On limit periodicity of discrete time stochastic processes*

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Abstract

We consider a discrete time dynamic system described by a difference equation with periodic coefficients and with additive stochastic noise. We investigate the possibility of the periodicity of the solution. In particular, we established sufficient conditions for convergence of the solution in mean square or almost surely to some stochastic periodic process.

Key words: discrete time dynamic systems, stochastic difference equations, periodic solutions.

AMS classification: 39A23, 39A30, 39A50

1 Introduction and problem setting

The a.s. asymptotic stability of stochastic discrete time processes has been widely addressed; see, e.g., [1] – [8], [12] and the bibliography here. However, periodicity of these processes are not so well represented. In [11, 14], conditions of periodicity in the distributions were obtained for discrete time systems; a review of periodicity for nonlinear discrete time equations can be found in [6]. In [7, 9, 10], conditions of periodicity in the distributions were obtained for continuous time systems. In this article we obtain sufficient conditions for convergence of solutions to a periodic process in a strong sense. We consider processes described by stochastic difference equations with

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periodic coefficients and with decaying additive stochastic noise. We investigate the asymptotic properties. In particular, we found some sufficient conditions when this convergence can be achieved almost surely.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, with elementary events $\omega \in \Omega$.

Consider a stochastic process $X_n = X_n(\omega)$, $\omega \in \Omega$, with the values in $\mathbf{R}^d$, where $d \geq 1$, that evolves as

$$\begin{aligned} &X_{n+1} = X_n + A_n X_n + \sigma_n \xi_{n+1}, \quad n = 0, 1, 2, .., \\ &X(0) = X_0. \end{aligned}$$

Here $X_0 \in \mathbf{R}^d$, $\{\sigma_n\}$ is a nonrandom sequence of matrices from $\mathbf{R}^{d\times d}$, $\{A_n\} = \{A_n(\omega)\}$ is a random periodic sequence of matrices from $\mathbf{R}^{d\times d}$, and $\{\xi_n\} = \{\xi_n(\omega)\}$ is a sequence of random vectors from $\mathbf{R}^d$.

We assume that $\{A_n\}$ is a bounded periodic sequence of random matrices with a period K, i.e. $A_{n+K} = A_n$ a.s. for all $n = 0, , 1, 2, ...$ We use the standard abbreviation "*a.s.*" for the wordings "almost sure" or "almost surely" throughout the text.

We assume that $(I + A_{K-1})(I + A_{K-2}) \cdots (I + A_0) = LI$ a.s., where $L \in \mathbf{R}$ and I is the unit matrix in $\mathbf{R}^{d\times d}$. This assumption is quite restrictive for $d > 1$; we consider this case for the sake of generality.

We assume that $\sigma_n \to 0$ as $n \to +\infty$.

We assume that there exists $p \in [1, +\infty]$ such that $\|\xi_k\|_{L_p(\Omega)} \leq 1$ for all k.

For the brevity, we denote $L_p(\Omega) = L_p(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^d)$.

We denote by $|\cdot|$ the Euclidean norm for vectors and Frobenius norm for matrices.

2 The main results

Theorem 1 *(i) If* $\operatorname{ess\,sup}_\omega |L| < 1$ *then* $\|X_n\|_{L_p(\Omega)} \to 0$ *as* $n \to +\infty$.

(ii) Assume that $p \geq 2$, $\operatorname{ess\,sup}_\omega |L| > 1$, ξ_n *are mutually independent and also independent on* $\{A_n\}$, *and either* $X_0 \neq 0$ *or* $\sup_n \mathbf{E}|\sigma_n \xi_n|^2 > 0$. *Then* $\operatorname{limsup} \|X_n\|_{L_p(\Omega)} \to +\infty$ *as* $n \to +\infty$.

(iii) If $L = 1$ *a.s. then* $\|X_n - X_{n+K}\|_{L_2(\Omega)} \to 0$ *as* $n \to +\infty$.

(iv) If $L = 1$ *a.s. and* $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$ *a.s. then* $X_n - X_{n+K} \to 0$ *a.s. as* $n \to +\infty$.

Remark 1 *Condition when, a.s.,* $\lim_{n\to\infty} \sigma_n \xi_{n+1} = 0$, *are given e.g. in [3, 4].*

Starting from now and up to the end of this paper, we assume that $L = 1$ and that the following condition is satisfied:

Condition 1 *One of the following conditions is satisfied:*

(i) $\sum_{k=0}^{\infty} |\sigma_k| < +\infty$; *or*

(ii) $p = 2$, $\sum_{k=0}^{\infty} |\sigma_k|^2 < +\infty$, *and* $\{\xi_n\}$ *is a sequences of independent on* $\{A_n\}$ *and mutually independent identically distributed random vectors such that* $\mathbf{E}\xi_n = 0$.

Lemma 1 *The sum*

$$\bar{Y} = \sum_{k=0}^{\infty} \sigma_k \xi_k.$$

belongs to $\in L_p(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^d)$; *it is defined as the limit of the partial sums in this space.*

Let

$$b_{k,n} = (I + A_{n-1})(I + A_{n-2}) \cdots (I + A_k), \quad 0 \le k < n, \qquad b_{n,n} = I,$$

and let

$$\bar{X}_n = b_{0,n} \bar{Y}.$$

Note that $b_{0,K} = LI$ a.s., $b_{k,K+k} = LI$ a.s. for all k, and $\bar{X}_n$ is a.s. a K-periodic process.

Theorem 2 $\lim_{n\to 0} \|X_n - \bar{X}_n\|_{L_p(\Omega)} = 0$.

The following second question arises: Is it true that $X_n - \bar{X}_n \to 0$ as $n \to +\infty$ a.s.? This question is addressed in the following theorem.

Theorem 3 *Let at least one of the following conditions is satisfied:*

(i) $\sup_{n\ge 0, \omega\in\Omega} |\xi_n(\omega)| < +\infty$, *or*

(ii) Condition 1(ii) is satisfied.

Then $\lim_{n\to 0} |X_n - \bar{X}_n| = 0$ *a.s. and* $\lim_{n\to 0} |X_n - X_{n+K}| = 0$ *a.s.*

Note that the assumption (i) in Theorem 3 does not require that ξ_n are independent and independent from $\{A_k\}$.

3 Proofs

For $n \geq 0$ and $m > 1$, let

$$\psi_{n,m} = b_{n+1,n+m}\sigma_{n+1}\xi_{n+1} + b_{n+2,n+m}\sigma_{n+2}\xi_{n+2} + ... + b_{n+m,n+m}\sigma_{n+m}\xi_{n+m}.$$

Proof of Theorem 1. We have that

$$X_n = b_{0,n}X_0 + b_{1,n}\sigma_1\xi_1 + b_{2,n}\sigma_2\xi_2 + ... + b_{n,n}\sigma_n\xi_n.$$

We have that

$$\begin{aligned} X_{n+K} &= b_{0,n+K}X_0 + b_{1,n+K}\sigma_1\xi_1 + b_{2,n+K}\sigma_2\xi_2 + ... + b_{n+K,n+K}\sigma_{n+K}\xi_{n+K} \\ &= b_{0,n+K}X_0 + b_{1,n+K}\sigma_1\xi_1 + b_{2,n+K}\sigma_2\xi_2 + ... + b_{n,n+K}\sigma_n\xi_n + \psi_{n,K} \\ &= L(b_{0,n}X_0 + b_{1,n}\sigma_1\xi_1 + b_{2,n}\sigma_2\xi_2 + ... + b_{n,n}\sigma_n\xi_n) + \psi_{n,K}. \end{aligned}$$

It follows that

$$X_{n+K} = LX_n + \psi_{n,K}. \tag{1}$$

Let $Y_i = X_{iK}$ and $\eta_i = \psi_{iK,K}$. By (1), it follows that

$$Y_{i+1} = LY_i + \eta_i, \quad i \geq 0, \qquad Y_0 = X_0. \tag{2}$$

Note that the set $\{b_{n+s,n+K}, s \in \{0, ..., K-1\}, n \geq 0\}$ is bounded in $L_\infty(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^{d\times d})$. By the assumptions, $\sigma_k \to 0$ as $k \to +\infty$. Hence $\|\eta_i\|_{L_2(\Omega)} \to 0$ as $i \to +\infty$ and $\|\psi_{n,K}\|_{L_2(\Omega)} \to 0$ as $n \to +\infty$. Then statement (iii) follows. Under the conditions of statement (iv), $\psi_{n,K} \to 0$ a.s. as $n \to +\infty$. Then statement (iv) follows.

Let us prove statement (i). Let $\alpha_i = \|Y_{i+1}\|_{L_2(\Omega)}$, $\beta_i = \|\eta_i\|_{L_2(\Omega)}$, $L_0 = \operatorname{ess\,sup}_\omega |L|$. We have that

$$\alpha_{i+1} \leq L_0\alpha_i + \beta_i, \quad i \geq 0, \qquad \alpha_0 = |X_0|.$$

Consider the equation

$$\bar{\alpha}_{i+1} = L_0\bar{\alpha}_i + \beta_i, \quad i \geq 0, \qquad \bar{\alpha}_0 = |X_0|.$$

By the properties of the solutions of this equation, we have that $\bar{\alpha}_i \to 0$ as $i \to +\infty$. Since $0 \leq \alpha_i \leq \bar{\alpha}_i$, we obtain statement (i).

Let us prove statement (ii). Consider the event $Q = \{|L| > 1\}$. By the assumptions, $\mathbf{P}(Q) > 0$. We have that

$$Y_{i+1} = L^i X_0 + L^{i-1}\eta_1 + \cdots + L\eta_{i-1} + \eta_i$$

and

$$\mathbb{I}_Q Y_{i+1} = \mathbb{I}_Q L^i X_0 + \mathbb{I}_Q L^{i-1}\eta_1 + \cdots \mathbb{I}_Q L\eta_{i-1} + \mathbb{I}_Q \eta_i.$$

Note that $\mathbb{I}_Q$ is non-random on the conditional probability space given $\{A_k\}_{k=0}^{+\infty}$, and the random variables $\mathbb{I}_Q L^{m-1}\eta_m$ are independent on the conditional probability space given $\{A_k\}_{k=0}^{+\infty}$. Clearly, if $X_0 \neq 0$ then $\mathbf{E}|Y_i|^2 \to +\infty$ as $i \to +\infty$, and Theorem 1 is proved in this case.

For a vector $x \in \mathbf{R}^d$, we will use notation $\mathrm{Var}\, x = \mathrm{Cov}\,(x, x)$.

Assume that $X_0 = 0$. In this case,

$$\begin{aligned}
&\mathrm{Var}\,(\mathbb{I}_Q Y_{i+1}|\{A_k\}) = \mathrm{Var}\,(\mathbb{I}_Q L^{i-1}\eta_1|\{A_k\}) + \cdots \mathrm{Var}\,(\mathbb{I}_Q L\eta_{i-1}|\{A_k\}) + \mathrm{Var}\,(\mathbb{I}_Q\eta_i|\{A_k\}) \\
&= L^{i-1}\mathrm{Var}\,(\mathbb{I}_Q\eta_1|\{A_k\}) + \cdots L\mathrm{Var}\,(\mathbb{I}_Q\eta_{i-1}|\{A_k\}) + \mathrm{Var}\,(\mathbb{I}_Q\eta_i|\{A_k\}) \\
&= \mathbb{I}_Q(L^{i-1}\mathrm{Var}\,(\eta_1|\{A_k\}) + \cdots L\mathrm{Var}\,(\eta_{i-1}|\{A_k\}) + \mathrm{Var}\,(\eta_i|\{A_k\}).
\end{aligned}$$

Here $\mathrm{Var}\,(\cdot\,|\{A_k\})$ is the conditional variance given $\{A_k\}_{k=0}^{+\infty}$. By the assumptions, there exists m such that $\mathbf{E}|\sigma_m\xi_m|^2 > 0$. If $|L| > 1$, we have that $\det b_{k,n} \neq 0$. Hence there exists j such that $\mathrm{Var}\,(\eta_j|\{A_k\}) \neq 0$ a.s. given that $|L| > 1$. We obtain immediately that $\mathbf{E}(|Y_i|^2|\{A_k\}) \to +\infty$ as $i \to +\infty$ a.s. given that $|L| > 1$. Therefore, $\mathbf{E}|Y_i|^2 \to +\infty$ as $i \to +\infty$. This completes the proof of Theorem 1.

□

Proof of Lemma 1. Let

$$\bar{Y}_i \triangleq \sum_{j=0}^{i} \sigma_j\xi_j.$$

We have that

$$\bar{Y}_i - \bar{Y}_{i+m} = \sum_{j=i+1}^{i+m} \sigma_j\xi_j.$$

Let us assume first that Condition 1(i) is satisfied.

$$\begin{aligned}
\|\bar{Y}_i - \bar{Y}_{i+m}\|_{L_p(\Omega)} = \Big\| \sum_{j=i+1}^{i+m} \sigma_j\xi_j \Big\|_{L_p(\Omega)} \le \sum_{j=i+1}^{i+m} |\sigma_j|\|\xi_j\|_{L_p(\Omega)} \le \sup_{k\ge 0} \|\xi_k\|_{L_p(\Omega)} \sum_{j=i+1}^{i+m} |\sigma_j| \\
\le \sum_{j=i+1}^{\infty} |\sigma_j|.
\end{aligned}$$

By Condition 1(i), $\sum_{j=i+1}^{\infty}|\sigma_j| \to 0$ as $j \to +\infty$. Hence $\{\bar{Y}_i\}$ is a Cauchy sequence in $L_p(\Omega)$. Then the statement of lemma follows in this case.

Let us assume first that Condition 1(ii) is satisfied. Let

$$\bar{Y}_i \triangleq \sum_{j=0}^{i} \sigma_j \xi_j.$$

We have that $\mathbf{E}Y_i = 0$ and

$$\bar{Y}_i - \bar{Y}_{i+m} = \sum_{j=i+1}^{i+m} \sigma_j \xi_j.$$

Hence

$$\begin{aligned}\mathbf{E}|\bar{Y}_i - \bar{Y}_{i+m}|^2 &= \mathbf{E}\Big(\sum_{j=i+1}^{i+m} \sigma_j \xi_j\Big)^{\top}\Big(\sum_{j=i+1}^{i+m} \sigma_j \xi_j\Big) = \sum_{j=i+1}^{i+m} \mathbf{E}\Big(\sigma_j \xi_j\Big)^{\top}\Big(\sigma_j \xi_j\Big) \\ &\le \sup_{k\ge 0} \|\xi_k\|^2_{L_2(\Omega)} \sum_{j=i+1}^{i+m} |\sigma_j|^2 \le \sum_{j=i+1}^{\infty} |\sigma_j|^2.\end{aligned}$$

By Condition 1(ii), $\sum_{j=i+1}^{\infty}|\sigma_j|^2 \to 0$ as $j \to +\infty$. Hence $\{\bar{Y}_i\}$ is a Cauchy sequence in $L_2(\Omega)$. Then the statement of lemma follows in this case. This completes the proof of Lemma 1. $\square$

Proof of Theorem 2. Let an integer $m \in (0, K]$ be given. We have that

$$X_{nK+m} = b_{nK,m} X_{nK} + \psi_{nK,m} = b_{0,m} X_{nK} + \psi_{nK,m}.$$

By the definition, $\bar{X}_n = b_{0,m}\bar{Y}$ and $\bar{X}_{nK+m} = b_{0,m}\bar{Y}$. It gives

$$\begin{aligned}X_{nK+m} &= b_{0,m} X_{nK} + \psi_{nK,m} = b_{0,m} Y_n + \psi_{nK,m} \\ &= b_{0,m}\bar{Y} + \psi_{nK,m} + b_{0,m}(Y_n - \bar{Y}) \\ &= \bar{X}_{nK+m} + \psi_{nK,m} + b_{0,m}(Y_n - \bar{Y}).\end{aligned} \tag{3}$$

Clearly, $\|b_{0,m}(Y_n - \bar{Y})\|_{L_p(\Omega)} \to 0$. Further, we have that $\|\psi_{nK,m}\|_{L_p(\Omega)} \to 0$. This completes the proof of Theorem 2. $\square$

Up to the end of this paper, we assume that the assumptions of Theorem 3 are satisfied.

Let $\{M_n\}_{n\in\mathbf{N}}$ be a square integrable martingale, $M_0 = 0$ and $M_n = \sum_{i=1}^{n} \rho_i$, where $\{\rho_n\}_{n\in\mathbf{N}}$ is a sequence of independent random vectors with $\mathbf{E}\rho_n = 0$ and $\mathbf{E}|\rho_n|^2 < \infty$. We define

$$\langle M_n \rangle = \mathbf{E} M_n M_n^{\top} = \mathrm{Cov}\,(M_n, M_n) = \sum_{i=1}^{n} \mathbf{E}\rho_i \rho_i^{\top}.$$

A detailed exposition of the definitions and facts of the theory of random processes can be found in, for example, [13].

Lemma 2 *Assume that Condition 1(ii) is satisfied. In this case,*

(i) there exists a.s. finite random variable $\bar{Y}$ *such that, a.s.,*

$$\bar{Y} = \lim_{i\to\infty} Y_i.$$

(ii) $\lim_{i\to\infty} \mathbf{E}|Y_i - \bar{Y}|^2 = 0$ *and* $\mathbf{E}|\bar{Y}|^2 < \infty$.

(iii) $\mathrm{Cov}\,(\bar{Y}, \bar{Y}) = \lim_{n\to\infty} \mathrm{Cov}\,(Y_n, Y_n) = \sum_{i=0}^{\infty} \sigma_i \sigma_i^\top$.

The Proposition below is a variant of the martingale convergence theorem (see e.g. [13]).

Proposition 1 *Let Condition 1(ii) be satisfied, let* $\{\bar{\sigma}_n\}$ *be a sequence of matrices in* $\mathbf{R}^{d\times d}$ *such that* $\sum_{n=0}^{\infty} |\bar{\sigma}_n|^2 < +\infty$*, and let*

$$M_0 = 0, \qquad M_n = \sum_{i=0}^{n-1} \bar{\sigma}_i \xi_{i+1}, \quad n > 0. \tag{4}$$

Then

(i) there exists a.s. finite random variable $\bar{M}$ *such that, a.s.,*

$$\bar{M} = \lim_{i\to\infty} M_i.$$

(ii) $\lim_{i\to\infty} \mathbf{E}|M_i - \bar{M}|^2 = 0$ *and* $\mathbf{E}|\bar{M}|^2 < \infty$.

(iii) $\mathrm{Cov}\,(\bar{M}, \bar{M}) = \lim_{n\to\infty} \mathrm{Cov}\,(M_n, M_n) = \sum_{i=1}^{\infty} \bar{\sigma}_i \bar{\sigma}_i^\top$.

Proof of Proposition 1. Item (i) is a variant of martingale convergence theorems (see e.g. [13]).

For $n > k$, we have

$$\begin{aligned} \|M_n - M_k\|_{L_2(\Omega)} = \mathbf{E}\left|\sum_{i=k}^{n-1} \bar{\sigma}_i \xi_{i+1}\right|^2 = \mathbf{E}\sum_{i=k}^{n-1} |\bar{\sigma}_i \xi_{i+1}|^2 \le \mathbf{E}\sum_{i=k}^{n-1} |\bar{\sigma}_i|^2 |\xi_{i+1}|^2 \\ \le \sum_{i=k}^{n-1} |\bar{\sigma}_i|^2 \to 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

It follows that $\{M_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $L_2(\Omega)$. Fix $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$\|M_n - M_k\|_{L_2(\Omega)} < \varepsilon, \quad \text{as} \quad n, k \ge N.$$

Then, by the Fatou's lemma and the fact that $M_k \to \bar{M}$ a.s., we have, for $n \geq N$,

$$\begin{aligned}
&\|M_n - \bar{M}\|_{L_2(\Omega)} = \mathbf{E}|M_n - \bar{M}|^2 = \mathbf{E}|M_n - \lim_{k\to\infty} M_k|^2 \\
&= \mathbf{E}\left\{\lim_{k\to\infty} |M_n - M_k|^2\right\} = \mathbf{E}\left\{\liminf_{k\to\infty} |M_n - M_k|^2\right\} \\
&\leq \liminf_{k\to\infty} \mathbf{E}\left\{|M_n - M_k|^2\right\} = \liminf_{k\to\infty} \|M_n - M_k\|_{L_2(\Omega)} < \varepsilon.
\end{aligned}$$

By Minkosvki inequality this also implies that $\bar{M} \in L_2(\Omega)$, which completed the proof of (ii).

To prove (iii) we estimate,

$$\begin{aligned}
&\mathbf{E}M_nM_n^\top - \mathbf{E}\bar{M}\bar{M}^\top = \mathbf{E}M_nM_n^\top - \mathbf{E}M_n\bar{M}^\top + \mathbf{E}M_n\bar{M}^\top - \mathbf{E}\bar{M}\bar{M}^\top \\
&= \mathbf{E}M_n(M_n^\top - \bar{M}^\top) + \mathbf{E}(M_n - \bar{M})\bar{M}^\top \\
&\leq \sqrt{\mathbf{E}|M_n - \bar{M}|^2}\sqrt{\mathbf{E}|M_n|^2 + \mathbf{E}|\bar{M}|^2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Proof of Lemma 2. By (2), we have that $Y_{i+1} = X_0 + \sum_{j=0}^{i} \eta_i = X_0 + \sum_{n=0}^{(i+1)K} \sigma_n\xi_n$. Hence $\{Y_i\}_{i\geq 0}$ is a subsequence of $\{X_0 + M_n\}_{n\geq 0}$. Then the proof follows immediately from Proposition 1 applied with $\bar{\sigma}_n = \sigma_n$.

Proof of Theorem 3. Let $m \in (0, K]$. (i) Let assumption (i) holds. In this case,

$$|Y_n - \bar{Y}| \leq \sup_{n,\omega} |\xi_n| \sum_{k=n}^{\infty} |\sigma_k| \to 0 \quad \text{as} \quad n \to +\infty \quad \text{a.s.},$$

since $\sum_{k=0}^{\infty} |\sigma_k| < +\infty$, by Condition 1. It follows that $b_{0,m}(Y_n - \bar{Y}) \to 0$ a.s. Similarly,

$$|\psi_{nK,m}| \leq K \sup_{n,\omega}(1 + |A_k|)^K \sup_{n,\omega} |\xi_n| \sum_{k=n}^{\infty} |\sigma_k| \to 0 \quad \text{as} \quad n \to +\infty \quad \text{a.s.}$$

By (3), the proof of Theorem 3 follows for this case.

Let assumption (ii) holds. By Lemma 2, it follows that $b_{0,m}(Y_n - \bar{Y}) \to 0$ a.s. Further, we obtain that $\psi_{nK,m} \to 0$ a.s., by Proposition 1 applied with $\bar{\sigma}_{n+k} = b_{n+K,n+m}\sigma_{n+k}$ on the conditional probability space given $\{A_k\}$. By (3) again, the proof of Theorem 3 follows. □

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