0.07, 0.06)$. $u^* = (0, 0, \dots, 0)^T$ is an equilibrium point of system (10).

Since there exist an matrix

$$P = \begin{bmatrix} 0.34 & 0 & -0.02 \\ 0 & 0.32 & -0.01 \\ -0.02 & -0.01 & 0.42 \end{bmatrix}$$

such that $\alpha = 0.8156$ in Theorem 3. Let $\gamma_k < 0.6586, k \in Z$. Then by Theorem 3 we see that the equilibrium point u^* of system (10) is globally asymptotically stable for $t_{k+1} - t_k \leq \frac{0.6586 - \gamma_k}{2.5585}, k \in Z$.

5 Conclusions

The problem of global asymptotic stability analysis for impulsive high-order Hopfield type neural networks have been discussed in this paper. By means of Lyapunov functions, some global asymptotic stability criteria have been derived. These criteria are easy to verify and can be used to analysis the dynamics of biological neural systems or to design globally stable artificial neural networks.

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