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Being Mathematical: An Exploration of Epistemological Implications of Embodied Cognition

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This thesis is presented for the Degree of
Doctor of Philosophy
of
Curtin University

December 2012

DECLARATION

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgment has been made.

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ABSTRACT

In this thesis I explore epistemological implications of embodied cognition in the hope of developing my apprehension of what it means to think mathematically. I allow my understanding of embodied cognition to emerge in stages, early in the piece laying contrasts against which it may be set, infolding elements to the purpose of qualitatively interpreting data as my thesis finds form. I use the language of autopoiesis to frame an understanding of change in the context of an individual's learning and also within broader constructs, such as in mathematics classrooms. I recognise dualisms and set them aside in an attempt to reread what it means to think mathematically.

Research from a variety of fields constitutes one part of my data, the second part being a selection of experiences drawn from mathematics classes I have taught. In balancing the two, I find that an embodied account contributes a means of interpreting mathematical experience wherein received boundaries, such as between you and me, and categories, such as "number", are not globally robust, and intentionality pervades and shapes the worlds we create.

The perspective that embodiment affords my apprehension of mathematical thinking is consistent with a formulation in which judgements of what is good are aligned in part with a kind of aesthetic, whereby being moral is founded in innate dispositions. The question of what one is to do with an embodied epistemology is therefore focused on a consideration of how I am to orient myself to teaching mathematics.

Throughout all of this, the locus of my attention remains within the classroom, fixed upon the goal of eliciting perspective and on developing skill in interpreting experience; on becoming a tactful teacher, sensitive to the tacit language of the body.

ACKNOWLEDGEMENTS

I am grateful to my supervisors, Dr Bevis Yaxley and Dr Roya Pugh, for their practical wisdom and enlightened, sage advice; and for the culture of scholarship they so carefully nurture. Also to my school and principal for ongoing interest and encouragement within a community of learning; to my colleagues with whom I share conversations, ideas, frustrations and who have been my prop; and to my students, from whom I learn what it is to learn.

More than all, however, I thank my wife, Janet, and daughters, Jessica, Eliza and Laura, for tolerating my ways and allowing me the space in which to think and write.

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INTRODUCTION

> The inmates of my cottage, all at rest,
> Have left me to that solitude, which suits
> Abstruser musings: save that at my side
> My cradled infant slumbers peacefully.
>
> Samuel Taylor Coleridge (1957),
> *Frost at Midnight*

Some years ago I found a copy of *An Essay on the Psychology of Invention in the Mathematical Field* by the French mathematician, Jacques Hadamard (1954). The text was originally published in 1945 by Princeton University Press, but I cannot recall if the old copy I had found in a library was a Princeton edition or some later version. In any event, I have since acquired a Dover reprint of the original and it sits well on my bookshelf.

The text, as the name suggests, treats with the question of how mathematicians think. More precisely, Hadamard addresses himself to mathematical creation. In this, Hadamard acknowledges that he owes much to the observations of others, including the French mathematician, Henri Poincaré, whose work I, too, have found cause to reference in the construction of this thesis.

In his work, Hadamard discusses four stages of mathematical creation: preparation, incubation, illumination, and verification. Preparation refers to that period of work in which the mathematician consciously thinks about a problem; incubation to a largely unconscious meditation in which the processes of the mind seem to collect and organise about the problem with no input of will; illumination to that dawning, or burst of recognition that signals a solution; and verification to the labour of introducing rigour to validate the solution and prepare it for communication.

The themes discussed in that book have stayed with me. While I might not be sympathetic with all Hadamard wrote, the attention given to the processes of the mind as one wrestles with mathematics have influenced me in my occupation as a teacher of mathematics. How we think, how we learn, what it is we are doing when we suppose we are thinking mathematically—these are matters of abiding interest to me. This thesis is a mark of the influence of that little edition on me and upon on my work.

WHY THIS? ANGELS AND CLAY

From an early age, my interests have coincided with interests of science. I have long felt what I might call a natural resonance with things of the world, with the ecology of life; and methodical scientific inquiry has been my guide. I was raised, however, in a Catholic tradition, wherein I was taught that there are things of flesh and things of spirit and that the things of the spirit are the more significant. As a student of mathematics I walked, as it were, with a foot in each of these worlds: mathematics is often used to solve practical, physical problems, but yet retains something of an almost mystical lustre. Mathematics, logic, *thought* itself: these seemed things beyond this world.

The “natural” world, however, knows nothing of this. When I looked at life, I saw such richness in form and yet such preservation of structure through evolved lines, that I could not dissociate myself from questions of continuity. Does an ant feel? A worm? A lizard? A mouse? Is a bird that notices an egg to be missing able to count? Can cats think, choose? Does a dog have a sense of self? Is it as rich as that of a chimpanzee? How important is language in presenting or maintaining my own sense of self? When I think I am doing mathematics, what is *really* happening? Can these questions be addressed without recourse to dualism? Assertions that render such musings prey to terms akin to “consciousness”, “soul”, “spirit”, “energy” or even “thinking” never satisfied me as being more revealing than ignorance itself, and usually worse, for they created further intractable difficulties. Awareness of ignorance, as Plato has Socrates recall, seemed the wiser path: "'He thinks he knows something when he does not, whereas when I do not know, neither do I think I know'" (Plato, 2002, p. 26, *Apology*, 21d).

Abstruser musings, indeed; but then I am a teacher. I have a professional interest in exploring how we do mathematics, and yet I do not want to replace the difficult problem of how we think mathematically with what seem to me to be the less desirable difficulties of dualism.

My purpose in constructing this thesis, then, is to search assiduously for an epistemological framework consistent with embodied cognition research because I desire to build an understanding of what it is I and my students do when we learn mathematics. This, in brief, is the impetus which drives my project. This, so that I might become more sensitively attuned to the abiding realities of my students' experiences of mathematics. I hope to learn something of how it is that we come to have mathematical concepts and to move closer to understanding sources of mathematical meaning.

This work can be read, then, as an attempt to draw mathematics and mathematics teaching into a naturalistic paradigm in such a way that a teacher of the subject can learn to interpret mathematical behaviour. More significantly, I hope that the work I create remains susceptible to refinement rather than abandonment as new technology and research enable further discoveries to be made about our ability to think and reason. I do not know of any other, like work, written by a teacher of mathematics for teachers of mathematics.

A naturalistic understanding of mathematics permits the possibility that we will recognise epistemological models that fail adequately to take account of how children are able and disposed to learn. As Stanislaus Dehaene (1997) has observed,

> If educational psychologists had paid enough attention to the primacy of intuition over formal axioms in the human mind, a breakdown without precedent in the history of mathematics might have been avoided. I am referring to the infamous

> episode of "modern mathematics," which has left scars in the minds of many schoolchildren in France as well as in many other countries. (p. 241)

I was a child in the 1970s and experienced modern mathematics, a method of teaching in which formal set theory was imposed on young minds, as if we were small information-processing computers, awaiting input, and mathematical knowledge was data. We were not small computers then, neither today do I teach young information-processors.

> On a quiet street where old ghosts meet I see her walking now
> Away from me so hurriedly my reason must allow
> That I had wooed not as I should a creature made of clay –
> When the angel woos the clay he'd lose his wings at the dawn of day.
>
> Patrick Kavenagh (2005), *On Raglan Road*

Forty years on I think it timely that we make the effort to describe ourselves as we are, to let go the angel wings and cease imposing shape upon the clay.

THEMES AND INSPIRATIONS

Michel de Montaigne (1996), the sixteenth century essayist, in his essay *Of Repentance* wrote that

> Others form man; I only report him: and represent a particular one, ill fashioned enough, and whom, if I had to model him anew, I should certainly make something else than what he is: but that's past recalling. (p. 65)

Montaigne was among the first to "only report him," a worthy challenge still for the modern essayist; better, perhaps, we assign the scientist such a task of data management. We who are teachers can fulfil another role, however, for we are gifted to witness in every child the complex and marvellous being of being that is the crux of our profession. We, in runnels and rills of communication, participate in a cultural experiment that has enabled us to create our modern society: that of the education of our young. In learning we strive to perpetuate the good and bring about the better. *What* is good and what will be better remain, as always, to be seen, but we are preconditioned to learn—this much seems clear—and, while George Edward Moore's (2004) admonition against the perniciously seductive naturalistic fallacy resounds yet, we might allow that this much, at least, is good.

Those of us who teach mathematics can find particular interest in exploring questions of how it is we are able to learn to enumerate: what is the source of this ability to count? What, indeed, is it to count, and whence come higher mathematical concepts? Mathematics has long been regarded as paradigmatic of rational thought, the signature of *res cogitans*, we might say, borrowing Descartes' "thinking thing" categorisation, and yet modern analytical research techniques suggest a more nuanced interpretation of this capacity is warranted. This is not a revolution—a careful reading of history shows that mathematics has long undergone revisionary interpretations that, while not perhaps reverberating among the general public, have inspired factions and splinter groups, from Pythagoreans to Intuitionists. As Buldt, Löwe and Müller (2008) note,

> The conceptual framework of mathematics has changed so dramatically that, say, identifying Greek numbers with modern axiomatic characterizations just seems outrageous. That it could seem to be otherwise is basically the product of a modern myth, Cartesian dualism. (p. 6)

The myth of Cartesian dualism is far-reaching and pervades mathematics so that one speaks of ideas and concepts as if they were a species apart from Descartes' second ontological category, the *res extensa*, or extended (corporeal) thing. To do otherwise, however, is to risk invoking cumbersome language, for our very sense of self drives us to this distinction; and yet I seek to find a voice with which to describe a more profound appreciation of the subtleties of mathematical creativity.

Jacques Hadamard remarked that, “Among philosophers, there seems, indeed, to be a certain tendency to confuse logical thought with the use of words” (1954, p. 95). John Searle (2007) emphasises the same and extends the point:

> I think it is fair to say that the philosophy of language is no longer at the center of philosophy. It was the center of philosophy for nearly a century, partly because many people felt that other philosophical problems could only be resolved by using linguistic methods, but also because it was widely accepted among analytical philosophers that all thought requires language. This is a mistake. Human language is an extension of more biologically fundamental forms of intentionality such as perception and action, as well as belief and desire, and we need to see language as derivative from these more basic, biological forms of intentionality. (p. 29)

It is in our use of words that we betray ourselves, but what alternative do we have? The poets know that words serve the senses and yet we of words forget. We writhe and write and believe in our worlds, knowing not whence they come. “‘Beauty is truth’,” said Keats, “‘truth beauty’—that is all / Ye know on earth, and all ye need to know” (Keats, 1956), but no such assertion can quell our desire to know. What, after all, constitutes beauty? What, truth? Among mathematicians there is a tendency to appreciate finely wrought, esoteric forms. An elegant proof is a thing much to be sought and cherished, but it is a rarefied beauty. And even here we can ask, by what lights do we judge this or that to be beautiful? What is the source of our aesthetic?

Mathematicians are familiar with definitions. When I was a student it seemed that much of my study was dedicated to adopting in very precise terms the language of a definition and applying it to tease out consequences for the objects implicated in the definition. Words and symbols were co-opted in rigorously constrained ways to guide explorations; consistency held sway. There was, however, an element of arbitrariness about the process. Had a definition been otherwise, the exploration would likewise have proceeded along quite different lines. A mathematical definition says less about the world than we might suppose—and more about ourselves than we might imagine.

In all this we are drawn to a consideration of what it is to be engaged in thoughtful reflection and sensual appreciation of concepts. So now as ever; but we who reflect on such matters now have a greater—and growing—need to account for mundane facts. We know things today that we did not know yesterday. We can, in broad relief, see the brain at work; we can identify the deep connections between the movement of our bodies in space and the emergence of language and mathematics; we can see how emotions impel cognition. Studies in neurobiology, anthropology and palaeontology combine forcefully to enrich our picture of what it is to be human. Sheridan (2005), for example, reports that skull fossils reveal an increased blood circulatory capacity about early hominin brains, such as would be necessary to provide a cooling effect to compensate for the increased energy expenditure concomitant with the development of symbolic language: "Brains capable of talking and writing needed a new kind of vascular system"; and "cooler brains became bigger brains" (p. 418).

Once I had decided to pursue my thesis—but before it had assumed even an embryonic form—I read three books. Each was to inform my subsequent work in differing ways. The first was Stanilas Dehaene's *The Number Sense* (1997). This work alerted me to neurobiological research that was taking place into questions of how we enumerate. Dehaene made little attempt to question *what numbers are* as such. His work was less a philosophical inquiry into the nature of mathematics, than a bottom-up account of discoveries about how we actually process number. Here was an ingredient I had been seeking: *observations of what is the case* that could inform my deliberations, rather than a top-down imposition of *theories of what might be* to determine the shape of my thoughts. As Dehaene (1997) observed, "any five-year-old has an intimate understanding of those very numbers that the brightest logicians struggle to define. No need for a formal definition: We know intuitively what integers are" (p. 240).

Dehaene (1997) investigated sources of our number sense and found it not so much "ours" as we might have imagined. We now know that very young children and various animals share an innate capacity to recognise small numerosities; we know that there are systematic similarities in the way we process number; we can point to neurological conditions that impede our capacities to process number; and we know that some of the problems to which we apply numbers are variously solved the world over, according to local, pragmatic and cultural concerns (see, for example, Butterworth, Reeve, Reynolds, & Lloyd, 2008; Gordon, 2004).

The second book I read was George Lakoff and Rafael Núñez's *Where Mathematics Comes From* (2000). This work initiated a minor furore, in that it stirred feelings among mathematicians who seemingly felt their subject matter invaded by non-mathematicians. I, too, remember frustrations with the text, and a sense that the authors had "gone too far". And yet, the work was not a mathematical treatise, nor should ever have been read as such. What it offered me was a means of thinking about mathematics within a naturalistic paradigm, a second ingredient in my thesis.

Lakoff and Núñez (2000) applied what they call "mathematical idea analysis" to address the foundational question of where mathematics comes from. At the core of their approach is the understanding that conscious, deliberative thought-making is rooted in unconscious processes that derive, ultimately, from the way in which we engage with the physical world. Entailments of such engagement, such as the simple fact that many objects can be held and grouped, transfer via a process of inference-preserving metaphorical mapping to ideas: objects in the mind can be held and grouped, too. Through an aggregation of conceptual processes, the body of mathematics is constructed on the framework of a real body with a minimal innate ability to be aware of small numbers and a sophisticated capacity for negotiating a complex environment.

Meaning, under this scheme, inheres ultimately in a groundedness, a being in the world; but it is experienced by the individual—according to the extent of her familiarity—as thinking, as a gathering together (for this is the root meaning of cognition) of associated ideas, valenced with emotional seams. This construal of meaning brings us, *as teachers*, to a consideration of what we ought to do. That is, to the ethical question: within the embodied cognition framework, are there normative guides to elicit moral behaviour? What are to be the foundations of morality?

Here I found indirect guidance from my third key author. When I read Shaun Gallagher's *How the Body Shapes the Mind* (2005) I found a third ingredient, something that would bind the first two and permit my thoughts to grow. The ingredient is not so easily described, however, for it is less a thing than an attitude. Gallagher eruditely folds phenomenological sensibilities through scientific research by drawing on wisdom where he finds it to illuminate what science can possibly be saying. Chief among his determinations is a recognition of the fundamental significance of the physical qualities of the growing body on our perceptions of self and of our entanglement with others, a coupling so rich that it I felt it bereft to speak of notions such as "good" without recourse to and appreciation of our evolved, embodied intentionality.

I recall feelings of excitement as I contemplated that the arbitrariness of mathematical definitions with which I was familiar and comfortable might be mirrored in arbitrariness of received concepts such as "you" and "me". Even within my own body, the scope of my boundaries seemed to shift, to blur in time, and the continuity I had recognised in the natural world engaged my philosophical considerations: maybe there are no boundaries, not really; we bring our worlds into being according to our momentary intentionalities. And yet, there are stabilities, or equilibria, and these we recognise, name and so reify. But the naming is a convenience, a fiction; no name can fix the world in time. No naming can make absolute what is ephemeral and intimate with our evolved condition, be it a number or a moral good.

For Charles Taylor (1989, p. 105), "Our visions of the good are tied up with our understandings of the self," but then Gallagher (2005), on analysing the experiences of the body, asserts:

> In all these aspects of experience and *before you know it*, the body is working to shape the mind.... [To] find a viable interpretation of experience, a viable science of cognition that is consistent with these facts [about the way the body experiences phenomena], requires that we begin to redraw the map that guides our understanding of how precisely embodiment contributes to human experience. (p. 245, author's italics)

Our visions of the good, our sense of self, our embodied being: to approach what we ought to do necessitates a recognition of who we are. And in this, continual vigilance is required so that as we learn new facts about the world we remain open to the possibility of living a moral life—of perpetuating the good and bringing about the better.

To the extent that the development of this thesis has represented something of a creation for me, it does not seem so out of place that Hadamard's stages apply here, as they do to mathematical creation. This work has, moreover, unfolded over time as I have gone about my daily business of teaching, and so it represents a somewhat autobiographical reflection pieced together over several years of active engagement in teaching and research. By way of a small tribute to Hadamard, then, and to aid the reader, I have partitioned the work into four parts, each named to represent one of the four stages: preparation; incubation; illumination; and verification.

Part One, Preparation, introduces the reader to various fragments that underlie the subsequent development of my thesis. They include elements of fixity and change and a dualistic worldview, a background as it were, to the elaboration which follows. Autopoiesis, or "self-making", is introduced, preparatory to learning to reinterpret the world and our place in it without recourse to dualisms. The human face of mathematical activity is foregrounded through a reflection on mathematical proof, and the first part rounds out with a recognition that there is an ethical implication when one attempts to reconstitute an understanding of learning.

In Part Two, Incubation, I focus on the question of what knowledge is. This entails ranging through themes that have underwritten our conceptualisation of knowledge, including syntax and semantics. The traditions developed since Renee Descartes and formulated by Alan Turing are considered, and the view that having knowledge of something involves abstracting general principles is investigated. The question of how knowledge could be instantiated in the brain closes out this part, leading us, as it does, from knowledge as abstract possession to knowing as embodied behaviour.

Part Three, Illumination, turns more to knowing as an orientation, in which language, including number, is used to indicate that which we wish to express, but is now considered secondary to the knowing of the body. The felt experience of being mathematical is highlighted throughout this part. Metaphorical analysis is described as a means of interpreting how the body increases its ways of knowing, and processes of thinking are illustrated through examples of mathematical creativity.

The final part, Verification, moves through themes associated with ethical questions. Does an embodied account of knowledge force us to reconsider how we ought to act? I consider the possibility that an understanding of our ethical orientations as having an evolutionary track can influence interpretations of our feelings about how we should act. In terms of the classroom, I move toward a view in which my understanding of what I feel drawn to do is of a kind with an aesthetical response to my situation, in which feelings of fitting and shared involvement with my students are pre-eminent.

The final part of this introduction is given over to an elaboration of the methodological processes and sensibilities that I have attempted to employ in the construction of my thesis.

THESIS AND METHOD: BRINGING FORTH A WORLD

My thesis is that an epistemology consistent with theories of embodied cognition can—and does—inform and enrich the teaching of mathematics. It affords a means of being with students such that one can become a more tactful teacher, deeply conscious of the subtleties of experience. The development of this thesis represents something of a journey, a living-with-an-idea that has pervaded my professional practice and swelled my private moments with contemplation. The pages that follow reflect this working through of ideas. My method is to interpret data qualitatively through the lens of embodied cognition, thereby to grow an understanding of the implications of what the data says. The development of this thesis is, in effect, *my search for a naturalistic epistemology proper to the philosophy of mathematics education*.

The data itself can be described in two parts. One the one hand, it is my experience of teaching students over nearly twenty years, lessons I have recorded, moments that have impressed themselves on my mind. These data serve to illustrate and inform ideas and to impel me toward new thoughts. The second part of my data consists in scientific explorations, reported in literature, that provide new knowledge which is to circumscribe and inform my emerging epistemology.

In such a way, this research project represents something autobiographical. I am a teacher of mathematics and my drive to develop a way of understanding my world that is consistent with scientific research has required me to suspend and question terms and entities that I suppose we often take for granted, from "number" to "knowledge", but none the more so than my very sense of self. In large measure, I am attempting to follow the exhortation of Rosiek and Atkinson (2005) to engage as a reflective practitioner:

> Teachers are increasingly seen as professionals who must be prepared to be reflective practitioners, who engage in inquiry that informs their teaching practice, and who occasionally publish original research on their teaching that can inform other teachers' practice. (p. 421)

In conducting this inquiry, I steer a wide course, bound by the concern that what I learn can inform what I do in the classroom and beyond. The scope of the project is limited to that purpose, although it cannot be denied that the effect of attempting to construct an epistemology is more profound for me, at least. I make no claim to describe the world as it is but only attempt to craft a way of fitting.

It is to be understood that the interpretations I place on literature are selective. I am in no position to test scientific reports for myself and so I can but weigh the data with something approaching a balance between credulity and cynicism. In any event, the breadth of the data is too wide to permit sustained critical testing. Rather, I attempt to triangulate the scientific data by ensuring that it is consonant with emerging practices within the fields of study and is weighed against alternative data sets.

By presenting observational data I am hoping to sustain the attachment of the research to the goal of informing teaching practice, to prompt and illuminate the discussion of scientific research, and to emphasise the felt consequences of our teaching behaviours, in sympathy with what Mason (1998) describes as the process of inner research: “Inner research is about developing sensitivity, whether to mathematical ideas, to pedagogical possibilities, or to the thinking of other people. Sensitivity means noticing (making distinctions)” (pp. 362–363).

In this, I find myself in a somewhat compromised situation, writing in a philosophical, positivist tradition with my left hand but with a hermeneutic interest with my right; and yet each form of sensitivity must support the other. Tony Brown (2001) describes the relationship between positivist and hermeneutic inquiry:

> Positivism is concerned with “is” statements rather than normative “ought” statements and as such offers no guidance to revising ways of doing things. Meanwhile, the interpretivist or “insider” perspectives focus on the world as experienced by inhabitants in particular situations. The hermeneutic approach underlying this concerns a reconciliation between experience and ways of describing it. (p. 7)

In practice, I am attempting less to *describe* experience as an insider than to *notice* it and inquire into its source, to learn something of how we can come to have (mathematical) experience at all, in the hope that I will be able to return to my role as insider, the better equipped to encourage healthy experiences of learning mathematics.

Among the aids I adopt, the concept of autopoiesis, discussed presently, provides a useful mechanism with which I have attempted to organise my thoughts. Put briefly, autopoiesis means "self-making", appropriate to the process of educating oneself, but consonant in a broader sense: to see ourselves as mathematical thinkers in autopoietic terms is to posit a means whereby we are able to broaden our autopoietic domain. More pithily, as Humberto Maturana and Francisco Varela, the authors of autopoiesis, have put it, "*every act of knowing brings forth a world*" (Maturana & Varela, 1992, p. 26, authors' italics). The development of this thesis, then, is a bringing forth, a broadening of my world as teacher, imbued with the hope that as it grows, I learn to engage tactfully with my students and to enhance their experience of learning mathematics.

I shall go on now to consider briefly the notion of interpreting science, or the relationship between science and hermeneutics, and to recognise that there is a tension between general and particular observations. The role of the teacher as researcher shall then be discussed through an emphasis on the practices of listening.

Science and Hermeneutics

> Everything except language
> knows the meaning of existence.
> Trees, planets, rivers, time
> know nothing else. They express it
> moment by moment as the universe.
> Even this fool of a body
> lives it in part, and would

have full dignity within it
but for the ignorant freedom
of my talking mind.

Les Murray (2011), *The Meaning of Existence*

Naturalistic, or scientific, attempts at reasoning are liable to be categorised as apodictic—in which truths are *demonstrable*, that is to say—but this is to understand such reasoning in reduced terms. Whilst scientific practice aspires to rational transitions to guide its path, it need not be the case that a scientific approach rebels against an interpretative attitude toward phenomenal accounts. A mature science will recognise the ad hominem character[1] of its own speculation and impetus; will recognise, indeed, that to do science is to engage with a form of hermeneutic circle, wherein rational scientific transitions are balanced ineffably by interpretations sourced in the characters, interests and motivations of the scientists and the broader community. As Shaun Gallagher (2004) puts it:

> One often gets the impression that if one is doing hermeneutics, then one cannot be doing science, and vice versa. I think, however, that there is no question that if you sit down with practicing scientists who are at the cutting edge of their fields, they will be the first to admit as an obvious fact what Gadamer, among others, has suggested. The practice of science is itself hermeneutical. That is, scientists make interpretations, and their interpretations are biased in a very productive way by the scientific tradition to which they belong, and the specific kinds of questions that they ask. Explanation is no less interpretation than understanding. (p. 3)

That is to say that within the constraining parameters afforded by a particular paradigm, scientists practising what Kuhn (1996) calls “normal science” are compelled to interpret data, and that this interpretation, at its best, is hermeneutical in character. To the extent that I will be interpreting science and classroom scenarios as I develop my thesis, bounded by the traditions to which I belong, which is to say to that of mathematics and science and education, this thesis is written with a hermeneutical attitude.

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[1] By which is meant the biased, emotionally laden quality that may characterise human activity.

This approach, applied with regard to scientific data, opens the possibility to interpretations that make new demands on the practising teacher, even to the extent of promoting what Kuhn describes as paradigmatic crisis.

> So long as the tools a paradigm supplies continue to prove capable of solving the problems it defines, science moves fastest and penetrates most deeply through confident employment of these tools. The reason is clear. As in manufacture so in science—retooling is an extravagance to be reserved for the occasion that demands it. The significance of crises is the indication they provide that an occasion for retooling has arrived. (Kuhn, 1996, p. 76)

The development of a new epistemology is, for the teacher, a retooling of high order.

The Dialectic of the General and the Particular

> Welcome to Lake Wobegon, where all the women are strong,
> all the men are good-looking, and all the children are above
> average.
>
> Garrison Keillor (Prairie Home Productions, 1974)

Scientific understanding, for Friedman (1974), is “global rather than local”:

> Scientific explanations do not confer intelligibility on individual phenomena by showing them to be somehow natural, necessary, familiar, or inevitable. However, our over-all understanding of the world is increased; our total picture of nature is

> simplified via a reduction in the number of independent phenomena that we have to accept as ultimate. (p. 18)

The influence of this scientific approach is felt in educational research. In the classroom, on the other hand, the teacher is required to call on other resources creatively to accommodate the gap between the general, predicted behaviour and the observed, actual, noisy behaviour of students.

How, if at all, are we to apply scientifically generated, general knowledge in the context of the classroom? Pamela Moss (2005) reports on the related matter of warrant: how are we to choose rightly when to apply a general finding to a particular context? She applied a hermeneutical attitude to the task of reviewing responses to the National Research Council's publication of *Scientific Research in Education (SRE)*. She found that the authors and critics agreed that context "matters deeply" (p. 275), but that there were differences: while the authors concerned themselves with locating the boundaries of generalizations, critics identified the problems with applying generalizations in given contexts with unique features. On this account, identifying the constraints of a generalization is insufficient to provide a warrant for that generalization in any particular case:

> The number and variety of interactions with other features of the context make it impossible to know whether a particular theory will work in the same way here and now for the students in a given classroom, the teachers in a given school, and so on. Therefore, on this view, a program of research that culminates in well documented (valid) generalizations culminates at the wrong place. The warrant for any particular case—any particular classroom, school, district, student, and so on—is weak, perhaps too weak, to serve as a basis for consequential decision. (pp. 275 – 276)

When faced with a generalisation, we must decide whether to enact it in our given situation. On what basis do we make that decision? That is, how do we decide whether the general result will apply to our particular situation? Beneath the status of the generalisation lies a deeper judgement: we weigh our knowledge of the generalisation against the particularities of our situation and determine if there is a match. How do we do this? We do not rely on the general result, we judge it. We do not rely on the particulars, we balance them. What is brought to the judgement, to the balance? This is the locus of the hermeneutical response, the hermeneutic rule, as Gadamer (2004) reminds us, that "we must understand the whole in terms of the detail and the detail in terms of the whole" (p. 291). Which generalisations are germane to our particular situation? How do the particularities of our situation relate to generalisations? What interpretations are brought to bear, and in any event, what interpretations predetermined the generalisations themselves?

This is not to argue against the value or power of the generalisation. It is, rather, to suggest that one is able to accept or reject its application. It is in the interplay between the general and the particular that the creative agent finds expression and action. By way of making a generalisation, a guiding principle is that one should apply generalisations in situations of generality but consider particular features when confronting unique cases. A school district, for example, will make use of generalised data about expenditures and educational programs, but the teacher will consider each child as a particular. The challenge for the teacher, evident throughout this thesis, is to assimilate and interpret the general in order to distil what can be applied in each particularity.

So, while we arm ourselves with scientific generalisations, as teachers we might perhaps do well to draw on van Manen (1990), for whom

> pedagogy requires a phenomenological sensitivity to lived experience (children's realities and lifeworlds). Pedagogy requires a hermeneutic ability to make interpretive sense of the phenomena of the lifeworld in order to see the pedagogic significance of situations and relations of living with children. And pedagogy requires a way with language in order to allow the research process of textual reflection to contribute to one's pedagogic thoughtfulness and tact. (p. 2)

Phenomenology itself, however, is not unconcerned with universals. Indeed, intuiting the essence of a phenomenon—that universal, defining aspect which can be said to underlie any particular expression—was very much the purpose of the phenomenological practice, as developed by Edmund Husserl (1970), for whom phenomenology had “as its exclusive concern, experiences intuitively seizable and analysable in the pure generality of their essence” (p. 166).

In this sense, phenomenology has something in common with science. Indeed, phenomenology, carefully practiced, is scientific to the extent that it seeks to take little for granted, is self-critical and draws attention to assumptions. Where it may differ, particularly in its application to education, is in the means of finding universals or essences. For the phenomenologist, subjective reflections are a means of inquiry; for the scientist, they are to be controlled[2]. For the phenomenologist, meaning is to be found in interpreting the lived experience of the subject: the scientist seeks less to elicit meaning than to describe, but must always consider the twin questions, what to describe and why?

> Phenomenology is the systematic attempt to uncover and describe the structures, the internal meaning structures, of lived experience. A universal or essence may only be intuited or grasped through a study of the particulars or instances as they are encountered in lived experience. (van Manen, 1990, p. 10)

So we embrace a dialectic between the particular and the general:

> Phenomenology consists in mediating in a personal way the antinomy of particularity (being interested in concreteness, difference and what is unique) and universality (being interested in the essential, in difference that makes a difference). (van Manen, 1990, p. 23)

Mathematicians enjoined to practise concision are accustomed to excising examples and particular applications of their findings in favour of general statements, thereby rendering their texts dense. In teaching, on the other hand, one may learn that generalisations are inherently other-directed and so adapt practices to appeal to the particular. Teaching demands of the teacher a phenomenological sensitivity: always and ever dialogically to seek ways between too little and too much; too narrow, too broad; too fine, too coarse. Teachers are uniquely placed to analyse learning, to weave a story from the weft of particular experiences on the warp of grander, more general theories and expectations.

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[2] Although scientific research in fields that touch on qualitative experience necessarily grant a certain privilege to subjective reporting.

Teaching, Noticing, Listening

I now turn to the perspective of the teacher-researcher and consider what it is to listen. The perspective afforded educational research by the teacher has been noted by Rosiek and Atkinson (2005):

> Over the past twenty-five years a research movement has emerged that seeks to bridge the gap between educational theory and practice by taking seriously the intellectual dimension of the practical work of teaching. This work includes, but is not limited to, research on teachers' practical knowledge, craft knowledge, personal practical knowledge, and wisdom of practice, as well as teacher research, action research, and the scholarship of teaching. Researchers contributing to this literature ask questions about the content, production, epistemology, and representation of teachers' practical knowledge. In a short span of time this movement has made a lasting impact on teacher education research and practice. (p. 421)

In a similar key, van Es and Sherin (2007) emphasise the importance of teachers reflecting and learning to notice. They have developed a *Learning to Notice Framework* consisting of three main aspects:

> (a) identifying what is important in a teaching situation; (b) using what one knows about the context to reason about a situation; and (c) making connections between specific events and broader principles of teaching and learning. (p. 2)

They make the observation that at the heart of much research on teacher reflection

> is the claim that reflection is key to improving one's teaching. Engaging in reflection allows teachers to make sense of their experiences and to then use this knowledge to inform future decisions. We argue that learning to notice is one important dimension within the process of reflection that deserves additional attention. (pp. 3–4)

Drawing on this research they highlight the importance of interpreting classroom events: "How individuals reason about what they notice is as important as the particular events they notice" (p. 4). They see that there is a

> goal then for teachers to look at teaching situations for the purpose of understanding what happened, for example, to consider what students understand about the subject matter or how a teaching strategy influenced student thinking, as opposed to examining a situation for criticism to take action. While teaching certainly involves making judgements about what went well or poorly in a lesson, we believe it is critical for teachers to first notice what is significant in a classroom interaction, then interpret that event, and then use those interpretations to inform their pedagogical decisions. (p. 4)

As a reflective teacher, I ruminate on perceptions, expectations, understandings and judgements of what mathematics is, what it means and what it signifies, as encapsulated in the actions, statements and expressions of students, parents and colleagues, as surely as it is in my own utterances and actions. To reach students, to empathise with them in the hope of communicating, I try to become aware of ways in which children see mathematics, see me and see themselves; and how we each of us understand our roles in the classroom. I endeavour to implement what Davis (1997) describes as hermeneutic listening, "a title intended to reflect the negotiated and participatory nature of this manner of interacting with learners" (p. 369):

> Hermeneutic listening demands the willingness to interrogate the taken for granted and the prejudices that frame our perceptions and actions. Further, drawing from the traditions of hermeneutics … the notion of hermeneutic listening is intended to

> imply an attentiveness to the historical and contextual situations of one's actions and interactions. (B. Davis, 1997, pp. 369-370)

We can observe that listening can take place "in the moment", as the teaching unfolds, in which case it is practice; and in the reflection, after the event has opened itself, in which case it is research. We do well to ask ourselves whether we are listening hermeneutically to what is happening, or listening to, or listening for. Listening hermeneutically to teaching episodes would imply that one is not to be listening for occasions in which hermeneutic listening was employed; that would not be hermeneutic, but pedagogic. The listening, rather, would be characterised by listening *with*, by moving into the episode and making deeper inquiry into the circumstances that pertain, constrain and structure the event.

This thesis, then, is infused with the spirit of learning to notice and listen. The hermeneutic locus, however, is underwritten by a naturalistic philosophy, built upon embodied cognition in particular but inclusive of naturalistic accounts in general, in which meanings may be elicited and out of which a sense of tactful appreciation of the significance of teaching may unfold. The autopoietic model serves as more than an analogy for the hermeneutic circle—in the quest for levels of description proper to phenomena, I shall be seen to be acting autopoietically. I shall draw information from scientific literature and so create a domain of interpretative action. I shall reflect on phenomena and so build and maintain that domain within a hermeneutic horizon. I will strive to become that which I seek to know: a teacher sensitive to the meaning of what it is to be mathematical.

Henry David Thoreau, in his 1862 essay *Walking*, wrote of walking in terms that celebrate the art of patient, quiet noticing. To *saunter* captures the sense of strolling with loose purpose, all the while bending toward some perhaps unforeseen end:

> I have met with but one or two persons in the course of my life
> who understood the art of Walking, that is, of taking walks, —
> who had a genius, so to speak, for *sauntering*: which word is

> beautifully derived “from idle people who roved about the country, in the Middle Ages, and asked charity, under pretence of going *à la Sainte Terre*,” to the Holy Land, till the children exclaimed, “There goes a *Sainte-Terrer*,” a Saunterer, — a Holy-Lander. They who never go to the Holy Land in their walks, as they pretend, are indeed mere idlers and vagabonds; but they who do go there are saunterers in the good sense, such as I mean. Some, however, would derive the word from *sans terre*, without land or a home, which, therefore, in the good sense, will mean, having no particular home, but equally at home everywhere. For this is the secret of successful sauntering. He who sits still in a house all the time may be the greatest vagrant of all; but the saunterer, in the good sense, is no more vagrant than the meandering river, which is all the while sedulously seeking the shortest course to the sea.
>
> (Thoreau, 2006, author's italics)

If at times it seems that this thesis saunters, then let it be in the good sense of Thoreau—not with idle vagrancy, but finding its passage midst experiences of my teaching life and being “at home” in all the various fields of inquiry to which I turn, for “a path exists only in walking” (Varela, Thompson and Rosch, 1991, p. 241).

Thoreau also wrote—more or less famously, though often misquoted—that “in Wildness is the preservation of the World” (Thoreau, 2006). I am not so far from him in saying this, though for me the wildness is perhaps better cast as a deep respect for what we are and how we came to be—for our wild heritage, I might say. Throughout my thesis runs this thread: that I value first an understanding of who we are over knowledge of what others might say we ought to be.

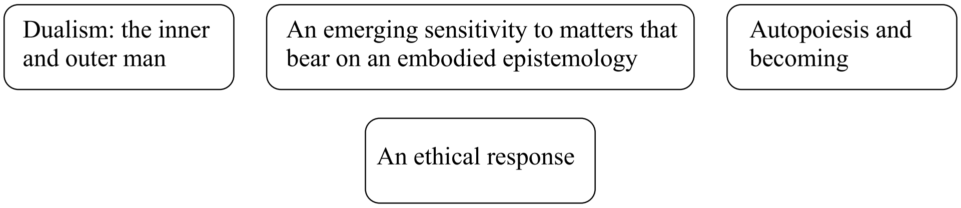
So then, thus prepared, let us walk.

PART ONE PREPARATION

In this part I prepare a way by laying out key themes. I present a classroom scenario to introduce several motifs that will be developed in the pages to follow, and also to illustrate the process of learning to notice and question structures. I then provide a brief introduction to autopoiesis and consider mathematics as an evolved system. The question of our sense of self underlies much of my thesis, particularly insofar as it encourages a dualistic worldview, and so to contextualize my work I describe Augustine's understanding of the inner and outer man, the more to present a contrast against which an emerging understanding of embodied mathematics can be set. I reserve a more full description of what is to be understood by "embodied mathematics" for later.

Among the ideas that emerge as significant in this chapter are that a hermeneutic attitude underlies scientific questioning; that our sense of self, which includes our habits of thought, has the potential to inveigle us into interpreting the world dualistically; and that a move away from acceptance of fixed forms toward a dynamic of becoming opens the possibility of reinterpreting what it is to have knowledge. Above all, it emerges that we are implicated in making our world, so that an understanding of what mathematics is will require an understanding of how we think.

I round out this part by foreshadowing the question of how one is to respond to the possibility of a new epistemology: if we adopt an embodied epistemology are there implications for our understanding of what is moral?


Dualism: the inner and outer man
An emerging sensitivity to matters that bear on an embodied epistemology
Autopoiesis and becoming
An ethical response

CHAPTER ONE THE TERRAIN

My life has been the poem I would have writ,
But I could not both live and utter it.
Henry David Thoreau (1868, p. 364)

What Will the World Allow?

The word "prepare" shares its origins with the word *pare*, from the Latin *parare*, to make ready, as in to pare away—typically the skin of a fruit or vegetable—before (*pre*) eating. The word is related to *parere*, to produce or bring forth. In taking this for my opening stanza, then, I indicate a desire to trim away an outer skin in order to reveal a finer fruit, so that I might bring forth a fresher understanding of what it is we do when we teach mathematics.

Treatments of mathematical philosophy sometimes focus on mathematics as communicated. Kitcher (1984), for example, rejects the apriorism of mathematics maintained by some philosophers, and focuses on the question of how people have mathematical knowledge. This necessitates deep considerations of warranted belief and the processes by which rational transitions in knowledge can occur. Kitcher appreciates Kuhn's (1996) reference to *scientific practice* as a means of coming to grips with the processes underlying scientific change, and whilst jettisoning Kuhn's concomitant paradigm concept, adopts *mathematical practice* as central to a thesis about mathematical change:

> I suggest that we focus on the development of *mathematical practice*, and that we view a mathematical practice as consisting of five components: a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important, and a set of metamathematical views (including standards for proof and definition and claims about the scope and structure of mathematics). (Kitcher, 1984, p. 163)

These components are developed to account for the growth of mathematics, via rational interpractice transitions, within community. For those with a foundational bent, the question immediately arises as to what ultimate grounding mathematical beliefs can have, to which Kitcher advises, “A set of (mathematical) beliefs is *directly warranted* if its members are all perceptually warranted” (p. 225, author’s italics). The perceptual warrant, in turn, finds its source in experience of the world:

> Children come to learn the meanings of ‘set,’ ‘number,’ ‘addition’ and to accept basic truths of arithmetic by engaging in *activities* of collecting and segregating. Rather than interpreting these activities as an avenue to knowledge of abstract objects, we can think of the rudimentary arithmetical truths as true in virtue of the operations themselves. (pp. 107–108, author’s italics)

Kitcher summarises his view pithily thus: “We might consider arithmetic to be true in virtue not of what *we can do* to the world but rather of what *the world* will let us do *to it*” (Kitcher, 1984, p. 108, author's italics).

We might conclude that this is a purely naturalistic view and worry that belief founded on *the way things are* is belief perilous. Yet Kitcher speaks of a perceptual warrant: that is, of a basis on which we might reasonably proceed. The warrant rests, moreover, less on things as they are, than on a relationship between ourselves and things: on our interactions with the things of the world, not forgetting that we are to be counted among those things. That is, the perceptual warrant is founded in our sense-making engagement with the world, on which basis mathematics can be read as a “working-through” or “being-in-the-world”.

so much depends
upon

a red wheel
barrow

glazed with rain
water

beside the white
chickens

William Carlos Williams (2011), *The Red Wheelbarrow*

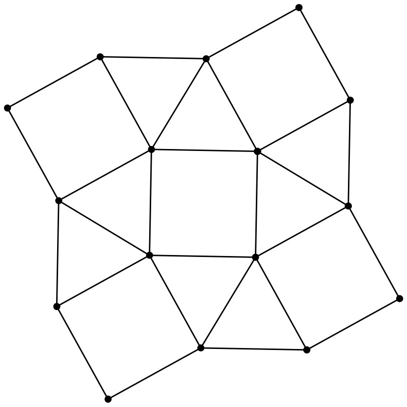
So much depends... It might be said that we have become accustomed to responding intellectually to questions philosophical, and yet in this homely scene summoned by Williams we find a warrant to peel back—to pare away—sophisticated accretions and consider the foundational importance of water, of chickens, and to reappraise what a wonder of the ages a wheelbarrow is, what an earthy thing it is *to count*: and then ponder what a *human* thing it is to turn counting into mathematics.

In recent years this realist orientation has become susceptible to subtle yet deeply significant elaboration. The question of what the world will allow—and indeed, of what it has already wrought—informs the kernel of this study.

ENCOUNTERS WITH FORMS: FIXITY AND CHANGE

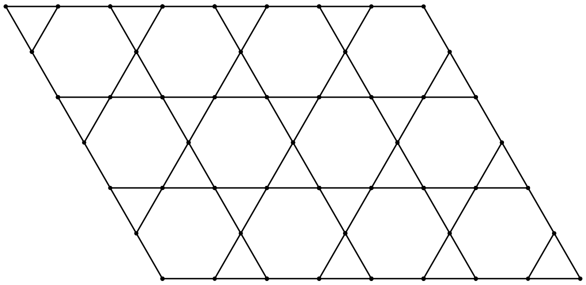
I teach mathematics in an independent school for girls. My classes range from specialist sessions with Year Fives through to university preparation lessons with Year Twelves. Among the specialist classes I have taught was one with a group of Year Seven girls who had been selected as showing mathematical promise. On one occasion when I arrived to meet with this group, one among them, Claire, was behaving in a manner that I described at the time as “silly”. Claire is an exuberant child, but this particular morning she was acting out in a manner that threatened to disrupt my teaching purpose. About half the girls had arrived at this time, but in order to prevent Claire from encouraging an unhelpful dynamic I adopted a serious tone and said to her, “I won’t have that.” She understood the implication immediately, and, chastened by the rebuke, quietened as the other girls arrived.

We sat together in a circle, no desks, a small group in a space outside the regular classroom. I presented a problem. It was to: (i) find a pair of "possible" semi-regular tessellations by using the angles of regular triangles and hexagons; and (ii) to attempt to construct these semi-regular tessellations. A semi-regular tessellation is a covering of the plane with regular polygons with common side length such that the arrangement at each vertex is identical. An example using squares and triangles is shown below, together with a reduced symbolic presentation, the Schläfli symbol (Watson, 1973).



[4, 3, 4, 3, 3]

The description of the problem with which I had provided the girls included an example of a semi-regular tessellation using triangles and hexagons:



[3, 6, 3, 6]

By referring to the angles of the triangle and hexagon, the girls were required to show that there were two other arrangements to consider that might tessellate.

The structure of the question as it was posed presented its own challenges for the girls, and energy was expended on understanding what was meant by “semi-regular tessellation”. Some questions asked were definitional: “What does ‘same rotational order’ mean?” Others were what we might call operational, or consequential: “Do the triangles have to be the same size as in the examples?” The suggestion that there were two “possible” tessellations prompted the girls to believe that there were, in fact, two *valid* tessellations to be found. The wording of the question was inherently ambiguous on this point, but the girls did not notice this difficulty and embraced the challenge of finding the tessellations.

Initial forays were more directed to drawing rather than calculating. That is, rather than attend to the structure of the question and attempt to apply the heuristic of being guided by that structure, the girls expressed their understanding of “tessellation” by drawing diagrams in an exploratory mode. Their drawings were somewhat imprecise, however, which had the consequence that it was often unclear whether a drawing represented a valid tessellation or not; inaccuracies in drawings led some girls to believe that they had found a tessellation when, in fact, they had not. Since the accuracy of their drawing was a limiting factor, I provided isometric dot and grid paper. This had the effect of constraining and binding their searches.

I had weighed the option of turning to a software environment, such as *The Geometer’s Sketchpad*, but there were reasons that I did not do so. I was interested in observing the manner in which the students coped without significant scaffolding and environmental aids. The provision of dot and grid paper was a considered compromise. Even this may have had the effect of endorsing the abandonment of the heuristic, “follow the question structure”, although the girls were set on their path before the paper was introduced. Further to this, the introduction of a software environment introduces the cost of a technological distracter. I had wanted to control for features of the lesson that would contribute to the girls’ experiences; the addition of a technological effect would likely have resulted in a quite different outcome.

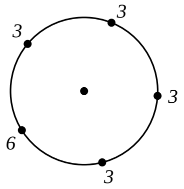
Attention was initially confined to a more or less trial and error searching process, as the girls drew diagrams, checked and searched. After a period of this activity, however, several girls gave more attention to the clue in the question, and began to consider the importance of the polygonal angles. The significance of 360º and of the interior angles in regular triangles and hexagons (60º and 120º) emerged, as the girls realised that the example given was, in part, a demonstration of the mundane fact that $60^\circ + 120^\circ + 60^\circ + 120^\circ = 360^\circ$. With this in mind, another possibility (using both polygons) was soon discovered:

$$60^\circ + 60^\circ + 60^\circ + 60^\circ + 120^\circ = 360^\circ.$$

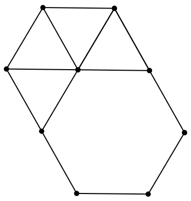
The other sums, $60^\circ + 60^\circ + 60^\circ + 60^\circ + 60^\circ + 60^\circ = 360^\circ$ and $120^\circ + 120^\circ + 120^\circ = 360^\circ$, while each giving a trivial semi-regular tessellation, did not qualify as candidates in this exploration as each required but a single polygon.

Victoria had been the first to realise that it was impossible to construct a semi-regular tessellation with precisely one triangle at each vertex and assumed the role of a teacher at the whiteboard to explain this to a classmate. Her appropriation of the structures of teaching caught the attention of others in the class, and it was not long before her finding had filtered throughout the group so that interest resolved around the cases [3, 3, 6, 6] and [3, 3, 3, 3, 6], understood as “two triangles” and “four triangles” at a vertex.

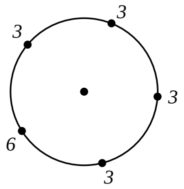
The question remained whether varying the *cyclic order* of the sum would lead to a different tessellation. Would rotational variations lead to replications? For example, the construction [6, 3, 6, 3], once drawn, was found to be identical to the given [3, 6, 3, 6]. Variations such as [3, 3, 3, 6, 3] and [3, 3, 6, 3, 3] likewise came to be seen to be identical to [3, 3, 3, 3, 6]. The drawing of the tessellation appeared to be necessary in most cases in order to recognise such rotational congruences; the Schläfli symbol did not convey full representational information. It is likely that the linearity of the Schläfli symbolic representation shrouded an essential feature, that of cyclic congruence. A cyclic presentation, with the suggestion of rotation,



might enhance recognition of congruence under rotation. This draws our attention—out of the lesson for but a moment—to the semiotic study of the degree of separation between symbols and the objects to which they refer; between symbols and referents. If the mathematician privileges concision and brevity, and if the conditions of type-setting impose themselves so that the linear [3, 3, 3, 3, 6] becomes the received representation, then one may wonder whether teacher and student become victims. It is arguable that greater attention needs to be given to the question of how we interpret symbolically codified information and on the relationship of that process to learning and remembering. Is it the case, for instance, that symbolic representations, or signs, which are in some sense more concordant with the information they represent lead to better learning outcomes? Duvall (2006, p. 107) makes the observation that, "The part that signs play in mathematics is not to be substituted for objects but for other signs! What matters is not representations but their transformation," on which ground we can ask how a student might transform [3, 3, 3, 6, 3] into 60º + 60º + 60º + 120º + 60º = 360º, or



and whether

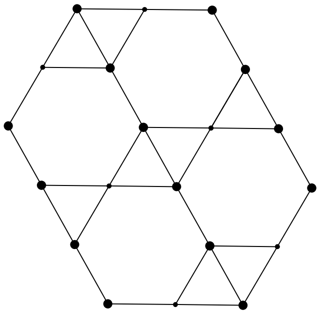


or some alternative, would lend the transformation process greater fluency. As in most transactions, the question is one of balancing gains and losses: is the sign *necessary*? Is it *useful*? As teachers, we can interrogate our practice by asking whether we privilege the received traditions of practising mathematicians or typographers over the needs of the learner. These "received traditions", it might be noted, are public performances, which can be distinguished from the private practices of mathematicians. As Paul Ernest (2008) asserts,

> it is a myth that informal and multi-modal texts disappear in higher-level mathematics. What happens is that they disappear from the public face of mathematics, whether these be in the form of answers and permitted displays of "workings", or calculations in work handed in to the school mathematics teacher, or the standard accepted answer styles for examinations, or written mathematics papers for publication. (p. 5)

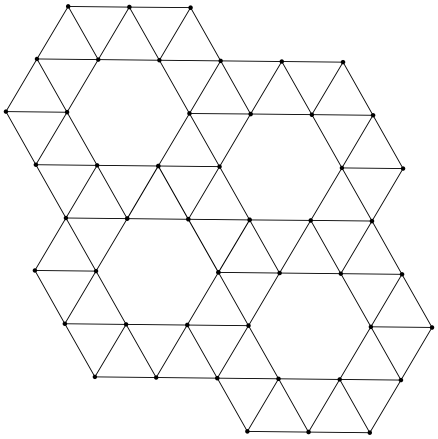
Is there, on the other hand, some counterintuitive effect whereby having to work harder to interpret a symbolic structure leads to better recall? Diemand-Yauman, Oppenheimer and Vaughan (2010) examine so called disfluency effects in text whereby the font used to present the text was found to affect recall: "Student retention of material across a wide range of subjects ... and difficulty levels ... can be significantly improved in naturalistic settings by presenting reading material in a format that is slightly harder to read" (p. 114). In another study, Kidd, White and Aslin (2011) present evidence that when toddlers hear speech disfluencies—pauses and sounds such as "Err" and "Ahm"—they use them to infer a speaker's intentions through a mechanism whereby the disfluencies direct attention to words that the speaker intends to use in anticipation of the words being spoken.

Our lesson continued. The search for the "two triangles" tessellation that would correspond with [3, 3, 6, 6] led to initial cries of success, as the following tessellation was discovered:



The joy was short-lived, however, as it was revealed that while this is, indeed, a tessellation, it is not a semi-regular tessellation. The structure [3, 3, 6, 6] does manifest itself at the bold vertices, but at other vertices the structure is [3, 6, 3, 6]. With this discovery the girls came to understand a significant aspect of the problem to be dealt with. The definition of a semi-regular tessellation now assumed substantive form and attention shifted from arrangements of 60° and 120° that summed to 360°, to arrangements of objects that would tessellate a plane, to finding tessellations that met the criteria of the definition.

The construction of [3, 3, 3, 3, 6], the "four triangles" case, proved difficult. While the notation is generative in the sense that it describes how to construct the polygons about an arbitrary vertex, the girls did not fully appreciate this advantage. They simply searched for a semi-regular tessellation with four triangles and a hexagon at each vertex. It was something of a victory for each girl who successfully made the construction, while for some it remained elusive:



Louise had been reluctant to participate in the search. The task had loomed before her, daunting in its complexity and size. She did engage in the process, but each unsuccessful attempt at a construction elicited a roll of her eyes. It was as though she was caught between "I can't do this!" and "I'll try again!" She once was heard to exclaim, "It's impossible!"

Claire, usually so bold, was today more circumspect. Perhaps it was that our early contretemps had left her in a subdued frame of mind. She had met with less success than her peers in the initial searches—unusually so, for she was often the quickest to see the point of an exercise—and this also might have contributed to her quietude. Julia, a friend of Claire's, was very determined. She did not want to hear others tell her "how it should be done" or of "a better way," but was keen, rather, to share her "own thinking," or "her theory," as she called it.

By the end of the lesson the question of the "two triangles" case was unresolved. Was it possible to find a second semi-regular tessellation using two triangles and two hexagons at each vertex, or was Louise's declamation correct? I had been asked for the solution during the lesson, but most of the girls had felt a tension between wanting to know and wanting to find. I had, in fact, made almost no contribution during the lesson, save to clarify word meanings or judge the correctness of putative tilings when asked. I had not studied the problem prior to the lesson—in and of itself, the problem was immaterial—save to determine that it was suitable for the girls, preferring rather to diminish the role of the teacher as an "expert" with technical knowledge in the hope of encouraging a more egalitarian dynamic to emerge (Healy & Hoyles, 2001). By downplaying my status as teacher-expert *and* by ensuring that the problem was approachable yet demanding, I had hoped the girls would play a greater role in determining the progress of the activity; that they would learn *what can only be learned through experience*. To use Brousseau's (1997) term, this was an *adidactical* situation:

> The modern conception of teaching therefore requires the teacher to provoke the expected adaptation in her students by a judicious choice of "problems" that she puts before them. These problems, chosen in such a way that students can accept

> them, must make the students act, speak, think, and evolve by their own motivation. Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. Not only *can* she do it, but she *must* do it because she will have truly acquired this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. (p. 30)

In this case the new piece of knowledge was entailed in the process of exploratory searching, the balancing of anxiety and anticipation; and the production of a convincing argument. The need I had felt to stabilise Claire prior to the commencement of the lesson was balanced by my desire to have a minimal impact on the unfolding of the activity, and yet to assist the girls to attain a level of constructive flow.

I met Rowena, one of the students in the class, the following day as we entered the school hall for an assembly. We shared a moment in which she recalled the morning's exploration with warmth. And was there yet a second tessellation? No, the second two-triangle case is impossible to construct. I knew this by now, for I had taken time to look at the problem, much as Julia would have done had she the chance—on my own, in my private space. Rowena laughed, head back, and I could but wonder, what had the morning's work meant to you? Had you experienced mathematics? Had you learnt how to do mathematics? Had you felt something of what it is to do mathematics, of the uncertain searching-for that can lead, the not-knowing-yet-doing that permeates practice?

Valerie DeBellis and Gerald Goldin (2006) would have us notice examples of intimate behaviours during this lesson. In their theoretical framework in which affect is developed as an internal representational system, *mathematical intimacy* involves “deeply-rooted emotional engagement, vulnerability, and the building of mathematical meaning and purpose for the learner” (p. 137). Among the intimate behaviours that could be observed we can highlight the way in which Julia craved sanctity for her thinking space, the breathy exhalations that accompanied discoveries, the groans that ensued when the discoveries were found to be flawed. DeBellis and Goldin develop this theme further with reference to intimate experiences that can “*build a bond* between the personal knowledge constructed and the mathematical content” (p. 138, authors’ italics), and thence to express the danger that “a solver may feel disappointed, angry, or *betrayed in intimacy* by unexpected outcomes, failures, negative reactions from loved ones, rebuke from a trusted teacher, or scorn from peers” (p.138, authors’ italics).

Talk of bond building and betrayal highlights the emotionally laden quality of engaging fully with a problem: there is a commitment in play that carries the risk of betrayal but also the hope of fulfilment. Faced with such risk, we might understand a reluctance to commit—the difficulties of a mathematical problem are more than merely logical.

For some of us, learning to manage this fear is part of becoming a mathematical thinker. I have seen students smile, even laugh, as the complexities of a problem become apparent, but I have also seen expressions of grave uncertainty and trepidation. Successful problem solvers seem able to corral not-knowing-uncertainty into an attitude of interest and curiosity. There is an act of faith implied in this: faith that the problem is within solvable reach, although quite possibly only just barely so; and faith in the teacher, who surely would not place students in a position from which failure is the likely outcome? DeBellis and Goldin describe this capacity to interpret fear or frustration favourably as a productive meta-affect, and urge its importance in the classroom:

> A supportive classroom culture provides a sense of *safety* in being “stuck”. Then frustration *coupled with productive meta-affect* suggests the problem is worth pursuing, and motivates further exploration rather than disengagement. (p. 137, authors’ italics)

This provides teachers with an interpretive stance: do I foster a culture in which meta-affect operates to "provide a sense of safety in being stuck"? Have I encouraged productive meta-affect, which is to say, have I equipped my students to think and feel positively about the feelings that derive from their engagement in mathematical situations?

Claire, who I had known for a couple of years, has a history of deeply intimate mathematical behaviour. She regularly becomes engrossed in mathematical problems and her meta-affective capacities are such that when difficulties emerge her excitement level heightens and she redoubles her engagement. She experiences exhilaration in the mathematics classroom. Does this explain her immersion in the problem after the initial rebuke? Possibly, although it is difficult to be sure. What we can say is that she exhibited behaviour that indicated a narrowing of attention, a close coupling with the problem, suggesting that for her, the problem became at once a haven and a threat: a haven in that she could revert to behaviour that she *understood*—how to work mathematically—and a threat in that it carried the risk of an unsatisfactory solution. Her meta-affective abilities were, I suggest, sufficient to subdue the threat and render deep engagement her most viable response. She had, moreover, a commitment to what DeBellis and Goldin call *mathematical integrity*.

DeBellis and Goldin see "affect functioning as an *internal representation system* exchanging information with cognitive systems" (p. 132, authors' italics). Their hypothesis is that "affect is fundamentally *representational*, rather than a system of mostly involuntary, physiological side-effects of cognition" (p. 133, authors' italics). The suggestion, more or less implicit, is that affect can be thought of as being *like* cognition and contributing to cognition. DeBellis and Goldin are attempting to construct a framework in which states of emotional feeling function like cognition in that they "carry meanings for the individual ... they encode and exchange information in interaction with other internal systems of representation" (p. 133). As my study progresses, it might seem that I come from another direction in that I broaden my understanding of "cognition" so that it becomes somewhat like affect. In both cases, there is recognition that one cannot be deeply understood in terms that preclude the other. The development of an embodied account remediates the need to invoke representational affect in that both affect and cognition are seen as fundamental elements of what it is to be embodied (see Hannula, Evans, Philippou, and Zan, 2004, for a dialogue concerning various theoretic affective frameworks; and Hannula, 2012, for an overarching review of theoretical approaches to mathematics-related affect).

Alfred Adler, mathematician and author of *Mathematics and Creativity*, a provocative, somewhat hyperbolic essay in the New Yorker (February 19, 1972), wrote that,

> there is no such thing as a man who does not create mathematics and yet is a fine mathematics teacher. Textbooks, course material—these do not approach in importance the communication of what mathematics is really about, of where it is going, and of where it currently stands with respect to the specific branch of it being taught. What really matters is the communication of the spirit of mathematics—a spirit of disciplined search for adventures of the intellect. Only an adventurer can really tell of adventures.

In the course of a crowded academic year the "communication of the spirit" may be seen as a luxury, but it must, nonetheless, be brought to the table. It is the spice that renders the meal palatable, without which there can be no feast, and for the sake of which students will return, hungry for more. The dry crumbs that characterise mathematics of unpopular sentiment are but the residue of ill prepared, unseasoned fare.

If we more closely consider the forms[3] that characterize this lesson we can gain a perspective from which to consider it in terms of an embodied paradigm, or, perhaps more correctly at this early point in my exploration, we gain an insight into the kinds of questions that an embodied account might seek to address. Recalling my experience of the lesson, I am left with an impression of having wrestled with the dialectic of fixity and change. The "fixed" forms of the class impress themselves upon us: the room, the teacher-student dyad, the mathematical definition of a semi-regular tessellation, the time constraint of the lesson, the expectation that there be teaching and learning. These constitute the formal aspect of the lesson in deference to the Latin *formare*, meaning to shape or mould. They shape the space in which the lesson has its being. None is fixed in virtue of its own constitution; rather, each is a type of construction and could have been otherwise were we in another place or at another time. Of some interest is the thought that in terms of its own being, the element that is "most fixed" is arguably the mathematical definition, since were it to change, then it would be a definition of something else. It is also possibly the most "abstract" of the various formal elements: does abstraction promote fixity? The question of what is meant by "abstract" is a matter to which I will return.

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[3] I do not mean to suggest any association with Platonic Ideas, or Forms, by my usage of this word.

On another reading, the very idea of a semi-regular tessellation may yet be an expression of arbitrariness, an art form or geometrical artefact that is *interesting* to us, where yet we can wonder why. What is the source of the interest?

This is a matter partly for aesthetical study, of course, and yet we can consider it directly relevant to this exploration, for the experiential basis of our lives might be driven in part by aesthetical considerations, a proposal I shall come to. To balance any assertion that mathematics is important *because* it is useful, we can pose a question inspired by the popularity of games and puzzles: "Why do we so enjoy mathematics that is *not* useful?" The likelihood that Rowena will exploit her brief experience with semi-regular tessellations to some considerable purpose later in life is slight, I guess, and yet she derived something from the lesson: a sense of shared exploration; an appreciation of the mechanics of searching, pleasure in the pursuit of mathematics. Are any of these inconsiderable?

The forms of the lesson that represent "change" include interpersonal and affective phenomena, such as the relationships between students, between each student and the teacher, and perceptions of the self. The weighting the curriculum gives to aspects of mathematical behaviour and to content, the “what” and “how” of the teaching program, is also a formal yet variable component of the discourse. The rate at which these elements change is variable: while the curriculum may be seen as relatively stable, the attitude of a student (to her peers, or to the lesson itself) can alter within the space of a moment, such as when Louise flicked from despair to determination, before collapsing in defeat. Longer standing, less rapidly changing emotions can be witnessed, as in the case of Claire, who maintained an uncharacteristically restrained posture throughout.

What is not clear is what constitutes “boundaries” in this structure. Fixity and change can be seen as opposites, or they can be regarded as poles of perpetual flux, where what distinguishes one from the other is the rate of change. The “objects”, while at first blush apparent, are revealed as determined by the resolving power and intentions of the observer, who is, herself, subject to the same phenomenology. Each student interprets herself multiply: as a student, as a friend, a girl, a daughter—and these are crude descriptions given simply to illustrate the point. Each sees the other according to her needs at the time. This applies equally to the way in which the teacher is understood. I was, for some, a potential source of a solution, for another a disciplinarian, perhaps for Louise a tormentor; and these perceptions coexist and vary. I wondered if by minimising my didactic persona I would be understood less as a teacher and more as an observer; or, to the extent that the girls knew that I had not yet solved the problem, as a participant in the challenge, a co-puzzler.

The “forms” of Victoria as robust interlocutor, as reflective thinker, as persuasive speaker, as vigorous protagonist, as rebellious adolescent, are undistinguished by clearly marked boundaries. One has, rather, to “scent the air” to determine which of the manifestations is active. Neither is one a true observer in this case, for as surely as Victoria’s form is affected by you, so yours is affected by her. The same is true, one suspects, of Victoria’s *self-determination*. How she perceives herself will be a reflection, in part, of how she understands that she is perceived by others. There are no individuals, here. Hans-Herbert Kögler (2010) describes this as an inter-subjectivist grounding of agency: "the agent's capacity for a reflexive self-relation and self-directed action emerges from a dialogical process of perspective-taking" (p. 1). Consider further that there are a number of students in the room and the degrees of freedom of the decidedly non-linear, dynamic system that is "the class", taken as a unitary whole, can be seen to increase.

Each girl, then, is imagined as an agent of change for each other girl's sense of self-identity. The teacher, too, is implicated in this dynamic making of selves. The inter-binding of selves thereby determines the cohesive system that would be recognizable to an external observer as the class, a system that transcends individuals. The elaboration of structure that this process represents is not cognitive in the sense of being deliberately intended and maintained by the participants, and yet it is partly shaped by forms, some of which—the room, desks, chairs, timetables − have been established with the express intention of creating "classes". To the extent that a problem was being jointly investigated and solutions proposed and debated; that artefacts were deployed to mediate efforts; and that this was happening in a bodily-dispersed yet cohesive manner, one can argue that a variety of extended cognition was in evidence.

In this regard, the embodied cognition paradigm as elaborated in this exploration needs to be seen as embracing a variety of research orientations, such that the reference to being embodied is not to be taken to imply that cognition is necessarily bounded within any solitary body. The problematic point of what constitutes a body, subject as it is to the perspective of the inquirer, not to say the question of what constitutes cognition, must underlie any such consideration[4].

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[4] Andy Clark and David Chalmers (1998), for example, propose and defend the thesis of "the Extended Mind", whereby the environment is actively implicated in driving cognitive processes, in which the "human organism is linked with an external entity in a two-way interaction, creating a *coupled system* that can be seen as a cognitive system in its own right" (p. 8, authors' italics). Matthew Barker (2010) analyses this and a variant extended mind thesis and urges the wisdom of

Linguistic forms are present, too, in shades of fixity and flux. The turns of tone, the rise and fall of modulation, the silence of bated tongues and the garbled ejaculations of imminent discovery echo the many mouths of speech, while the words and phrases we utter are but modern phonemes in a verbal lineage that extends aeons past. Language serves a unifying purpose (Garvin, 1959; Garvin & Mathiot, 1956; Mathiot & Garvin, 1975), at once reinforcing a group identity, and encouraging a mutual awareness that each is indebted to each for the recognition of self. Kögler (2010) expresses this by asserting that "the linguistic mediation of one's existence both entails the danger of falling victim to a dangerous self-reification *and* the promise of articulating one's most crucial value-orientations and life-projects in a common medium" (p. 10, author's italics). This sense of debt-for-selves, then, steers us toward an ethical orientation that arises from our corporate togetherness; a waypoint to which I shall return in Part Four.

This small body of girls represents a larger world. I can imagine, were they assembled in some late Pleistocene age, that Victoria would have wielded power, that Louise would have been bound to comply by forms of fixity as she was this day, that Julia would go her own way and ruminate, that Claire would eventually release her bound energy and celebrate life in full exuberance, and that Rowena would laughingly enjoy the being-there.

Stopping to reflect on the constitutive forms and interpretative challenges of so commonplace a lesson as this brings to mind Jean Follain's poem, *Music of Spheres*. I am clicking my pen-like-keys, remembering Victoria's forceful quickness, laughing with Rowena; the sky, meanwhile, overarches and holds sway. We must, from time to time, locate our gaze and direct it beyond our fragmentary situation.

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relaxing the question of *where* cognition is to "more directly and explicitly debate the issue of what cognition is" (p. 365).

He was walking a frozen road
in his pocket iron keys were jingling
and with his pointed shoe absent-mindedly
he kicked the cylinder
of an old can
which for a few seconds rolled its cold emptiness
wobbled for a while and stopped
under a sky studded with stars.

Jean Follain (1996), *Music of Spheres*

The day following this lesson I was fortunate to attend a production of Transient Theatre's *Café Floriani* with our Year Nine students. This was a performance in the tradition of La Commedia dell'Arte, a form in which the actors wear traditional masks and demonstrate skills of comedic improvisation. While free improvisation is central to the performance, the production itself is bound by certain forms, as evident in the meaning of the verb, to *per-form*. The cast consisted of the traditional characterisations of two high-status old men, The Doctor and Pantelone, their servants, or *zannis*, and a young man of adventure, the Captain. Each character was a form symbolised by a mask that the actor wore. Each actor, in fact, played a variety of roles, where a change of mask made it clear to the audience who was portraying whom at any given time. The play struck me as a model of a teaching lesson: somewhat scripted, but continually improvised, bound by traditional forms yet able to twist at any moment. The classroom characters are readily identified by their masks—uniforms, hair styles, professional clothes—and how they occupy the stage. The stage itself is a fixed form; fixed, yet ephemeral, a backdrop to the real action. What binds the act is the interplay of language, plot and characterisation.

A performer afterwards described how in preparation for rehearsal he would sit at a mirror with a mask before him and attempt to mimic the mask with his face. Then, on placing the mask on his face, turning to the mirror and seeing the "face" in the mask, he would assume the character's persona. I heard one teacher remark, "Teachers wear masks!" The performer said that he thought stage-actors would find Commedia dell'Arte very difficult, because they were not trained to "think on their feet". They deliver a staged performance. Is the teacher's performance staged? Is it improvisational?

As part of the play, the formal distinction that normally exists between stage and audience was relaxed, and two Year Nine girls were invited to take part in the play. Emma and Jenny were selected from the audience and brought to the stage where they were provided with costumes and invited to adopt roles. I thought of Victoria and Julia sharing their explanations during our lesson, speaking to the assembled class, teaching, and reflected that the forms of the class can be subtle, less explicit than the extravagant costumes and masks of Commedia dell'Arte, but they are forms nonetheless. The essence of a form, after all, is that its function be recognised by those for whom it exists as a form. It need not be recognised *as* a form, but merely for what it does. The forms of the stage are overt, to be sure, but the forms of the class are no less powerful.

EMBODIED MATHEMATICS AND AUTOPOIESIS

I will now briefly introduce embodied mathematics before exploring the mechanism of autopoiesis. I will later contrast these with a few reflections of St. Augustine, the more to see what must be discarded if one is to develop an epistemology consistent with embodied cognition.

With the publication of his book *The Number Sense*, Stanislas Dehaene (1997) drew attention to the embodied roots of mathematics. Research indicating that aspects of number sense are innate encourages one to contemplate the view that *all* mathematics is fundamentally bodily-based. On adopting this idea, conceptualisations of number that require dualisms—numbers as Platonic ideals, for instance—are laid aside in favour of exploring the question of how one moves from an ability to recognise small numerosities to deft manipulation of fine grained, highly abstract mathematical structures. The use of the word *manipulate* is highly suggestive of the direction such questions bend researchers seeking clues.

George Lakoff and Mark Johnson (1999) introduced their significant book, *Philosophy in the Flesh*, with three statements: (i) “the mind is inherently embodied”; (ii) “thought is mostly unconscious”; and (iii) “abstract concepts are largely metaphorical” (p. 3). These represent empirical findings, and are significant for the future direction of philosophy: “A philosophical perspective based on our empirical understanding of the embodiment of mind is a philosophy in the flesh, a philosophy that takes account of what we most basically are and can be” (Lakoff and Johnson, 1999, p. 8).

Building on these findings and applying them to mathematical knowledge, George Lakoff and Rafael Núñez (2000) investigated the question of where mathematics comes from through a process they call *mathematical idea analysis*. Under their theory, which will be explained more fully later, direct physical engagement with the world provides a grounding from which, through a cascade of “inference preserving metaphors”, we unconsciously learn to treat mathematical ideas as though they were real, physical objects. We “manipulate ideas”, we “gather our thoughts”. This approach celebrates our evolved condition and aims to account for what we can do by looking closely at what we are, in line with the general principles of evolutionary educational psychology:

> The principles of evolutionary educational psychology will provide a much needed anchor for guiding instructional research and practice. An evolutionarily informed science of academic development is in fact the only perspective that readily accommodates basic observations that elude explanation by other theoretical perspectives. (Geary, 2002, p. 340)

In the course of such an analysis idealistic views of mathematics as somehow privileged are set aside in favour of dispassionate questioning. Lakoff and Núñez criticise the “Romance of Mathematics”, although for some, including Burton Voorhees (2004), they go too far:

> With apparent ideological motives, they try to discredit what they call the 'Romance of Mathematics'. As defined by them, this is a straw man, combining an extreme version of mathematical Platonism with an elitist view of mathematics as the ultimate science, and mathematicians as the ultimate experts on rationality. (p. 84)

We stand, then, as with Poe, feet upon the shore, grains of golden sand sifting through our fingers, wondering the while if our mathematics is but dreams within dreams:

> I stand amid the roar
> Of a surf-tormented shore,
> And I hold within my hand
> Grains of the golden sand—
> How few! yet how they creep
> Through my fingers to the deep,
> While I weep—while I weep!
> O God! can I not grasp
> Them with a tighter clasp?
> O God! can I not save
> *One* from the pitiless wave?
> Is *all* that we see or seem
> But a dream within a dream?
>
> Edgar Allen Poe (1871), *A Dream Within A Dream*

We speak at times of mathematical gems, beautiful proofs and demonstrations worthy of admiration: are these "grains of the golden sand" to be saved? Or are they to drift slowly, inexorably through our fingers as we lose hold of the romance of mathematics? Is our treasure to be dross?

We are left to ponder whether our mathematics, once so solid and reliable, is less some sure footing for aspirations than an exposition, potentially fractured, of our frail selves. Are we still to turn to mathematics to confer credibility upon our social theories, our economics, our science, if we discover a mathematics shaded less with cosmological lustre than earthy hues?

To appreciate the subtle significance of the turn to embodied mathematics requires us to understand how we have become accustomed to thinking about mathematics—or whether we have avoided the question of how to regard it at all. An account of mathematical ontology would seem a necessary goal of such an enterprise, but the terms on which that account might proceed will not be commensurate with the Platonic variety of realism in which objects of mathematics are taken to have real "existence", where our relation to that variety of existence is likened to that of prisoners seeing only shadows cast by unseen objects in Plato's famous cave allegory. The names the prisoners give to shadows—which they believe to be real objects, since they are all they have ever seen—refer instead, we are told, to objects hidden from view and unavailable to direct experience. As Plato has Socrates say in his *Republic*, "All in all, then, ... the shadows of artefacts would constitute the only reality people in this situation would recognize" (Plato, 1993, p. 241); and yet the artefacts abide. Socrates likens our experience of "reality" to the situation of the prisoners, on which account numbers are to us as shadows to the prisoners: what we name "numbers" are but shadows cast by "real" numbers. These real numbers inhabit the realm of Forms, or Ideas.

The turn to embodied accounts of mathematics, then, will require a philosophical restructuring of the very notion of what it is *to think*. To begin the process we shall now ground ourselves in a brief exploration of autopoiesis, the biological theory of self-making and sustenance of being.

Autopoiesis

When speaking of embodied mathematics the concept of the autopoietic being can be helpful. Autopoiesis refers to a process of becoming, within a defining or bounding space, in which that very becoming both depends upon and determines the unitary coherence of that space. It captures the sense in which life is dynamically self-forming and self-informing. It is a term that can be applied to organisms, to ecosystems, and possibly even to social systems, although John Mingers (2004), among others, remains agnostic on this last assignment.

An autopoietic system, introduced in a biological context by Humberto Maturana and Francisco Varela (1992), is one in which there exists both a boundary (a membrane, say, in the case of a biological cell) and dynamic transformations (or metabolism, say, to persist with the biological scenario) such that the two are mutually co-dependent. In the case of cellular chemistry,

> on the one hand, we see a network of dynamic transformations that produces its own components and that is essential for a boundary; on the other hand we see a boundary that is essential for the operation of the network of transformations which produced it as a unity. (p. 46)

In more general terms, Varela (1992) gives the following:

> An autopoietic system is organized (defined as unity) as a network of processes of production (synthesis and destruction) of components such that these components:
>
> (i) continuously regenerate and realize the network that produces them, and
>
> (ii) constitute the system as a distinguishable unity in the domain in which they exist.
>
> Thus, autopoiesis attempts to capture the mechanism or process that generates the *identity* of the living, and thus to serve as a categorical distinction of living from non-living. This identity amounts to self-produced coherence: the autopoietic mechanism will maintain itself as a distinct unity as long as its basic concatenation of processes is kept intact in the face of perturbations, and will disappear when confronted with perturbations that go beyond a certain viable range which depends on the specific system considered. (p. 5)

Care is required here. The point is not to be able to identify that "this is a cat" or "that is a dog," but rather, to understand that the cat has a perspective on the world—it has *its* world—and it is to this that the autopoietic definition points:

> The autopoietic unity *creates a perspective* from which the exterior is one, which cannot be confused with the physical surroundings as they appear to us as observers, the land of physical and chemical laws *simpliciter*, devoid of such perspectivism. (Varela, 1992, p. 7)

Rocks may be said to be rocks, but to a skink, those same rocks may be shelter from pursuit or basking platforms on a warm morning. Lichen knows no rock, only substrate.

Maturana applied this concept of autopoiesis directly to the question of cognition:

> If whatever takes place in a living system is specified by its structure, and if a living system can only have states in autopoiesis because it otherwise disintegrates (and stops being a living system), then the phenomenon of cognition, which appears to an observer as effective behavior in a medium, is, in fact, the realization of the autopoiesis of the living system in that medium. For a living system, therefore, to live is to cognite, and its cognitive domain is as extensive as its domain of states in autopoiesis. The presence of a nervous system in an organism does not create the phenomenon of cognition but enlarges the cognitive domain of the organism by expanding its domain of possible states in autopoiesis. (Maturana, 1978, p. 37)

Varela (1992) elaborated this description by stating that the operational closure of the nervous system

> brings forth a specific mode of coherence which is embedded in the organism. This coherence is a *cognitive self*: a unit of perception/motion in space, sensory-motor invariances mediated through the interneuron network. The passage to cognition happens at the level of a behavioral entity, and not, as in the basic cellular self, as a spatially bounded entity. The key in this cognitive process is the nervous system through its neuro-logic. In other words the cognitive self is the manner in which the organism, through its own self-produced activity, becomes a

> distinct entity in space, but always coupled to its corresponding environment from which it remains nevertheless distinct. A distinct coherent self which, by the very same process of constituting itself, configures an external world of perception and action. (p. 10)

To live is to cognite—I am, therefore I think, we might say, echoing an earlier existentialist reflection. Martin Heidegger (1996), for example, had identified the need to revisit Descartes' *cogito ergo sum* in *Being and Time*:

> He [Descartes] investigates the *cogitare* of the *ego*—within certain limits. But the *sum* he leaves completely undiscussed, even though it is just as primordial as the *cogito*. Our analytic raises the ontological question of the being of the *sum*. Only when the *sum* is defined does the manner of the *cogitationes* become comprehensible. (p. 43)

The nature of the *sum*, of the "I am" that is, lies at the root of the attempt in this thesis to account for mathematical thinking in an embodied context, for it becomes necessary to understand the activities of thinking without recourse to dualities.

The process of living necessitates a spatial definition within a being-defining space[5] and behaving effectively—sustaining life, that is to say—in that space. The appearance of such behaviour is taken to be cognition; a broad rendering, perhaps, but then, embodied cognition requires that we suspend customary habits of thinking about cognition. At this preparatory stage in the formulation of my thesis, let us allow that a kind of cognition consists in effective being in an environmental medium, calling to mind Heidegger's Dasien, for which "*The 'essence' ... lies in its existence*" (Heidegger, 1996, p. 40, author's italics). Heidegger had noted that the simple ordinariness of everyday coping (Dreyfus, 1993) was a feature of being that was too readily passed over for being too present-at-hand to be noticed: "What is ontically nearest and familiar is ontologically the farthest, unrecognized and constantly overlooked in its ontological significance" (Heidegger, 1996, p. 41). In like manner, the significance of sustaining existence—of effective behaviour—is not to be understated, although, not wanting to draw links too tightly, it is well to note that whereas Heidegger's notion of Dasein was intimately tied to that particular being who questions its being, which is to say, to humans, autopoiesis applies more broadly to life per se, and with varying degrees of confidence, to other complex systems.

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[5] Heideggerian considerations notwithstanding, for lack of space I shall leave undiscussed questions of temporality.

Thus Maturana's point may flow: we do not find cognition in the nervous system, but the nervous system operates to determine the breadth of autopoietic states—the autopoietic domain—in which the organism can persist, and so express the cognitive self relevant to that domain. For a cell, cognition captures the way in which the cell appears to cope with its environmental conditions, allowing some chemicals to permeate its membrane, synthesising and secreting others; for the scholar, we could identify a very different autopoietic domain in which to seek the appearance of effective behaviour, including, not least, research and publication. It is suggested that such behaviour is evidence of cognition relevant to the scholar's *umwelt*, or lived world. In both cases, it is a suite of behaviours of the organism that determines the coherence of the cognitive self. In the cellular case, this coherence is expressible largely in spatial terms—it is here, this shape, with such and such a volume. In the case of the scholar, whilst "coupled" to the environment, the coherence configures a very different world "of perception and action": the world, indeed, of the professional thinker.

Weaving the threads together, Evan Thompson (2004), in a tribute to Francisco Varela given at the Sorbonne in, brings us to the point where the hard problem of consciousness, exemplified in Nagel's (1974) question, "What is it like to be a bat?" (p. 435) embedded as it is within a Cartesian mind-body dualism, is given over to a realisation: "The problem of what it is for mental processes to be also bodily processes is thus in large part the problem of *what it is for subjectivity and feeling to be a bodily phenomenon*" (p. 384, author's italics). What it is, as he puts it, for a physical living body to be also a lived body. Thus we arrive at a phenomenological imperative: to discern the reciprocally entwined emergence of the lived experience of teaching and learning mathematics and the habituation of beings in the world. To explore, that is to say, how the experience of living evokes the cognitive world of mathematics, and how in that evocation, the lived world of mathematics alters the breadth of what it can mean to be mathematical.

Mark Johnson (2007) recognises in this endeavour the hallmarks of pragmatic philosophy, edified in the writings of James and Dewey:

> At the heart of all pragmatist philosophy is the fundamental understanding that thinking is doing, and that cognition is action. Pragmatism recognizes that thought can be transformative of our experience precisely because thought is embodied and interfused with feeling. Thinking is not something humans “bring” to their experience from the outside; rather, it is *in* and *of* experience—an embodied dimension of those experiences in which abstraction is occurring. Our ability to conceptualize is our chief means for being able to respond to the problems we encounter, to adapt to situations, and to change them when it is possible and desirable, via the use of human intelligence. This conception of mind and thought is the basis for James’s famous pragmatic rule of meaning, which states that the meaning of a concept is a matter of its consequences for our present and future thought and action. (p. 92)

The integration of being, body and cognition is difficult to express when one must resort to symbolic modes of communication. Language, particularly, is deeply infused with the apparently dualistic quality of thought and action; that is, of thought as mind-idea and action as body-oriented. Jennifer Thom and Wolff-Michael Roth (2011) recognise as much as they study the way in which Owen, an eight year old boy, manipulates blocks in order to discover and compare their properties:

> In this study, we observe the utter irreducibility of knowing and acting. It is not possible and does not provide insight into conceptual growth to decontextualize, disembody, or part out what Own [*sic*] knows, how he knows, what he does, and how he works. (p. 281)

The body, then, emerges as the core of the self-sense-symbolic system, not in distinction to thought and language but intrinsic to and indistinguishable from such possibility.

The Dynamic of Becoming

> Wisdom lies neither in fixity nor in change, but in the dialectic
> between the two.
>
> Octavio Paz (1991, p. 10), *The Monkey Grammarian*

To regard ourselves in autopoietic terms is to realise a creative tension—a natural dialectic—between the specificity and plasticity of growth in the exercise of building lives, or *bildung*. Specificity refers to the observation by which it is held that we are similar to each other, built to a shared plan, as it were, so that our organs appear and function much the same from person to person. An arm is an arm; arsenic is poisonous to you and to me. Plasticity, on the other hand, refers to the capacity of the growing body to accommodate itself to selective pressures as it grows, so that, for example, damage to one region of the brain might lead to compensating functionality emerging in another region of the brain (see, for example, Ramachandran and Blakeslee (1998) on the question of phantom limbs), and every day we are able to accommodate ourselves to circumstances with which we are presented in such a manner as to exemplify a constitutional flexibility. We are in the process of forming ourselves. As Steven Rose (2005) suggests,

> it is specificity and plasticity rather than nature and nurture that provide the dialectic within which development occurs and both are utterly dependent on both genes and environment. (p. 64)

This places an emphasis on *process*—the autopoietic dynamic of becoming—that can be contrasted with a more architectural interpretation of the brain, in which our modern behaviours are determined in large part by modules that evolved in our

Pleistocene past (see, for example, Tooby and Cosmides (1992) and Tooby, Cosmides and Barrett (2005) on the hunter-gatherer adaptations of the brain and the making of modern psychology). While the issues involved are matters of considerable contestation, it will suffice here to recognise that there are questions over how to balance the somewhat complementary theories of modularity and plasticity—between what is fixed and what is changeable—and also over whether or not the brain behaves as a processor of information—a form of super-computer, that is to say (a representation of such views can be found in Carruthers, Laurence, & Stich, 2005). Edelman and Tononi (2000b), noting the inherent neuronal variation of brains, for example, state that the brain is fundamentally *not a computer*: "The data provide strong grounds for so-called selectional theories of the brain—theories that actually *depend* upon variation to explain brain function" (p. 47, authors' italics). Under their theory, Darwinian selection operates at the level of groups of neurons such that a growing individual interacting with its environment contains within itself a responsive elaboration of that interaction, structured by weakening or strengthening synaptic connections within and between groups of neurons, so that the individual is ever-becoming (see later, and also Edelman and Tononi, 2000a).

When we understand that fixity and change present lenses through which we can interpret phenomena, we are drawn to an appreciation of the significance of *becoming*, of transitions. This dynamic of becoming encourages us to revisit Kitcher's suggestion that we see "rudimentary arithmetical truths as true in virtue of the operations themselves" (Kitcher, 1984, p. 108), because the operations themselves, read autopoietically, play into what we can do. On a grand timescale this speaks of our species' very evolution. On a more confined temporal scale, the suggestion is that we form our individual apprehension of mathematics subject to the specificity and plasticity dialectic. The question of operations reminds us that it is through engagement that we imbue meaning into the process so that our evolved constitution, taken together with our lived expression of that constitution, informs what can possibly qualify as a rudimentary arithmetical truth.

Mathematics as an Evolving Autopoietic System

To speak thus is to find a basis for mathematics in the evolutionary emergence of *Homo sapiens*, to seek its elaboration in subsequent cultural engagement, and to descry its import in the lives of its practitioners. Tracing modern behaviours through evolutionary roots is a tempting past-time and can lead to interesting theory. The urge to religiosity, for example, exemplifies a field that attracts theorising and challenges traditional orthodoxies (see for example, Pascal Boyer (2001) and Scott Atran (2006)). David Lewis-Williams and David Pearce (2005) apply neurological discoveries to the purpose of reinterpreting archaeology, and so re-conceive the nature of our relationship with myth and religion. The emergence of art has been explored by researchers. It is proposed, for example, that our earliest known examples of rock and cave painting are expressions of the workings of brains under trance conditions (Lewis-Williams, 2002). Denis Dutton (2009) mounts a powerful exposition of the significance of our evolutionary history for our modern artistic sensibility. Such explorations draw on studies in neuroscience, anthropology, ethnology and evolutionary psychology. They represent a genuine attempt to describe and understand our condition in parsimonious terms, and if there is a temptation to elaborate data into plausible stories to account for our ways of being in the world, at the heart of each story is a kernel that we might find preferable for being the less confabulated of alternatives.

The dynamics of evolution are themselves the subject of mathematical investigation. Martin Nowak (2006), for example, surveys the field, ranging through population dynamics, quasi-species theory, evolutionary game dynamics, the Prisoner's Dilemma, features of finite populations including game theory, spread of virulence, dynamics of cancer and the evolution of language, and describes how simple mathematical rules produce or describe complex landscapes. In a concise primer of what is required for evolution to occur, he states:

> The main ingredients of evolutionary dynamics are reproduction, mutation, selection, random drift, and spatial movement. Always keep in mind that the

> population is the fundamental basis of any evolution. Individuals, genes, or ideas can change over time, but only populations evolve. (p. 4)

And in case there was any danger that the reader might section the study as irrelevant to much of the experience of lived human life, "Every living system, and everything that arises as a consequence of living systems, is a product of evolutionary dynamics" (p. 7). Nowak is in no doubt:

> I do not know what the "ultimate understanding" of biology will look like, but one thing is clear: it will be based on precise mathematical descriptions of evolutionary processes. Mathematics is the proper language of evolution, because the basic evolutionary principles are of a mathematical nature. (p. 292)

This throws a symmetry into the problem of being mathematical, of course, for here we see that mathematics is being used to describe the means whereby—or at least the broad conditions under which—mathematics, as a "consequence of living systems," is brought into being. This non-linearity is in keeping with the very nature of evolutionary dynamics, however, and hints at autopoiesis. If we take "mathematics" to be a system that is a consequence of a living ecology then the network of processes of production become the daily work and peer review of working mathematicians on the one hand, and the educational practices of teachers and students on the other. The fact that we can identify and speak of the mathematical community, that there are ways of becoming a part of that community or of leaving it, that the community, taken to include mathematics per se exhibits both stability (think Euclid) and flux (the boundaries of mathematics are ever wider), and that mathematics draws influences in from outside its unitary boundary and returns information in kind, and even that mathematicians have a way of seeing the world that makes it *their* world, *their* way, suggests that we are able to think of mathematics as an autopoietic system. The physicist Eugene Wigner (1960), in his essay "The Unreasonable Effectiveness of Mathematics in the Natural Sciences", famously suggested that the "enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious" (p. 2). And yet, when we read mathematics as an autopoietic system encompassing our interactions with the natural world, we sense that there would be greater mystery in mathematics *not* being useful.

This is not to be blind to the practical possibility—even probability—of mathematics being poorly or imperfectly applied. Robert May (2004) observes that while "mathematical models have proved to have many uses" (p. 793), it is also the case that

> perhaps most common among abuses, and not always easy to recognize, are situations where mathematical models are constructed with an excruciating abundance of detail in some aspects, whilst other important facets of the problem are misty or a vital parameter is uncertain to within, at best, an order of magnitude. (May, 2004, p. 293)

Mathematics itself, seen as a population of entangled theories, is a system, moreover, that can be said to evolve. The mechanisms of change—the agencies, that is to say, of reproduction, mutation, selection, random drift, and spatial movement—can be identified. Reproduction consists in "copying ideas" from one person to another—it is the communication of thought and theorem that binds the community. It takes place through journals, in coffee rooms, lecture theatres and classrooms. It is an imperfect mechanism—that is to say, there is mutation. Most mutations are "mistakes", deleterious, leading to a variety of failure. New ideas are a kind of mutation, too, not in the limited sense that they are imperfect copies, but in the sense that they contribute something new to the pool of thought. Selection occurs when ideas are tested and found wanting, or thought good. Random drift is like mathematical fashion; an ill-defined body of thought slowly assumes significance, becomes the latest thing and is eventually replaced by some other trend in a cycle that might take hundreds of years or mere months. Local or regional features encourage clinal variations within individual universities or countries.

This portrayal may be viewed analogically, but on the whole, it does not seem so far from a true or reasonable description. What is clear is that at every point in the account the mechanisms of change are inextricably involved with human mathematicians. As such, the characteristics of humans infuse mathematical evolution.

AUGUSTINE OF HIPPO AND NEOPLATONIC COGITATION

To permit a greater appreciation of what it is that is being pared away, I turn now to Augustine of Hippo, that early philosopher and archbishop of the Church. In the development of my thesis, Augustine represents a dualistic way of thinking that contrasts with embodied accounts. By no means was Augustine the author of dualism, but his influence, especially in the development of Christian theology, has been significant. By the time of Augustine's writings, Platonic dualism had been modified through various interpretations, but the syncretism which came to be known as Neoplatonism, attributed in large measure to Plotinus, was perhaps most influential on his work (Meredith, 1986). There is, for example, a move in Neoplatonism to understanding Plato's ideals as pointing to a hierarchically supreme unity called the One; Augustine would have found strong resonance in this with the Christian idea of God.

The force of dualism is such that it has persisted to this day, although from the late nineteenth century onwards, phenomenology has opened the possibility of breaking with the traditional separation of appearances and Forms, of exemplars and ideas. Edmund Husserl is credited with developing phenomenology as a method, but it was perhaps Heidegger, a student of Husserl, who first maintained attention on the situated being of the body in the world as the fundamental way of understanding its nature. As such, these figures represent significant waypoints in the movement of ideas, and for this reason I draw on their works to illustrate the background context of embodied cognition.

In the tenth book of his *Confessions* Augustine, reprising themes described by Plato's Socrates in his *Phaedo* (Plato, 2002), writes that "the memory contains those reasons and laws beyond counting of numbers and measurements, none of which have been imprinted on it by the senses, for they have neither colour, nor sound, nor taste, nor smell, nor feeling" (Augustine, 2009, p. 264).

Here we confront the difficulty of the dualist. Augustine was committed to a world-view of flesh and form, a world informed by Manichaeism and Neoplatonic philosophy, and so the problem of how the body has awareness of things that seemingly are not of the flesh, or sensible—numbers, lines and the like—was pressing. The problem cuts both ways: how does the immaterial soul, or the intellect, have knowledge of things that are sense impressed? As Aristotle notes in the *Metaphysics*, "Above all one might discuss the question what on earth the Forms contribute to sensible things" (Aristotle, 2005, p. 23). For Augustine, the central problem was more the concern of how one comes to know God (see, for example, O'Daly, 1999).

Augustine, who walked the paths of the senses before finding a more introspective route, devoted the tenth book of his *Confessions* to an elucidation of the process of coming to know God through the agency of the "outer man" as interpreted by the "inner man". To the self-directed question, "Who are you?" he answered that he was a man, and that "in my person are body and soul, one without, and the other within" (Augustine, 2009, p. 257). Thus are the inner and outer man discerned to be distinct entities within the one man. And by which of these ought one seek God? Augustine determines that the inner man comes to knowledge through the agency of the outer:

> Better is the inner self. For to it all the body's messengers made report, as though to a president and judge, about the replies of heaven, earth and all the rest which they contain, saying: 'We are not God, he made us.' These things did my inner self know by the service of the outer self. I, the inner man, understood these things, I the soul, through the body's sense. (Augustine, 2009, p. 257)

Not only are we dual creatures, then, but we are dual creatures with an implied inequality of parts: the inner man is better than the outer; the mind is better than the body, which must report to it. Understanding is the prerogative of the inner man.

Sensation, therefore, is distinguished from knowledge. Augustine observes that animals fail to have knowledge, for though they may see the semblance of God in the world, “reason does not preside as judge over the senses which make report” (Augustine, 2009, p. 257). Animals, that is, lack the inner being. Even men can fail to know God if they are too subject to love of the world, for “subjects cannot be judges” (Augustine, 2009, p. 257). A distancing of the man from the world, of the inner man from the outer, bridged by some force of communication, seems to be required if one is to nurture knowledge. And so men withdraw from the world, and judge it base.

Augustine deemed that powers animate the body, fill it with life and organise the senses. The body is understood as something other than the “I”, which is more closely akin to the mind, or soul, and revealed in memory and thinking on things that are not easily understood as having been sensed. This epistemology, however consonant with our general sense of self, is to be surrendered if we are to move toward a new epistemology consistent with theories of embodied cognition. An obstacle on this route rests in what Aristotle would call the *habitus*[6]. We are inclined or disposed to see things as they are presented to us, where an unquestioning existential privilege is given to the “us”.

One helpful technique in moving away from the dualism of Aristotle is to become more acutely aware of how we—as beings—experience things. This move is reminiscent of the epoché of the phenomenologists. Edmund Husserl, in the first volume of his *Ideas*, wrote that “We have to abide by what is given in the pure mental process and to take it within the frame of clarity precisely as it is given. The "actual" Object is then to be 'parenthesized'" (Husserl, 1983, pp. 219-220). This aims to reveal the Object itself, and to bracket out the attachments of valence incumbent upon our receipt of it as an object of interest. It is almost as if one were to see without perceiving, but to know that one has seen; tacitly to know, we might suggest. The attachments are not to be denied, however:

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[6] *Hexis*, in Greek, translated into Latin as *habitus*, particularly by the scholastics. See, for example, Ockham (1990): “Knowledge as a *habitus* (‘habitual’ knowledge) is no less such a quality [which exists in the soul] than an act of knowledge is; but an act of knowledge is such a quality; therefore knowledge as a *habitus* is such a quality also” (p. 3).

> They are indeed there, they also essentially belong to the phenomenon. Rather we contemplate them; instead of joining in them, we make them Objects, take them as component parts of the phenomenon—the positing pertaining to perception as well as its components. (Husserl, 1983, pp. 219-220)

Whilst Husserl has presented an interest in things as they are and in acknowledging that things become objects when we cast them with intention, emotion and purpose, it is not clear that his world view is ultimately any less dualistic than Augustine's. In both cases, there is a sense that the one perceiving, however phenomenologically disciplined, is ontologically distinct from that which is perceived. When attachments—or properties—of an object are understood as discernible parts of the object, moreover, it becomes easy to veer toward a Platonic dualism in which we begin to believe that properties such as "threeness" confer distinct ontological status to forms such as "three". To develop sensitivity to a non-dualistic ontology requires that we try to understand our felt sense of self *as part of experience*, even as it contributes to the feeling of experiencing something "other".

For both Husserl and Augustine the role of conscious deliberation is significant, whether as obfuscator or elaborator. We can imagine that Augustine might have been sympathetic to the view that we saturate the world with meaning as we perceive it, but his knowledge of the world is gained less through an epoché than through a process of *gathering* in the mind that which is already given:

> We therefore discover that, learning those things whose images we do not ingest through the senses, but perceive them, as they are, in ourselves without a medium, is simply to receive by perception things which the memory already contained unclassified and unarranged, and to take knowledge of them and be careful that

> they should be placed near at hand, in the same memory where before they were scattered, unheeded and hidden, and so should readily come into the mind, now aware of them. (Augustine, 2009, p. 263)

Plato (1993) had spoken earlier of the “capacity for knowledge [being] present in everyone’s mind” (p. 245), suggesting that education should be “the art of orientation” (p. 245), that it “should proceed on the understanding that the organ [the mind] already has the capacity, but is improperly aligned and isn’t facing the right way” (pp. 245–246). Augustine’s view that the mind already contained things, but that an effort was required to arrange them into knowledge can be read, therefore, as a continuation of the Platonic perspective. For both Plato and Augustine, the work of realignment ultimately consisted in turning toward some ultimate good.

Augustine (2009, p. 264) identifies the relation between *cogo* [collect] and *cogito* [re-collect] whence the ongoing, re-collections of the mind comes to us as *cogitare*, cogitation, or thinking. Remembering, then, in Augustine’s scheme, is a form of gathering or collecting. This gathering of what is already in the mind is, indeed, suggestive of the experiences of remembering and thinking with which we are familiar, and yet it is far removed from Maturana’s notion of cognition as effective behaviour, a situation which reflects the different view points from which Augustine and Maturana have studied their worlds.

While it is possible that *cogito* itself is derived from *co-* [*com*] and *agito*, as in agitate, whence *cogito* is to be shaken together, an alternative derivation has *co-* join with a form of *aio*, where the reference here is to speaking or affirming. Taken all in all, *cogitare*, in practice, came to refer to actions of calling to mind, gathering memories, and, allowing a certain etymological license, self-talk. In short, it came to refer to the actions that we now regard as thinking. Cogitation can thus be construed as an active self-speaking gathering of things, which is to say that cogitation can be interpreted as a self-realisation of the process of living. Taken thus, Maturana’s broader “effective behaviour” can be seen to provide the material basis upon which the sensation of Augustine's cogitation reveals itself and becomes known.

Inspired by Maturana and Varela, I shall turn aside from the Platonic realm and venture to formulate an alternative to Augustine's mathematical reflections. Numbers, lines and all the objects of mathematics will be taken as proper to the autopoietic realm of the individual for whom they become significant. The path to explore will open us to the subjective feeling of engaging in mathematics as a bodily phenomenon.

Later in his life, Varela preferred to say that "living is sense-making" (Thompson, 2004, p. 386). As Thompson elaborates:

> Living isn't simply a cognitive process; it's also an *emotive* process of sense-making, of bringing signification and value into existence. In this way the world becomes a place of *valence*, of attraction and repulsion, approach or escape. (Thompson, 2004, p. 386, author's italics)

Thus, it is not just toward the origins of mathematics that this inquiry turns, but to me, as sense-maker. The lived experience of cogitation—*to live is to cognite*—captivates me, but how I approach the question of mathematical cognition will, in large part, require a working through of that which I bring to the question. As Husserl's student, Heidegger, put it in his inaugural 1929 lecture, "Every metaphysical question can be asked only in such a way that the questioner as such is present together with the question, that is, placed in question" (Heidegger, 1977, pp. 95-96). Now, while it will not serve any direct purpose to be drawn into a discussion of what "metaphysics" might entail, we can find in Heidegger's assertion at least a reminder that questioning being as such is done by beings. The reflexive quality of this activity suggests that rather than ask in expectation of direct answers, one may enter a cycle of recalibration and refinement.

Valence demands a relationship, so that by conducting myself through this investigation I inform myself and thereby, it might be hoped, extend the breadth of the autopoietic domain in which I make my world. This is a theme to which I shall return, but for now I want to dwell a little longer on the question of the self-voice.

The Situated Self-Shaping Self

The entanglement of the "self-voice" in mathematical thinking requires us to confront what is meant by the self. The Platonic mathematician will be capable of believing that our sense of conscious self underlies a distinct presence and let pass the question of whether that presence is *of* or *in* the body. The proponent of embodied cognition, on the other hand, might subject this very sense of self to question. This is not to say that the *experience* of self is not acknowledged—there are circumstances in which we patently feel ourselves to be intentional agents—but there is uncertainty over whether this *sense* of agency relates to actual conscious, free agency. It may be that our sense of intentional self is more characteristic of after-the-fact self-talk, by which it is suggested that we act and afterwards believe that our actions were caused by our intentions, that we unknowingly apply *post hoc, ergo propter hoc* (after this, therefore because of this) rationales, we might say.

I do not want to presume that there is an either-or determination to be made here. It seems clear enough that there are occasions in which I am not a free agent—when my actions are autonomic or reflex based, for example. Actions of this type we would probably not regard as intentional. More interesting is the observation that I just reached for my cup of coffee. I do not quite remember deciding to take a sip, but there I was, cup in hand, reading the previous sentence, imbibing hot, black coffee. Now, it is true that I had earlier decided to make the coffee—or had I? There was an element of routine in the actions that led to the making, and I cannot really know whether I *found* myself in the coffee-making-groove or whether I had freely chosen to put myself there. As for the sip I had taken, I can claim the placement of the cup within easy reach as I sat at my desk, deeply immersed in writing, as mitigation for my culpability in the final sipping. I was framed. Merleau-Ponty (2007) is surely right to draw our attention to *attention*:

> Consciousness must be faced with its own unreflective life in things and awakened to its own history which it was forgetting: such is the true part that philosophical reflection has to play, and thus do we arrive at a true history of attention. (p. 36)

My inattention to the coffee on my desk marks the occasion of sipping as different from the occasion of making, or at least from some point in the making where I had the opportunity to desist but yet continued. Orientation of attention, however, is subject to the same questions: do we choose that to which we attend? Is there a grading of involvement? Is it that I sometimes freely choose, but otherwise my attention may be taken, as something "grabs" me?

Such considerations were given impetus by the results of the famous experiment of Benjamin Libet in which it was discovered that there is an issue of timing between observed cerebral activity and reported awareness of an intention to act. It was found that cerebral activity—the readiness potential originating in the supplementary motor area of the brain—preceded subjects' reports of their subjective experience of wanting to act by an average of about several hundred milliseconds: "It is concluded that cerebral initiation even of a spontaneous voluntary act, of the kind studied here, can and usually does begin unconsciously" (Libet, Gleason, Wright, & Pearl, 1983, p. 640). More recent work has found an even longer preparatory time period suggesting a "tentative causal model of information flow, where the earliest unconscious precursors of the motor decision [to press a button] ... remained unconscious for up to a few seconds" (Soon, Brass, Heinze, & Haynes, 2008, p. 545). It is very difficult to disentangle a determination of intention from action, however, particularly when subjective reports are implicated. Judy Trevena and Jeff Miller (2010) have since explored cerebral activity associated with decisions *not* to move, and reported "no evidence that prevailing conditions in the brain just before a spontaneous decision can predict the outcome of that decision" (p. 454).

The question of intentionality is bound inextricably with an understanding of *who* or *what* intends. To say that my brain exhibits signs of my readiness to act before I am aware of my intention to act threatens to sunder *me* from at least part of my body. It begs a search for self, to seek it somewhere in the body, and calls to mind Descartes' suggestion that the pineal gland was a likely seat of the soul: "My view is that this gland is the principal seat of the soul, and the place in which all our thoughts are formed" (Descartes, 1991, p. 143).

An alternative, deflationary view, on the other hand, renders the Libet timing gap consequential of a particular theoretical understanding of the conscious *you*. Daniel Dennett (2003) summarises:

> What you do and what you are *incorporates* all these things that happen and is not something separate from them. Once you can see yourself from that perspective, you can dismiss the heretofore compelling concept of a mental activity that is *un*consciously *begun* and then only later "enters consciousness" (where *you* are eagerly waiting to get access to it). This is an illusion since many of the reactions *you* have to that mental activity are initiated at the earlier time—your "hands" reach that far, in time and space. (p. 242, author's italics)

The *you* that exists (somewhere, somehow) within your body is, in other words, a throwback to a Platonic understanding of being. This does not remove problems of understanding agency, but forces a realignment of our understanding of what the problems are. Theories of intentional action begin with the problem of how the emergence of the reflective being relates to the process of being. We can be wary, too, of saying that the intentional self is the *consciously aware* part of the larger *you* Dennett identifies, for this is still to postulate a partitioning in which I locate *me* within my body. We can speak of an intentional agent, but that agent must not be understood as a self in Platonic, dualistic terms, so much as an involved being with pragmatic concerns. The challenge is to surrender notions of self and embrace an understanding of being-in-the-world.

A salient feature of being in the world of the mathematics classroom is the relatively mundane observation that actions that demand considerable conscious effort from a novice can be automated with practice so that for the expert these same actions become largely unconscious and unmediated by deliberative thought. A Year Eight student might struggle to factorise an algebraic expression that the teacher scarcely attends to in pursuit of a solution to some broader problem. The actions of an expert are still actions, however, and freely chosen to the extent that the expert has selected his or her situation[7], but the priming effect of training and experience relegate the "control" of the actions to a more pre-reflective aspect of self, to what Merleau-Ponty has called motor intentionality (Merleau-Ponty, 2007, p. 127).

The foregoing draws attention to the need to re-imagine the idea of the self if one is to pursue an embodied account of cognition: to no longer seek a self within the body, to abandon the self-body dichotomy; rather to work to understand the body within the world. To add to the sense of complex realignment that is emerging, we can take heed of Steven Rose's (2005) observation that even the brain-body dichotomy ought to be abandoned. Noting the complex integration of the nervous system throughout the bodily organs, Rose cautions that

> no account of the development of the brain can be complete without understanding that what we need to interpret is not just the growth of an organ but of an organism of which the organ in question—important though the brain is—cannot be understood in isolation. No brains in bottles here. (pp. 64–65)

The tight integration of our senses, and the interplay of innate propensities and environmental stimuli as we grow, indicate that isolating and disentangling threads might distort a picture that is intricately interdependent on every part of itself for its coherence. Thompson and Varela (2001) urge this upon us in forceful terms:

> Because they are so thoroughly enmeshed—biologically, ecologically and socially—a better conception of brain, body and environment would be as mutually embedded systems rather than as internally and externally located with respect to one another. Neural, somatic and environmental elements are likely to interact to produce (via emergence as upward causation) global organism-environment

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[7] The batsman, for example, may choose to bat, but when the ball approaches at 140 kmh^{-1} there is no time for reflective considerations over what to do.

> processes, which in turn affect (via downward causation) their constituent elements. (pp. 423–424)

There is, in other words, a circularity of engagement where every level informs every other—and much of this engagement, moreover, is with other beings. Since we do not grow in isolation from each other, nor have evolved as solitary beings, our social interactions can be seen as crucial elements in the forging of our sense of selfhood. Shaun Gallagher (2005) has found that movements of others are fundamental to the formation of the primary embodied self:

> Quite literally, in any particular instance, it may be the other's movement that triggers my own proprioceptive awareness. There exists in the newborn infant a natural intermodal coupling between self and other, one that does not involve a confused experience. Rather than confusion, a self-organizing collaboration between visual perception and proprioception, between sensory and motor systems, and between the self and the other is operational from the very beginning. Body schemas, working systematically with proprioceptive awareness, constitute a proprioceptive self that is *always already* 'coupled' with the other. (p. 81, author's italics)

To be always already coupled with the other is to be a functioning organism in a milieu within which a sense of self becomes a way of seeing and of interacting with oneself and others, but not necessarily a veridical pointer to any Cartesian *res cogitans* or Augustinian inner man. Such, then, is the challenge of embodiment: traditional descriptions, together with concomitant assumptions, are pared back to allow a complex, integrated, autopoietic interpretation to emerge.

Under Gallagher's description, the prenoetic body (by which Gallagher means the body that acts before cognitive or reflective awareness) becomes the source of coherence of the conscious experience. The body, however, is not conceived in merely mechanical terms: rather, it "*actively organizes* its sense experience and its movement in relation to pragmatic concerns" (Gallagher, 2005, p. 142, author's italics). How the body experiences the world, that is to say, is influenced by the intentions of the subject. If I am thirsty then I might perceive splashing water as welcome and relieving drink, but if I am cold and tired that same water might appear threatening. The touch of a surface might be understood as smooth or hot, depending on whether my interest was in texture or temperature. In short,

> physiological processes are not passively produced by incoming stimuli. Rather, my body *meets* stimulation and organizes it within the framework of my own pragmatic schemata. (p. 142, author's italics)

Here, too, we see the circle of engagement: intentions play a part in the determination of how phenomena are experienced, and the experience of phenomena must play into our subsequent intentions. We learn through engagement; but not only this, we *make ourselves* through engagement. That which we want to do plays a role in determining what we will be able to do. We find form in this world by sculpting ourselves and feeling the caress of space. To define the world in our image ever tempts, therefore, and a special effort of mind is required to touch the earth repeatedly so that we do not yield to anthropomorphic descriptions.

AN ESSAY IN PROOF: WHOSE MATHEMATICS IS IT, ANYWAY?

The felt, human quality of mathematics is paramount, here—there is no disentangling of form from flesh. In what may be a surprising revelation to those less familiar with the Sunday speculations of mathematicians, mathematics is not entirely the rigorous, formal construct it might appear. We can consider the question of what counts as proof by way of illustration. Reuben Hersh (1997), alert as ever to the teacher's pedagogical plight, identifies two meanings of mathematical proof:

> In practice, it's one thing. In principle, it's another. We *show* students what proof is in practice. We *tell* them what it is in principle. The two meanings aren't identical. That's O.K. But *we never acknowledge the discrepancy.* How can that be O.K.? (p. 49, author's italics)

This question offers a glimpse into the undercurrents of conflict that can simmer within the community of researchers, exemplified by an exchange of views in the Bulletin of the American Mathematical Society, known as the Jaffe-Quinn debate, which followed the publication of "Theoretical mathematics: Toward a cultural synthesis of mathematics and theoretical physics". In this paper, Arthur Jaffe and Frank Quinn (1993) expressed their concerns that experimental mathematics, such as that facilitated by the use of computers, which encourages speculation and conjecture but not always rigorous proof, is potentially dangerous to mathematics. They observed that the use of rigorous proof "has brought to mathematics a clarity and reliability unmatched by any other science" (p. 1). The cost, they acknowledged, is that this practice can make mathematics "slow and difficult" (p. 1), and that groups have in the past sought to disengage from such rigour, with occasionally disastrous results. "Yet today in certain areas there is again a trend toward basing mathematics on intuitive reasoning without proof" (p. 1).

Jaffe and Quinn suggested that the reliability of the mathematical literature is of central importance to the ongoing development of knowledge and the avoidance of error. They prescribed that credit for those who make conjectures should not be given in the same measure as to those who do the rigorous work of proving those conjectures. They were not against experimentation and speculation, but required that such activity be clearly distinguished from theorem and proof.

There ensued a collection of letters written in response to the paper (Atiyah et al., 1994), and for a time the roles of experimentation and proof were much discussed (see, for example, Epstein & Levy, 1995; Horgan, 1993; Thurston, 1994). Michael Atiyah (1994) agreed with the need for distinction "between results based on rigorous proof and those which have a heuristic base" (p. 178), but found the overall tone of Jaffe and Quinn's view too authoritarian. Jaffe and Quinn, he said,

> present a sanitized view of mathematics which condemns the subject to an arthritic old age. They see an inexorable increase in standards of rigour and are embarrassed by earlier periods of sloppy reasoning. But if mathematics is to rejuvenate itself and break exciting new ground it will have to allow for the

> exploration of new ideas and techniques which, in their creative phase, are likely to be as dubious as in some of the great eras of the past. Perhaps we now have high standards of proof to aim at but, in the early stages of new developments, we must be prepared to act in more buccaneering style. (p. 178)

René Thom (Atiyah et al., 1994) made the point that

> rigor is a relative notion, not an absolute one. It depends on the background readers have and are expected to use in their judgement. Since the collapse of Hilbert's program and the advent of Gödel's theorem, we know that rigor can be no more than a local and sociological criterion. (p. 203)

He went on to associate rigour with rigor mortis, and to suggest that work claiming full rigour is "graveyard mathematics" (p. 204).

In response, Jaffe and Quinn (1994) appreciated these comments and added:

> There is a very real sense in which mathematics which has been successfully rigorized is dead, and the real life is, as Atiyah suggests, in speculation. We might think of mathematics as a tree: only the leaves and a thin layer around the trunk and branches are actually "alive". But the tree is supported by a large literature of published "dead" wood. Our position is that material should be really sound before being allowed to die and to be incorporated into the wood. (pp. 208–209)

What is significant for us in the above is that there is and likely always will be debate over what constitutes mathematical proof, on what is regarded as sufficient rigour, and over the extent to which proof is necessary, anyway. Debate, indeed, over the extent to which mathematics and "rigorous proof" are synonymous[8]. In any case, who has the right to say? As Benoit Mandelbrot stated, "For its own good and that of the sciences, it is critical that mathematics should belong to no selfappointed group; no one has, or should pretend to, the authority of ruling its use" (Atiyah et al., 1994, p. 193).

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[8] Kleiner and Movshovitz-Hadar (1997) provide a historical context for this debate, grounding our appreciation of the ever-so intermingling of heuristics, rigour, intuition, formalism, synthesis, analysis, idealism and empiricism in mathematics.

More recently, Richard Foote (2007) has observed that mathematics has become so complex that it is approaching limits of human verifiability. He alerts us that

> the human element is an essential agent in the evolution of mathematics as a complex system, and the "layers" of complexity mirror the "knowledge states" in this adaptive process. Computers are now taking an increasingly important (and controversial) role in both verifying and discovering new mathematics; and so we stand on the threshold of a new dynamic where mathematics, the very foundation of science, may produce and build on results that are humanly unverifiable by even the combined effort of the community, and the veracity of certain results may only be known with some degree of probability. (p. 412)

Foote makes this observation in the context of a consideration of the principles and characteristics of complex systems, of which mathematics might be regarded as "the ultimate" (p. 410). Among the principles he discerns, "A complex system must have a substantial impact on systems other than itself; from its study, some larger principles, insights, techniques, or connections should accrue" (p. 412). Thus the scope of such study exceeds narrow parameters that fragment science and learning in general, and so motivate explorations such as this, my inquiry. His conclusion, reminiscent as it is of Mandelbrot, is an exhortation that we can take as a form of rallying cry:

> Finally, the study of complex systems should be the exclusive purview of no one but the responsibility of everyone: Each scientist, mathematician, or researcher unfurls the mysteries of nature and humankind in small, deliberate steps. Science embodies the ability to verify, reproduce, and convince others of the veracity of one's discoveries, so the work of scientists is inherently incremental and precise. On the other hand, it is incumbent on us all to work toward enhancing the understanding of "big picture" issues within our own disciplines and beyond; each of our disciplines must itself exhibit the inherent facets of a complex system, or our research is surely nugatory. (Foote, 2007, p. 412)

Buldt, Löwe and Müller (2008), recognising that mathematics is neither one thing nor the other, which is to say that it is neither fully formal nor utterly socially constructed, assert that it is "a desideratum that philosophers of mathematics develop a mediating position that strikes a balance between the special epistemic character of mathematics and the social embedding of mathematical practice" (p. 312). By the unfolding measures of this thesis, there is no sense in which the one can be contemplated without the other.

WHAT IS LOST? THE MORAL QUESTION

Do we lose something when we realign our understanding of ourselves? To surrender Platonic ideals, even if they were only ever loosely held, is surely a loss of some kind. "Their links were made so loose and wide, methinks for milder weather," wrote Henry Thoreau in his poem, *Sic Vita* (Fuller, Emerson, & Ripley, 1842, p. 81). Too loose, too wide; but Thoreau's poem begins, "I am a parcel of vain strivings, tied by a chance bond together," and it is perhaps this fear that we are held together by chance, vainly striving, that underscores sentiments of anxiety that arise when we invoke embodiment.

Rainer Maria Rilke might have had a sense of our need to maintain our acquaintance with simple things when, with a trace of an Augustinian echo, he wrote in his first *Duino Elegy*,

> I choke back my own dark birdcall, my sobbing. Oh who can we turn to in this need?
>
> Not angels, not people, and the cunning animals realize at once that we aren't especially at home in the deciphered world. What's left? Maybe some tree on a

> hillside, one that you'd see every day, and the perverse loyalty of some habit that pleased us and then moved in for good. (Rilke & Young, 1978, p. 27)[9]

This evokes the sense of loss that we can feel in letting go a long held set of beliefs or practices, and calls us to consider, "What's left?" The deciphered world—our *umwelt* or cognitive domain, in which we make our way—occupies us, but this occupation fails to quell a desire for foundational explanation. We persist in asking the perennial questions, "Why, how?" The animals, over which we thought to hold dominion, seem to belong more to the natural world than we, yet embodiment reminds us that we, too, are in this natural world. We, however, have discovered *noesis*—knowledge not just of self, but of an awareness of being in it. Of ourselves as Dasein, as Heidegger (1996) might be telling us; and yet this awareness cannot allow us to dissociate from the world. For we are not simply in the world, we are *of* the world, and though we might fear returning to its fold, we cannot beg angels for relief.

What, then, do we have? The tree on the hillside expresses all that seems natural and clear. It is our touching-of-earth. We hold to that tree. It is sufficiently distant, un-us-like, to be trustworthy; yet, like us, it is evolved and spatially situated. It is, like us, in a process of becoming. It changes. If we cannot fix ourselves by what we are, perhaps we can find ourselves in what we do? When we struggle to find ourselves in theory, then, it may be that we can discover ourselves in habits of action: we can be what we do in the deciphered world, and by accepting that we belong in the natural world, find solace beside that tree. Heidegger (1996) tells us that "familiarity with the world does not necessarily require a theoretical transparency of the relations constituting the world as world" (p. 81), suggesting that whereas philosophy has tended to focus on theorising intentional activity, a taken-for-granted-familiarity with the world constitutes a primordial core of our sense of being: "This familiarity, in its turn, helps to constitute Da-sein's understanding of being" (Heidegger, 1996, p. 81).

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[9] I have not retained the triadic line structure preferred by the translator, David Young.

And what does this familiar world allow? What will we dispose ourselves to learn? "Nothing is gone, dear, or no one that you knew. The forests are at home, the mountains intimate at night and arrogant at noon. A lonesome fluency abroad, like suspended music" wrote Emily Dickenson (Dickinson & Bianchi, 1924, p. 63). And yet, if we seek foundations in a scientifically informed, embodied mien do we perhaps expose ourselves to the risk of moral loss? In his essay, *Explanation and Practical Reason*, Charles Taylor (1995) alerts us to the danger:

> The model for all explanation and understanding is the natural science that emerges out of the seventeenth-century revolution. But this offers us a neutral universe; it has no place for intrinsic worth or goals that make a claim on us. Utilitarianism was partly motivated by the aspiration to build an ethic that would be compatible with this scientific vision. But to the extent that this outlook has a hold on the modern imagination, even beyond the ranks of utilitarianism it militates in favour of accepting the apodictic model, and hence of a quasi-despairing acquiescence in subjectivism. (p. 38)

Does the apodictic model (that in which things are demonstrably so) condemn us to "acquiescence in subjectivism"? The link between naturalism and subjectivism (read relativism) is shown in the post seventeenth century destruction of the Platonic-Aristotelian Forms, which provided standards by which things were to be judged, says Taylor. Once gone,

> the only plausible alternative construal of such standards in naturalistic thought was as projections of subjects. They were not part of the fabric of things, but rather reflected the way subjects react to things, the pro-or-con attitudes they adopt. Now perhaps it's a fact that people's attitudes tend to coincide—a happy fact, if true; but it does nothing to show that this point of coincidence is righter than any other possible one. (C. Taylor, 1995, p. 38)

Taylor here draws attention to the naturalistic fallacy—that as things are, so they ought to be. The "neutral stance toward the world" that science affords is unable to lead to insight into moral ontology, says Taylor (1989, p. 8). We should, rather,

> treat our deepest moral instincts, our ineradicable sense that human life is to be respected, as our mode of access to the world in which ontological claims are discernible and can be rationally argued about and sifted (C. Taylor, 1989, p. 8).

The mode of thinking in which Forms provide standards of rightness is not now to be contrasted against embodied accounts, however. Orientations in which the world, including its people, were objectified, might or might not have been typical of practical thought following the efflorescence of scientific thinking from which the enlightenment emerged, but this mode of thinking is not that which is represented in the embodied account that is emerging here. It would not be going too far, indeed, to state that the embodied account de-objectifies being. The *practice* of science, moreover, affords no neutral stance toward the world, but finds itself perpetually implicated in ad hominem adjudications in virtue of its being done by embodied beings. By such means, indeed, is science impelled.

The fear that a coincidence of attitudes fails to satisfy a criterion of "rightness" must be addressed, however. There are two matters that bear. First is the *reason* that commonly held attitudes are common. Taylor himself touches on this when he appeals to our deepest moral instincts. For to speak of our instincts as foundational, provided one makes no concomitant appeal to a dualistic ontology, is to recognise a potential natural basis for moral behaviour.

The second matter is the question of *rightness*, itself. Taylor is correct, of course, when he asserts that even if our shared instincts inculcate common behaviour, this is no reason to suppose that this behaviour is any more right in virtue of that fact, but the assertion itself threatens implicitly to lead one to the presupposition that there is an ontologically discernible criterion of rightness. Taylor might feel obliged to provide that criterion, but it is not the task of a proponent of embodied cognition to do so. For the embodied account does not seek to impose such a criterion upon behaviour as such, so much as to discern—or admit—an explanatory basis for behaviour.

This is not to say that a sense of what is right or of what is good is not a feature of the natural condition. It demonstrably is the case that we feel a sense of moral rightness about certain actions and that we acknowledge taboos. It is, rather, to suggest that a moral sense arises as a consequence of our evolved way of being in the world, as we shall describe later. It comes with us, as part of the deal; it is neither something we aspire to nor something imposed upon us. This feature of our condition stands to inform the question of a criterion of right behaviour, if only because it obviates the necessity of recourse to metaphysical dualisms for explanations. The question shifts from, "How ought I act?" to "How will I act; and why that way?" The discernment of the "why" might leave open ethical judgements, but an understanding of who is judging, of the nature of the being, is foundational to making judgments, because ultimately, the judgments we make will be of a one with our constitutive nature.

It is dangerously easy to confuse explanations with prescriptions. A simple thought experiment, inspired by an observation made by Patricia Churchland (Campbell, 2012), might illustrate this point. Suppose that it was found that a foundational reason for our altruistic behaviour, our kindly behaviour toward others, was that such behaviour resulted in the production of a chemical in our bodies that caused us to feel good, that helping others was known to produce a "natural high". Now suppose further that it was possible to isolate or synthesise this chemical and to deliver it at will via a nasal spray. Would we consider it good or right to use the spray as a means of delivering that natural high, even if it meant we would no longer perform the actions that formerly produced it? That is to say, would we consider the end as more significant than the means? I doubt it. The foundational explanation—the natural high—does not supplant the argued-for value of the actions we have customarily associated with the good feeling, even if that very argued-for value is itself founded upon a desire to make judgments that accord with our ineluctable sense of what is right.

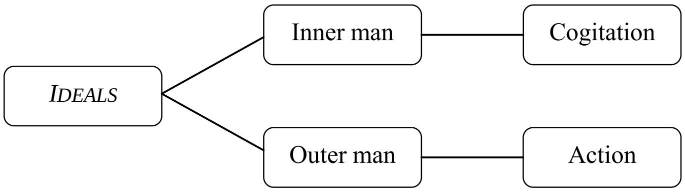
In other words, the knowledge that a *feeling* underlies our behaviour in an antecedent sense might play significantly into arguments over whether it is *right* to behave so. If the foundational seed of a behaviour is less other-directed than inner-satisfying, less suggestive of an aspiration to some ideal of good and more derivative of an evolved adaptation, any claim to rightness, as such, is at least contingent upon circumstances, and in that sense pragmatic and arbitrary. It might yet transpire that sophisticated justifications for the morality of certain behaviours find their ultimate foundation in the humble condition of our being according to which certain behaviours less than so much *are* right, as *feel* right. But to take this further, if we believe we are being moral for other-directed reasons, if our rationally constructed justifications represent genuinely intended deliberations on the welfare of ourselves and others, can we really gainsay our intentions even if it transpires that the foundations of our deliberations are grounded in chemical facts? Any attempt to apply knowledge of chemicals to rational sifting of moral choices would implicate an over-stretching of levels of description and an under-appreciation of how we are constituted in relation to others and, indeed, to ourselves.

While it is not my intention to press the point or make the case, it will not escape the attentive reader that there is a tight analogy to be found between the emergence of mathematical thought on the one hand and the emergence of what is understood to be norms of good behaviour on the other as features of our embodied condition. Even as I say there is no ideal “three” or “four”, and yet count, I incline to refute an ideal good while yet desiring to act well. In the end, my actions will surpass my argument.

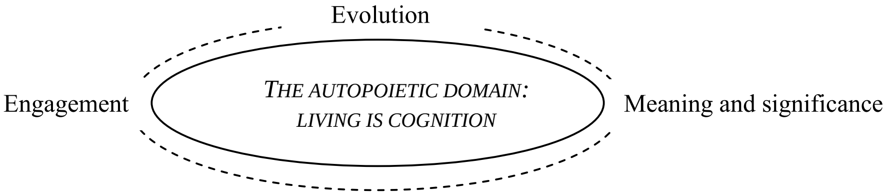
PART TWO INCUBATION

Our sense of self, our very feeling of agency, is a remarkable feature of our being. It dominates our philosophy; it is the ground from which much of our educational practice is sourced. And yet, if we are to pursue an embodied theory of mathematics we will need to re-imagine the self. Before we become attentive to our possibilities, we—as embodied beings—are alive with thought. “Thought” here takes on a broad meaning, a saturation of our being with potential; an experience of sensation—whether generated internally or experienced from without the body—resolving in a course of action. *To live is to cognite*, as I have been noting.

To this point I have suggested that a realignment of our understanding of ourselves is needed in order to develop an understanding of what it means to be mathematical within the parameters of embodied cognition. I have contrasted a linear, Platonic variety of dualism with a more circular, autopoietic model of being, in which engagement is emphasised.


IDEALS
Inner man
Cogitation
Outer man
Action

A MODEL OF DUALISM
in which the separation of the inner and outer is fundamental


Evolution
Engagement
THE AUTOPOIETIC DOMAIN:
LIVING IS COGNITION
Meaning and significance

A MODEL OF AUTOPOIETIC EMBODIMENT
in which the maintenance of life and the dynamic of becoming are fundamental

In this part I shall consider more closely the process of learning mathematical ideas, including questions of how we process mathematical thought. In particular, I consider the relationship between syntax and semantics, and consider the valence of ideas, or their associated significance for us, as significant to an embodied epistemology.

CHAPTER TWO WHAT IS KNOWLEDGE?

> Long I thought that knowledge alone would suffice me—O if I
> could but obtain knowledge!
>
> Walt Whitman (2009, p. 354), *Calamus*

HOW DO WE KNOW WHAT WE KNOW?

Meg asks how we know that the pattern 52, 77, 102, … can be described by stating that the nth term is given by $25n + 27$. She is not asking whether the description is correct—she can see that it is. What she wants to know is *how we come to know it*. Here too, she can be misunderstood. It is not just that she seeks a method that will enable her to find the description. She does want that, yes, but more, she wants to understand the means by which we come to recognise, even within a method, that $25n + 27$ is an accurate description. What is the source of the recognition that 25 is one number and 27 the other? When pushed with a simpler example, she acknowledges that she "just knows" that five times six is thirty or two plus four is six, but this does not quite satisfy her. She knows she can recall certain number facts and combine elements to produce new facts: but the process of combining entails a certain intention, a deliberate directedness of thought. Meg, however, had seen a fellow student *realise* that $25n + 27$ worked, almost "out of the blue". And that, she indicated, was somehow different to remembering or deliberately searching and finding.

Felicia is on exchange from Germany. She is finding her senior mathematics course something of a challenge. She regularly asks me to explain this or that detail, and she has me look over her efforts with a critical eye. I ask her after class one day how she feels she is coping, whether the mathematics "makes sense." "Oh yes," she says, "it does make sense! It is just hard to get the understanding."

Her emphatic response, “It does make sense,” is a statement of conviction. Felicia knows that the mathematical derivations she has been shown are correct in the terms of the subject, that they are *sensible*. She believes that what is being done is valid. For all that, however, she finds that when she attempts problems on her own, it is difficult to “find the way,” as she told me.

These examples underline distinctions between a belief in the validity of the process of doing mathematics, the conscious ability to calculate and a kind of unconscious knowing. As a teacher, I think it good to be aware of the distinction, and perhaps to help students become aware.

I enter class one day and ask my Year Eight girls to construct a rod of 90 cm by taking suitable fractions of a 180 cm red rod and a 120 cm blue rod. They understand and proceed to find suitable cases: take one sixth of the red and half of the blue, for example. One student asks if she is allowed to take one half of the red and none of the blue. “Yes, that is fine. In that case you are considering zero as a fraction.” After the girls have compiled a collection of solutions, I ask them to consider the problem as an exercise in numbers, to divorce the numbers from their associations with rods. That is, I ask the girls to solve $120a+180b=90$. Once they understand the implication that they are now able to use negative numbers and fractions greater than unity, they attack the problem with vigour. I try to draw their attention to what is happening—what I think is happening: that they are abstracting a mathematical description from a real context, that they are engaging in number theory so that the exercise is now more general in that it does not only pertain to rods, but to any situation in which 120 and 180 function meaningfully. I do not push this too far; I wish only to plant a seed. I could say that the problem creates its own meaning: that it is *self-determining* in that it does not require 120 and 180 to relate to any particular fact of the world, and that it is generalisable in the sense that 120 and 180 can be replaced with other numbers or objects. The problem, taken in concert with our experience, determines the possibility of its own interpretation within the scope of its structure, and the extent of its application is realised in these interpretations and subsequent reinterpretations. In mathematical terms, this is represented in the search for homomorphisms—structures that share an identical underlying form even though they may arise in quite varied contexts.

The girls simply enjoy the process. They do not trouble themselves overmuch with theory, but search assiduously for numbers. They have discarded the rods without the least regret.

On another day I ask these same girls to perform the basic calculation $\frac{2}{5} \div \frac{3}{7}$. There is some hesitation and indication of uncertainty. We are a small group, relaxed in our little room and we are familiar with each other, having worked together in a previous year; yet even so we do not want recklessly to show ourselves to be ignorant. Ellen begins to frame a suggestion: “Is it—you multiply the two and the seven and the five and the three?” motioning with her hand an imaginary multiplication cross that would pair the two and seven, the three and five. Jenny is encouraged to chip in, “Oh yeah, you flip the fraction!”

“Which one?”

“The second one, and make it multiplication.”

There are exclamations of agreement, memories burble. “Ours is not to reason why, just invert and multiply,” I incant, recalling a former teacher—not a mathematician—who exemplified the “it doesn’t make sense, just do it” attitude.

I ask “Why?” The question is scarcely intelligible. “Because that’s how you get the answer,” explains Lara, betraying a faith in authority that would thwart a mathematician’s curiosity. She accepts the process as valid.

We spend the next half an hour developing an explanation that will become an account for the method. I lead the girls with numerical questions and observations. They are respectfully attentive, gracefully suspending immediate concerns to allow several points to become established, to be seen to be sensible, to be believed. We learn that multiplication is associative, that dividing by a number is equivalent to multiplying by a suitable fraction, that there are seven sevenths or five fifths in 1, and so on. We proceed with an implicit faith in the power of inductive reasoning—I do not impose an algebraic formalism on these girls, not yet. We extend ourselves to realise that doubling a divisor halves a quotient, tripling a divisor thirds a quotient and so on. We realise, therefore, that since $1 \div \frac{1}{6} = \frac{6}{1}$ then $1 \div \frac{5}{6} = \frac{6}{5}$. Other examples follow. We strike the final blow:

$$\frac{2}{5} \div \frac{3}{7} = \left(\frac{2}{5} \times 1\right) \div \frac{3}{7} = \frac{2}{5} \times \left(1 \div \frac{3}{7}\right) = \frac{2}{5} \times \frac{7}{3}.$$

Lara anticipated the denouement. Her face lit up with joy as she realised how the pieces she had been compiling aligned to resolve the question. There is pleasure to be had in achieving a feat of reasoning, the more so if it renders sensible what formerly has been obscure. Now the mathematical process is more than valid: it is substantiated. If I were to press further, I could ask Lara to reflect upon the basis upon which she accepts the substantiation. That is, I could ask her if the “proof” was sufficiently convincing. Convincing for whom? *Whose proof is it, anyway*? This would force a movement toward the foundational questions that underlie this inquiry.

For now, however, the girls are relaxed, happy. They joke about showing “all this working out” when they are next called upon to perform such a division. “Mr Carmichael wants us to show full working!” laughs Serena, and so we spend a moment talking about where this new development fits in our scheme of knowledge. Do we need to reproduce such detail each time? Can we now employ the “invert and multiply” method as we did before? Are we somehow different, though, because we now have a deeper understanding of why we use that particular method? What has changed?

Analysis of lessons such as these can proceed variously. Valerie Walkerdine, for example, might draw our attention to the way in which I appear to privilege abstraction over reference to real, or “concrete” objects. In drawing my students toward that goal I can be seen to participate in—or even encourage—a view in which mathematics in particular but science more generally “props up a fantasy of omnipotence of scientific discourses that can control the world” (Walkerdine, 1994, p. 74). She might remind me of the example of non-European children calculating change in the context of selling goods for the family wellbeing, and the difficulty such children may have in extracting general concepts from their practice:

> What they refused to do was to separate reasoning from the meanings in which thinking was produced. Western logical thinking demands a certain discourse in which reference is actively suppressed. (Walkerdine, 1994, p. 71)

I had presumed that teaching children to learn to abstract is a necessary imperative, a responsibility with which I am charged. I had not considered that in this I am complicit in a hegemony of forgetting:

> such an approach does in my view teach ... the forgetting of which the post-structuralist theorists have spoken in relation to an understanding of the constructed nature of consciousness. When we treat the world as abstract in this way we forget the practices which form us, the meanings in which we are produced, we forget history, power and oppression. (Walkerdine, 1994, p. 71).

This is a powerful alert to the hidden imperatives that govern our expectation that abstraction is desirable or to be encouraged. It warns of a teleological interpretation and of the inherent social implications in forgetting meaning. It focuses our reflections on the existence of a meta-narrative that is being served and begs us to consider the ethical question: *ought* I do this?

> Is it then that so-called abstraction is a forgetting by those who believe themselves autonomous, free and who have enough money and power to treat the world as a logical game, not a matter of survival. What does this removal of meanings do? (Walkerdine, 1994, p. 72)

What indeed? One thing it does is open the thinking mind to new thoughts, thoughts that are impossible when referencing is mandated. The subtle art of mathematical creation *relies* on a kind of forgetting, but whether we who are well fed ought to be making mathematical creations while others go hungry is a difficult question. Perhaps, at the very least, our mathematical creations ought to be directed toward some public good. But then, who determines what is good, and who pays; and who can predict the purpose to which a mathematical invention will be put?

Walkerdine's alert reveals that our participation in the mathematical exercise is a culturally laden undertaking, neither devoid of ethical considerations nor immune to criticism at that level.

On the question of how we participate in the exercise, Hans Freudenthal might have us consider the effect of imposing algorithms as distinct from developing them out of common sense. The degree to which the learner is able to participate in the process would seem to be a relevant factor:

> At any developmental stage of common sense it may be significant how much the learner has contributed to this progress [of cognitive development]. One extreme is learning without being intentionally taught, and the other is learning what has been bluntly imposed; and since the first is more deeply rooted in previous common sense, it may matter some time in the future how the development took place in the past. It matters whether, for instance, an arithmetic algorithm was acquired as an abridged and streamlined version of former common sense activities, or whether abridging and streamlining (or even the algorithm itself) were imposed. (Freudenthal, 1991, p. 7-8)

Several of my students indicated that a previous teacher had spent time explaining why we invert and multiply, but that they had forgotten the details. The others had had no such elaboration. Could such a technique arise out of "common sense" notions? Well, to some extent, yes, but we might venture to guess that most students would require some deliberate teaching to render transparent the connections that govern the technique. In any event, acquiescence in "common sense" is a form of concession that we might hesitate to admit.

Those familiar with the writings of Richard Skemp might recognise in this vignette the relevance of the idea of a mathematical schema (Skemp, 1993) together with illustrations of both instrumental and relational understanding (Skemp, 1976). "To understand something means to assimilate it into an appropriate schema" (Skemp, 1993, p. 43), wrote Skemp in his influential book, *The Psychology of Learning Mathematics*, first published in 1971. This work continued the tradition of attempting to understand mathematical learning in relation to psychological knowledge. The idea of a schema has passed into cognitive orthodoxy. It refers to a conceptual structure by which we understand that newly learned concepts will be integrated more or less successfully with earlier or pre-existing concepts. So, in the above example, we can reason that the girls were able to relate ideas about associativity to previous knowledge about multiplication; that the concept of division as multiplication by a suitable fraction "fitted" with an existing awareness of the idea of halving as dividing into two; that the *new* idea that unity divided by a fraction equalled the fraction inverted was rendered consonant via a process of dialogical reasoning that depended upon existing schemas for its coherence. Felicia's touching faith in the intelligibility of a mathematical technique can be seen as arising out of a level of acceptance of the validity of the process of building successively complicated and further-reaching schemas even while she herself lacked aspects of the appropriate schema. It is conceivable that, in her case, the primary ingredient missing was the time required to accommodate with confidence the barrage of succeeding concepts within her naïve mathematical schema.

An important consideration in such a discussion is the question of whether the learner need be aware of her mathematical schemas. The issue is not so much whether it was it necessary that my young students were aware that they were calling upon previous knowledge—they surely were so aware—so much as whether they were aware of the manner in which that previous knowledge resided in their very selves. At one point in the preliminary discussion with the girls I asked them to visualise the fraction $\frac{2}{5}$. They dutifully closed eyes and imagined—what? They reported two images: a circular pie chart, nearly half coloured; and a rectangle, with a similar proportion coloured. This was hardly surprising, but we must now be careful not to suppose that just because the girls imagined, at a *conscious, cognitive* *level*, that the fraction $\frac{2}{5}$ was a reasonably coloured circular pie or rectangle, that this is precisely the manner in which the concept of the fraction $\frac{2}{5}$ is actually embedded, or schematically organised, in their bodies. Memories of exposure to such fractions would doubtless constitute such images, and so the question of recall can plausibly be understood to occlude knowledge of an underlying bodily integration.

How are mathematical ideas such as fractions represented in the body? It is not even clear that "represented" is a helpful word, here. The cognitive science program is interested in knowledge encoding, and there has been an inclination to expect that knowledge is truly represented as encoded information in some manner. On the other hand, an emerging view, less inspired by computing analogies, is that knowledge is less *encoded* than *structurally instantiated* or *grounded* in the body proper. Such a re-conceptualisation urges sensitivity to the possible meaning of "knowledge". I shall return to a consideration of grounded cognition in due course.

Both accounts, the informational and the grounded, acknowledge that neuronal connections via synapses provide a fundamental note on which the overtones of learning redound. Learning, at the level of primitive memory realised in bodily action, has been demonstrated by Nobel laureate Eric Kandel to occur in the snail *Aplysia* via new synaptic connections that are integrated into existing brain structures (Kandel, 2006). Arguing by analogy and allowing for the massive number of synaptic connections in the brains of large mammals such as ourselves encourages one to speculate that learning is fundamentally a matter of connections between neurons and groups of neurons. We encapsulate this notion in the maxim "nerves that fire together wire together."[10]

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[10] See Edelman and Tononi (2000b) for an account of their theory of neuronal group selection, or Neural Darwinism, in which a mechanism to achieve the effects of consciousness is propounded.

THE FOUNDATIONS OF MATHEMATICAL KNOWLEDGE

A significant feature of mathematics is that it can be communicated from one person to another: but what is communicated? Is there something that is transmitted, or is the sharing more akin to a form of encouragement? Do I give you something, or do I urge you to produce something within yourself? There are, as ever, several levels at which we can approach this question. To help maintain an interpretive awareness (Neuman, 1997) I shall briefly revisit both the purpose and stance of the thesis.

I am attempting to develop an understanding of how we think mathematically and to forge an epistemology within the framework of embodied cognition. To that end, the experience of a student can pose questions and illuminate possibilities, but an attempt to deal with questions requires that attention be given to the way in which we interpret certain actions, words and behaviours. In order that any response be intelligible and credible, it is necessary that it accord with known facts about the way we are[11]; or, in the absence of known facts, that speculation is acknowledged as such and remains putative awaiting further evidence.

So let us delve a little deeper by considering another mathematical experience. Alison is in Year Eleven. She is a dutiful child, who studies hard and tries her best to learn all that is required of her. One day, on returning a marked test paper on applied algebra, I remarked that her interpretation of a correlation coefficient and comments on the validity of extrapolating from data were excellent, that her words flowed easily and with accuracy. She smiled and informed me that they were my words; that she had merely taken the text I had provided in a class example and

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Neurologist Jean-Pierre Changeux and mathematician Alain Connes (1995) turn this to a discussion of the means by which mathematical ideas might be created.

[11] Inasmuch as facts can be said to be known. I take a pragmatic approach, for the greater part.

returned them to me, adjusted for context. I returned the smile and said that they were no longer my words, that I had given them to her as a gift and now, if she chose, they were hers to use. I said that this was part of what learning was.

Let us picture that scene for a moment. I have no doubt that Alison's performance, if that is the right word, would please an examiner and earn her good grades, but had she learnt about the interpretation of a correlation coefficient? Was my pleasure at perceiving her work any more than a form of self-adulation, albeit veiled by Alison's text? More generally, what do we mean when we say that an idea has been learnt? Can a concept "move" from one person to another; and if so, in what sense does any person "have" or hold a concept? If language is the tool we use to communicate, when do we stop short of saying "She is mimicking" and accept that "She has learnt"? If knowledge is encoded, why can we not just transfer the code between ourselves and be done?

Much of the above is grist for the mill of educational philosophy, but we shall remember our backdrop and review features of the scene in terms of embodied cognition. A casual witness would tell us that there were two participants, the teacher and the student. A more post-modern, sociological analysis might observe residue of greater influence and suggest that we have evidence of a power play, not just between the teacher and the student, but in the very idea that it is desirable to draw an abstract mathematical concept (such as correlation) out of a real situation (the specific problem with which Alison was faced). Even presenting Alison with a text book problem drawn by an author from the real world of human activity—or an imaginary illustration of this world—can be seen as a betrayal of implicit motivations rooted in the structures of the prevailing economic powers. Returning to Valerie Walkerdine (1994), we are implored to recognise that

> Thinking is produced in practices, replete with meaning and complex emotions, that thinking about thinking is deeply connected to the way that power and regulation work in our present social order. We therefore need to construct new and different narratives which recognize specific practices, which can see the place of those stories in the construction of us all. (p. 74)

Mathematical activity, then, even the seemingly innocuous goal of teaching a student to abstract, appears bound with power structures and implicated in the maintenance of those structures.

Whether or not the circumstances that led to Alison and me discussing the ownership of words were artefacts of the social power base, and whether or not my interest in teaching Alison about abstract concepts is evidence of the same sociological circumstance, casts but a little light on the question of what it means to share abstract concepts, however[12]. To participate in the process of abstracting and learning could be seen as promoting certain structures, so that the meaning of learning becomes implicated with the maintenance of broad social conditions. This is not to say that one does not improve[13] one's personal conditions through education, but rather that in a society a certain form of education is part of a process of conservation. It seems necessary that there be at least an implicit agreement between teacher and student that certain structures be maintained, although we need not require that their separate motivations, whether overt or not, be identical.

This approach would seem to focus attention on the underlying meaning of the *sharing* of concepts, and on the circumstances that put us in a position in which we can share at all. I, however, shall focus a little more tightly on the very idea that concepts can be shared. As Núñez, Edwards and Matos (1999) observe,

> the nature of situated learning and cognition cannot be fully understood by attending only to contextual or social factors considered as inter-individual processes. Thinking and learning are also situated within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world. These characteristic ways of

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[12] This is not permanently to set aside the significant matter of emotional commitment, which will be considered presently.

[13] My use of this word rather betrays me: who is to say what constitutes an improvement? Walkerdine is clearly alert to precisely this.

> understanding, talking about, and acting in the world are shared by humans by virtue of being interacting members of the same species, coexisting within a given physical medium. (p. 46)

If I claim to have a mathematical idea "in my head", as it were, and later observe that Alison uses language that makes me believe that she too, despite her protestation, has the *same* idea "in her head", then what has transpired between us? What light, moreover, do theories of embodied cognition bring to bear on this question?

What, that is to say, are mathematical concepts? Since I am working within an embodied cognition framework, I am committed to a certain stance on this question. I theorise that pre-reflective sense experience, honed via evolutionary means, has endowed each of us with the capacity to interact with our environment so that neurological analogues of key aspects of environmental interactions are somehow physically instantiated. On this reading, concepts will be both derivative of physical reality and present in reality—at least, in the kind of reality that admits *properties* of physical objects.

The notion of a concept, though, ought not be dissociated from the bodily substrate in which it may be instantiated, any more than a word is to be isolated from the ink in which it may be registered in text. Neither need we accord any metaphysical characteristics to concepts; nor any ontological status beyond a recognition that they are a properties of some other real entities, such as humans. John Searle (2007), with characteristic bluntness, cuts through dualistic temptations:

> It is a kind of mystification to suppose that because we can write poems and develop scientific theories, somehow or other these inhabit a separate realm and are not part of the one real world we all live in. (p. 22)

He makes the point that the introduction of abstract entities such as greenness provides a way of speaking, but does not make any ontological demands upon us. So, to take his example, whether we say, "This is green," or say, "This object has the property of greenness," makes or identifies no difference in the world. In the same way, if we see three horses in a field then we can identify that each horse has the property of being a horse, and that the *collection* of horses has the property of being three, or of "threeness" (Searle, 2007, pp. 24-26).

The recognition and naming of that property as threeness constitute two separate acts, as can be seen by noting that: (i) there is considerable evidence that humans and some animals innately recognise small numerosities; and (ii) not all human cultural groups adopt number names, and varying numbers of number names are adopted across groups.

This second observation highlights the cultural valence of number and the way in which innate capacities are not automatically made manifest in sophisticated social behaviour. The absence of a word for "three" among the Amazonian Pirahã, for example, does not imply that the members of this group do not recognise or are not capable of recognising three, but rather that in their particular circumstance, there is insufficient need to encourage its usage as such. As Frank, Everett, Fedorenko and Gibson (2008) discuss,

> the use of a discrete, symbolic encoding to represent complex and noisy perceptual stimuli allows speakers to remember or align quantity information with much higher accuracy than they can by using their sensory short-term memory. Thus, numbers may be better thought of as an invention: A cognitive technology for representing, storing, and manipulating the exact cardinalities of sets. (p. 823)

Here, then, we see a move for number as praxis to supplant number as theory. There will be more to say on this later, but for now we can recognise that it is one thing to be capable of having a concept, another to have it, another yet to recognise it as such, and a further stretch still to consider it worth using and sharing.

Is Zero a Number?

Sophie asked me whether a number divided by zero was equal to “infinity” or not. She had read somewhere that it was. We looked at some examples, dividing unity by progressively smaller, but positive, numbers. We saw that the smaller the divisor, the larger the quotient. It seemed that there was some sense in the suggestion that division by zero would be a suitable description for infinity, and yet I knew of another way to approach division by zero. I suggested to Sophie that we would need to think carefully about what numerical “facts” we would accept, before teasing out their consequences. For example, if we were to accept that zero multiplied by any number must yield zero, what then, if anything, would this say about division by zero?

“But zero isn’t a number,” said Sophie. “Ah, but mathematicians have pretty much decided that they want to say that it is a number,” I returned, and we entered into a discussion about zero as a marker of absence in our place-value numerical system contrasted with zero as a number.

In situations such as these we encounter mathematics rasping against our interpreted world. Can we decide, as if by fiat, that zero should be a number? By what right would we do so? Is it merely a language game, so that our labelling is arbitrary, or does zero represent, or correspond, with some aspect of reality? How am I to respond to students such as Sophie? I can say, as I have frequently heard it said, that division by zero is “not allowed!” or I can appeal to a higher authority: the calculator.

“Try to divide by zero,” I said to Sophie, indicating that she use her calculator.

“It says, ‘math error’.”

“Why does it say that?”

“I don’t know,” she conceded.

"People programmed it to say that," I offered, hinting that it gets us no closer to unravelling the difficulty. I deferred the matter to another day, promising to teach the class something about division by zero. I can demonstrate algebraically to Sophie that division by zero produces a number that is indeterminate[14], which is to say, it could be any number, but this demonstration would require us to weigh in the balance the significance of certain other number notions: we would be compelled to accept that $\frac{x}{y} = z \Rightarrow x = yz$ and that $0x = 0$ for any rational number x. The consequence of these assertions is that division by zero yields no *single* value, so that we say division by zero yields an indeterminate expression. This seems unacceptable for most purposes, and inconsistent with a nascent sense of what a number is.

Such a development would appear to require us to transition through David Tall's (2008) "three worlds" of mathematics:

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[14] A demonstration could proceed as follows. We take it that $0 \times z = 0$ for *every* rational number z (theorem 1) and that $\frac{x}{y} = z$ means the same as $x = yz$ (theorem 2). Suppose that division by zero was allowed. That is, suppose $\frac{x}{0}$ did have a value. It would be some rational number, so call it z. That is, say $\frac{x}{0} = z$. By 2, then, we have $x = 0 \times z$. But then by 1 we see that x must be equal to zero. That is $x = 0 \times z = 0 \Rightarrow x = 0$. So we see that if $\frac{x}{0} = z$ then $z = \frac{0}{0}$ (by replacing x with the value zero). But now, by 2 we see that $z = \frac{0}{0}$ means the same as $0 = 0 \times z$. We know from 1, however, that this works for *every* rational number z. Therefore, z, which is to say $\frac{0}{0}$, could be *any* rational number. In other words, $\frac{0}{0}$ is *indeterminate*, and as such lacks 'value' or meaning.

> • the conceptual-embodied world, based on perception of and reflection on properties of objects, initially seen and sensed in the real world but then imagined in the mind;
>
> • the proceptual-symbolic world that grows out of the embodied world through action (such as counting) and is symbolised as thinkable concepts (such as number) that function both as processes to do and concepts to think about (procepts);
>
> • the axiomatic-formal world (based on formal definitions and proof), which reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set-theoretic definitions. (p. 7)

Any apparent transition through these worlds, however, need not entail a departure from the body. The embodied concern, to which we shall return, is to develop an understanding of how we appear to "think formally" within a grounded paradigm.

Sophie has an understanding of zero as "nothing"; she can think with this concept and she can employ it in processes of addition, subtraction and even multiplication; but under division she is led to rely on upon a reversal of sense—the formal properties of division force an interpretation of zero upon her that is not "sensible". Whether such a formal consequence tells us anything important about the "real" world depends to some extent on what numbers are in relation to that world. In response to Sophie's question, I can shift the point of inquiry, but I cannot dispense with it entirely.

We have here an example of the shift from what appeals to the intuition as reasonable (multiplication by zero yields zero) leading to intuitively surprising consequences (division by zero is indeterminate). This foregrounds the role of "normal intuitions" in determining mathematical content. When mathematics produces theorems that cannot be readily interpreted on physical grounds, how can we explain our knowledge of them? How can such knowledge be instantiated in our brains? We, as children of the age of information technology, are primed to suppose representations will be amodal and abstract—which is to say, dependent on no particular medium or mode for their expression, or, in the stronger sense, independent of all media and modes—but need they be so?

LOCATING THOUGHT: CARTESIANISM AND THE TURING TEST

Information representation is a prominent topic in cognitive research. As an indicator of this, the Cognitive Systems Research Journal ran a special edition in 2002 on situated and embodied cognition. Tom Ziemke (2002), in the introduction to the issue, noted that:

> While there is much agreement in the situated /embodied cognitive science community that the traditional view of cognition as computation is flawed, or at least incomplete, there is less agreement on what exactly the fundamentals of the new approach are. (p. 271)

The paradigmatic way of thinking about thinking as computation can be seen as an outgrowth of Cartesianism, in which cognitive man was distinguished by his rational capacities, and by which particular Descartes was able to affirm that self-awareness of thinking—of doubting, to be more precise—underpinned the duality of our ontological status.

In Part V of his *Discourse on the Method*, Descartes reports that through his studies in vivisection he was able to find bodily properties in man in accordance with those of animals, but nowhere could he find the seat of the very rationality that defined us as men:

> when I examined the kind of functions which might ... exist in this body, I found precisely all those which may exist in us independently of all power of thinking, and consequently without being in any measure owing to the soul; ... but ... I could not discover any of those that, as dependent on thought alone, belong to us as men. (Descartes, 1850, p. 88)

The rationality that is characteristic of men Descartes ascribed to the existence of a soul:

> I did afterwards discover these as soon as I supposed God to have created a Rational Soul, and to have annexed it to this body in a particular manner which I described. (Descartes, 1850, p. 88)

In this we read how Descartes, who had a good practical understanding of the physical operations of the body (Grayling, 2005), was moved to the view that it was a rational soul created by God that distinguished man from other animals, and even from machines. The annexation of the soul to the body was not without its own problems, however. Even in Descartes' time concerns existed over questions of how such a soul could interact with the body.

Once the source of rationality (the soul) is understood as being distinct from the body, the properties of rationality (ideas, deliberations and the like) are readily held to be likewise distinct. One benefit of this epistemology has been the focus it has brought to the operation of logic and the consequent development of computing, as the interpretation of cognition as logical manipulation of disembodied symbols has encouraged the development of computing technology. It may be said that in this very success, however, lies the greatest difficulty in re-imagining cognition: we are disposed to view the mind as if it were a computer, primarily because computers are now the model of cognition with which we are predominantly familiar[15].

It is quite a thing: that which was derived as an abstraction of our acts of cognition—computing—now turns to influence how we understand the processes of our own cognition. It is timely, indeed, that we recognise the computing model as an abstraction of the whole and return to the body. To look once more at the body with the hope of making new discoveries, one must first reorient one's interpretation of what thinking is. That is a significant point of this, my thesis: to understand cognition as embodied requires a reinterpretation of cognition as a property of that very body.

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[15] Descartes, lacking computers, drew on the inspiration of water powered statues in the royal gardens of St. Germain, Paris. Such machines encouraged him to the view that we are, in part, as machines with pipes for nerves, but that it is the rational soul seated in the brain which controls the body (Grayling, 2005).

Later in the *Discourse*, Descartes presages the Turing test. After speaking at length on the movement of the blood through the body and on the various mechanical properties of bodies, he turns to muse on automata. He speculates that even if it were possible to construct machines that were imitations of men in all appearances and actions, there would remain two tests to show they were not really men.

> Of these the first is that they could never use words or other signs arranged in such a manner as is competent to us in order to declare our thoughts to others: for we may easily conceive a machine to be so constructed that it emits vocables ... but not that it should arrange them variously so as appositely to reply to what is said in its presence, as men of the lowest grade of intellect can do. The second test is, that although such machines might execute many things with equal or perhaps greater perfection than any of us, they would, without doubt, fail in certain others from which it could be discovered that they did not act from knowledge, but solely from the disposition of their organs. (Descartes, 1850, pp. 97-98)

The first test reveals that Descartes could not imagine a machine able to communicate in a manner that suggested it were able to interpret its situation. The second test follows directly from his understanding of rational thinking as disembodied: the capacity to reason confers flexibility because it is "a universal instrument" (Descartes, 1850, p. 98) whereas the organs of the body are, by design, constrained to particular actions. The existence of a disembodied, interpretative faculty is central to Descartes' epistemology.

Moving forward three hundred years, the Turing test says, in a nutshell, that if it looks and smells like a rose, it's a rose. More properly, that if in conversation you are unable to determine that your interlocutor is not invested of its own intelligence, then you have no basis on which to claim that it is not intelligent (Penrose, 1989).

Alan Turing, writing in 1950, modified a question that was popular at the time, "Can machines think?" to ask whether a suitably constructed, finite computer could play an interrogation game in which it could not be distinguished from a man; which is to say, he returned to Descartes' musing. He wrote,

> I believe that in about fifty years' time it will be possible to programme computers … to make them play the imitation game so well that an average interrogator will not have more than 70 per cent. chance of making the right identification after five minutes of questioning. The original question, "Can machines think?" I believe to be too meaningless to deserve discussion. Nevertheless I believe that at the end of the century the use of words and general educated opinion will have altered so much that one will be able to speak of machines thinking without expecting to be contradicted. (Turing, 1950, p. 442)

Turing's test is founded on a very pragmatic basis: if the notion we are questioning—"thinking" in this case—is itself somewhat vague, then perhaps the best test we can apply is to ask a question such as, "Does it matter?" or, "What would be different if things were otherwise?" Such questions can haul us back when we risk drifting away on spurious or ill-grounded supposition. In other words, even asking, "Can machines think" is to pose a particular type of thinking that is largely accepting of Cartesian dualism. The implicit formulation of the question is, "Given that our thought is the action of our disembodied rational agency, a universal instrument, can a brute machine think?"

We also note that Turing reminds us it is "by their fruits that we shall know them." To extend his prediction, we can ponder whether in another fifty or one hundred years, by which time our descendants have become accustomed to walking and conversing and sharing lives with "artificial" companions, anyone would pause to ask any such companion, "Are you really thinking?" A more astute question might be, "Do we think the same way?" Even better, "Do we negotiate our world in the same way?" In other words, "How alike are our autopoietic domains?"

By way of an aside, it is noteworthy that Turing proceeded to say,

> The popular view that scientists proceed inexorably from well-established fact to well-established fact, never being influenced by any unproved conjecture, is quite mistaken. Provided it is made clear which are proved facts and which are conjectures, no harm can result. Conjectures are of great importance since they suggest useful lines of research. (Turing, 1950, p. 442)

This draws attention once again to the ad hominem character of scientific progress, and, in the current context of questioning whether machines can think like us, merits attention if we are drawn to speculate on the potential of machines to direct their own research.

We can also ask why it is that Turing's prediction appears to have fallen short of the mark. We have passed the end of his century and we are still asking whether machines can think, and, moreover, what it is that *we* do when we think. The artificial intelligence program has not progressed so far that we can take its achievements for granted. Does this suggest that there has been something in the way we have pursued thinking machines that has constrained progress? In short, have classical views of representation been so attractive that we have been reluctant to conceive of alternative models?

In the remainder of this chapter, a more substantial role for the body in the action of thinking is considered.

TACIT THOUGHT AND SELVES

It has been noted that one can have a concept such as threeness but not recognise or name it, as such. To help develop this further, I now turn to a consideration of tacit thought. David Bohm (2004) speaks of coherence and the tacit process of thought. "Coherence," he tells us, "includes the entire process of the mind—which includes the tacit processes of thought. Therefore, any change that really counts has to take place in the tacit, concrete processes of thought itself. It cannot take place only in abstract thought" (p. 78). These tacit processes of thought may be distinguished from abstract thought, but we ought to be careful before we infer that the distinction is anything other than a stylistic convenience. Thought may be imagined in two ways: as language imbued and consciously expressive; and as corporeally contained, unconsciously wrought. Thought, taken as a whole, that is to say, can be envisioned as a spectrum of bodily-dispersed activities. By way of analogy, we are familiar with the electromagnetic spectrum, of which the visible portion constitutes a very small, but for us, very significant fraction. So, too, we can imagine that the spectrum of thought includes within itself a portion that we experience as abstract, representational thought. Yet for all its significance, we might do well to allow that this remains but a portion of the whole; the greater part being, to adopt Bohm's term, tacit.

Koch (2004) hypothesises the presence of a nonconscious homunculus to help explain the view that not all thoughts are conscious. He introduces the term *supramental* to denote processing beyond conscious perception, in contrast to more primitive, or *submental* processing, likewise held to be beyond consciousness. The purported nonconscious homunculus, consisting of neuronal networks in the frontal lobe, "receives massive sensory input from the back of the cortex, … makes decisions, and feeds these to the relevant motor stages" (p. 298). The suggestion is that this nonconscious homunculus is responsible for high level cognition, such as planning and creativity, a consequence of which is that "you are not directly conscious of your thoughts" (p. 302):

> Supramental processing—thoughts and other complex manipulations of sensory or abstract data and patterns—occurs in the upper stages, the habitat of the nonconscious homunculus. Its content is not directly knowable by consciousness, which arises at the interface between the representations of the outer and inner worlds. (Koch, 2004, p. 302)

Thus put, Koch advances the consequence that we can *knowingly know* neither the outer world nor our innermost thoughts, but that we are "aware only of the sensory representations associated with these mental activities" (Koch, 2004, p. 302).

An understanding of *who* we are within ourselves, of what we allow within the designation *I*, is fundamental to how we read such statements. With Searle (1984), we might allow that when I choose to raise my hand, *I* really do raise my hand and that it is my free choice to do so; and this belief is mediated by a sense of agency that is grounded in neurological structures and can be disrupted (Gallagher, 2005). The question to confront here is, "Who are you?"

Varela, Thomson and Rosch (1991) frame the question that a cognitivist account must address thus:

> If consciousness—to say nothing of self-consciousness—is not essential for cognition, and if, in the case of cognitive systems that are conscious, such as ourselves, consciousness amounts to only one kind of mental process, then just what is the cognizing subject? Is it the collection of all mental processes, both conscious and unconscious? Or is it simply one kind of mental process, such as consciousness, among all the others? (p. 50)

Referring to the massive neurological re-entry processes dispersed in time across the brain as a loop, and to the desire to find oneself in the brain, as it were, Dennett (2003) provokes us with: "You are not out of the loop; you *are* the loop. You are that large" (p. 242). When Koch says that "you are not directly conscious of your thoughts" he is prompting us to interrogate what is intended by "you".

We are familiar with the variability of the first person pronoun "*we*". Its usage can include the second person interlocutor, as in "we are here"; it can exclude the second person interlocutor in preference to a third person: "Did you know we were here?" It can also indicate a grandiloquent inclusiveness, as we see in this sentence. The first person "I" is apparently more constrained, and yet if we care to undertake the exercise we can discern usages that indicate its reflexive character. "I am on top of my game" might sound odd when spoken by a decrepit man, but if he had just solved a difficult crossword then we would interpret the meaning in light of intellectual acumen. A tennis player speaking the same sentence, however, would more likely be referring to his athletic prowess. *I* is a plastic pronoun—it can refer to I the thinker of conscious thoughts, I the agent of action, or I the soul, or I the being that is all this and more. Reflections are like that: what we distinguish depends on the lights by which we see.

For Gerald Edelman and Giulio Tononi (2000a), subjectivity emerges as levels of what they call primary consciousness, or mental life I, become enmeshed with products of higher order consciousness, or mental life II:

> With the emergence of a higher-order consciousness through language, there is a consciously explicit coupling of feelings and values, yielding emotions with cognitive components that are experienced by a person—a self. When this coupling occurs, the already complex events of mental life I become intercalated with those of mental life II, which is even more complex. A true subjectivity emerges with narrative and metaphorical powers and concepts of self and of past and future, with an interlacing fabric of beliefs and desires that can be voiced or expressed. Fiction becomes possible. (p. 205)

Thus the lending of voice to our subjective awareness arises midst a swelter of tacit self-being of which we remain largely unaware. “Tacit” derives from the Latin *tacitus*, to be silent. To be silent here is to be without voice, to lack formal language. Within this silence, however, is no quiescence. This silence is the thought of the body; the heaving, thrusting, muscular, nervous and emotional responses to stimuli that engulf, permeate and define our very being. Neither is to be tacit to be without influence. Tacit thought imbues our prelinguistic intentionality. Tacit thought impels us, shapes us: we frame conscious, representative thought only within the affordances of the tacit dimension.

We suppose our conscious, language-driven thought—our intentional agency, that is to say—can influence our bodily expressiveness (Gallagher, 2005), and in this interplay we find our free being, bounded yet creative; one, yet dually conceived as body and spirit. This is to describe thought as coupled with our environment in an autopoietic turn, whereby within our bodies we are perpetually shaping and being shaped by our thoughts, as we within in our communities and wider environments are at once creators and created. The paradoxical ouroboros[16] falls, twists and becomes our symbol for infinity (Hughes & Brecht, 1975). We are ever made and making ourselves, until, “Consum’d with that which it was nurrisht by,” (Shakespeare, 1956) we expire.

Francisco Varela (1992) submitted that

> The organismic dialectic of self is a two-tiered affair: We have on the one hand the dialectics of identity of self; on the other hand the dialectics through which this identity, once established, brings forth a world from an environment. Identity and knowledge stand in relation to each other as two sides of a single process: that forms the core of the dialectics of all selves. (p. 14)

Thus are we ever in the process of becoming, not merely situated and surrounded, but implicated and involved; as we learn we become learners, as we become learners we create a world in which we make meaning and identity.

Valence and Semantics: the Significance Signified

The valence of a concept is a key by which the concept as such may be read against the material in which it is instantiated. Ink and paper, in and of themselves, lack the degree of valence that the concept explicated thereby holds for the reader. What is the source of this valence? It is the acquired life experience of the reader that renders meaningful the text and that distinguishes it from the paper. To the unschooled, the squiggles on the page carry no meaning, no more than the page itself. It is only by living a particular kind of life that the significance of the text emerges. This draws the distinction between syntax and semantics: semantics are an emergent property of text, where the degree and quality of emergence are determined by a life led; syntax are the rules or physical constraints that govern the manner in which the text is instantiated in some medium including, for example, the grammar of a language, but also the biological realities that govern the growth and possibilities of interactions between neurons and groups of neurons.

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[16] The ancient circular symbol of a serpent swallowing its tail.

We can distinguish syntax and semantics in students' algebraic writings. Consider the problem of expanding $(x+9)^2$. There are several attitudes one may adopt when confronted with this task. If the problem is seen as consonant with expansions of the type $(xy)^2 = x^2y^2$ then there is every danger that the incorrect procedure "append the two to each term" will be applied, yielding $x^2 + 9^2$. This may also arise as an application of a linearity heuristic, "append the outer to each of the inner", so that $2(b+c) = 2b+2c$ suggests $(x+9)^2 = x^2 + 9^2$. If the problem is recognised as $(x+9)(x+9)$ then it is more likely that an expansion rule will be applied, leading to the correct $(x+9)(x+9) = x^2 + 9x + 9x + 81$, which may be reduced to $x^2 + 18x + 81$. As a fourth case, the correct result can be obtained most directly by application of the "perfect square" rule: $(x+y)^2 = x^2 + 2xy + y^2$.

From one of my classes of Year Nine girls, 12 out of 23 students correctly employed the expansion rule to convert $(x+9)^2$ into $(x+9)(x+9)$ and thence to $x^2 + 18y + 81$. Of the eleven students who made errors, only one attempted to use the expansion heuristic. All ten others employed the linear "append the outer to each of the inner" heuristic, leading to $(x+9)^2 = x^2 + 81$. Of these ten girls, however, six successfully employed the expansion rule to $(x-6)(x+6)$, three expanded with errors and one did not attempt the question. It is apparent that the $(x+9)^2$ presentation interfered with the heuristic selection, or, to put it another way, for approximately half the students the linearity heuristic trumped the expansion rule when the syntax of the presentation obscured the semantic cue.

(d) $(x-6)(x+6)$ $= x^2 + 6x - 6x - {}^{+}36$

$= x^2 - 36$

(e) $(x+9)^2$ $= x^2 + 81$

In cases such as these we witness the wrestle that takes place to draw semantics out of syntax. For about half the girls, the superscript 2 in $(x+9)^2$ did not mean what it means to an algebraist. It lacked valence: what was signified by the superscript was not significant to the girls. On the other hand, the experience of “expanding brackets”, from cases such as $2(3+5)=2\times 3+2\times 5$ to $x(3x-5)=3x^2-5x$ has capitalised upon the students’ familiarity with the operation of multiplication, a process that is semantically rich, we might say. Semantics trumps syntax: when forced to interpret a mathematical structure the student can defer to what seems most familiar, what carries greatest valence.

The distinction between syntax and semantics is made stark in John Searle’s Chinese Room Experiment (Searle, 1984). In this thought experiment an English speaker is placed in a room and provided with a book of rules for associating Chinese characters. Strings of Chinese symbols are forwarded to the English speaker who applies the rules assiduously and produces new strings of Chinese characters, which are then passed out of the room. As it transpires, standing outside the room are Chinese speakers, and they are feeding questions into the room. The English speaker, using only the rules of syntax, is constructing what to the Chinese speakers appear to be answers to the questions. Does the English speaker understand Chinese?

Searle is in no doubt: the English speaker does not understand Chinese. The motivation behind this thought experiment is the question of whether a digital computer could think. Searle sees the importance of distinguishing between syntax and semantics. He argues that while the computer is a machine that can apply syntactic rules, there is no way of getting from syntax to semantics. No matter how good the technology is, he says,

> If it really is a computer, its operations have to be defined syntactically, whereas consciousness, thoughts, feelings, emotions, and all the rest of it involve more than a syntax. Those features, by definition, the computer is unable to *duplicate* however powerful may be its ability to *simulate*. The key distinction here is between duplication and simulation. And no simulation by itself ever constitutes duplication (Searle, 1984, p. 37, author's italics).

That which the computer cannot duplicate is that which one acquires in a life led: the valence-conferring experience through which semantics can be said to emerge. Syntax and semantics are not opposites, as they are sometimes presented: they are to one another as ink to story. Alison, we might recall, had taken the ink by which correlation could be described: had she the story? To share words requires little; to share a concept requires that lives converge. Imagine a young man reading Yeats's "Prayer for My Daughter", below. How would he understand the poet's concerns? Contrast that with the impact of the words upon the father of a young girl:

May she become a flourishing hidden tree
That all her thoughts may like the linnet be,
And have no business but dispensing round
Their magnanimities of sound,
Nor but in merriment begin a chase,
Nor but in merriment a quarrel.
O may she live like some green laurel
Rooted in one dear perpetual place.

William Butler Yeats (2003), *A Prayer for my Daughter*

To wrestle with a mathematical problem is to be engaged in thought; abstract and tacit. To solve a mathematical problem is to create coherence. Coherence and simplicity are indications that "sense" is found: a solution is at hand. This might be a consequence largely of conscious effort, but at some stage it would seem that tacit knowledge is required in order to bring to fruition that which amounts to recognition. We cannot by words alone—by abstract thought—render meaningful mere symbols. This was the point of Searle's Chinese Room Argument: "*Syntax is not sufficient for semantics*" (Searle, 1984, p. 39, author's italics). Symbols are, after all, only signifiers when they stand for something. As Searle (1998) insists:

> The reason we cannot analyze consciousness in terms of information processing and symbol manipulation is that consciousness is intrinsic to the biology of nervous systems, [whereas] information processing and symbol manipulation are observer relative. (p. 386)

The observer, that is to say, supplies the meaning.

It is through a shared immersion in place and time that we are we able to synthesise that which is seen or heard or felt or otherwise sensed into a meaningful constitution. Tacit thought, if it is to enable us to elicit meaning, will need to involve a process in which lived experience is at once employed and generated. Meaning is to be construed, not found; made, not bought; interpreted from the perspective of experience, not articulated from abstraction. Such a process could fire the recognition that underlies problem solving, the "seeing" that elicits the "Aha!"

Sophie once asked me about the expression $-\frac{5}{6}$. She demanded some clarity about the negative sign and the two numerals: "Which is negative: 5 or 6, or both?"

I asked her to consider $-\frac{6}{12}$. It can be read as $\frac{-6}{12}$ or as $\frac{6}{-12}$, as both are essentially the same as -0.5. It is a question of perspective, I suggested. If you look at $-\frac{6}{12}$ then you are perhaps seeing it as the negative of one half, whereas if you see it as $\frac{-6}{12}$ then you are seeing it as -6 divided by 12.

The idea of perspective seemed to resonate with Sophie. When I suggested that $-\frac{6}{12}$ is like $-\left(\frac{6}{12}\right)$, where we are looking at the negative of a calculation to be done, she exclaimed, "Aha, it makes sense to me now!"

ABSTRACTION AND GENERALISATION

We speak of coherence as a metaphor for processes that bring us to recognition. In an embodied framework, these processes need not be limited to conscious, representational thought, or what might be called abstract thought. "Abstract" is generally taken to imply that symbolic language is co-opted to facilitate cognition of a type that expands upon or exceeds the body simpliciter. Abstract representation enables communication in the temporal and spatial absence of another self. That is, abstraction transcends presence.

Abstraction, moreover, is sometimes regarded as being amodal. This means, in effect, that the information that is expressed abstractly is not constrained to a single means of expression, but can be recapitulated in other ways. It is this feature of an abstraction that encourages Platonic dualisms. I can speak a mathematical formula, for example, or I can write it; I can draw it and maybe I can even act it out. In each case, the information that is being expressed through the formula is faithfully communicated. At least, that is a disembodied interpretation. An embodied view would question whether it is really proper to speak of a formula as such, without linking it to a given expression. There is the formula as spoken and there is the formula as written, and while they are intended to represent the one abstract idea, they are, in and of themselves different expressions.

That is not so interesting. More worth consideration is the question of whether, in the embodied view, there is any "formula" as such that is variously represented. That is, whether there is any actual abstract idea encapsulated in a formula. This goes straight to the question of dualities. Working within the embodied epistemological frame, I take it that the "formula" is, at root, evidence of our recognition of something that is a feature of the world as we perceive it, something about the world, which is, in effect, a selective re-presentation of some isolated properties of the world. It is not something that could be said to exist in the absence of beings capable of intuiting it, any more than "green" as a concept can be said to exist in the absence of beings who so label that particular property of the veritable "grass of home". The grass itself, as bearer of that property that we recognise as "green", does not require our acknowledgement. In like manner, the world, as bearer of the property that we come to describe with a formula does not require our presence, whereas any expression of the formula, as such, does; and without expression, the formula, which is an expression, does not inhere.

The formula, moreover, recognised as an expressed abstraction of selected properties of the world, is a form of model. This is to say that abstraction is a form of *model making*: a compression of information relating to some phenomena.

The dialectic of model making consists in the tension between size and accuracy, or quantity and quality. The infinite sequence 111111..., for example, could, in theory, be modelled with an infinite string of 1s. This would be a particularly unexciting model, it would be infinitely large, but it would also be perfectly accurate. An alternative model could be expressed in natural language as "an infinite string of ones." This model has the benefit of a significant reduction in size—it is not infinite. What is more, the very simplicity of this formulation forces no loss of accuracy on the model provided the compression system—the deployment of words—is perfectly understood. On this proviso, the model permits an accurate reconstruction of that which is modelled, the string 111111....

We can therefore identify two aspects to abstraction: generalisation and decontextualisation (Ferrari, 2003). When, in an elementary mathematics class, we seek to demonstrate that a class of examples such as

$$\begin{gathered}2+2+2=3\times 2\\5+5+5=3\times 5\\11+11+11=3\times 11\end{gathered}$$

leads to the generalisation $x+x+x=3x$, we are attempting to elicit a shift of attention from the calculations as such, with their "answers", 6, 15 and 33, to the induction that *any* number, *x*, (or at least, any *natural* number, *x*) added thrice over can be expressed as 3*x*. It is arguable here that the attempt to mathematize what seems implicit in language, that "a number taken three times gives three lots of that number," is unwieldy or strangely tautological in comparison and adds nothing to the development of number per se (Heike Wiese (2003, 2007) discusses numbers as words, as we will see below), but the generalisation here draws attention to the inductive step that this is true of *any* number. It could also be said to be signalling the meaning of multiplication by relation to addition. Once made overt, this extends in a meaningful way so that if *z* is *a* number, then we can write $z+z+z=3z$, regardless of the use to which *z* is being put, if to any use at all. That is, the generalisation facilitates decontextualisation in the sense that where the numbers once were understood to stand for something, they now become objects of study in their own right[17].

The question of what is *general* ought not be taken too lightly, however, as the following example illustrates. I once introduced my Year Twelve Mathematics Specialised class to the derivative of a logarithm of any base *a*, $a>0,\ a\neq 1$: $\frac{d}{dx}\log_a x=\frac{1}{x\ln a}$. Therese suggested that the proof of this could be achieved by employing the change of base rule [C. O. B] in conjunction with a result of the previous year's study, that $\frac{d}{dx}\ln x=\frac{1}{x}$, in the following way:

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[17] The examples given, $2+2+2=3\times 2$ and so on, already indicate a decontextualisation from enumerated objects to numbers themselves. The treatment $x+x+x=3x$ could, therefore, be said to represent a second order decontextualisation.

$$\begin{aligned} \log_a x &= \frac{\ln x}{\ln a} \quad [\text{C. O. B}] \\ \Rightarrow \frac{d}{dx}\log_a x &= \frac{d}{dx}\frac{\ln x}{\ln a} \\ &= \frac{1}{\ln a}\times\frac{1}{x} \\ &= \frac{1}{x\ln a} \end{aligned}$$

Seen in this way, the desired result $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$ could be read as an *application* of the apparently more general result $\frac{d}{dx}\ln x = \frac{1}{x}$. But this does not seem quite right. After all, the $\ln x$ case is limited to base *e* whereas the other case, $\log_a x$, permits any positive base other than unity.

When I posed the obvious question to Therese, "How do you know that $\frac{d}{dx}\ln x = \frac{1}{x}$?" she responded assuredly, "You told me so!" as, indeed, I had done in her previous year of study. Here, then, we have a case where, from Therese's point of view, the first result, $\frac{d}{dx}\ln x = \frac{1}{x}$, though given at the time with a simple hand-waving type of argument and based on a too-hurried sense of "it looks about right, and that's enough for our current purposes," was given by a teacher with at least enough authority to be believed, and had passed into what we might please to call knowledge[18]. She had no concern about *using* it as such to press onward to further mathematical discoveries. The brilliantly buccaneering eighteenth century mathematician Leonhard Euler would have been proud of her (Dunham, 1999)!

We agreed to return to first principles and so recalled our earlier conception of the derivative of a function $f(x)$ as the limit $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, upon which we proceeded to develop the standard result. Once $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$ had been established, the special case that applies when $a = e$ yielded the aforementioned $\frac{d}{dx}\ln x = \frac{1}{x}$ as a specific instance. Seen in this light, $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$ appears as the more general expression. Here the generality rests on the indeterminacy of the base, a. In this sense, the expression $\log_a x$ determines a *class* of expressions, including paradigmatic cases such as $\log_{10} x$ and $\ln x$. That is, what is taken to be the general concept seems to be a name for a class, and the members of that class are specifics.

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[18] There were, of course, other authoritative sources for her to rely upon; not least, her textbook.

We can take a second example to illustrate this notion. Looking out the window at the street I see cars of various shapes and sizes, some moving, some stationary. I see small cars, large cars, and I see a truck and then a bus. What is it that I call "a car"? What is the *general* notion? Is there a set of features all of which must be present, or is it sufficient that a number among several features be present? Must there be wheels? How many? Must there be accommodation for passengers? Is an engine a necessity? Is there a size constraint? I am sure I do not know, but I suspect that this is an example where there is some inductive, more or less commonly held sense of "what will suffice" and "what is necessary" for the term to apply—a case of Wittgensteinian family resemblances (Wittgenstein, 2009).

We need not fall into the trap of assuming a generalisation necessarily exists, however. We are drawn to categorise by observing similarities, but the similarities we observe say as much about us as they do about the broad sweep of phenomena under our consideration, a basis on which George Lakoff (1987) develops an experiential theory of categorization. To the extent that we declare a generalisation to consist in such and such features, it may be an arbitrary property-based description that could conceivably have been otherwise, not an ontological feature of the world. Me referring to "cars" makes no difference to the world, as such, as we have already noted John Searle might say. And we seem disinclined to make such declarations when we cannot discern sufficient degrees of "likeness", even supposing some other perspective might possibly afford a reasonable basis for determining a category.

We regularly witness such categorisation tensions in the mathematics classroom. One day, in an impromptu moment, I realised that it was necessary to introduce several students to the basic algebraic idea of "like terms" and had to consider what approach to take. Recalling my university days, where the plot seemed to run definition – example – lemma – proof – theorem – proof, I was reluctant merely to *say* what like terms were. I remembered the fragility of definitions presented in dense text: I would be occupied with trying to apprehend the essential features while the lecture proceeded, so that I was regularly placed in a condition of manipulating literal symbols without having the sense that I knew what the symbols meant. Lectures were held in the Chinese room, of course. This can be reasonable when the definitions relate to highly theoretical constructs—one of the challenges of learning mathematics is to suspend the desire to recognise correlates in the physical world—but it is surely too much to ask of a Year Eight student new to the game.

So rather than present merely a definition we can call on an inductive strategy: we can present examples and allow the students to elicit the essential features of the concept for themselves. That is, we can allow the children to abstract the generalisation. Needless to say, the choice of illustrative cases becomes central to the success of this approach. I do not *tell* what like terms are, I show examples of like and unlike terms. The articulation of a definition proceeds as a process of inductive refinement: "Ah, but xy and $3yx$ are both multiples of xy when we allow commutativity." Even in simple cases, however, definitions can be tested: "Are x^2y and xy like terms?" we might ask, and we would expect the answer to be "No," but then this would be anticipating that we were to try to add the terms as presented. We *can* write $x^2y + xy = xy(x+1)$, of course, in which case we have *implicitly* identified like terms in xy, revealing that the early algebraic activities can involve a confusion of procedures such as addition and factorisation. The abstraction of general principles—field elements and operations, for example—

seem to require a drawn out process of gaining experience. We can regard this as a recapitulation of mathematical phylogeny. The evolution of the notion of a *function*, for example, demonstrates a search for essential properties within the history of mathematics, comparable to searches in the mathematical life of a student.

The example of like terms demonstrates the need for the teacher to be aware of the perspective he is encouraging his students to take. *What* we notice when we see objects is likely to be determined in part by our intentions and also by a balancing of contrasts. If we notice that here is a collection of cars, one blue, one white, one red; and there is a collection of pens, two blue, one black; and if we also see a collection of chairs, one hard, two soft, what commonality are we likely to draw out of the observations? The general concept of "threeness" would be one possibility, particularly as the number is small enough as to enable subitization: we would "see" the threeness without knowing it[19]. The markedly contrasting shapes, textures and sizes of the objects might render further class formation unlikely. The word "three", the number, indeed, becomes the name for a class of collections, each specific member (each collection, that is) necessarily having the property of containing three members.

We see that the class "three" is determined by a minimal set of conditions, and details such as colours or surface properties of objects counted are extraneous to the class definition. The generalisation *three* consists in the minimally necessary conditions placed on the term[20]. On this reading a generalisation, far from being a widening or broadening, becomes a refinement, an elucidation of what it is that is required to define that thing: an *abstraction*, indeed, of the necessary elements, as in a *drawing out*, consonant with the Latin roots *ab* for away and *trahere,* to draw.

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[19] On the other hand, this might mean that it would be the very thing we would *not* notice. We shall consider what brings objects to conscious awareness presently.

In terms of the foregoing, the essence of abstraction is the compression of information so that a minimal set of conditions can be said to pertain. Once such abstraction has occurred, we have the possibility of recognising a generalisation. Here we are casting generalisation as the noun and abstraction as the verb, but there is something more to a generalisation, for its merit lies with its potential application to unanticipated cases. This is the sense in which decontextualisation is employed as a term.

Gray and Tall (2007) pursue the same themes, noting that abstraction occurs in various ways and concentrating on three: "A focus on *properties* of perceived objects" (leading to the kind of compression described above); "a focus on *actions* on objects, which leads through compression to the computable symbols in arithmetic, the manipulable symbols in algebra and symbolic calculus"; and a focus on "the properties of mental objects, compressed through several stages leading to a third form of abstraction by formulating a set-theoretic *concept definition* to construct a thinkable concept from the definition using mathematical proof" (p. 27, author's italics). Gray and Tall provide a summary of the earlier work in this field, as do Pegg and Tall (2005), to which Bettina Dahl (2006) provides an overarching commentary. From the viewpoint of embodied cognition, we retain the concern of what it really means to formulate a concept in terms of the body.

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[20] This is not to suggest that *all* categorisations necessarily have minimal conditions imposed on membership. Many "natural" categories do not and hence the appeal to "family resemblances" (see, for example, D. G. K. Nelson, 1984). Mathematical categories, on the other hand, tend to be constructed in just such a manner. Neither are we currently attempting to address the empirical question of *how* children abstract generalisations from data.

This process of creating thinkable concepts, or mental objects, has been variously described in the philosophy of mathematics education literature. Ed Dubinsky (1991), for example, refers to encapsulation as one form of construction in reflective abstraction (the main other being interiorization), as part of his genetic theory of mathematical knowledge acquisition, the structures of which he and his colleagues have categorised as *action, process, object,* and *schema*, or APOS (Dubinsky & McDonald, 2001; Dubinsky, Weller, McDonald, & Brown, 2005a, 2005b), and which we may summarize diagrammatically as

$$\text{action} \xrightarrow{\text{interiorization}} \text{process} \xrightarrow{\text{encapsulation}} \text{object} \xrightarrow{\text{organization}} \text{schema}.$$

Ana Sfard (1991) discusses the “peculiarity of mathematical thinking investigated through reflections on the epistemological and ontological status of mathematical constructs” (p. 2) and proposes three stages in the process of concept formation:

> First there must be a process performed on the already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired. We shall call these three stages in concept development *interiorization*, *condensation* and *reification*, respectively. (p. 18, author’s italics)

Interiorization and condensation involve gradual processes of familiarisation, but reification represents a new way of seeing. Aspects of a concept are reified into a new construct:

> Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. (pp. 19–20)

The value of the objectification process can be summarised by David Tall (2006), for whom, “The fundamental idea in powerful cognitive growth is the compression of ideas into thinkable concepts that can be connected together in increasingly flexible ways” (p. 199).

At this point we might ask whether anything can be said about the way in which these thinkable concepts, or mental objects, are instantiated within the cognising agent—the thinker, that is to say. On the face of it, the Piagetian inspired models of Dubinsky and Sfard could be as well accommodated within a classic representational paradigm as within an embodied paradigm. The difference would lie in *how* that accommodation takes place.

I soon shall discuss ways in which information is maintained and acted upon by the thinker, but I wish first to reflect upon an illustration of the process of abstracting information in the course of solving a problem in the mathematics classroom.

GUESS THE AGES: AN EXERCISE IN ABSTRACTING THE ESSENTIAL

Learning to abstract salient features in a mathematical problem solving context requires some particular skill. There is a general rule shared among those who construct mathematical problems which sets these problems apart from the practical or applied variety. It is that all information presented shall be useful and no information that is not useful shall be presented in the problem. This provides budding puzzlers with an invaluable heuristic: use all the information presented in the problem.

The following is a variation upon a well-known challenge[21] that requires puzzlers to exploit this heuristic. I dressed the problem in a form that I anticipated might appeal to my eight Year Seven Mathematics Enrichment girls by adding a picture of the puppet, Elmo (not included below). Since Elmo is a favourite of quite young children, I had hoped that my “big Year Seven” students would sense the lightheartedness intended. This simple act encouraged an interesting deviation from the general heuristic referred to above.

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[21] While the text of this particular version is widely available on the internet, the earliest version of this problem that I have is Problem 65.1 in George Polya’s and Jeremy Kilpatrick’s (1974) *Stanford* collection.

GUESS THE AGES

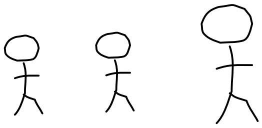
> The host at a party turned to a guest and said, “I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?”
>
> The guest rushed to the door, looked at the house number and informed the host that he needed more information. The host then added, “The oldest likes strawberry pudding.” The guest then announced the ages of the three girls.
>
> What are the ages of the three daughters?

We were seated around a rectangular table in the centre of a small classroom. I asked Rosie, who is often the quietest in our group, to read the question. She did so, with a single, potentially significant substitution: she said “*a* house” for “*my* house”.

When Rosie had finished there was a general outcry, “It doesn’t make sense!” I asked the girls to spend a few minutes trying to wrestle with the information individually, after which we would reconsider our positions. We have employed this strategy before, so the girls knew what to do. They took some paper and pens and silently occupied themselves with their own attempts to make something of the problem, to overcome their initial puzzlement.

After a minute or two, Diane came to me and whispered that she had “done it.” She had found that 2, 4 and 9 is a combination of ages that worked to give a product of 72. Kelsey, sitting close by, overheard and said she had found the same. I asked Kelsey where she saw herself in the problem: at the beginning, in the middle or at the end? Both girls looked thoughtful, and then returned to their seats to resume the search for other possible triples.

After four minutes I signalled the end of the personal “getting to know the problem” phase of our exploration by asking the girls what they were doing. All were now searching for triples that multiplied to 72, though some had begun searching for factor pairs. Ella had drawn a picture: three stick figures, but one larger than the other two, because as she said, “One is the oldest.”



Ella and Casey wondered if the picture of Elmo was relevant. This, they told me, was because Elmo is a character designed to appeal to young people, and they wondered if that meant something—perhaps at least one of the daughters were quite young. I was intrigued by their thought, but decided not to tell them that I had simply added the picture to lighten the text. I realised that I had inadvertently broken faith with puzzlers!

Kelsey showed me her collection of triples. “Have we found them all?” I asked.

“Some,” said Kelsey.

“Will we need them all?” I pressed.

“Maybe, no, some ....” Kelsey spoke without assurance, watching me for clues to the answer to my question. I suggested that we find them *all* and then check to see if the remaining information in the question would prove helpful.

Jayne asked, “Can one of the [ages of the] daughters be zero? Zero times something times something equals 72?” Zara answered her with, “Zero times one is zero, times zero is zero, times 72 is zero.”

“Oh,” said Jayne.

I suggested that we pool our searches. Kelsey went to the board and invited the others to follow her with their numbers. Zara suggested, “Just write yours, Kels, and we’ll see what you don’t have.”

Casey then drew attention to missing information. “We don’t know what the house number is.” Gesine answered, “It can’t be over 72, because when you add them [the factors] up they need to be less than 72.”

“It can’t be zero,” said Rosie. To this stage no one other than Gesine had made much of the *sum* of the factors being the house number.

Kelsey drew the following table on the whiteboard:

2	4	9
3	2	12
12	3	2
3	*3*	*12*
1	9	8
6	3	4

The other girls then contributed these triples:

1	4	18
1	6	12
1	2	36
1	1	72
2	6	6
2	2	18
3	3	8

Neither the error, presented here in italics, nor the repetition of 3, 2, 12 were commented upon.

Before the table was completed, Ella called out, “Wait! 2, 6, 6 can’t be one because there’s no oldest!”

“Yeah,” said Kelsey, acknowledging the point, but then she halted Ella’s drive with, “Just wait, we’ll write them all down.”

“Do we have all of them?” asked Ella, when it seemed Kelsey had finished writing.

“I don’t know,” said Zara. “Do we?”

A few moments passed while they checked for more. The triple 1, 3, 24 was never found.

Zara, now picking up on Ella's observation, suggested a strategy: "Cross out the ones that have the same ages." She did not restrict her suggestion to cases where the repeated ages were those of the older siblings, but this was the understood intention.

"The oldest one likes strawberry pudding," said Diane. "Maybe she lost her teeth." In saying this, Diane was attempting to explain that 1, 1, 72 might be a possible solution. She was still attending to distracters in the problem.

Gesine, meanwhile, had become a little frustrated that no one had attached any importance to the point she had made earlier. "The 72 can't be there, because when you add them up it has to be lower [than 72]." She was incorrect in this assertion, but she was trying to incorporate an arguably more sophisticated analysis of the properties of numbers than were her peers.

"How are we supposed to narrow them down?" asked Casey, uncertain of how to proceed. She addressed the question to me, not to the group at large.

"What information do you have?" I asked in reply.

"We don't know the house number," she said, a contrary answer.

"*We* don't know the house number, but—" I halted; they sat motionless, breaths suspended. It was time to provide a little guidance, so I dropped the big hint: "*The guest* does, and *knowing isn't enough to let him decide*."

Casey fairly leapt in excitement. "But wait! Isn't that—, you have to find the one [sum] that has *two* of them!" Her words were somewhat jumbled, hurried, garbled. "I can't say it," she said, when she attempted to explain to the group. "I think it's 1, 8, 9," she said.

"Why?" asked Zara.

Casey explained. “2, 6, 6 and 1, 8, 9 have the same sum.”

“Oh!” exclaimed Zara, as she recognised the significance of this, even though Casey’s selection of triples was erroneous. Kelsey, however, was puzzled. “What?” she blurted.

“Because there’s two, so—” began Zara.

“And when he says the *oldest* …” continued Jayne.

While this last conversation was proceeding, Diane and Kelsey were writing the sums on the board beside the factor triples. The girls could see that the only sum to be repeated was 14, for each of 2, 6, 6 and 3, 3, 8; the error and the duplication having by now been identified.

			Sum
2	4	9	**15**
3	2	12	**17**
~~12~~	~~3~~	~~2~~	~~**17**~~
~~*3*~~	~~*3*~~	~~*12*~~	~~***18***~~
1	9	8	**18**
6	3	4	**13**
1	4	18	**23**
1	6	12	**19**
1	2	36	**39**
1	1	72	**74**
2	6	6	**14**
2	2	18	**22**
3	3	8	**14**

Casey reflected for a moment. “Oh, I made a mistake, but anyway ….” She knew that even though she had made an error, her method was, nevertheless, sound. She adjusted to the data in front of her and settled on 3, 3, 8 as satisfying the condition that there be an oldest child.

“So the strawberry pudding doesn’t matter,” said Ella. “It was just ‘the oldest’!”

Being there while the girls teased this problem out was quite a privilege. The searching for data, the suspension of uncertainty and the struggle to relate clues to observations all signalled the difficulty of abstracting information. The distractions in the problem, which is to say the picture of Elmo and the reference to strawberry pudding confounded efforts to find the salient features, while the crucial clues were hidden in unremarkable text: "The guest … looked at the house number," and "the *oldest* likes strawberry pudding."

Experiments conducted with young children over the past ten years or so suggest that babies are equipped to behave as proto-scientists, gathering data experimentally and weighing probabilities to "test hypotheses" and form causal inferences (see, for example, Gopnik et al., 2004). A baby learns differently to an adult, says Alison Gopnik (2010), having a more flexible brain with less prefrontal control than an adult.

> The prefrontal area inhibits irrelevant thoughts or actions. But being uninhibited may help babies and young children to explore freely. There is a trade-off between the ability to explore creatively and learn flexibly, like a child, and the ability to plan and act effectively, like an adult. The very qualities needed to act efficiently—such as swift automatic processing and a highly pruned brain network—may be intrinsically antithetical to the qualities that are useful for learning, such as flexibility. (Gopnik, 2010, p. 81)

By noticing the way the girls in this class managed themselves, we can observe signs of the maturation toward the thinking of adulthood. In the initial moments, despite being uncertain about what the problem meant or how to solve it, all the girls proceeded to act in the one way that made sense to them at the time: they all adopted a search for triples with a product of 72. Granted that this scenario was already an intrinsically constrained form and that the time for getting to know the problem was a condition imposed by their teacher, they were, nevertheless, free to behave as they chose in those first few minutes. They could have drawn diagrams, or acted out the scene, or behaved in any number of ways, but they did not. They did what we would expect adults to do, not what we would anticipate babies might do (anything but what we wanted!) Here, then, is a sign that they are maturing toward focused, adult problem solving behaviours.

What is more, the girls selected from among the text the one sentence that made clear unambiguous sense to them: the product of the ages is 72. They did not linger, once a time constraint was imposed, on unclear factors such as the relevance of the house number or strawberry pudding. We can identify a simple heuristic in action here: do something with the numbers. This can lead to humorous errors, such as when young children will add numbers presented in a problem even if there is no relevance to the summation.

The executive control exhibited in the searching process was not thoroughly refined, however. The searching for triples was more or less unstructured. Diane and Kelsey felt that they had achieved something significant when they had found one triple, but then needed prompting toward the thought that there might be relevance in knowing if there were more.

By the time I had decided to open the problem to the group, most of the triples had been found, but not quite all. One triple remained undetected, even to the end of the lesson. This begs consideration of a related difficulty, as Ella and Zara noted: how do we know when a search is complete?

Attention turned toward the rest of the problem. Ella had made a start by recognising that one daughter was *uniquely* oldest, which she had encoded in the stick diagram shown above. It was not clear whether the full significance of this was appreciated by all. It was Ella, after all, who drew the attention of the group to the need to exclude 2, 6, 6 from the list of triples. Of even less certain value were the picture of Elmo and the reference to strawberry pudding. A potential reason for Elmo's inclusion was offered—his appeal to young children, suggesting that one of the daughters might be quite young—but what possible value could there be in knowing about the strawberry pudding? The distracter in a puzzle is like a magician's misdirection: it has to be attention grabbing otherwise it will not work. The presence of strawberry pudding was so ridiculous in the context of this problem that nobody thought to ask the question, "What do we actually know about it?" It is somewhat ironic that the implication of the answer, that there was an oldest daughter, was already known to the group.

It was Gesine who was perhaps most closely tuned to the *mathematical* character of the problem. Although she was erroneous in thinking that the three ages had to sum to less than 72, perhaps because she had overlooked the 1, 1, 72 possibility, she was, nevertheless, persistent in sticking to an apparent numerical constraint. This illustrates the ability to focus that may be contrasted against the ranging flexibility of the more naive puzzlers.

In a similar way, we can recognise Casey's relative indifference to her trivial error in the light of the more significant realisation that she knew how to solve the problem. Here we see the hallmark of a true puzzler: what mattered to Casey was not the answer to the problem so much as the means of obtaining it. Once she had that, the actual answer paled in contrast.

CHAPTER THREE EMBODYING KNOWING

I now turn to the question of *what thought is* and how we come to have thoughts. The representation of ideas is important in this development and so I give space to reflections on how we use words and numbers.

In stages, I find that mathematical concepts are usefully imagined less as things than as orientations; that mathematical concepts represent relatively stable associations of modes of thinking couched within a dynamically organising body; and that our forms of representation, whether words or numbers, come to function as aids to thought and communication. What is more, our very intentionality acts to influence how we perceive and organise what are to become our thoughts.

WHAT DO WORDS REPRESENT?

I am moving beyond the conception of thought as the disembodied action of a universal instrument; but what of *words*? How do they fit in the emerging epistemology? The question of how we think with words is hardly modern, of course, and even in historical cases we can find indications that an embodied account was desired.

Consider the following extracts from Edmund Burke's *On the Sublime and Beautiful*, first published in 1756, in which he introduces us to two blind men, Mr. Blacklock, a poet, and Mr. Saunderson, a professor of mathematics in Cambridge. Of Mr. Blacklock he wrote:

> Few men blessed with the most perfect sight can describe visual objects with more spirit and justness than this blind man; which cannot possibly be attributed to his having a clearer conception of the things he describes than is common to other persons. (Burke, 1909, p. 141)

He made the following observation of Mr. Saunderson:

> What was the most extraordinary and the most to my purpose, he gave excellent lectures upon light and colours; and this man taught others the theory of these ideas which they had, and which he himself undoubtedly had not. (Burke, 1909, p. 141)

Burke deduces that since blindness, for these gentlemen at least, is no impediment to the communication of ideas of visual objects, this communication cannot be based solely upon images: "A man may hear words without having any idea of the things which they represent, and yet afterwards be capable of returning them to others, combined in a new way, and with great propriety, energy and instruction" (Burke, 1909, pp. 140-141). He reinforces this point by arguing that we do not necessarily form images even in simple discourse.

> If I say, "I shall go to Italy next summer," I am well understood. Yet I believe nobody has by this painted in his imagination the exact figure of the speaker passing by land or by water, or both; sometimes on horseback, sometimes in a carriage; with all the particulars of the journey. Still less has he any idea of Italy, the country to which I propose to go; or of the greenness of the fields, the ripening of the fruits, and the warmth of the air, with the change to this from a different season, which are the ideas for which the word *summer* is substituted: but least of all has he any image from the word *next;* for this word stands for the idea of many summers, with the exclusion of all but one: and surely the man who says *next summer,* has no images of such a succession and such an exclusion. (Burke, 1909, p. 142, author's italics)

Burke's observations are quite discerning and force us to reflect carefully before we succumb to the notion that the words we use conjure images of things. How we represent "red", much less "I shall go to Italy next summer," is hardly well understood, even today, but there are theories of representation drawn about empirical evidence that bear consideration, as we shall soon discuss.

Burke's examples are different to, but reminiscent of the March, 1693 letter[22] of William Molyneux to John Locke, in which the following problem (to which Molyneux's name was to be ever-attached) was posed:

> Suppose a man born blind, and now adult, and taught by his touch to distinguish between a cube and a sphere (suppose) of ivory, nighly of the same bigness, so as to tell when he felt one and t'other, which is the cube, which the sphere. Suppose then, the cube and sphere placed on a table, and the blind man to be made to see; query whether by his sight, before he touch'd them, he could now distinguish and tell which is the globe, which the cube. (Locke, Molyneux, & van Limborch, 1708, p. 37).

Molyneux thought that the hypothetical man would not be able to distinguish the cube from the sphere by sight alone, "For tho' he has obtain'd the experience of how a globe, how a cube affects his touch; yet he has not yet attain'd the experience, that what affects my touch so or so, must affect my sight so or so" (Locke et al., 1708, pp. 37-38). A recent report (Held et al., 2011) provides empirical evidence that, indeed, the newly-sighted require a period of time in which to learn to match visual stimulation with haptic experience—but only a few weeks. Thus, as Shaun Gallagher (2005) has noted, Molyneux and Locke and the empiricists in general who answered the question in the negative were "correct in their negative response to the Molyneux question, understood as the empirical question, but wrong with respect to their reasons" (p. 169), since they had failed to appreciate the intermodal character of perception; neither had they been aware of empirical facts concerning neuronal organisation in neonates, most notably the likelihood of neuronal deterioration in the absence of visual stimulus. Gallagher, therefore, distinguishes what he judges to be an *in principal* Molyneux question from an *empirical* one; and determines different outcomes for each.

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[22] The problem was first posed in an earlier letter (1688), but little came of it.

Turning to the question of how our mathematical perceptions and reality relate, we can find a variety of historical glosses, two of which here will suffice as illustrations. Henri Poincaré, in his essay *Science and Method*, wrote with some passion of the significance of mathematics, but proffered the understanding that a reductionist analysis could reveal only so much. Some other degree of accommodation, which he called intuition, was required to bind mathematical facts to reality:

> The principal aim of mathematical teaching is to develop certain faculties of the mind, and among them intuition is not the least precious. It is through it that the mathematical world remains in contact with the real world, and if pure mathematics could do without it, it would always be necessary to have recourse to it to fill up the chasm which separates the symbol from reality. (Poincaré, 1913, p. 437)

Another mathematician of renown, G. H. Hardy, in his 1940 reflection piece *A Mathematician's Apology*, wrote, "I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations" (G. H. Hardy, 1992, pp. 123-124). To this, the anthropologist Leslie White (1947) responded:

> There is no more reason to believe that mathematical realities have an existence independent of the human mind than to believe that mythological realities can have their being apart from man. The square root of minus one is real. So were Wotan and Osiris. So are the gods and spirits that primitive peoples believe in today. The question at issue, however, is not, Are these things real? but Where is the locus of their reality? It is a mistake to identify reality with the external world only. Nothing is more real than an hallucination. (p. 290–291)

White's prescient assertion that it is a mistake to associate "reality" with exogenous phenomena, particularly so when we are exploring thinking and meaning-making, reminds us of the dialectic of fixity and change: it is too easy to see that which impresses itself upon us from outside, as Augustine or Plato might have said, as a reflected shade of some fixed, ideal world. The proponent of embodiment attends to a process of becoming, rather, as we shall now see as we now turn our attention to the question of how words, symbols and bodies combine to elicit and instantiate concepts as thinkable objects.

REPRESENTATION AND EMBODIMENT

Classic views of representation of thought include symbols that function within the constraints of cognitive systems to mediate thought. The question of what one understands by the term "symbol"—how it is instantiated, how it functions—in large measure determines whether one is forming an embodied account or a dualistic account of cognition. Whether one is able to instantiate symbolic thought across different cognitive systems—in a body and also in a computer, for example—and still consider that the symbols maintain the same representational function also bears on the question.

There is no in-principle requirement that symbols either represent or are themselves anything holding the ontological status of a Form. A symbol in the brain might consist of an arrangement of neurons, a symbol on paper might be built of ink, but as such the symbol is not a significant thing: it is neurons, it is ink. Its importance inheres in what it does: it stimulates and facilitates cognition. Let us consider what this means.

Non-classic[23] accounts of cognition, or what I have to-date been grouping under the general title of "embodied" accounts, have developed varying shades, according to the emphases given to different aspects of representation. Perceptual symbol systems highlight modal aspects of thought; situated action accounts emphasise the environmental context of cognition; embodied cognition accounts (proper) emphasise the role of sensori-motor and perceptual systems; and dynamic systems employ continuously changing state descriptions rather than discrete, enduring representations (Markman & Dietrich, 2000). Lawrence Barsalou (2008) suggests the phrase *grounded cognition* be used to cover these descriptions in preference to "embodied cognition", for:

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[23] Gallese and Lakoff (2005) refer to classic accounts as "first-generation cognitive science" (p. 455), on which basis, we could refer to embodied accounts as "second-generation" accounts.

> "embodied cognition" produces the mistaken assumption that all researchers in this community believe that bodily states are necessary for cognition and that these researchers focus exclusively on bodily states in their investigations. Clearly, however, cognition often proceeds independently of the body, and many researchers address other forms of grounding. "Grounded cognition" reflects the assumption that cognition is typically grounded in multiple ways, including simulations, situated action, and, on occasion, bodily states. (p. 619)

On this basis, what I have been calling "embodied cognition" would broadly be designated by "grounded cognition". It is not my purpose to choose between these varying accounts but rather to maintain that arising from grounded accounts—embodied accounts, as I have been saying—an epistemological framework can be discerned that enriches teaching. I will return to discuss sensori-motor embodied cognition accounts more fully in the third section of this thesis, with particular reference to George Lakoff and Rafael Núñez's (2000) presentation, but for now I want to focus more closely on perceptual systems, as the question of how thought is represented is fundamental to all accounts.

Barsalou's statement that "cognition often proceeds independently of the body" is not, incidentally, a suggestion that thought can take place outside of or in the absence of the body, but rather a calibrated refinement of a possible error: that evidence indicating that bodily states are not directly involved in *all* processing constitutes evidence against embodiment. Barsalou, Simmons, Barbey and Wilson (2003), for example, report evidence that "when bodily states are compatible with cognitive states [as when your thoughts coincide with your actions], processing is optimal" (p. 87). Markman and Brendl (2005), on the other hand, report evidence to the contrary and conclude that "perceptual and motor representations alone may not be sufficient to account for cognitive processing, because phenomena that at face value seem prime examples of lower-order perceptual and motor processing may nonetheless involve higher-order symbolic processing" (p. 10). This is likely to remain an active field of research for some time to come.

One very interesting aspect of such research consists in trying to understand the shift from thinking in terms that are readily anticipated to engage sensori-motor systems, such as when we group objects for counting, to what is called higher-order thinking, such as when we find roots of complex numbers. For George Lakoff and Rafael Núñez (2000), metaphor is central to this purpose, as we shall come to see.

Perceptual symbol systems, as noted above, are modal, meaning that the mode of perception implicated in an experience—auditory, visual, haptic, and so on—is central to the manner in which that experience will be remembered and recalled. This overcomes a difficulty of amodal representations: namely, that they lack flexibility in the face of ambiguous properties, exemplified by the usage of the preposition "in" (Markman & Dietrich, 2000, p. 471)[24]. An apple may be "in" a bowl, for example, even if it sits upon other apples and is actually situated *above* the bowl. Barsalou (1999) proposes that perceptual systems are intrinsically implicated in the process of thinking:

> A perceptual state can contain two components: an unconscious neural representation of physical input, and an optional conscious experience. Once a perceptual state arises, a subset of it is extracted via selective attention and stored

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[24] On the other hand, Noordzij, Neggers, Ramsey and Postma (2008) report fMRI evidence which suggests that locative prepositions are associated with cerebral activity in the supramarginal gyrus in the left inferior parietal lobe. Because the observed activity was not associated with context, they propose that this forms evidence for a general amodal representation of locative prepositions.

> permanently in long-term memory. On later retrievals, this perceptual memory can function symbolically, standing for referents in the world, and entering into symbol manipulation. As collections of perceptual symbols develop, they constitute the representations that underlie cognition. (pp. 577–578)

This system relies on perceptual memory—experiences that are maintained as perceptions and recalled as perceptual symbols, or "records of the neural states that underlie perception" (Barsalou, 1999, p. 582). That is to say, the same, or a subset of the same neurological systems that were initially involved in an experience are co-opted in the recall in such a way that the recalled perceptions can function as symbols and so enter into higher level thinking, as mental objects or thinkable concepts: "On this view, a common representational system underlies perception and cognition, not independent systems" (Barsalou, 1999, p. 578). Symbols are symbols *of* the cognising being, not disembodied, ontologically distinct atoms of information.

I see a cup before me. I close my eyes. A trace of the cup lingers, and then fades; it is gone. How shall I now bring this cup to my thoughts? I have a word, "cup", which is useful, but the word is too general to convey a sense of *this* cup. It fails to resolve ambiguity. In the absence of any other cup—on the table, for example—"cup" will suffice, but I am at one with Burke in acknowledging that when I say "cup" I parse the word with a rapidity that does not permit any imagery or sense that I have done anything more than articulate an abstraction. How shall I contemplate *this* cup? I re-cognise; I re-perceive. I try to feel the smooth width of the brim, but I cannot literally *feel* the brim with my lips. I remember that brim, however, as central to what that cup means to me. I remember, too, the balanced weight of the cup in my hand, but I cannot feel its weight in my hand, running tension down the length of my arm. When I want to remember this cup, I summon memories of what this cup is to me; I reconstitute my perceptions of this cup, but imperfectly, not in fullness. Only a perceptual part is reconstructed; not the touch of the cup but a shade of my perception of the touch; not the curve of the handle in my grasp, but some understanding that the curve was well formed and suited my fingers. The more I dwell, the more I create *this* cup; but then, of course, this cup of my memory is *not* this cup, but some creation inspired by its recall. *Ceci n'est pas une tasse*, as Magritte might have said.

Under the perceptual symbol system theory, the cup of my memory would be regarded as the product of a "simulator". A simulator draws on a subset of mode-specific memories to produce a particular, attenuated instance of recall: a simulation, that is to say, of what is being remembered or thought upon. Which subset of mode-specific memories will be utilised depends upon the circumstances surrounding the act of recall. In this way, simulators can create any number of possible memories, according to the conditions that prompt remembering (Barsalou, 1999, 2008; Barsalou et al., 2003).

This position profoundly upsets Cartesian dualism. What is more, it puts the engagement of the being at the epicentre of every act of cogitation and representation. Every thought, every act of recall, at some stage in its enactment, signifies the implicate coupling of man and his world; signifies, indeed, the *inextricability* of man from his world. When we reflect that the "inner" world of a man is a part *of* his world, then we realise that this coupling is iterative in character, and we become motivated to turn to a dynamic systems analysis.

The question of how concepts are acquired[25] and contemplated is further nuanced by the dynamic systems approach, since here any allusion to concepts or mental objects as *static* is given over to the *emergence* of thoughts in time. The complexity of the neuronal system is such that it is regarded as a self-organizing system wherein states are characterized by their relative stability or instability. States evolve in time under perturbations, but the evolution is non-linear, in that small shifts in one state may lead to little or no alteration (stability) or to significant change (instability), depending on the interplay between the current state and the nature of the perturbation.

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[25] The difficulty of being wedded to English is that we become accustomed to uses of language that are not necessarily in the interests of precise communication. I do not want to say that we *acquire* concepts—since I am engaged in the question of what it is we do when we think—but neither do I want to be so abstruse as to be unreadable. Let it rest, uneasily, for now.

Some researchers take this phase transition to be highly significant in the process of thought. Stephen, Dixon and Isenhower (2009), for example, suggest that cognition involves an interplay between component and interaction dynamics, with component-like dynamics dominating when the system is in a stable configuration, "when higher-order cognitive structures are the crucial variables governing the behavior of the cognitive system" (p. 1829); but that the emergence of new structures occurs as moments of instability are approached: "Interactions dominate when the system becomes unstable and approaches a transition. Once the system self-organizes from those interactions into a new stable mode of functioning, component-dominant dynamics re-emerge" (p. 1829).

Under this suggestion, *concepts* as such can be thought of as relatively stable configurations of an emergent system, robust under perturbing stimuli. This would be consistent with Gerald Edelman and Giulio Tononi's (2000a) theory of neuronal group selection, under which large scale associations of cross-modal neurons are stimulated to function as a group, where the dynamics of group selection are driven by Darwinian selection pressures. When the brain maps stable clusters of neuronal groups *into itself* as a schema—that is, when higher level structures bind in relation to lower level structures—we have a candidate for a relatively stable pattern or organization, of the type I have been describing as "thinkable concepts".

Linda Smith (2005) emphasises the time-locked multi-modal interactions that characterise perception: "Experience is inherently multi-modal—actions time-locked to sights, sights time-locked to sounds. Each unique modality with its individual take on the world is bound to other modalities and educates and alters them, and in so doing permanently changes us" (pp. 292–293). Concepts, read as ideas that we *have* and *acquire*, have no significance in this model except as emergent and ever re-emergent stable phase states:

> Concepts are merely hypothetical constructs. They are theoretical entities that have been proposed to explain stabilities in behaviour. ... It is not at all clear that concepts are necessary. From a dynamic systems point of view, dynamic stabilities and instabilities (the potential for change) emerge as a consequence of how complex systems composed of many heterogeneous components self-organize in context and in time. The unitary, context-free, and timeless theoretical entity of a concept has no role to play in this endeavour. (Smith, 2005, pp. 295-296)

Tony Brown (1996) approaches this general theme from the perspective of social phenomenology. He observes that "mathematical phenomena are understood differently by each individual, where the distinction between such phenomena and the perception of them is softened with phenomena and perception evolving together through time" (pp. 117–118). The "evolution in time" is modulated by the personal experiences of the individual, to be sure, but also—and significantly for application of dynamic systems theory—by previous states of understanding: "The hermeneutic process can be seen as shaping and flavouring mathematical notions in the mind of the individual. As we adopt successive new perspectives, our way of seeing the world and our expectations of it are renewed" (T. Brown, 1996, p. 143). By this we see that a dynamic systems model is well accommodated within the hermeneutic tradition. Brown proceeds to destabilise the notion of a universal concept:

> In engaging in mathematical tasks we are generally faced with deciding where and when to use particular ideas, especially in problem-solving situations, and we often remain unsure until we have checked things out. Our phenomenologies, and in particular, our mathematical phenomenologies and the ideas contained within them, evolve as we experience successive situations. These ideas never become 'fully formed', rather they are subject to successive modifications through time as they are encountered in new situations—or perhaps become fragmented as memories fade. (T. Brown, 1996, p. 143)

Let it be clear, however, when we are speaking of an individual and when we intend a community. The foregoing relates to the individual in society, and, to the extent that each community consists of individual members, there is likely to be some carry-over influence to the community. The community taken as a whole, however, expresses a different degree or scale of concept stability, wrought largely by inertia of mass and reification in text. A concept can approach universality by being recorded in a text shared among the members of a community, and might even persist in that representative form for hundreds of years unchanged; but two hermeneutic forces will apply selective pressure on that putative concept: each reader will bring his or her own historicity to the task of interpreting that text; and the slow, inexorable swing of social fashion will ensure that it can never stabilise except it become a fossil. But a concept cannot remain a static entity: either it lives perpetually reincarnate, or it fades and is lost.

The question of communication concerns a tacit response and a reinterpretation of an abstraction. If a phenomenon experienced primarily as, say, an aural, haptic or visual sensation is to be communicated through words, it must be reinterpreted at some level of abstraction. That is to say, it must be simulated in formal language. This is what language permits: the construction of a model of a tacit abstraction conducive to communication through words—to dialogue, that is to say. As students of a Western philosophical tradition, we privilege these voices and risk confusing them with reality, but tacit abstractions must always be closer to the truth of experience. What the voiced abstraction gains in communicability the tacit abstraction retains in faithfulness. The question is, are we able to relax our reverence for the voiced, the better to apprehend the tacit?

THE ELUSIVE WORKING OF KAN

Can an appreciation of our tacit qualities inform our teaching? Consider the following extract from Kisshomaru Ueshiba's *The Spirit of Aikido*[26] in which he describes how a German scientist came to Japan before World War II and became an admirer of Japanese swords. He wished to see how the swords were made, but was not allowed:

> Something bothered him: the air of mysticism that enshrouded the traditional method of forging the steel blade, for the swordsmith, dressed all in white to symbolize purification, does his work before a Shinto altar. This seemed to him very, very primitive, and he also had a low opinion of the sacred awe with which the Japanese regard the sword. He wanted to penetrate the mystery and unravel the secrets, but no matter how earnestly he requested it, he was never permitted to watch the swordsmith at work. (Ueshiba, 1985, p. 72)

He decided to apply a scientific approach. He analysed swords in a laboratory and applied the best technology available to the task of manufacturing swords.

> The outcome was utter failure. Collecting scientific data posed no problem, of course, but when he actually tried to make a sword, the result was just another commonplace sword. His use of scientific knowhow, even with repeated experimentation and modification of the production method, ended in disappointment. Finally, he was forced to give up his attempt and acknowledge the superiority of the "old fashioned, esoteric" method of Japanese swordmaking. (Ueshiba, 1985, p. 73)

This story echoes the difference between syntax and semantics. The scientist was focussed on the syntax of sword production—the technical details, the *how* of the process. The traditional masters, on the other hand, were consumed with the meaning. Indeed, if pressed to explain their techniques, it would be found that the process was largely tacit:

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[26] Kisshomaru Ueshiba (1921–1999) was the son of Morihei Ueshiba, founder of modern Aikido. He became the chairman of the Aikikai Foundation in 1967. He succeeded his father as hereditary head of the Aikikai in 1969.

> In traditional technology much of the achievement is due to an intuitive quality, known as the working of *kan*, and this can be acquired only through the accumulation of years of training. For *kan*-intuition to work, one must experience a creative tension stemming from single-minded concentration on the work in progress. This opens the way for a higher power, *kami* in Japanese, to enter into the process. Much of the success depends on becoming filled with divine consciousness or *kami*. The Japanese craftsman in making a single sword relies on *kan* both to select the proper materials and to combine them in the precise way transmitted in his family. It is not too much to say that the entire process of firing, forging and cooling is dependent on the elusive working of *kan*. (Ueshiba, 1985, p. 73)

In terms of our inquiry, we might venture to suggest that the working of *kan* is not so dissimilar to learning to hear the unvoiced voice of the experienced body; learning to interpret, that is, the tacit language of being.

It is a commonplace to hear a teacher of mathematics speak of the need for students to learn some or other concept. It seems an innocuous thing to say. And yet, if we analyse what can be intended by such a statement we begin to align ourselves with a more nuanced, tactful attitude. We can begin our analysis like the German scientist and seek to apply a positivistic attitude to a craft that is subtle; or we can embrace the mystery of teaching and concentrate ourselves on the working of *kan*.

Learning an art such as Aikido can bring these polarities into focus. As a martial art with deep roots in Japanese culture, it shares a form of mysticism with the process of swordmaking. And yet, its effectiveness as a means of deflecting and redirecting attacks can be understood in physical, mechanical terms. I remember hearing my Sensei say that one must open one's mind and be uncommitted before an attack, and speak of redirecting the *ki*, or energy flow of the attack. I, like the German scientist, was bothered by this. Sensei understood and recast some language for my benefit: we spoke of vectors and the physics of rotation. Emptying one's mind I understood not in spiritual terms, but in the same way that I was taught as a youngster playing cricket to wait for the ball before playing it, an early commitment to a shot being almost certain to put the feet and body in the wrong position to receive what actually comes. I had to find my own blend of *kan* and explanation. Pure analysis helped me to appreciate how movements achieved their purpose, but in the end, I had to make the movements, and that could be achieved only through hours of training. There had to be a balance between an articulation and a sustained physical engagement in the activity.

Azarello, Robutti and Bazzini (2005) identify two such polarities in teaching: the symbolic-reconstructive and the perceptuo-motor. The first approach requires learners to construct mental objects from symbols whereas the second "produces learning based on doing, touching, moving and seeing" (p. 57). Of the symbolic-reconstructive approach, they claim:

> It is a sophisticated way of knowing and requires awareness of the procedures and the appropriation of the symbols used and their meanings. In this regard, we note that "traditional" teaching in mathematics, which is usually characterized as "transmissive", is based, almost exclusively, on a symbolic-reconstructive approach. (p. 56)

As I have been indicating, the qualities of mental objects thus (re)constructed will vary between learners, and will be determined in part by pre-existing experiences. To reconstruct an object in the mind from a description given in symbolic form is fundamental to what it is to communicate in language, but the suggestion here is that, "The risk of using symbols in a mechanical way is great, and can cause misunderstandings and mistakes" (p. 56).

Transmissive teaching, it might be noted, can be read as a partial consequence of a teacher's belief about the character of mathematics itself. As Leone Burton (1998) has observed, "If a teacher is convinced that mathematics consists of a fixed body of knowledge which learners must acquire, and that the only way to do so is by being told, then transmission teaching is a natural outcome" (p. 140).

I recently checked to see whether a group of Year Eight girls understood Pythagoras' Theorem. The name "Pythagoras" caused a ripple of reflection, as each girl struggled to recall something to mind, and then Lana announced, "Oh, is it that $a^2 + b^2 = c^2$ thing?" The letters a, b, c are memorable, of course, a fact exploited in the traditional labelling of a triangle, but had Lana done any more than recall a rote "fact"? Well, yes and no. She did have a sense—as did others in the group—that these letters referred to sides of a triangle, although it was not uniformly appreciated that they represented the *lengths* of the sides. Nor was it certain *which* sides they represented, nor even that the triangle must have a right angle. Some of the girls recalled the theorem with greater precision, but none had related the squares of the formula to actual geometric squares based on successive sides, and certainly none were aware that the theorem can be cast in terms of the *areas of any similar* figures, not solely of squares.

Azarello goes on to explain that, "The perceptuo-motor approach, on the other hand, involves action and perception and produces learning based on doing, touching, moving and seeing" (p. 57). I introduced Lana and her peers to the dynamic geometry software *The Geometer's Sketchpad* and we proceeded to construct right angled triangles with squares on their sides, and we used the functionality of the software to explore the relationships that unfold (inductively) as Pythagoras' Theorem. My hope was that the use of such artefacts would encourage robust and veridical mental objects, partly because it engages the body more intently than symbolic reflection, but also because it lends greater emotional valence to the process.

While evidence for the use of manipulatives is mixed (see, for example, Raphael & Wahlstrom, 1989; Sowell, 1989), paramount among the positive emotions appears to be a sense of pleasure brought on simply by understanding what is being done. Sounding a note of caution, however, Schlöglmann (2002) refers to the implicit emotional memory system that can be activated by the problem solving process, leading in some cases to negative reactions beyond cognitive control that effectively block learning, a point to which we shall return later. In a related vein, Healy and Hoyles (2001) discuss issues related to tool selection. They report that less successful students can “find themselves in a position where they are unable to use the tools they have in mind, even if they are convinced that their use would make sense mathematically, and they are familiar with how the tools should work” (p. 252). They note that, “the mediation of students’ activities by the software is not necessarily positive for their engagement and for their learning” (p. 252).

We see that there is a need for tact. It is one thing to have an epistemological guide; it is another to discern how one should employ its lessons.

PART THREE ILLUMINATION

The body is emerging as the foundation of thought. Throughout this third part of my thesis the qualities of the body—from its capacities to orient in space and to manipulate with grasping fingers, to its emotional responses—rise in significance.

A conception of number as a language tool shaped by our intentions, themselves formed and constrained through our active engagement in our world, emerges as an element in the embodied epistemology that is slowly shaping. Close contact is maintained with the classroom through excursions in teaching encounters by which I maintain an orientation toward my students, as I build my understanding of knowledge as deeply implicated in our shared way of being.

I describe a metaphorical way of building mathematical knowledge and reflect upon the experience of what it feels like to act mathematically.

CHAPTER FOUR ORIENTING TO KNOW

THE COUNTING BOARD: MIDDLE SCHOOL OR MIDDLE AGES?

Does the epistemological shift to the tacit qualities of cognition afford pedagogical insights? The risks of transmissive teaching are frequently exposed. Consider the following case in which Isabelle, a student in my Year 5 enrichment class, explains a difficulty she was experiencing. I had gone over to the Year 5 rooms prepared to present a problem designed to investigate aspects of place value. While I was collecting the girls, however, Isabelle spouted, “I’ve got some sums that, I know the answer, but I can’t get the theory that we’re meant to do. Mum and Dad tried to explain it, and I’m going, ‘What?’”

The girls were settling in readiness for class, but Isabelle pressed me. “I know the answers, but I don’t get the method.” Within the emerging epistemology of this thesis, “knowledge” consists more in action than in possession. Isabelle’s recognition that the method surpassed the answers signalled her recognition of the significance of doing in distinction to having.

It might seem that Isabelle is not your average student. She was acutely aware that there was more to be had than the mere production of correct answers. She wanted to know, perhaps because of her own nature or because it has been impressed upon her that it was important, why the method she employed had yielded success. Our class times together were frequently marked by a somewhat free spirit, where capitalising upon meaningful mathematical issues was a part of my general plan, and so I decided to investigate Isabelle’s concern to see if her difficulties were shared among the group.

The problem we focused on was

$$\begin{array}{rccc} & 3 & 0 & 4 \\ - & 1 & 8 & 6 \\ \hline \end{array}$$

I invited Isabelle to describe the problem prior to seeing if any other child could explain the theory. “Now, this is what I was stuck with—I am told this is correct. You go, okay, make that [4] a fourteen, cross that [3] out [and] make that a two, cross that [0] out [and] make that a nine. Is that how you’re meant to do it?”

$$\begin{array}{rrrr} & \cancel{3}2 & \cancel{0}9 & {}^{1}4 \\ - & 1 & 8 & 6 \\ \hline \end{array}$$

“That will work,” I said. Phrases such as, “How you’re meant to do it” are frequently heard, and say much about the arrangements of power structures within mathematics, schools and families. It is clear that Isabelle trusted in a certain level of authority.

“But I can’t see how a zero can turn into a nine, ‘cause you’re borrowing a hundred, not a ten from there, and how can it turn into a nine? In theory, it would be ninety,” she said, describing her problem. “But Mum and Dad were doing all this, and you give it back and have to borrow constantly again—”

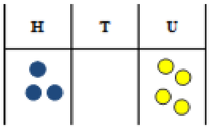
Girls interjected, eager to show their knowledge of the algorithm and, to some extent, their understanding of why it worked. I was quietly pleased, because Isabelle was asking for help with a problem that required a considerable appreciation of place value, the very thing I had been intending to work with, and, moreover, she had done the job of stimulating interest among her peers!

“How can a zero turn into a nine,” Isabelle persisted, “when zero is at the bottom and nine is at the top?” She was employing a spatial metaphor that has zero and nine placed at opposite ends of some spectrum containing the ten digits of base-ten arithmetic. She was also using the language of numbers “turning into” other numbers, an almost magical process which neatly conveyed the crux of her confusion: she was an intelligent girl but was suffering under the impression that she was “meant to” accept something akin to numerical transmogrification.

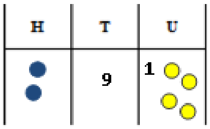
Isabelle was undergoing something of a crisis of thinking. While she is relatively adept at the instrumental level, her relational understanding was flawed (Skemp, 1976). What makes her position interesting is that she was very much aware that something was wrong. She was relatively confident that her algorithm was correctly executed, but it jarred with her understanding of theory of number. She *nearly* had a relational understanding; she was emotionally prepared for it, certainly, in which ground we find a variety of frustration many parents will have witnessed, that of the child whose ambitions exceed her capacities.

It seemed that it was necessary to help her make that transition and so we turned to the kind of concrete symbols that have been employed across eras and cultures in order to facilitate calculations on counting boards (Menninger, 1969). That is to say, we used counters in columns, or ranks.

I arranged counters to illustrate 304, by placing three counters in a "hundreds" column (or rank), no counters in a "tens" column, and four counters in a "units" column. The counters served as a focus for the following exchange in which I encouraged the girls to share explanations while I observed the to-and-fro. In what follows, I present a transcript of the remainder of the lesson so that the flavour of the exchanges is faithfully conveyed, together with elaborations on elements that bear on an epistemology of mathematical knowledge. I play the role of the teacher, Tch. Where a student cannot be identified from the transcript, or where several students speak at once, S is used.



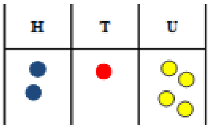
Isabelle Well I think that, you're borrowing a hundred from there [H rank]—you're taking away a hundred and putting one [in the U rank], and putting nine down there [in the T rank, where "putting" here has a double meaning of moving and writing].



Abby	No, you’re putting it there, and then you take away ten from there so it becomes a nine…
Isabelle	If you take away ten, in theory it would be ninety, because that’s a hundred. One, two, three hundred, and you’re taking away…you’d take away 81…[to make nine; she’s thinking of 90 – 9, not 100 – 9].
Abby	It is ninety, Isabelle.
Tch	Isabelle, I can see what you’re thinking. You’re talking about taking nine away and making…
Isabelle	Yes, because you’d have to take a lot more than one away to make nine.

Here we have a clue that Isabelle has not appreciated the significance of the ranks of the numbers, that a counter in the H rank signifies 100, and that were the counter to be removed the need to represent 100 would persist. We could say that while Isabelle is aware that 304 = 300 + 4, or even 3H + 0T + 4U, the language employed in the algorithm of “borrowing” and “changing” has occluded the elementary transitions: 1H = 10T and 1T = 10U. She is thinking of 100 – 91 = 9. What does Abby think? I invited her to describe how zero becomes a nine.

Abby	You can’t take six from four, so you need to make that into 14 to be able to take it, so you need to borrow from there—you borrow a hundred and you need to put it there. Then you need to take ten away from that and make it ninety, and you know how you, ‘cause you can’t just write 300 and 90 and 4.
Tch	So, Abby, you’ve put a blue counter away and put a red counter in the tens column.



Isabelle	But ninety stuffs the sum, because it’s not 394, it’s 304.
Tch	At the moment there are two blues in the hundreds column, so at the moment that would be 200…
S	No, it’s still 300.

Tch Isabelle has a point to make here. I think we all need to agree that we must not turn this number into something other than 304.

Isabelle Because we are, because Abby is saying taking ten which equals 90… but on the sum she's saying it's nine—so you're twisting your theory.

Abby So you cross out the ten and make it a nine… and just pretend that's a nine [the red counter], and so then you'd have a ten to put to this, take six from four…

Tch At this stage I am feeling like I want to jump in and explain it, but I also am anxious for you three as well to explain what's going on. So Eliza I am going to pass to you for a moment…

Eliza I have no idea.

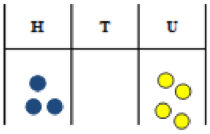
Tch You're not sure what we're doing?

Eliza No.

Tch Alright, we're going to make this our goal for the morning, to understand how this works. Diane, do you know how to do this?

Diane Yes.

Tch Sophie, I'm going to ask you to try to have a go as well. Diane, over to you. We've got three blues in the hundreds, we've got four yellows in the ones, no tens, so this number is 304 and it must stay 304, but we don't like the way it is, because that zero makes some problem—



Diane Well, you've almost got to buy a hundred, because… [she pauses].

Tch "Buy" a hundred did you say?

Diane Borrow, because then you've got the three hundred, well two hundred—[Pause, into which jumps Isabelle]:

Isabelle And ninety four, and 294 isn't the same as 304.

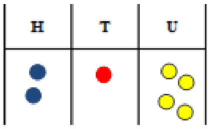
S No, and then you borrow that—

S And cross out that, and make that a nine.

[At this point Abby and Diane and Isabelle move the counters around to try and reinforce their positions].

S No you don't put the ten back on.

Tch [Describing for the record] So the girls are playing with the counters; at the moment there are two blues in the hundred, there's a red in the tens, and there's still [four in the ones] so that's 294.



[Isabelle and Abby become quite involved in the dispute]:

Isabelle But you've changed the number.

Abby Of course you have to change the number, you can't take 8 from nothing.

Isabelle Yes, but ... we are taking 304, not 200, 9 and 4.

There is evident confusion over what constitutes legitimate transformations. The language of "borrowing", "buying" and "changing" coupled with the use of a single counter to represent nine bespeaks transmissive experience and a privileging of the symbolic-reconstructive over the perceptuo-motor pole. That is, Isabelle, Diane and Abby have been operating with procedural knowledge but they have not fully appreciated the relationships that make the procedures work. That is not a surprising finding; what is more concerning is that there are signs, as in Abby's comment about changing the number, that the procedural elements are threatening to override the fundamental bases on which they are predicated.

[I turn to Sophie to encourage her to join in].

Sophie Ah well, you can't take 400, ah, the 4 from the 6—

S No, you can't take 6 from 4, you can take 4 from 6.

Sophie Yeah, so there's nothing there, so I have to borrow from here.

Tch Okay, so you're borrowing from the hundreds into the tens…

Isabelle But there’s nothing there, so you need to borrow from there [the H rank] first.

Sophie Yeah, so I borrow from there, and then I get that, and borrow from that, so then that—

Isabelle But you need to cross that out to borrow, and make it a nine.

Sophie It’s easier to write down.

I could see that Sophie was finding it difficult to relate her understanding of a procedure generally expressed in a strict, written form to numbers presented with counters.

Tch Yes, look, that’s right. And now, what happens in school is this: you get these ideas and then you start—it may be a little confusing, but then you get a system—you start crossing things out, and often times you can follow the system without really knowing what’s going on, but [turning to Isabelle] you’re not happy with that, you really want to know what’s going on, I think.

Isabelle I think we’re all getting the same idea about what’s going on.

Tch What do you think, Diane?

Diane We’re focussing more on the tens than that, we’re trying to pull from six.

Tch You’re right; in the problem we have four ones take away six ones which is a problem right at the start.

Isabelle Can I say the answer to make it easier?

Tch I don’t know if the answer will make it easier for us. Eliza? [Eliza looks reluctant to contribute]. Well, let Diane continue for a second and then… I can help you with this—and I will help you with this—I’d just really like you to see the problem and wrestle with it a fraction [longer].

Diane Well, you could take the nine away and make that—

Abby Pretend light blue is ten, light blue is just ten, and so you give a ten to that to make it fourteen

Isabelle Yeah, but how?

Tch The “how” is the question.

Abby Because you borrow.

Isabelle That's ninety. You're taking ten, so that would equal eighty!

[Laughter ensues, and flurry about moving things. The girls work at moving the counters and trying to convince each other and themselves. There is difficulty articulating the process, there is energetic disagreement. At dispute is whether the number is changed by the borrowing technique. Abby is enforcing the borrowing process, Isabelle is stressing the change that is being forced on the number. To an extent, Abby is not engaging with the difficulty that Isabelle is noticing].

Abby I took 10 from the 100…

Isabelle Which made it 90…

Abby And the ten that I took I gave over four.

Isabelle So that's still 90, but that's not the sum. The sum isn't three hundred, two hundred and ninety four.

Abby Yes, but it's written like that.

Tch Now Abby, when we do these problems, in fact, whatever we write, we have to make sure that we don't change the problem, really.

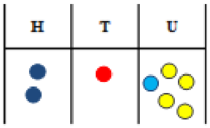
Isabelle Because that's what she's doing! She's changing the number.

Abby Because look, it's 2 and 9 and 4, see?

Tch I can certainly see. Well, in fact it's 2 and 9 and fourteen, and that makes a difference.

[I invite Eliza to take a turn].

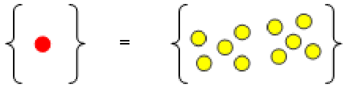
Eliza Um, well, it doesn't have to be, it doesn't have to, it has to be a 2—I mean it has to change, because there's also the six in there, so it's got to change, it's impossible not to, because otherwise it's impossible to do the sum, so you have to move it.



Tch	Eliza, now there are two in the hundreds, you've put a red counter in the tens column, and a light blue counter over here with the ones. What's this red counter in the tens column?
S	It's ninety!
Tch	That's ninety, is it?
S	The red is nine tens and the blue is one ten.
Tch	So when I read that number are you telling me to read it as 200 and nine tens (ninety), and—is this fourteen over here?
Eliza	Yeah…I'm... wasn't sure, but [indistinct].
Isabelle	Yes, you don't know how you did it. That's what I'm getting at! No one knows how they're doing it, they're just doing it!

Indeed, it seems Isabelle was onto something. It seemed to me that it was time to attempt a degree of clarification.

Tch	Here we go. I would like you to reflect for one second on something very important that is happening here. Whenever we write numbers in this part of the world, and in other parts of the world too, we have a little rule that we follow, and the rule is, in every column we don't go past a certain digit…
Isabelle	Which is nine.
Tch	Which is nine, okay, the biggest digit that we are allowed to use in any column is nine.
S	[In comic voice] The largest number in the world!
Isabelle	It's not the largest number, it's the largest digit.
Tch	So we go from zero to nine in any column. But—and here's the thing, and I think you'll all get this—everyone knows that ten of these ones makes one of these tens, so I'm going to get these counters [I count out ten of the ones so that I can represent this]. If that's ten ones, and this red is one ten, this is completely matched by that.



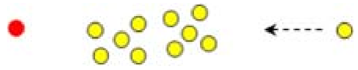
S So they're equal.

This represents a reversion to a form of peasant reckoning. The red counter might as well be a Roman X and each yellow counter a Roman I. Since the essence of the algorithm that the girls are wrestling with lies in the groupings of quantities, we returned to that level of operation. The application of place value to *numerals* is one step removed from its application to groups of quantities—that is the bridge to cross. Menninger (1969) points out that this was the prevailing condition in Europe throughout the Middle Ages ahead of the introduction of the Indian numbers that (eventually) led to our familiar numerals:

> In computation people used a mature place-value system with gradations based on 10; in writing the old [Roman] numerals were retained. The two went along hand in hand for many centuries. The price paid for the "popular" simplicity of the Roman numerals was their uselessness in computation, but this disadvantage was again nicely circumvented by using the counting board with its more advanced gradational arrangement. The mutually complementary use of the numerals and the counting board thus created a fully adequate and convenient tool for simple computation, which people were therefore extremely reluctant to part with. For what is in fact the essence of the new Indian number system? That it combines the two aspects, writing numbers and making computations, into one single procedure, by extending the more advanced principle of the counting board—its place-value notation—to the numerals as well. The separation between computation and the writing of numbers was never realized during the Middle Ages; people wanted utility, not intellectual or spiritual perfection. It never occurred to anyone even to try to take the step which the Indians had taken. (p. 298)

Isabelle, on the other hand, did desire a form of intellectual perfection. We explored the use of the counters.

Tch Here's the heart and soul of the borrowing idea. Just to recap: here on the table we have ten ones, a red as well, so that's the number twenty. And if I put another one there, it's the number…



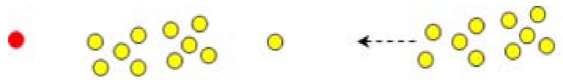
S 21.

Tch Everyone agrees, it's twenty one.

S [Thinking about place value] But you couldn't do that with the ones; you'd have to put that with that, otherwise …

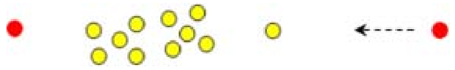
Tch At the moment I'm not doing place value, I'm just putting piles of symbols down.

Here's another ten ones. If I put them over here, what number is it now?



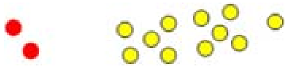
S 31.

Tch This ten ones I could just do that with, though, couldn't I? I just replace those ten [yellow] ones with a red [replacing ten ones with a red ten].

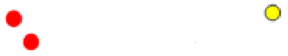


S Yes.

[I organise the counters into clear piles, as if in columns].



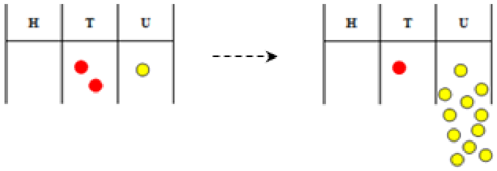
Tch Alright, well this is what happens. All that happens in the whole place value thing with all the borrowing and so on is merely this. [I check again that they see what I am doing with the counters] This number, if I remember rightly is 31, is it? And let's just say I'm going to move these ten ones away, is the number now 21?



S Yes

Tch All that happens with the place value calculations is that you allow yourself to break one little rule, and the one little rule that you allow yourself to break is that you can't go past nine in a column.

So what happens is this, if you were doing a problem, if you didn't want to have a two in this tens column, because maybe you were trying to subtract a five, for example, all you would do is, instead of putting a single counter there, let's put ten counters there instead [speaking while doing it with the counters]. Now just remember, is that 21?



S Yes.

Tch Is that still 21 [having replaced a red ten counter with ten yellow ones]?

S Yes.

Tch Now, how many ones are in that in that column [pointing at the ones column]?

S Ten!

Tch No.

S Eleven.

Tch One, two, three, ..., eleven.

S Oh, yes, eleven.

[We now record on paper the process undertaken with the counters].

Tch So what's happened is [writing on paper] I'll put a tens column and a units column. We started with 2 and 1, and then I took away one of the tens, so now there's only one, and I made 11 over here, and to show eleven [on paper] I just put a little superscript "1" there [showing the "borrowed' notation].

$$\begin{array}{cc} T & U \\ 2 & 1 \end{array} \rightarrow \begin{array}{cc} T & U \\ \cancel{2}1 & {}^{1}1 \end{array}$$

S So that you can subtract…

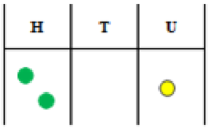
Tch So that I could now subtract a nine or whatever it might be.

S Yeah.

Tch So I'm breaking the "don't go past nine" rule, but it's still twenty one. One ten and eleven ones is still twenty one. Does that make sense to you?

Isabelle Can I do my theory?

Tch Well, just before you do, if you don't mind, in the case where we had zero in a column …. So now, if you imagine for a moment that [getting another colour to represent hundreds], say I use green for hundreds. Now look at the number on the board [table top] now. I've got two in the green column, nothing in the tens column, and one in the ones column. So what number is this?



S 201.

Tch Do you all see that? Ok, how many tens are there in a hundred?

S Ten.

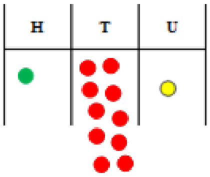
Tch So I need ten reds. [I start counting out ten reds].

Isabelle Oh, I'm starting to get it now. So you can do that now. OK.

S Who was right?

S I don't think either of us is right.

Tch Well, you were all on the right track. So here we go. I would like there not to be a zero in the tens column. Can you see the process now? Instead of having this [two green counters for 200], I'm going to replace this [green counter] with [ten red counters]. That's a hundred, it's equivalent to ten tens, and I'm going to put all ten tens there, in the tens column. So that number now, if you were writing it in columns is:



Isabelle But you can't put ten tens in one column.

Tch But we're allowing ourselves to break the nine rule.

Abby Yeah, yeah [there ensues some light commentary about breaking rules].

Tch So you have one there [the H rank], ten there [the T rank], and one there [the U rank]. And you know, that's effectively what's going on. So the way you do it in your little symbols, you cross out the 2, make it a one, and you'd bring that one up here like this. One 100 and ten 10s and one 1 is 201.

$$\begin{array}{ccc} \text{H} & \text{T} & \text{U} \\ 2 & 0 & 1 \end{array} \rightarrow \begin{array}{ccc} \text{H} & \text{T} & \text{U} \\ \cancel{2}1 & {}^{1}0 & 1 \end{array}$$

Isabelle OK, I get it, now then can I show my theory?

Abby So with that number [referring to our starting problem], it would still be 304, but written a different way. So it's still 2 hundred and ninety and fourteen, so it's still 304.

Tch That's the thing. It's a question of seeing what you were moving. So you were allowing yourself to put more than nine in a column, but what you were moving across [to the tens column] was an entire hundred, ten tens. So for three hundred, you left two hundreds in the hundreds column, I'll write the two there now, and what you moved across were ten tens, so that's why you said there were ten tens there.

$$\begin{array}{ccc} \text{H} & \text{T} & \text{U} \\ 3 & 0 & 4 \end{array} \rightarrow \begin{array}{ccc} \text{H} & \text{T} & \text{U} \\ \cancel{3}2 & {}^{1}0 & 4 \end{array} \rightarrow \begin{array}{ccc} \text{H} & \text{T} & \text{U} \\ \cancel{3}2 & \cancel{1}\cancel{0}9 & {}^{1}4 \end{array}$$

Isabelle Yeah, I get it! Now I can do my homework.

Diane Oh, I get it now! I seriously get it now! I get it, I get it!

Abby So in effect it's still the same number but written differently, like fractions if you write them differently.

There was a sense of excitement among the girls, but Eliza was still somewhat restrained. She had not felt the sense of clarity that was shared by the others. We applied our understanding to a new example to see if we had progressed. Isabelle, it might be noted, no longer pressed to give her theory. I asked Eliza to try to explain the process of using counters with the subtraction 308 – 289.

$$\begin{array}{rccc} & 3 & 0 & 8 \\ - & 2 & 8 & 9 \\ \hline \end{array}$$

She began hesitantly.

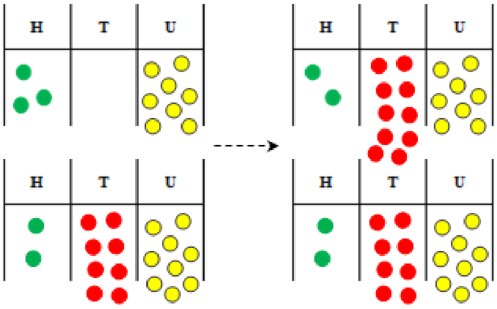
Eliza I don't know what to do.

[I focus her on the task by reading the problem and highlighting the difficulty in the tens rank. She moves a green counter from the hundreds rank to the tens].

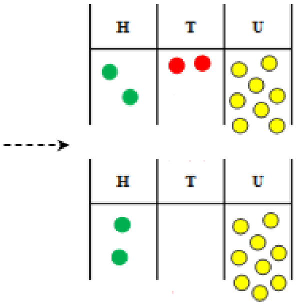
Tch How many tens are there in that green counter, Eliza?

Eliza Ten.

Tch So you'll need to put ten [red counters] in that red column. That number is two hundred and ten tens and eight ones, so it's still 308.



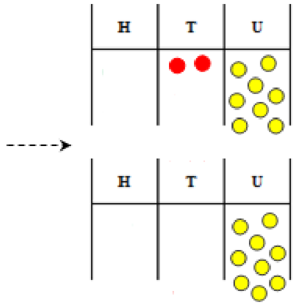
[Eliza proceeds to take away the eight tens].



[I note for the girls that this might not be the "usual order" in the standard algorithm, but there is no real reason not to do it this way. I ask Abby to remove eight tens from the tens rank in the paper version, leaving 228].

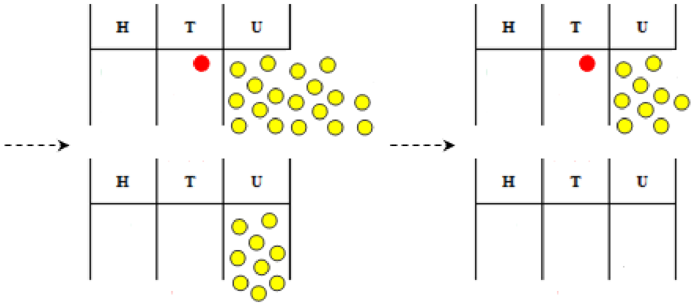
$$\cancel{3}2 \quad {}^{1}0 \quad 8 \dashrightarrow \cancel{3}2 \quad 2 \quad 8$$

[Eliza gains confidence, and removes the two hundreds. Abby crosses out the two hundreds on paper, leaving 28].



$\cancel{3}2 \quad 2 \quad 8 \dashrightarrow \quad 2 \quad 8$

[Eliza replaces a single ten with ten ones. She puts the ten ones in the ones rank, so that there are 18 ones in the ones rank. From this she is able to subtract the nine ones].



[This leaves eight ones in the ones rank and one ten in the tens rank. The solution she obtains is nineteen].

Tch The borrowing that you were doing was to get more than nine counters in a column. Once you had done that the order did not really matter.

There is something quite peaceful about the process of manipulating groups of counters in ranks. Provided one is able to understand that a counter in any rank is equivalent to ten counters of the rank to its right and can be relocated accordingly, then one requires only minimal counting skills to perform calculations. The requirement for computing skills, it would seem, are minimised. Even the order in which the counters are processed is relaxed. In fact, one can sense that computers[27] who use counting boards such as illustrated above are in a position akin to the English speaker in the Chinese room.

It might be clear enough that a quality of manipulating counters is to decrease tension; or if not that precisely, the effect of tension is reduced, since the working memory is less implicated in the processing. The calculation, we might say, has been off-loaded from the mind to the fingers. One can imagine a computer developing great fluency, such as we sometimes hear of expert abacus operators. The process can be quite tacit, though active.

Still, it is difficult to imagine one developing a great *feel* for numbers even though one becomes an expert computer on a counting board. Indeed, one lesson of history is that computational skill is not the same as mathematical sophistication. The example above demonstrates that the absence of a numeral for zero, for example, is no impediment to computation provided one has a means of registering the absence of counters in a rank. The real impetus for the numeral zero can be adduced to the movement away from counting boards and the advent of written digits in a strict place-value system, a point made by Menninger (1969):

> Only the abstract place-value notation with no symbols for the different ranks needs to introduce zero, a symbol for a missing rank. Since none of the ranks are written down, it must somehow indicate that one of the ranks which are never written is missing. (pp. 391–392)

The question of whether we can say that Isabelle and her peers will ever come to understand the calculations they were performing can only be addressed by distancing ourselves from the expectation that following an algorithm, or moving counters, will be sufficient to determine the matter. We have already noted that electronic computers may be able to simulate semantics, but that they cannot duplicate semantics, because that requires consciousness and emotions; that one cannot get to semantics from syntax alone, but that semantics emerge from text through the agency of lived experience. The question is, are *human* computers able to move from the syntax of computation to semantics, and if so, how?

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[27] "Computer" is here used in its original sense, meaning one who computes. The word *com-pute* is derived from the Latin *com*, for with, and *putare*, meaning to reckon; or an earlier version has it meaning to prune. Both derivations accord with the use of counting boards, since reckoning is achieved through a process of pruning away.

In the following illustration, Victoria puzzles over the application of a known characteristic of numbers to a new context in which her semantic sense was disrupted.

KNOWING NUMBERS

It is sometimes a condition of learning that a student suspends uncertainty; or, in a more active mien, that she willingly accepts as true statements for which she has no means of verification. This is the essence of what it is to accept a voice of authority. It demands trust, the more so when the voice scrapes at one's beliefs.

The following is a discussion I had about divisibility rules in base-five with Victoria, then in Year Six. She was unsure about base-five notation, so I indicated to her the nature of the place value system by drawing attention to what she already knew, at least implicitly, about base-ten. Victoria explained that she understood how the rule for divisibility by three worked in base-ten, using examples[28]. She was unclear, however, about a question that was asking her to check if the same rule worked in base-five.

"Now I need to know what they're asking us to do," she said.

"Normally speaking when we write a number it's in base-ten, so your columns, your number columns, are all powers of ten: ones, tens, hundreds," I began.

Victoria joined with me as we moved through hundreds, then thousands, after first indicating her familiarity by saying, "Oh yeah, like...," She had spoken as though to begin the list, but paused, before joining with me as I recited. Note that this was not a turn-taking conversation, but an instance of talking together when ideas were able to be shared. I did not expect that Victoria knew about "powers" or about details such as $10^0 = 1,\ 10^1 = 10$; yet at the time it seemed that these are features that it is reasonable to present as integral to a generalisable pattern. I did explain these features to a point—but a point that was really little more than a naming.

"We call that 'Base 10'. Base 10 is those columns. The first column is the 'ones' column; that's actually 'ten to the power of zero'." As I say this I point to 10^0.

"Oh, okay," says Victoria.

"The next column is the tens, which we call 'ten to the power of one'," I say, indicating 10^1. "The next column is the hundreds; you might be more used to calling that 'ten to the power of two'."

---

[28] The divisibility rule for three in base-ten is easy to apply because of a particular relationship between powers of ten and multiples of three: $10^k \equiv 1 (\mathrm{mod} 3)$, for positive integral k. That is, every power of ten is one greater than a multiple of three. For example, 10 is 1 greater than 9; 100 is one greater than 99, and so on. The consequence of this is that a number written as *abc*, to give a three digit example, will be a multiple of three provided the sum of its digits is also, for $abc = 100a + 10b + c \equiv a + b + c\ (\mathrm{mod} 3)$.

Victoria then makes her contribution: “And then it goes ten to the power of three, ten to the power of four, ten to the power of five.” Thus she has signalled a sense of the pattern, and an acceptance of the unknown interpretations of $10^0 = 1,\ 10^1 = 10$.

We can pause and consider what is involved in this simple exchange. What is the more salient: the *naming* of $10^0 = 1,\ 10^1 = 10$, or the consistency of the pattern, with which Victoria had already a passing familiarity? How important was her acceptance of my role as a teacher with (at least) mathematical authority over her? Had she been less accepting of me, either as a teacher or as an interlocutor, would she have reacted less favourably to my posing such arcane statements as if they were to be taken as granted and not questioned? I have certainly seen cases of antagonistic students who will not accept minor conventions from teachers without aggressive questioning. The relationship between student and teacher was important to allow the conversation to flow, with moments of delayed gratification, to a place from which the student’s goals in the exchange could be met. There appear to be two conditions in play: that she accepts me as her teacher; and *mathematics* as a learnable construct.

Consider, too, how Victoria’s capacity to react in a non-impetuous fashion fitted her well for assimilating new facts into an existing pattern, and, ultimately, for gaining the solution to the problem she had confronted. If the ability to resist impetuous behaviour is correlated with academic success, as Tangney, Baumeister and Boone (2004) have found, it might be that this exchange offers an example of a way in which such self-control confers a greater ability to learn and extend learning.

Resuming the thread, I motioned to several examples of numbers written in base-five. “In base-five, when you write a number down, notice there’s no digit that’s more than four?”

“Yeah…” she replied, her tone engendering in me the feeling that she was encouraging me to speak, accepting for the moment what I was saying but not yet committing herself to it. She was withholding her judgement.

“That’s like saying in base-ten that there’s no digit more than nine,” I continued.

“Oh, because otherwise you get the five and then you have to use the next column.”

“And the columns are headed not with one, ten and one hundred; they’re headed one, five, twenty five—”

“Powers of five!”

Victoria then anticipated the next column would be fifty, thus indicating that while she was following aspects of what was being put to her, the finer details were not yet completely clear.

We went over this again, looking at base-ten first, and then drawing the parallel more tightly about base-five, ensuring that it was understood what five cubed, for example, means.

The discussion then turned to our standard base-ten number representation, whereby 23 denotes $2\times10^1+3\times10^0$, or $2\times10+3\times1$. The same “23” presentation in base-five, we noted, would have a different interpretation: $23_5=2\times5^1+3\times5^0=10+3=13_{10}$, where the subscripts clarify the intended base.

This was confusing at first blush, but what was notable was the patience and persistence with which Victoria attended the explanation and attempted to derive the sense she sought. Once she had seen enough of this elaboration, we turned to her original question: if we can see that three goes into, say, 27 by forming $2+7=9$ and noting that three divides nine, can we likewise test a number in base-five for divisibility by three?

This, of course, presumes we know something about division processes in bases other than ten. We do, but only by extension; we naturally map what we know about base-ten into the new base and assume that the properties transfer. This is a form of metaphorical mapping from one mathematical domain into another, an idea we shall soon consider more closely.

The process of transferring properties from a familiar domain to another, less well known domain is, incidentally, not a practice restricted to mathematics. Pascal Boyer (2001), for example, has applied a more general theory of epistemological inference systems and ontological categories to the particular task of interpreting his observations of the religious behaviours of the Fang people of Cameron. Robert Boyd and Peter Richerson (2006), in a general critique of the application of evolutionary theory to culture, proffer a putative evolutionary value for social learning processes (epistemological inference systems including metaphorical mappings), suggesting that they allow human populations to

> accumulate reservoirs of adaptive information over many generations, leading to the cumulative cultural evolution of highly adaptive social institutions and technology. Because this process is much faster than genetic evolution, it allows human populations to evolve cultural adaptations to local environments, an ability that was a masterful adaptation to the chaotic, rapidly changing world of the Pleistocene. (p. 24)

There is a price to pay for this evolutionary advantage, however:

> To get the benefits of social learning, humans have to be credulous, for the most part accepting the ways they observe in their society as sensible and proper. Such credulity opens up human minds to the spread of maladaptive beliefs. (p. 24)

The credulity they cite is beneficial when a student such as Victoria inclines to accept a teacher's instruction, but it can be harmful, or simply fail to be beneficial if acceptance extends to impoverished ideas.

Victoria and I proceeded to convert numbers into base-five so that we could explore the question of divisibility by three. We considered 321_5, by way of an example. We had first to confront whether we were to look at $3+2+1=6$, and note that three divides six; or else to say that in base-five, $3+2+1=6=1\times5+1\times1=11_5$, then to ask if three divides 11_5. Our sense of what three "is" rebels at the thought that three could divide 11, and yet the number 11_5 is the same as 6_{10}, so that if three divides

6_{10} it must divide 11_5. Victoria's familiarity with 11_5 was very different from her familiarity with 6_{10}, however, and so while she could "see" that 3 divides 6 that same power was lost to her when she confronted 11_5. Adding the digits of 11_5 gives two, and it would seem that three does not divide two, so if the test was valid, one would conclude that three does not divide 11_5, nor, consequently, 321_5. What number is 321_5 in base-ten? It is $321_5 = 3\times 25 + 2\times 5 + 1\times 1 = 86_{10}$, which is not divisible by three.

A contra-example can be found, however: $423_5 = 4\times 25 + 2\times 5 + 3\times 1 = 113_{10}$. Here we note that $4+2+3=9$, which three does divide; yet three does not divide 113_{10}. There is, however, the scent of a confusion of bases about this. We could also have said $4+2+3=9=5+4=1\times 5+4\times 1=14_5$, and noted further that $1+4=5_{10}=10_5$, and finally that $1+0=1$, which is not divisible by three. On the other hand, three *does* divide 14_5, as can be seen with a simple list, written in base-five:

$$1, 2, \underline{3}, 4, 10, \underline{11}, 12, 13, \underline{14}, 20, 21, \underline{22}, \ldots$$

The underlined are multiples of three.

I remember wondering out aloud if the divisibility test might not rely on the "ten-ness" and "three-ness" of the numbers. What could this possibly have meant? And yet, Victoria was able to appreciate something in the question, that as there was something in the "being ten" or "being three" that was altered by a shift of base, so might there be something that is invariant under base change.

At every point here it is noteworthy that we are robbed of our number sense. We do not "know" much at all about numbers in base-five. Our only way forward is to convert to base-ten, as a novice learning a foreign language might translate into her mother tongue, or else we extend the list above and search for clues. A general theory might follow.

At a later meeting, I encouraged Victoria to construct a multiplication table in base-five, to help her develop a form of number sense, at least in terms of a faster lookup process. This begs the question, what is intended by “number sense”? It is one thing to know numbers; another to know *about* them, to penetrate their more subtle interrelationships.

To close this consideration, let us revisit the example of 423_5. There will be little lost in skipping to the final paragraph if you do not wish to pursue the mathematical details. We have

$$\begin{aligned} 423_5 &= 4\times 5^2 + 2\times 5^1 + 3\times 5^0 \\ &= 4\times 5^2 + 2\times 5 + 3\times 1 \end{aligned}$$

But if we observe congruences modulo three, we can write this as

$$\begin{aligned} 423_5 &\equiv 4\times 2^2 + 2\times 2 + 3\times 1 \left(\mathrm{mod}\,3\right) \\ &\equiv 4\times 1 + 2\times 2 + 3\times 1 \left(\mathrm{mod}\,3\right) \end{aligned}$$

This is because $5\equiv 2\left(\mathrm{mod}\,3\right)$, and in general, $5^k \equiv 2^k \left(\mathrm{mod}\,3\right)$.

Exploring this last congruence through the agency of the binomial theorem, we have

$$\begin{aligned} 5^k &\equiv 2^k \\ &= \left(3-1\right)^k \\ &= 3^k + k3^{k-1} - \binom{k}{2}3^{k-1} + \ldots + 3k\left(-1\right)^{k-1} + \left(-1\right)^k \end{aligned}$$

The last term in the binomial expansion determines the outcome. If k is even then 5^k is congruent to 1(mod3); if k is odd then 5^k is congruent to -1(mod3). This is the key observation: that the alternating powers of five are congruent successively to ± 1, modulo three. In the base-ten case that Victoria was familiar with, *all* powers of ten are congruent to one, modulo three. It is the relationship between the divisor and the base that is crucial to an understanding of a divisibility test.

Thus, $423_5 \equiv 4-2+3\left(\mathrm{mod}\,3\right) \equiv 5\left(\mathrm{mod}\,3\right) \equiv 2\left(\mathrm{mod}\,3\right)$. Since divisibility by three would require congruence with zero (mod3), we conclude that three does not divide 423_5.

In general, we can apply an alternating sum to determine if a number written in base-five is divisible by three. For example,

$$\begin{aligned} 43231_5 &\equiv 4-3+2-3+1 \pmod{3} \\ &\equiv 1 \pmod{3} \\ &\not\equiv 0 \pmod{3} \end{aligned}$$

so three does not divide 43231_5; but

$$\begin{aligned} 33124124_5 &\equiv -3+3-1+2-4+1-2+4 \pmod{3} \\ &\equiv 0 \pmod{3} \end{aligned}$$

so three does divide 33124124_5.

Such deliberations are abstruse, one might say. What practical value can there be in knowing a divisibility test for three in base-five? But that is to miss the point of such exercises. The question Victoria struggled with exposed her to a larger world than she had hitherto encountered. For a moment she was afforded an opportunity to broaden her hermeneutic horizon, her autopoietic domain, to expand what the world of numbers meant to her and what she could subsequently do. For as Gadamer (2004) tells us, “In the process of understanding, a real fusing of horizons occurs—which means that as the historical horizon is projected, it is simultaneously superseded” (p. 306). Gian-Carlo Rota (1997) makes a similar point about the significance of Andrew Wiles’s proof of Fermat’s conjecture: “The actual value of what Wiles and his collaborators did is far greater than the mere proof of a whimsical conjecture. The point of the proof of Fermat’s last theorem is to open up new possibilities for mathematics” (p. 190).

And so we can extract a principle from the exercise: that knowledge is contingent upon our condition and that an orientation to examine that condition is the key to understanding. To give the contrapositive of Socrates’ famous declaration, the examined life, is, indeed, a life worth living (Plato, 2002, p. 41, *Apology*, 38a).

Waiting for the Trick: Orienting to the Mathematical Drama

It was the last period of the day, Year Ten Mathematics. Anna was a little confused about the expansion $(x+3)(x-2)-(x+2)(4x-1)$. The difficulty was the subtraction of the second term. She did not seem to accept or understand that $-\left(4x^2+7x-2\right)=-4x^2-7x+2$.

I asked her to consider $10-6$. "That's four, right?"

"Yeah…," she said, awaiting developments.

"Well, if I write 6 as $4+2$, we have $10-(4+2)$. Now, it's still going to equal four, isn't it?"

Anna paused. She was clearly concerned and not prepared to go along with this obvious statement. It was obvious, wasn't it?

"Ye-e-s…" she offered, but her tone was clear.

"You're not happy with that, are you? It's all over your face," I said.

"Well, I'm waiting for the trick."

Anna and Victoria had very different expectations of the mathematical drama. Where Victoria was able to suspend uncertainty and trust that meaningful order would emerge, Anna was inclined to subjugate her natural sense—her number sense—to her expectations of the teacher-student drama.

On another day I presented my class of Year Nines with the Burning Rope problem: given two, not necessarily identical ropes, each of which would burn for precisely 30 minutes when lit, but not necessarily at a constant rate, use the ropes to mark off exactly 45 minutes in time.

The majority of the girls made the usual inquiries into the nature of the problem: "Why not divide the rope up into equal pieces?"

“Because the ropes don’t burn at a constant rate.”

“Just cut one in half.”

“But the ropes don’t burn that way. Half a rope won’t necessarily burn in fifteen minutes,’ and so on.

Emily, however, was different. She could not understand that there was a problem here at all. “But why not use a watch!” she insisted, over and over.

Emily could count, add, multiply and do all of the routine things that a moderately capable Year Nine student might be expected to do, but she could not extend her intellectual reach into the hypothetical game playing that is a part of the mathematical process.

When presented with a student’s solution, Emily could see that the calculations worked, but she did not understand why you would not use your watch to solve the problem. The hypothetical mathematical problem was simply not a feature in her world.

SAYING A WORLD: LANGUAGE AND NUMBER

> All peoples are at times cat in water with this language
> but it does promote international bird on shoulder.
> This foretaste now lays its knife and fork parallel.
>
> Les Murray (2002), *The New Hieroglyphics*

The earlier example of the Amazonian Pirahã reveals an inclination to the view that language can be used to refine and render precision to an innate sense of number. That is, that language can be used to help us *articulate* number and *communicate* number concepts, but numbers as concepts are not sourced in language. That they are, rather, explicated and expounded using language, but originate in our innate capacity to recognise, even inexactly, properties of small collections. This is not an uncontentious view. Some researchers would give language a greater role in the development of number, beyond its obvious significance in the communication and elaboration of number concepts (see Laurence & Margolis (2005) and also Gelman & Butterworth (2005), for example, for a discussion on this theme). It is not just a case of "chicken and egg"—which comes first?—but a question of whether the precise determination of positive integers is or is not achieved through the operation of language upon an innate sense of approximate numerosity; and further, whether the innate sense is of approximate integers or of an underlying continuous (real) variable, more akin to length or duration than number, which in a somewhat noisy process of being communicated to memory becomes reduced, or compressed, to a positive integer.

For Heike Wiese (2007), the use of number words as tools to identify positions in sequences affords a means of understanding how numbers and language could be related. Language, in her suggestion, does not merely give us words for numbers, so much as numbers themselves; and in this way the human capacity to link properties of relationships in language transfers directly to the number words *as numbers*:

> If ... we identify counting words as verbal numerical tools, this then means that language gives us *instances* of numbers, words that we can employ as number, rather than just as names that we employ to denote numbers and to reason about them. (p. 762)

This is not to deny an evolutionarily older basis to our number sense (see, for example Hubbard et al., 2008), but rather to cast "number" carefully as a tool wielded linguistically. The capacities to discriminate small sets, to subitize, to recognise serial order and even to distinguish objects, says Wiese, describe

> *prenumerical* concepts of empirical properties, rather than numerical capacities: they provide the cognitive underpinnings to grasp the relevant empirical properties in number assignments—set size (cardinal), sequential rank (ordinal), and identity (nominal)—but they do not support numbers themselves, the tools we use to assess

> these properties in a pattern of dependent linking between two relational structures. (p. 763, author's italics)

Brian Butterworth and Robert Reeve (Butterworth et al., 2008) make the further observation that counting words are not necessarily the only system for exploiting and refining a number sense. Their studies among Australian indigenous groups suggest that in the absence of number words an alternative mechanism is employed: that of spatial representation (pattern matching). On comparing the performance of three groups of children—one Warlpiri speaking, one Anindilyakwa speaking and one English speaking—on tasks designed to test for numerosity understanding, they found no language effects. This is significant because Warlpiri has only three generic types of number words: singular, dual plural and greater than dual plural; and Anindilyakwa has only four possible number categories: singular, dual, trial and plural. The children speaking these languages were shown to have the same numerical understandings as the English speaking children. Butterworth and Reeve conclude that "the development of enumeration concepts does not depend on possession of a number-word vocabulary ... and that using words to name exact numerosities is useful but not necessary" (Butterworth et al., 2008, p. 13182).

The idea of number as a pragmatic technology is further encouraged by the observation that within some cultures disparate collections with the same cardinalities are described using different words. Thus five flat stones might be described with one word, but five smooth stones with another, and five people with a different word yet (Barrow, 1992). This circumstance can, by contradistinction, alert us to the strong tendency within those of us immersed in modern culture to abstract general properties. While we might see "five" as descriptive of a property of both collections of stones, it is questionable whether this is an obvious thing to do. In our situation, it is a commonplace to recognise "five" because, after all, we are steeped in number-think. Were we not so groomed, however, we might identify some other feature of the stones as salient. This recalls to mind Walkerdine's (1994) observation, reported earlier, that our interest in abstraction betrays a forgetting of sociological structures.

Neither ought we presume that such object specific counting is necessarily indicative of an earlier stage of cultural evolution. Beller and Bender (2008), in what we might read as a precautionary tale, describe the evolution of number words among Melanesian and Polynesian people. They use linguistic studies to indicate that some object specific number sequences in current usage originated in more abstract counting sequences, but that the current, reduced systems afford a cognitive advantage over the older more expansive system when making calculations without notation. This signals that any predisposition to posit evolution as a march toward increased complexity risks misunderstanding an essential feature of Darwinian evolution: it is not a teleological drive to "improvement", nor to increased complexity, but a process whereby energy expenditure is corralled by the forces of survival and reproduction. Viewed thus, one can understand that features of a language—including number words—that do not contribute sufficiently to the needs of the community, risk becoming lost. The failure of a group to retain or use extensive number counting words should not be taken as an indicator of numerical ability, so much as a sign of the way in which our deployment of numbers as tools is linked to the question of what it is we seek to build. John Barrow summarises thus:

> Whilst not every society could count, they could all speak. Language predates the origin of counting and numeracy. Thus, any natural propensity the brain might possess for particular patterns of thought or analysis are likely to have evolved with greater bias towards effective general linguistic or gesticular communication rather than those features which focus upon counting practices. Counting appears to have evolved out of the general desire for symbolic representation which language first meets. (Barrow, 1992, p. 102)

SEEING A WORLD: INTENDING AND KNOWING

> The other diners didn't take much notice, even when Death leaned back and lit a rather fine pipe. Someone with smoke curling out of their eye sockets takes some ignoring, but everyone managed it.
>
> 'Is it magic?' said Mort.
>
> WHAT DO YOU THINK? said Death. AM I REALLY HERE, BOY?
>
> 'Yes,' said Mort slowly. 'I ... I've watched people. They look at you but they don't see you, I think. You do something to their minds.'
>
> Death shook his head.
>
> THEY DO IT ALL THEMSELVES, he said. THERE'S NO MAGIC. PEOPLE CAN'T SEE ME, THEY SIMPLY WON'T ALLOW THEMSELVES TO DO IT. UNTIL IT'S TIME, OF COURSE. WIZARDS CAN SEE ME, AND CATS. BUT YOUR AVERAGE HUMAN ... NO, NEVER. He blew a smoke ring at the sky, and added, STRANGE BUT TRUE.
>
> Mort watched the smoke ring wobble into the sky and drift away towards the river.
>
> '*I* can see you,' he said.
>
> THAT'S DIFFERENT.
>
> Terry Pratchett (1987, pp. 33–34), *Mort*

I once presented my daughters with a pair of mandarins and asked what it was they saw. They gave me several words (after the inevitable, "Mandarins!"): orange, fruit, dimpled, yummy, round. A word they did not give me, interestingly enough, was "two". We can be confident that they saw two mandarins, but they did not immediately identify the cardinal "two" as an adjective. Had they been presented with a collection of five mandarins there might have been more incentive to notice and name the number. That is, the size of a collection might influence the salience of the number-feature. It is likely that an optimal range of sizes is required to trigger attention to the cardinality property of a collection. When faced with a single object, there is little or no incentive to identify the solitary nature of its being as a salient feature; a pair of objects draws attention each to the other, but not in the sense that one is inclined to notice the number. The relationship is more akin to "I-thou", or "this-that" than "two". Indeed, the presence of two signifies the coming into being of a *single* relationship, in which sense "two" determines "one".

Were there *many* objects in a collection, too many to enumerate, as are the leaves on a tree for example, then we might see *one collection*. That is, we might see a leafy tree. "Many" can thus be attended to as one; but even if the *leaves as such* are of interest, they may be so plentiful as to defy characterisation by number. Collective nouns can be employed to manage the perceptual difficulty—we see a gaggle of geese and a flock of seagulls. I am sufficiently familiar with my hens, on the other hand, to say I have eleven chickens, although the enumeration "eleven" is never obvious as they fossick about.

There might be an additional language correlate here in the way we categorise persons in speech. We speak of I, you and he depending on the signification, but we do not employ a fourth or higher person. To say "I wondered what you heard her say to him" characterises the "her" in relation to me as the third person, but does not allow the "him" to be the fourth. There seems insufficient need in this well-evolved, mongrel language that is English to make the distinction between degrees of separation from oneself beyond the third person. This might be taken as a clue to the way we think, in the same way that the characterisation of groups by collective nouns when the members are sufficiently lacking in identifiable points of interest suggests something of the way in which we relate cardinality to relationships.

Investigations by Kinderman, Dunbar and Bentall (1998), corroborated by Stiller and Dunbar (2007), indicate that there appears to be an upper limit of about four or five to the levels of intentionality that we can ascribe to others: "He said that she felt that he thought she wanted to ...." More recently, Lewis, Rezaie, Brown, Roberts and Dunbar (2011) have demonstrated that mentalizing competencies such as the understanding of others' intentions, as well as the number of social relationships a person can maintain simultaneously, are each functions of grey matter volume in brain regions associated with conventional Theory of Mind (that theory which describes our abilities to ascribe intentions to others by treating them as if they have minds, as we ourselves seem to). Our language and social behaviour, that is to suggest, tracks our evolved condition.

Intentions, too, influence what we notice. Had my daughters been engaged in a game that required them to find a collection of two, they would certainly have seen “Two mandarins!”; had they been searching for shapes they would have seen “Round mandarins!” and had they been seeking examples of food types, then “Fruit!” is what they would have seen. Our intentions shape our perception; that which we notice is less than we see.

What we perceive, moreover, is not necessarily a veridical indication of what is presented to our eyes. We are familiar with simple visual illusions, for example, which suggest that our perception is always an interpretation. The way we interpret colour is also an example worth considering. When we see a rainbow we witness a spectrum of radiant energy. The spectrum is continuous, but we interpret it as banded. This is largely a function of the biology of our eyes in the first place, and of our neurology in the second. The rainbow, as we understand it, is an interpretation of reality.

We have previously noted Shaun Gallagher’s (2005) observation that our intentions influence not just our interpretation of the world but even the physiological responses of our bodies; that we tune ourselves into our intended perceptions: “My body *meets* stimulation and organizes it within the framework of my own pragmatic schemata” (Gallagher, 2005, p. 142). Thus we might hear expressed the view that different language speakers from around the world do not “see” the world in the same way; that to some extent, we speak our world into being. This idea represents the understanding that our capacity to use language is subordinate to the way in which we are primed by our previous experiences to apprehend.

We have heard the colloquial criticism that we sometimes “speak without thinking”, but we never make that assertion without expecting that the thing spoken *could not* be thought. Add to this physiological basis the foregoing observation that our higher-order intentions contribute to the way in which our bodies produce what we *experience*, and we begin to construct a cycle out of which language growth will be amplified and constrained by pragmatic, local interests and concerns. A nascent culture will have its own interests and having these interests will fuel what it is likely to conceive, will determine the structures that attain relative stability. This is why interactions with new cultures can lead to outbursts of creative invigoration.

So it is with number. It appears that our earliest sense of number is derivative of innate brain structures and capacities, realised through interaction with others and with the environment. Number is a hallmark of our involvement with each other and with our environment. We can “know” something of number independently of language, whether discrete or continuous, exact or fuzzy; but language becomes central to the ongoing learning and communication of higher-order numerical concepts, so that while it is probably the case that all speakers have the capacity to experience number in more or less the same way, the cultural interests of their particular communities direct the elaboration of number concepts and contribute to a determination of how any given individual will experience number in their lifetime. Gadamer (2004) tells us that “language is not just one of man’s possessions in the world; rather, on it depends the fact that man has a *world* at all” (p. 440, author’s italics). While we recognise the necessity of the spoken word in generating the *umwelt*, we have here been acknowledging the prior condition of evolved man as always engaged, perennially vexed and pragmatically concerned with the attractions and dangers of a very real world.

We are open, then, to the possibility that when Alison—introduced earlier—interpreted a correlation coefficient, she was merely repeating words and not showing evidence of having learnt any mathematical concept. On the other hand, we might expect that there will be an interplay, a hermeneutic circularity between spoken-word and felt-thought, that she will need language to foster the growth of an embryonic mathematical concept. Any expectation that she could conceive of such a high-order concept as correlation without language is surely misguided; but there remains the reasonable fear that she could apply the language without proper knowledge of correlation, à la the Chinese Room Experiment; and yet what alternatives are there for the assessing teacher? Alison will determine for herself when the language becomes truly hers and not a borrowing from her teacher.

Neither need we adduce the onset of knowledge from Alison's own determination, however, but rather allow that she might *know* before she becomes aware of it in herself. That is, we allow that the concept might become significant to her before she fully realises it as such. In any case, on what basis is Alison to make any such determination? The problem of assessment is not solved in this way, but merely shifted from the teacher to the student.

This difficulty foregrounds the problem of knowledge, but frames it less in terms of the *res cogitans* of Descartes than of the body scheme. We might be tempted to seek out the moment when one comes into possession of knowledge, but that search suffers a dualistic faith in knowledge as object, in concepts as fixed; a faith in facts distinguished from all appurtenances. We might, alternatively, imagine knowledge rather as a degree or type of engagement with the objects of the world: the *quality* of the engagement and the orientation of the student would become significant elements in assessment under this interpretation. Realigning metaphors of knowledge toward interest and away from possession ameliorates a child's assessment of her own abilities. It is a commonplace for the teacher to witness a child factorise a mathematical expression by employing the requisite techniques, for example, only to then hear her say, in concert with Felicia, "I can do it, but I don't understand it." What can be lost in such a circumstance is the possibility that the child's unease derives less from the mathematical processes in rehearsal than from the expectation that understanding is a thing which one can hold—is, indeed, *required* to have hold. Since she has yet to feel her mental fingers wrap around any particular revelation, she believes she does not yet have understanding. The language of grasping and possessing bespeaks a forceful attitude to knowledge; a gentler metaphor of interest and emergent sensitivity to a discipline's characteristic modes might alleviate such unease and permit an altogether more pleasant educational experience. Michael Polanyi (1966) has told us, "We can know more than we can tell" (p. 4). We might pause to consider the wisdom in this and ask whether there could be merit in concerning ourselves less with what our students can tell, and attending more nearly to how our students experience education.

UNDERSTANDING A WORLD: BEING THERE

> Until you understand a writer's ignorance, presume yourself
> ignorant of his understanding.
>
> Samuel Taylor Coleridge (1907, p. 160), *Biographia Literaria*.

Coleridge (1907) was well aware of the difference between understanding the ignorance of one and failing to understand the wisdom of another. Of the former, referring to a literary example, he states, "I see clearly the writer's grounds, and their hollowness" (p. 160). Of the latter (Plato's *Timaeus*) he writes, "Whatever I comprehend impresses me with a reverential sense of the author's genius: but there is a considerable portion of the work to which I can attach no consistent meaning" (p. 161). He adds, "I have no insight into the possibility of a man so eminently wise, using words with such half-meanings to himself as must, perforce, pass into no-meaning to his readers" (p. 161). It is this failure to understand Plato's inability to communicate to him, even in the face of contemporary scholarly adulation for the work, that compels Coleridge, in honour of his own maxim, to declare, "Therefore, utterly baffled in all my attempts to understand the ignorance of Plato, *I conclude myself ignorant of his understanding*" (p. 161, author's italics).

We, as teachers, might learn from Coleridge. Until we understand the ignorance of a child, the humbler path might be to declare ourselves ignorant of her understanding.

It was the first lesson of the year with my senior Mathematics Methods class. I introduced the idea of combinations and permutations to the girls. We discussed the two concepts, considered examples, used convenient props and so on. I distributed a paper I had written that more or less recapitulated the instruction. We made a connection to the binomial theorem, but I left it largely to the girls to read my writing on the topic so that they could have a break from the pattern of the lesson and from my voice. I moved around the room, checked progress and elicited details that were worth sharing—summation notation, for example. After a time we emerged with the practical aspect of the binomial theorem: using it to form expansions and to find particular terms in power series.

Toward the end of the lesson, Astrid called me over and said, "Does it matter if I don't get all this stuff? I can do these." The "stuff" to which she referred, waving her hand over the text, was the statement and general explanation of the binomial theorem itself, including examples, written using the standard algebraic notation proper to it. What she *could* do were the initial problems: actual expansions. That is, she claimed she could do the problems but not understand an articulation of the theory.

I suggested to her that "understanding" was perhaps less mysterious than we might believe, and that her ability to do the initial problems might be seen to constitute her initial understanding. Her challenge would be to do the problems, then return to the theory and see it in relation to what she had done; then to proceed to more problems, check the theory, and so on. By interlacing theory and problems Astrid stood a chance to gain technique and familiarity, and would that not be a form of understanding?

I have already noted Gadamer's (2004) hermeneutic rule, by which we "understand the whole in terms of the detail and the detail in terms of the whole" (p. 291), which relationship, he observes, is circular. There is also a sense in which the search for understanding can be understood as autopoietic. Each reading of a text contributes to and informs an emerging pattern of stability that, while never complete, determines its own boundary as a frame of understanding:

> The anticipation of meaning in which the whole is envisaged becomes actual understanding when the parts that are determined by the whole themselves also determine this whole. (Gadamer, 2004, p. 291)

When understanding is read as an end in itself, as a point of arrival, the process of learning becomes linear. Astrid's expectation that she might "get ... this stuff," signifies that for her, at least, and in my experience, for many students, understanding is something that you can grab and hold, that it is a fixed thing you can have. Since understanding is desirable, the corollary is that not having it counts as a deficit, a failing. Unless and until you feel the veritable penny drop, you are in a state of incompleteness. For every child thus disposed, the humble question, "Do you understand?" is a form of violence.

For all that, understanding is the currency of the school. On perusing school reports written to parents I have found that understanding is a central concern of the teacher—or, at least, teachers anticipate that understanding is of prime interest to students and parents. What precisely requires understanding varies among reports: *techniques*, such as how to write an essay or factorise a quadratic; *values*, from diligence to respect; *ideas*, and how we recognise and order them ; *consequences*, as when work is not completed or attention not given to the teacher. In short, if it is in the curriculum, even implicitly, reports suggest it appears susceptible to understanding. Is this reasonable? Does one *understand* that five times two is ten, or does one *know* it as fact? Sierpinska (1994) distinguishes an *act* of understanding, "an experience that occurs at some time and is quickly over," from *an* understanding, "which is a potential to experience an act of understanding when necessary" (p. 2). Are there distinctions to be drawn between knowing, doing and understanding?

Once, when my daughter was young, we practised memorising items. She had not yet learned her five or eight times tables, but she was able to calculate, by halving ten eights, which she knew, that five eights were forty. This had cropped up incidentally, in a natural context, and, having gone to the trouble of working it out, I suggested to her that she might like to remember it, that it was something worth knowing. So we looked briefly at how to "get things into memory" by comparing "short and long" forms of memories. We repeated the statements "five eights are forty" and "eight fives are forty" a few times, along with several other statements for practice. Later in the day we returned to them and made a few more repetitions. Next day, another repetition. A day later, a check, and yes, the memory was still there. So, it would seem that she now knew that five eights are forty and that eight fives are forty, but did she *understand* what is entailed by the statements? Remembering how she had worked the original problem out, it can hardly be said that her knowledge was naïve parroting of senseless words. Yet when I turned the statement around and asked, "How many fives are in forty?" she halted, thought a while, then made a false suggestion before resorting to counting in fives and recording the count with her fingers until she could correctly state that there were eight fives in forty. Even at this point there was not a recognition that the statement "eight fives are forty" held the necessary knowledge[29]. I asked how many eights there were in forty. She was unsure and proceeded to try and work it out.

We pursued this exercise, and eventually, by making concrete representations of numbers—handy pens represented fives, for example—I think that we made some progress toward developing a deeper appreciation of the entailments of the statement "five eights are forty," some of which had lagged behind knowing the verbal formulation.

This is typical of scenes that I encounter in the classroom. A child will know the quadratic formula, but not feel it meaningful. Another will solve a simple algebraic equation such as $3x+7=22$, but not relate the answer, $x=5$, to the original equation in a sense-making way. There appear to be shades of grey, here, between knowing as statement-making or action, and the underlying significance of the accomplishments. Meaning, such as might be attached to that which is to be known, appears to be associated with understanding; knowledge of "facts" without appreciation of their significance might be thought to be something less than understanding. We are back in the Chinese Room.

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[29] Although we could credit my daughter with not *guessing* that commutativity applied, it seems likelier that the thought that it might would not have occurred to her!

I was once asked to speak on the difficulty of assessing "deep understanding" in mathematics. Upon reflection, I realised that it was unclear to me what was meant by deep understanding in mathematics, much less how it could be assessed. In using the word "understanding" we confront the I-you dyad at its sharpest. In terms reminiscent of Meno's paradox[30], if I say "you understand" then I at once place myself in a position of possession. I cannot judge that you understand something that I do not, for how, in that case, am I to know whether you actually understand?[31] But then, how can I know that *I* understand? We are drawn toward a consideration of the relationship between the "I" and the thing that is to be understood. This consideration entails a form of distancing, a separation of the two, for they are distinguished by their contrast against each other. A prior difficulty is, then, not assessing *your* understanding, but contemplating *my* understanding and my relationship with that which is claimed to be understood. This, before we approach the question of inquiring into your knowledge.

Ricoeur (1981) tells us that, "the question of understanding is wholly severed from the problem of communicating with others" (p. 55):

> The foundations of the ontological problem are sought in the relation of being with the world and not in the relation with another. Understanding, in its primordial

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[30] Meno says to Socrates, "How will you look for it, Socrates, when you do not know at all what it is? How will you aim to search for something you do not know at all? If you should meet with it, how will you know that this is the thing that you did not know?" (Plato, 2002, p. 70, *Meno*, 80d)

[31] I can form the *belief* that you know, but then I am relying on some other, less proximate criterion, such as authority. This is a commonplace practice, but we are attempting to broach the question of assessment, not of warranted belief, as such.

> sense, is implicated in the relation with my situation, in the fundamental understanding of my position within being. (p. 55)

The experience of understanding, then, is supremely located in a relation within the situation in which learning is taking place. The moment at which the student feels that the thing is learned is the moment at which the thing reveals itself, as it were, not at which it was revealed or communicated by another. Attention shifts from dynamic transitions to a stable configuration. This is the “I get it!” moment, when the “penny drops” or the light comes on; the thing is there, where just a moment before all was obscure; or else it is the forgetting of wanting to know, upon which curtains are drawn back on a delitescent capability.

Even Coleridge, for whom there remained a necessary distinction between the subjective (self) and objective (nature), recognised understanding made apparent in the fusion of the two:

> During the act of knowledge itself, the objective and subjective are so instantly united, that we cannot determine to which of the two the priority belongs. There is no first and no second; both are coinstantaneous and one (Coleridge, 1907, p. 174).

Understanding thus unfolds as an awareness of entailments of something known and in the commitment to explore even more entailments. Viewed thus, understanding is never an attainment so much as a disposition to inquire further; an orientation rather than an acquisition. Understanding inheres less in the grasp than in the desire to extend one’s reach.

In this way we distinguish knowing from understanding. For one may acquire knowledge of things, one may gather together and record in memory or text facts about this or that which are susceptible to regathering, to thinking with; but to understand is to turn to see those things under new lights, to reorient oneself to new perspectives, to consider unfathomed consequences. So it is that we can, in all sincerity, bemuse the unappreciative with assertions that we do not understand what appear to be such simple things.

CHAPTER FIVE METAPHORICAL MATHEMATICS

Attention now shifts to a particular aspect of embodied cognition: metaphors and metaphorical entailment. I primarily focus on the embodied cognition framework of George Lakoff and Rafael Núñez (2000) in order to illustrate working with an embodied epistemology. I continue to maintain focus on teaching and learning, and draw in experiences of wrestling with mathematical thought from expert mathematicians, to which I add a personal reflection.

I begin with a classroom experience in which I reflect upon the entanglement of pedagogical presentation, affective responses and orientations.

TEACHING PROOFS: A PEDAGOGICAL PLIGHT

Understanding can be viewed as an autopoietic orientation to turn and return and so determine its own boundaries. A single "fact" can be seen in various ways and in each way be re-imagined, broadening its potential to elicit meaning. In the following passage I describe how one idea was variously presented. The intention at the time was not so much to develop this thesis, but to shift attention away from the result itself and toward proof as a process.

My class of Year Nine students had encountered the proposition

$$1+2+3+...+n=\frac{n(n+1)}{2}$$

in the context of solving a standard problem:

> There are ten people in a room. They each shake hands with one another, once only. How many handshakes will there be?
>
> Generalise to *n* people in the room.

The solution to the first part of the problem was found to be

$$1+2+3+...+10=55,$$

and, with inductive faith, it was clear enough to the girls that the answer to the second part must be

$$1+2+3+...+n.$$

I prompted the class, however, to consider that the answer, while correct, might turn out to be difficult to work with. What if there were a thousand people in the room? How would we calculate $1+2+3+...+1000$?

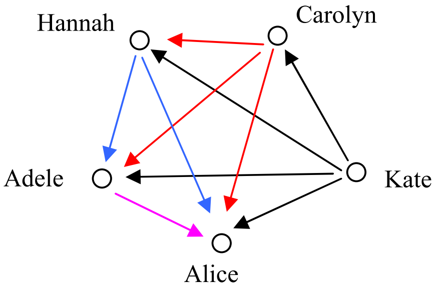
At this point we can recognise that there is a sense of a mathematical—or school—drama in play. Do they really believe that this problem exists to solve the difficulty of working out how many handshakes will eventuate if 1000 people gather at a venue? I guess not. It is likely that the girls realise that this problem is representative of something else, that while it is presented as a problem, it is more than likely given with other intentions. Their role in the game is to play the part of the student, to follow instructions and to try to accomplish the task set. The problem, that is to say, is contextualised in a greater whole, and if none the worse for that, it might be well to reflect that it is *not* simply a problem in mathematics.

So, how do we address the difficulty of arising at an efficient means of calculating the sum $1+2+3+...+n$? Note that now we have a new problem. No longer are we striving to count handshakes, but we are working with a hypothetical property of number. We have shifted attention, as Mason (1989) would say. We do not even know if there exists a solution to the problem. That is to say, in terms of the rules of the game, the girls probably believe that, since the teacher has asked them to consider finding an efficient means of calculation, there most likely *is* such a means, but there *might* not be one, not because numbers are like that, but because teachers can be devious!

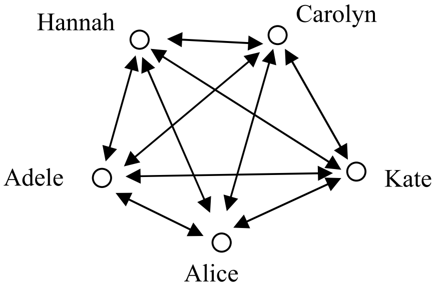
Even though the game has shifted, we can still draw on the given context of handshakes to explore the problem. We focus on the technique of modelling or acting out the scenario. Five girls come forward and prepare to shake hands. I ask

Kate to shake hands with her classmates while those seated count: “1, 2, 3, 4,” is recorded. Now that Kate has greeted her four friends she sits down; no more handshakes for her. Carolyn takes her turn: “1, 2, 3,” is the count. She sits down, having now shaken hands with all in the group. Hannah: “1, 2,” and down she sits. Two girls remain standing. Adele takes Alice by the hand and sits down. “1,” was the count. By now, Alice has also shaken hands with each of the other girls, so she, too, sits down.

Thus it is that our five volunteers have demonstrated the total count $1+2+3+4$. There were, apparently, ten handshakes. Ten, however, is not the focus of our attention. We focus, rather, on the structure before us. We have seen a representation of $1+2+3+4$. A diagram captures the essence of the model:



This is no closer to our goal of efficiently counting $1+2+3+4$, but here enters the shift of perspective that suggests a method. I ask the five girls to reassemble and shake hands again. This time, however, each will remain standing and shake each of the other four girls by the hand:



It became clear that each and every pair enjoyed two handshakes. Since five girls each shook hands with four others, that suggested $4 \times 5 = 20$ handshakes. Since we double-counted, however, the number of distinct handshakes was $\frac{4 \times 5}{2} = \frac{20}{2}$, in agreement with our previous result.

We had seen two ways of describing the one situation, one leading to $1+2+3+4$, the other to $\frac{4 \times 5}{2}$. Our conclusion was that

$$1+2+3+4=\frac{4 \times 5}{2}.$$

The implicit beliefs were not analysed at this point: that the number of handshakes must be constant under time changes; that assigning numbers to handshakes is a meaningful activity, and so on. We were not working at that level of description. There is, however, some interest in the observation that the girls were ready and happy to move from

$$1+2+3+4=\frac{4 \times 5}{2}$$

to

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$

That is, from a demonstration of the particular to a belief in the general.

Note that this is not a result about infinite sets, although it might be argued that it might as well be, for the girls had already discussed ideas such as “there is always a next number” so that the result has all the effect of being unbounded by restrictions on the size of *n*.

One of the reasons that I had the girls undertake this activity was that I wanted them to see the sometimes powerful effect of reconceptualising a situation from an alternative perspective. The result, while valuable in and of itself and certain to arise within many contexts to come, was not the point of the exercise. Had it been, I could have simply delivered it with the hand of authority: “Here is a result that is important!" Illustration through simple examples would likely be convincing enough:

$$1+2=3=\frac{2\times 3}{2}$$
$$1+2+3=6=\frac{3\times 4}{2}$$
$$1+2+3+4=10=\frac{4\times 5}{2}$$

A second reason for spending a little time on the exercise was to generate a powerful recognition of the algebraic structure: less a case of seeing $1+2+3+...+n=\frac{n(n+1)}{2}$, than of seeing, "The sum of consecutive integers is equal to half the product of the last with the next." This is by way of acknowledging that there is an issue concerning how we understand symbols in mathematical contexts. Does the benefit of algebraic concision price itself too steeply for the student to buy it?

I say "seeing" in the above, but perhaps a more accurate term would be "feeling". The kinaesthetic engagement—shaking hands, standing, sitting, pointing and so on—forces the locomotive and emotional brain centres to become involved in this derivation. The algebraic statement and even the text might now serve to label the memory of the event, to be associated with it, such that when next encountering a sum such as $1+2+3+...+670$, for example, the visual percept might be strongly associated with a memory of moving and being in a certain place, so that the result, $\frac{670\times 671}{2}$ can be recalled, or configured. This is the perceptuo-motor approach of Azarello, Robutti and Bazzini (2005).

This understanding of how memory works and of how abstract ideas are managed in thought has the quality of according with my personal experience. I know that I retain strong memories of what I have written on the classroom whiteboard for my students, for example, not so much because I remember *what* I wrote, as because I recall *where* I wrote it on the board. There is a spatial memory in the first instance, and the visual memory—the image of what was written, in all its erratic, coloured cumbrousness—seems to follow. The mathematical meanings of what is written is difficult to relate to my memories, because they are usually well enough known so that they seem ever present. What is generally recalled, however, are features of what was happening in the classroom—from my perspective—at the time of writing. I recall how I felt as I was writing, whether it seemed to me that the students were following well or not, if I felt I was communicating. Generally, the more unusual a response, the more likely I am to remember my emotional response to it. Kensinger, Corkin and Raichle (2004) report that emotional memory enhancement for arousing information depends on an amygdalar-hippocampal network whereas non-arousing information is supported by a prefrontal cortex-hippocampal network; and further, that secondary tasks reduce the effectiveness of the latter controlled processes whereas arousing words are more readily remembered. My experiences in the classroom, then, are suggestive of the idea that different neural networks are implicated in the various aspects of what I recall. Emotionally arousing information is recalled more readily than composed, controlled information.

For a teacher of mathematics, a discipline that specialises in information that seems to require significant amounts of encoding, key questions emerge. How do we associate arousal with encoding information? That is, are we able to capitalise on the more effective amygdalar-hippocampal network in our teaching so that mathematical ideas can be more successfully remembered? Is it even possible to co-opt this network for mathematical ideas, or are mathematical ideas by their very nature destined to be managed elsewhere in the brain? If so, can *recall* of mathematics benefit through associations with more arousing information? What if the arousing information is such that it promotes an *anxious* response? Does that impede the encoding processes of the prefrontal cortex-hippocampal network?

There is evidence to support the expectation that anxiety is negatively correlated with mathematical achievement, but it requires care to tease out the underlying mechanisms. Ho et al. (2000), for example, indicate in their study that the affective component of mathematics anxiety rather than the cognitive component reduces mathematical performance (p. 375). Ma and Xu (2004) report their findings that among junior and senior high school students a deficit model, in which low mathematics performance causes anxiety, is more strongly supported by data than an interferences model, in which high anxiety causes low achievement, or a reciprocal model, in which mathematics anxiety and mathematics achievement are reciprocally related. They also identify a small gender difference:

> We suggest that male mathematics anxiety comes from consistent poor mathematics performance in the past, whereas female mathematics anxiety becomes sensitive to poor mathematics performance only when girls are in critical transition periods. Once girls get used to the new phase of learning, their mathematics anxiety becomes less dependent on mathematics performance. But, how can one explain this fact that female mathematics anxiety is not much [*sic*] a result of their mathematics performance? We found that mathematics anxiety was more reliably stable from year to year among girls than among boys. Therefore, we consider it reasonable to suggest that female mathematics anxiety is largely a 'lingering effect' (prolonged previous mathematics anxiety). (p. 176)

Some researchers focus on self-efficacy beliefs (Hoffman, 2010; Hoffman & Spatariu, 2008; Pajares, 1996; Pajares & Kranzler, 1995); others on working memory (eg Adams, 1997; Barrouillet & Lépine, 2005; Camos, 2008; Rosen & Engle, 1998) and processing speed (Bull & Johnston, 1997) in order to explore factors affecting mathematical achievement. Miller and Bichsel (2004) report the intriguing finding that mathematics anxiety is not quite like other forms of anxiety, and that it is primarily related to visual working memory:

> Math anxiety was found to be the most important predictor of math performance, followed by both verbal and visual working memory. Math anxiety does not function entirely like other types of anxiety, at least as related to math performance. The finding that math anxiety is primarily associated with visual working memory, rather than verbal working memory, contradicts previous results in which more limited math abilities were tested. (p. 605)

Even so, we can anticipate that a cascade of interrelated effects will be implicated. It makes me feel bad, so I don't like to do it, so I spend no time on it; I can't remember, I don't want to do it; I do badly in tests, and I don't like tests; and my teacher doesn't like maths either; and so on.

An underlying question is whether it is reasonable to speak of dissociating a mathematical idea from its concomitant cognitive and affective aspects. This question is at the heart of what it means to speak of embodied cognition. To address it is to peer beneath assumptions that shape educational and scientific research programs. It is to allow the teacher to feel his way into moments, informed but open to the experiences within each new situation. It is to permit a re-reading of research such that its daily practice is strengthened through being subjected to a cycle of enquiry. This is in line with Kuhn's (1996) understanding of how normal science works and of what initiates paradigm change. It accords with what Ginev (1999) refers to as the hermeneutic fore-structure of scientific research:

> The interrelatedness of … these discursive practices informs the integrity of the scientific community's research everydayness. There are common meanings, implicit norms, ways of intersubjective experiencing, anticipations, inclinations, and orientations which are inextricable from this self-constituting totality of discursive practices. Looking at this everydayness, one can recognize within the interwovenness of practices a fore-having, a fore-sight, and a fore-conception of doing research. (p. 147)

To an extent, a teaching career can be seen in this light. The daily discursive practices of the teacher can be reinvigorated through judicious reflection and study, such that there is a shifting horizon of phenomenological awareness. Practices are able to be reinterpreted, reappraised. The goal is not to improve practice, so much as to eliminate it; to bring into one's self and reflect outwardly a being-there-ness, in which the authority of the teacher is a recognition of authenticity (Heidegger, 1996).

There was a third imperative urging me to spend time on the counting exercise. It concerns the question of what it means to prove something. Here we enter the realm of belief, mathematical meaning and social communication, not to understate rigour and recourse. What is it to be convincing? Is that ultimately what we mean by proof, or do we intend something more far-reaching? The Jaffe-Quinn debate referred to earlier (Atiyah et al., 1994; Jaffe & Quinn, 1993, 1994) exemplifies the kind of struggle that can take place within a mathematical community when the subject of proof—such a fundamental aspect of mathematics—arises.

This simple result of our lesson, $1+2+3+...+n=\frac{n(n+1)}{2}$, is subject to various presentations, and I was interested to see how the girls would react to seeing several alternatives. I had hoped that by presenting one result several times over, the focus would shift from the result itself to the idea of demonstrating, convincing, or, indeed, proving[32].

I had the girls write $1+2+3+4+5+6+7+8+9+10$. Below it we wrote the same series, but in reverse order:

$$\begin{array}{ccccccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & 6 & + & 7 & + & 8 & + & 9 & + & 10 \\ \underline{10} & \underline{+} & \underline{9} & \underline{+} & \underline{8} & \underline{+} & \underline{7} & \underline{+} & \underline{6} & \underline{+} & \underline{5} & \underline{+} & \underline{4} & \underline{+} & \underline{3} & \underline{+} & \underline{2} & \underline{+} & \underline{1} \end{array}$$

Now the girls were asked to add the two rows vertically:

$$\begin{array}{ccccccccccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & 6 & + & 7 & + & 8 & + & 9 & + & 10 \\ 10 & + & 9 & + & 8 & + & 7 & + & 6 & + & 5 & + & 4 & + & 3 & + & 2 & + & 1 \\ \hline 11 & + & 11 & + & 11 & + & 11 & + & 11 & + & 11 & + & 11 & + & 11 & + & 11 & + & 11 \end{array}$$

At this point some girls almost laughed out loud in recognition of the result. Alice, on the other hand, frowned with concern. To the girls who apprehended, it was clear enough that we had ten lots of eleven, and that we had double counted, so that the demonstration revealed the result:

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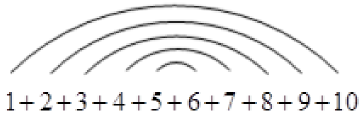
[32] In a different context, as we shall soon see, Mason (1989) identifies the shifts of attention that can occur in mathematics, even as a problem is solved. The teacher who is unaware of such shifts threatens to confuse his pupils.

$$1+2+3+4+5+6+7+8+9+10=\frac{10\times 11}{2}.$$

Several girls had been made ready for this, for they had previously noticed the following structure:

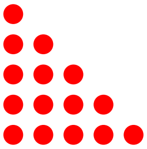


Adding in pairs shows that there is a collection of elevens. All that remains is to satisfy oneself that the number of elevens is given by half of ten. Alice had not seen this, and she looked upon the demonstration with doubt and worry. “I don’t get it,” she said, voicing the refrain of the ages. I had been making efforts to encourage the girls toward a method of reading mathematics, whether problems or explanations, whereby difficulties are acknowledged but held in abeyance pending subsequent reflections. “You will get this, Alice,” I said, confident, because I had already formed the view that she was very capable, if a little disinclined to extend herself. “But give yourself time, let it come to you when it can.”

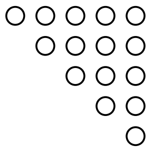
I wanted Alice to be relaxed. In line with the foregoing comments on mathematics anxiety, tension in thought seems similar to tension in muscle: it impedes the smooth flow that is characteristic of an expert in play.

Alice, you will be pleased to hear, was shortly to announce in clear tones, “I get it! Oh, that’s cool!” This, it might be said, while a third version of proof was being prepared. She had persevered, but I do not know how her mind had unravelled the shapes and seen the meaning. I doubt that she herself knew.

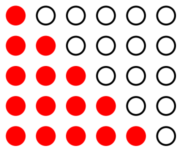
A picture is worth a thousand words, we have heard. Here, then, is a picture:



Not particularly revealing, although it does show $1+2+3+4+5$ in a triangular arrangement, suggesting triangular numbers, and figurate numbers in general. For our current purposes a significant other piece of imagery was required to complete the gestalt. Add a second copy of the same count



in just the right way, and a triangle becomes a rectangle:



How many dots can we see? Twice the sum of the first five numbers, certainly, for that is how the shape was constructed; but viewed as a rectangle, apparently we have 5×6 dots, meaning the original triangular arrangement has $\frac{5\times 6}{2}$ dots. So we are afforded the opportunity to see $1+2+3+4+5=\frac{5\times 6}{2}$.

What was the reaction of the girls to this image? Most saw its implications instantly, although several needed a little more time. I elaborated a little for their benefit, and we had ourselves a third reason to believe our result for the sum of the first *n* positive integers. Would such a visual demonstration prove more powerful than other methods?

The journal *Mathematics Magazine* is a treasure trove of such “proofs without words”, many of which are collated in Nelson (1993). I never tire of the experience of gazing at a series of diagrams, working to uncover their message, to suddenly see! This is surely a different process to decoding and encoding algebraic statements, but limited[33] in its application to situations that can be visualised and for which induction from small cases is valid[34]. Giaquinto (1994), for example, argues that while visual thinking in problems of analysis is useful, where matters of continuity and differentiability are central, then visualizations cannot be relied upon:

> The practical moral is not that we should avoid visualizing in analysis, but that we should try to improve our grasp of the conditions under which visualization is misleading. (p. 812)

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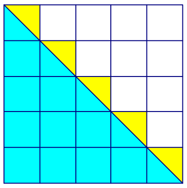
[33] Limited, that is, if one wishes to make the claim within the community of mathematicians that a diagram would constitute a *proof*. By other lights, such diagrams are pedagogically rich. Brown (1997) argues a positive role for pictures in proof that forces us to reflect on what "rigour" actually demands of us. Kajsa Bråting and Johanna Prejlare (2008), meanwhile, urge us to remember that *we* are implicated in any determination of meaning: "The meaning of the visualization is not revealed by the visualization. We should take into consideration that there is an interaction between the visualization and the individual. The visualization does not 'live its own life', so to speak. That is, if a visualization is removed from its mathematical context, it loses its meaning" (p. 357).

[34] David Sherry (2009), Matthew Inglis and Juan Pablo Mejía-Ramos (Inglis & Mejía-Ramos, 2009) and Zenon Kulpa (2009) are among the contributors to a special issue of the *Foundations of Science* journal, entitled "Mathematical Argumentation", in which the question of whether we can be justified in using diagrams to form rigorous mathematical arguments is discussed. Sherry concludes that "an account of diagrammatic reasoning that avoids abstract objects competes well with existing Platonistic accounts" (p. 71); Inglis and Mejía-Ramos report an experiment in which they found that both mathematicians and mathematics undergraduates regarded arguments as more convincing if text describing a diagram—though not creating a deductive argument—accompanied visualizations than if no text was provided, on which they suggest "the psychological study of mathematical behaviour could productively inform current philosophical discussions regarding the role of visualisation in mathematical practices" (p. 108); while Kulpa identifies a history of neglect of diagrams during a period when mathematics underwent a process of formalization that has led to a failure to develop rigorous diagrammatic methods. He develops arguments to resolve problems of generalizing from diagrams.

It is arguable that the gestalt image presented above actually shows that $2(1+2+3+...+n)=n(n+1)$, on which ground we might ask why it is that we present the conclusion as $1+2+3+...+n=\frac{n(n+1)}{2}$? It may be that the latter presents the article of interest as the subject of the equation, whereas the former presents the structure of the vision. Each has its merit, and while not, perhaps, such a difficulty in this rather simple case, doubtless there are circumstances where we fail as teachers to adequately ease the transition between *how we think* about a problem and how we present the conclusion; between the private and what Paul Ernest (2008) calls the "public face" of mathematics.

The following diagram (Richards, 1984) indicates a yet another way of seeing our result. It suggests $1+2+3+4+5=\frac{5^2}{2}+\frac{5}{2}$, and hence $1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$, although a little less transparently than the former cases:



At the risk of overburdening the girls with yet another proof, late one day and with little time to reflect on its merits, I decided to demonstrate a method of differences. I had it in mind that this version carried with it one very particular virtue, insofar as demonstrating a variety of proofs is concerned, of not being a simple, transparent elucidation. Here would be a less than clear demonstration of a now familiar result. The method is generalisable, but requires a little patience before it reveals its peculiar structure. While this confers the risk of it being presented as a "proof by authority", I was hopeful that it would be understandable. It is the case that mathematical techniques are sometimes self-discovered or created, but more frequently they are presented as the outcomes of the labours of others, extending sometimes over centuries. As a former colleague was fond of remarking, a trick, once seen, becomes a technique.

We began with the observation that $n^2-(n-1)^2=2n-1$. This says that the difference between the squares of two consecutive numbers is twice the larger number less one. As it turns out, this was not obvious to some of the girls in the class and so required a little development of its own. Below this equation, we wrote the same statement as it applied to the previous two consecutive squares. Continuing in this fashion we reached the difference between 1^2 and 0^2.

$$\begin{array}{ccccccc}
n^2 & - & (n-1)^2 & = & 2n & - & 1 \\
(n-1)^2 & - & (n-2)^2 & = & 2(n-1) & - & 1 \\
(n-2)^2 & - & (n-3)^2 & = & 2(n-2) & - & 1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
3^2 & - & 2^2 & = & 2\times 3 & - & 1 \\
2^2 & - & 1^2 & = & 2\times 2 & - & 1 \\
1^2 & - & 0^2 & = & 2\times 1 & - & 1
\end{array}$$

Adding the expressions on the left of the equals signs and likewise on the right of the equals signs yielded a further equality: $n^2=2(1+2+3+...+n)-n$. From here it was a comparatively small step to the result that $1+2+3+...+n=\frac{n^2+n}{2}$. For some of the less algebraically experienced, this was not obviously equal to $\frac{n(n+1)}{2}$, but it was soon shown to be equivalent.

This version of proof yielded the standard result, but it seemed to have the scent of a trick, as Anna of our earlier encounter might have anticipated. Why was $n^2-(n-1)^2=2n-1$ chosen, or, perhaps more pertinently, *where* did it come from? As it transpires, it is by no means the only identity that can be used in application to the method of differences, but it yields the result required simply enough. This air of mystery, this lack of clear sources, renders this proof a little less palatable.

The girls, nevertheless, were keen for the board displaying the proof to be left uncleaned so that the next class of girls could see what work had been achieved by their fellow Year Nines; particularly so when I hinted, more or less truthfully, that this technique was usually introduced to Year Twelves! So, they were impressed, but the pedagogical value of this presentation at this stage in their development surely would have been slight. It could be said that harm was done, since the girls had just been reminded that mathematics will always be one step ahead of them, that just when they think it makes sense something can come along and appear utterly mystifying. I did not introduce the girls to proof by induction, a proof that proves, but seldom explains (Hannah, 1989)[35]. For as Reuben Hersh (1997) asserts, "*Proof can convince, and it can explain. In research, convincing is primary. In high-school or undergraduate class, explaining is primary*" (p. 61, author's italics).

Looking back at this series of lessons, I wonder what was achieved. I hope that the girls understood that there are various ways to "see" the one thing. I hope further, that they realised that *proof* is a mathematical tool, one purpose of which is to be convincing, but that there is not necessarily one clear and correct proof that applies. Perhaps they began to understand that the linearity of mathematical presentations is not always best, that we can sometimes broaden our understanding by revisiting and reconstructing a situation, by being more circular.

For myself, I can reflect on what of my emerging epistemology was apparent, and what was lacking. While elements of my teaching were consonant with an emerging appreciation of the experiences of my students, I cannot but help feel that I privileged the result $1+2+3+..+n=\frac{n(n+1)}{2}$ with something approaching a distinct ontological status, as a Platonic thing. And yet, while this is an unfortunate feature of our current language disposition, it is to be expected within the epistemology. For a root feature of the embodied cognition framework, as we shall soon see, is to use the language of the body—grasping, holding and the like—to convey meaning, following the thesis that the experience of the body is inherent to the processes of thinking.

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[35] Marc Lange (2009) gives an analytical reason why induction is not explanatory. Alan Baker (Baker, 2010) critiques his conclusion. Both regard "explanation" in more or less analytical terms. A practicing teacher will understand implicitly and empirically that induction proves without revealing structural insights into that which it proves: it just convinces.

I earlier described the experience of understanding as “the forgetting of wanting to know.” The following illustrates how we can operate more or less efficiently without knowing *all* the details of what we are doing (thank goodness) but that returning and re-seeing can gradually increase our awareness of what we do.

My Year Twelve girls were undertaking a research based project in which they were required to come to terms with an element of mathematical theory or application by reading a mathematical text. Stephanie showed me a section of her resource text in which it was stated that if $P_1P_2P_3$ is a triangle, with $P_i = (x_i, y_i)$, $i = 1,2,3$, then the area of triangle $P_1P_2P_3$ is given by the matrix determinant

$$\pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the sign depends on the anticlockwise or clockwise orientation of $P_1P_2P_3$.

The text proceeded to demonstrate that this was indeed a valid formula for the area of a triangle by showing that

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left(x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix} + 1\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\right) = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}.$$

Stephanie had already become satisfied that $\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$ was, indeed, the area of a triangle $P_1P_2P_3$ (oriented anticlockwise), and so she accepted that the demonstration did confirm that $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ was the area of the triangle $P_1P_2P_3$.

And yet she came to me and said, “I don’t see where the ones come from,” referring to the ones in the third column of the matrix. As we looked at the text together it became clear to me that even though she accepted the statement, there remained, nevertheless, the feeling that something was lacking. We discussed the notions of “proofs that prove” and “proofs that explain”. I suggested to her that if we had proven the familiar $1+2+3+...+n=\frac{n(n+1)}{2}$ by induction, which method she had recently studied, then she would have been in a similar situation: that of believing in the validity of the statement but knowing not whence it had come.

She gave me her own example. “When we learned to derive by saying that $\frac{d}{dx}ax^n = nax^{n-1}$, I thought it was okay, I could do it, but I never really liked it. Then we learned about derivatives from first principles and I knew it was not just a pattern anymore.”

I asked her if she knew why the product of two negatives gave a positive number. She said she did not. I asked her why she had not asked me about that. “I think I just got used to it, and forgot to ask anymore.” We discussed that observation, insightful as it was, and agreed that her reaction to her current discovery was consistent with it: this determinant was so new to her that she had not yet got used to it and had certainly not forgotten to ask!

By way of closing this sojourn in seeing, re-seeing and understanding, a colleague once told me of a student, Emily, who was learning to complete the square. She would get to a point where she “loses it,” he said. From (x^2+6x+9) she developed $(x+3)^2$, but then, she wanted to know, “Where did the six go?” It bothered her. My colleague suggested, “Look at three times four. Now look at twelve. Where did the four go? It’s in there,” he said, pointing to the twelve.

Emily was not satisfied, but she resumed her routine exercises. After a few repetitions, she said, “Oh yeah, it’s okay now. I see it.”

EMBODIED COGNITION: SPATIAL BEING

I described earlier how embodied cognition, or "grounded cognition", can be presented with varying emphases. Perceptual symbols systems were discussed because the means by which information is represented—particularly abstract information—bears significantly upon how we are to understand what happens when children learn mathematics. I now describe that particular account labelled as "embodied cognition", and apply it specifically to the learning of mathematics.

The embodied cognition approach makes three assumptions (Bazzini, 2001):

> 1. the embodiment of mind: the structure of our body and of our brain influences the structure of theoretical concepts, including mathematical concepts.
> 2. the cognitive unconscious: most of our thinking is not accessible to direct introspection.
> 3. metaphorical thought: abstract concepts are mostly characterized in terms of real facts and reasoning and are based mainly in the sensory motor system. The mechanism by which an abstract concept is understood in terms of real objects is named a "conceptual metaphor". (pp. 260–261)

With regard to the first assumption, while the availability of modern imaging techniques encourages researchers to attempt to observe mathematical processing in action (see, for example, N. Davis et al., 2009; and Naccache & Dehaene, 2001), it can be misleading to locate mathematics in any particular brain region. Though certain parts of the brain might become strongly implicated with particular modes of thought, such as the angular gyrus or the left intraparietal sulcus with mathematical calculation, and might even be indispensible to that thought, as revealed in cases of patients who have suffered damage (Ashkenazi, Henik, Ifergane, & Shelef, 2008), this is a different thing to locating a source or seat of mathematical thinking (Kahn & Whitaker, 1991, trace the history of the localization of acalculia). Complex thought is generally a gestalt experience recruiting several systems in an integrated whole. This does not mean that all systems are always implicated. Mathematical calculations can proceed in the absence of language grammar (Varley, Klessinger, Romanowski, Siegal, & Purves, 2005), for example; and the degree of approximation or exactitude in calculations can affect the recruitment of brain regions (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999).

The movement of the body is significant to embodied cognition theory. Movement, in a broad, participatory sense, is thought to be important in mathematics education (see, for example, Nemirovsky et al., 2004), but the intention here is to consider the way in which our inherent capacity to experience movement contributes to the possibility of abstract thought. Gallese and Lakoff (2005) argue that "contemporary neuroscience seems to suggest that concepts of a wide variety make direct use of the sensory-motor circuitry of the brain" (p. 473). Gesture, for example, can be understood to provide a beneficial organisation upon propositional language production, thereby reducing demands upon working memory (Wagner, Nusbaum, & Goldin-Meadow, 2004); to function as a cognitive amplifier; a means of maintaining joint attention; and as an index of understanding (Reynolds & Reeve, 2001). With a grander sweep, though, the suggestion is that the qualities of *what it is to negotiate space* transfer to the qualities of thought. In a focussed illustration of this general point, Vandervert (2003) presents a theory in which the cerebellum is central to the process of mathematical discovery:

> The cerebellum constructs fundamental patterns of the framework of our cognitive consciousness of our bodily movement in its relation to our perceptual-cognitive world. Since this abstract model of the world is accessible to the central executive processes of working memory, it can be cognitively formulated into a 'science of patterns'. (p. 27)[36]

Spatial *representations* may play a role in how we process number. Granà, Hofer and Semenza (2006), for example, report the case of PN, a patient who had a damaged right hemisphere and was unable to perform multiplications. They determined that PN's errors were most likely spatial in nature:

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[36] See Paquier and Mariën (2005) for a comprehensive review of the role of the cerebellum in cognition.

> A speculative but likely conclusion is therefore that, vis-à-vis his pattern of performance in multiplication procedures, PN does know what, when and how to carry out the various steps, but he does not know where. What he may lack is a spatial schema of multiplication. (p. 2984)

Embodied cognition, in summary, represents an attempt to understand thinking as an integrated property of being and acting human, not by ascribing new ontologies but by relating thought to the being of the body in spatial contexts. One particular attempt to account for how abstract mathematical ideas can arise is the embodied cognition framework of George Lakoff and Rafael Núñez (2000). As it has a particular relevance to the concerns of my thesis, not least motivational, I will now proceed to describe it in some detail.

WHERE MATHEMATICS COMES FROM

Lakoff and Núñez (2000) promote the use of *ideas analysis* to explore metaphorical paths by which mathematical concepts such as infinity can arise. Their work on the origins of mathematical thought represents a continuation and elaboration of previous studies in cognitive linguistics (see Lakoff & Johnson, 1980; and Johnson, 1987). George Lakoff (1993), for example, had earlier made the empirical determination that metaphors are less novel or poetic ways of speaking than means of organizing thought: "the locus of metaphor is not in language at all, but in the way we conceptualize one mental domain in terms of another" (p. 203); and "The generalizations governing poetic metaphorical expressions are not in language, but in thought: they are general mappings across conceptual domains" (p. 203). Thus was laid the basis on which mathematical ideas could be analysed.

A central tenet of Lakoff and Núñez's (2000) approach is that our actions and opinions are not necessarily the consequence of rational deliberation, but are determined in part by frames of thought. Metaphor is central to the process of idea formation and to determining which frame will be "triggered".

Schiralli and Sinclair (2003), in critiquing this theory, caution that:

> Any adequate account of where mathematics comes from, we propose, would need to account for the ways in which a-cultural cognitive predispositions interact with various cultural and symbolic variables to produce mathematical cognitive structures. (p. 90)

An intriguing avenue of thought germane to this point arises from studies of synaesthesia, in which modes of cognition are apparently "cross-wired" so that, for example, one sees colour when prompted by the grapheme "5". Ramachandran and Hubbard (2003) propose that, to some extent, we are all synaesthetes and that the functional proximity of processing regions of the brain lend a path to the creation of certain metaphorical connections and bootstrap the evolution of language:

> Our studies of the neurobiological basis of synesthesia suggest that a facility for metaphor—for seeing deep links between superficially dissimilar and unrelated things—provided a key seed for the eventual emergence of language. (p. 58)

The idea arises from observations that humans are biased to associate certain sounds with certain shapes and gestures and is putatively explained by two neurological details: "The sensory areas for visual shapes and for hearing in the back of the brain can cross-activate specific motor areas in the front of the brain that participate in speech"; and "a kind of spillover of signals occurs between two nearby motor areas: those that control the sequence of muscle movements required for hand gestures and those for the mouth" (Ramachandran & Hubbard, 2003, pp. 58-59). The lips, tongue and mouth are wired, so to speak, to mimic the visual appearance of an object; and the mouth and hand are closely related. We see someone purse their lips whilst making a delicate operation with their fingers, such as threading a needle, for example. An expansive gesture is likely to be associated with an expansive sound, a tightly constrained motion with a tight vocalisation.

Evidence for such “sound symbolism”, a view whereby for at least some words there is a non-arbitrary correspondence between sound and meaning, is gradually accumulating. Ohala (1984), for example, has drawn observations that people use higher tones to denote questions, that voice frequency carries affective influences, and that high tones tend to be associated with words denoting smallness and low tones with words denoting largeness (with exceptions), together with cross-species data showing that size and aggressiveness is denoted by lower tones, while higher tones denote submissiveness or smallness, to argue that the “sound meaning correlations found in these cases adhere to the ‘frequency code’... where high F_0 [fundamental frequency] signifies (broadly) smallness, nonthreatening attitude, desirous of the goodwill of the receiver, etc., and low F_0 conveys largeness, threat, self-confidence and self-sufficiency” (p. 14). The shape of the mouth when smiling and the accompanying vocalizations is cited as an example of this code in application (see also Ohala, 2010, p. 457).

Zwaan and Taylor (2006) report experiments in which they ask whether language comprehension produces motor resonance, an activation of neural substrates as if one were making actions indicated by the language. Coupled with subsequent experiments, (Pecher, van Dantzig, Zwaan, & Zeelenberg, 2009), the data indicate that sensorimotor systems are involved in language comprehension, consistent with Barsalou’s (1999) model of perceptual symbols. Parault and Schwanenflugel (2006) demonstrate a link between sound symbolism and word learning; Aveyard (2011) shows that the sound of pseudowords improves object recognition when the sound is “congruent” with the object; and Parise and Pavani (2011) provide evidence that the visual attributes of shape and colour can influence the way in which subjects sound a vowel, thereby strengthening the argument that “sound-meaning associations can be reciprocal, affecting active (production) as well as passive (comprehension) linguistic behaviour” (p. 379).

If language is coupled in this way, or in some like manner, to actions and appearances, if metaphor also is associated with the way in which ideas in the brain cross-modally interact, then we do indeed have a basis on which to address Schiralli and Sinclair's caution, for cultural and symbolic variables are themselves products, or products of products, in a rather delimited fashion of the passage of beings evolving in the world.

Lakoff and Núñez begin their exploration of the cognitive structure of mathematical ideas by surveying the field that might be labelled "baby arithmetic"; that is, experiments designed to elicit what it is that young children and other animals know about number and order, either innately or from very early learning. It is known that very young children are able to discriminate between numerosities and subitize small numbers of objects. "Subitize" derives from the Latin *subitus*, for "sudden", although as Stanislas Dehaene (1997) indicates, the name is misleading,

> since subitization, however fast, is anything but instantaneous. It takes about five- or six-tenths of a second to identify a set of three dots, or about the time it takes to read a word aloud or identify a familiar face. Neither is this duration constant: it slowly increases from 1 to 3. Hence, subitization probably requires a series of visual operations all the more complex the greater the number to be recognized. (p. 68)

Lakoff and Núñez make clear a distinction between ordinary, conscious cognition and unconscious cognition—the cognitive unconscious. What has not been done in the past, they say, is to extend the study of the cognitive unconscious to mathematical cognition:

> A large part of unconscious thought involves automatic, immediate, implicit rather than explicit understanding—making sense of things without having conscious access to the cognitive mechanisms by which you make sense of things. Ordinary everyday mathematical sense-making is not in the form of conscious proofs from axioms, not is it always the result of explicit, conscious, goal-oriented instruction. Most of our everyday mathematical understanding takes place without our being able to explain exactly what we understood and how we understood it. (p. 28)

The challenge they set themselves is to study unconscious mathematical cognition and to ask,

> How much of mathematical understanding makes use of the same kinds of conceptual mechanisms that are used in the understanding of ordinary, nonmathematical domains? Are the same cognitive mechanisms that we use to characterise ordinary ideas also used to characterize mathematical ideas? (p. 28)

An immediate reaction might be to say, "Well yes, of course!" It will do to pause, however, over certain difficulties. Mathematics deals with phenomena that are beyond sense-experience. The idea of the infinite, to cite a key example, is not transparently a part of our everyday interaction with the world, so where does our understanding of the idea have its source? How does one "understand" the product of negative numbers, or the celebrated equation, $e^{\pi i} + 1 = 0$?

For Lakoff and Núñez, it is not enough to develop a folksy understanding of how everyday life and mathematical cognition are related. They ask, "*Exactly what everyday concepts and cognitive mechanisms are used in exactly what ways in the unconscious conceptualization of technical ideas in mathematics*?" (p. 29, authors' italics). The underlying mechanisms, on their analysis, turn out to be schemas and conceptual blends.

Schemas, as we have seen, are basic conceptual frames. They are thought to be descriptions of the way in which the brain perceives—and, we may say, makes sense of—the world. When, for example, we speak of the image of a mathematical graph as "moving" as we trace our pen along its curve—though the graph, in an objective sense, is static—we can be said to be enacting a force dynamic (Talmy, 1988) apparent in our language wherein a tendency to stasis (the graph is intrinsically still) and a force (the pen represents an antagonistic tendency) are weighed, much in the manner of vectors, to produce a resultant linguistic and cognitive effect; motion, in this case, of a (static) graph. Leonard Talmy (1996) develops a *fictive motion schema* in which such expressions of movement can be approached through the linguistic consequences of balancing competing cognitive representations. Even the phrase "can be approached", as I have just used it, is an example of a force dynamic in play. Motion, while an important element in our mathematical conceptualisation, may be less captured formally than enacted cognitively, as Núñez (2006) reminds us: "Motion … is a genuine and constitutive manifestation of the nature of mathematical ideas. In pure mathematics, however, motion is not captured by formalisms and axiomatic systems" (p. 168).

Schemas were elaborated in the succeeding decade (see, for example, Johnson, 1987; and Lakoff, 1987). They convey logical structures and it is a central feature of Lakoff and Núñez's system that mathematics is ultimately derived from the logical structures inherent in these schemas.

One variety of schema is the image schema. Image schemas operate to bridge visual perception and conceptual language:

> Image schemas can fit visual perception, as when we see the milk as being *in* the glass. They can also be imposed on visual scenes, as when we see the bees swarming *in* the garden, where there is no physical container the bees are in. Because spatial-relations terms in a given language name complex image schema, image schemas are the link between language and spatial perception. (p. 31, authors' italics)

A particular example of an image schema introduced by Lakoff and Núñez is the *Container schema*:

> The Container schema has three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside. This structure is topological in the sense that the boundary can be made larger, smaller, or distorted and still remain the boundary of a Container schema. (p. 31)

The Container schema is described thus:

> 1. Given two Container schemas *A* and *B* and an object *X*, if *A* is *in B* and *X* is *in A*, then *X* is *in B*.
> 2. Given two Container schemas *A* and *B* and an object *Y*, if *A* is *in B* and *Y* is *outside of B*, then *Y* is *outside of A*. (p. 31, authors' italics)

The suggestion here is that when we perceive situations to which this logic of containment applies, we reason through the application of this schema. In more general terms, image schemas having a spatial logical structure “function as spatial concepts and can be used directly in spatial reasoning” (p. 33).

This is a significant departure from the assumption that symbols are the atoms with which information is manipulated in the mind. Not with symbols, but through schemas are ideas understood. We co-opt structures that have evolved to facilitate our ability to negotiate space in order to negotiate thought.

Conceptual blends, another of the key elements in Lakoff and Núñez’s (2000) analysis, occur when two distinct cognitive structures, each with their own domain and inferential structure, work together to produce a new cognitive structure that draws on the inferential aspects of each domain. Lakoff and Núñez (2000) describe the unit circle as an example, in which the properties of the circle are combined with the properties of the Cartesian plane resulting in a blending of the inferential structures of both the Cartesian plane and the circle:

> The result is more than just a circle. It is a circle that has a fixed position in the plane and whose circumference is a length commensurate with the numbers on the *x*- and *y*- axes. A circle in the Euclidean plane, where there are no axes and no numbers, would not have these properties. (p. 48)

Blending, as this example suggests, is not simply a combination of inferential structures. As Giles Fauconnier and Mark Turner (1998) note,

> Conceptual blending is not a compositional algorithmic process and cannot be modeled as such for even the most rudimentary cases. Blends are not predictable solely from the structure of the inputs. Rather, they are highly motivated by such structure, in harmony with independently available background and contextual structure. (p. 136)

Fauconnier and Turner (2002) provide examples with detailed elaborations drawn a variety of sources. “The blend develops emergent structure that is not in the inputs” (p. 42) they say, and this feature enables Lakoff and Núñez (2000) to describe cognitive processes by which complex mathematical concepts are formulated. I shall provide an example shortly within a discussion of negative numbers.

This, then, is Lakoff and Núñez's account in a nutshell. Schemas are properties of neural circuitry that give rise to an "internal" sense of how the body relates to the world, complete with a logical structure derived from interaction with the world. There is one more key ingredient, however, that is required if properties about the "real" world are to be co-opted to enable the brain to formulate ideas about abstract concepts. That ingredient is the inference preserving, metaphorical mapping.

Metaphors, to be understood in the sense described by Lakoff and Núñez, are central to the construction of abstract thought: "Abstract concepts are typically understood, via metaphor, in terms of more concrete concepts" (p. 39). The assignment of metaphor to thought, moreover, is not arbitrary, but systematic, and it is this feature that gives rise to the ideas analysis that Lakoff and Núñez develop. A few examples, taken from everyday speech, are presented, one of which will suffice here:

> Affection … is understood in terms of physical warmth, as in sentences like "She *warmed* up to me," "You've been *cold* to me all day," "He gave me an *icy* stare," "They haven't yet *broken the ice*." As can be seen by this example, the metaphor is not a matter of words, but of conceptual structure. The words are all different (*warm*, *cold*, *icy*, *ice*), but the conceptual relationship is the same in all cases: Affection is conceptualized in terms of warmth and disaffection in terms of cold. (p. 41, authors' italics)

These conceptual metaphors are thought to arise from everyday correlations with physical phenomena and internal states, brought about by the way in which the brain conflates activation of two distinct regions into one complex experience. Thus the early childhood correlation of physical warmth with emotional security and affection (being held close by mother) results in co-activation of distinct areas of the brain in such a way that the overall effect becomes a conflation of two inseparable experiences into a single experience—affection with warmth: "It is via such conflations that neural links across domains are developed—links that often result in conceptual metaphor, in which one domain is conceptualized in terms of another" (p. 41). This is reminiscent of the synaesthetic mechanism proposed by Ramachandran and Hubbard, except that here the conflation arises out of a temporal co-activation rather than from a spatial contiguity.

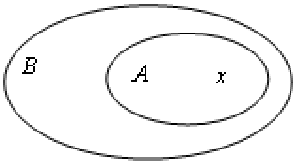
All such conceptual metaphors have the same structure, say Lakoff and Núñez, that of a mapping from one conceptual domain (the source-domain) to another (the target domain):

> Their primary function is to allow us to reason about relatively abstract domains using the inferential structure of relatively concrete domains. The structure of image schemas is preserved by conceptual metaphorical mappings. In metaphor, conceptual cross-domain mapping is primary; metaphorical language is secondary, deriving from the conceptual mapping. (p. 42)

This paragraph sets the basis on which Lakoff and Núñez proceed to show how mathematical thought can be derived from concrete experience. The concrete experiences form the "source-domain" of the above paragraph, and the mathematical ideas form the "target-domain". It transpires that the more complex mathematical ideas are built up, as it were, of ideas upon ideas, of blends of metaphorical mappings across and between target-domains. But all can be traced back to a central source: the response of the living being to correlated environmental stimuli and the conflation of the resulting experiences into a complex experience that confers the logical structure of a concrete domain to an abstract domain.

Meaning as a Metaphorical Entailment

Venn diagrams are often used to illustrate the logic of the Container schema in the classroom, and in my experience, while statements of logic, especially when expressed symbolically, can pose difficulties for students, Venn diagrams are "intuitively obvious". So, for example, while the modus ponens statement $x \in A$ and $A \subset B \Rightarrow x \in B$ (if *x* is an element of the set *A* and the set *A* is contained within the set *B*, then *x* must also be in the set *B*) might present a few initial difficulties, the Venn diagram is revelatory:



The intuitive quality of this diagram arises because it appeals directly to our understanding via a conceptual metaphor: that *categories* (in this case mathematical sets) *are containers* (Lakoff & Núñez, 2000, p. 43), and we have evolved a bodily understanding of the spatial logic of containers.

It is important to note, say Lakoff and Núñez, that conceptual metaphors are also able to introduce elements into the target domain. This is taken to be important in the development of some mathematical concepts. The element introduced will be one that is a feature of the source domain, but that was not previously belonging to the target domain: "Conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also *introduce new elements* into the target domain" (pp. 45–46, authors' italics). This becomes a basis on which a target domain can be said to "grow" and assume meanings that are not obviously inherent to it.

There is a potential ambiguity here, however. Lakoff and Núñez do not claim to be doing mathematics when they present their theories, so much as to be accounting for how we make sense of mathematics. To say that new elements are introduced into the target domain means, I take it, that the capacity to realise or make sense of logical entailments of existing elements in the target domain becomes tenable through the spatial logic of the source domain.

If, for example, I say to you that "numbers are objects," then it is likely that you will understand simple arithmetic, although fractions might appear mysterious, if they arise at all in your considerations. If I say "numbers are lengths" then fractions can represent parts of lengths and we have a metaphorical mapping between the spatial statement "halfway between two points" and the number $\frac{1}{2}$. The conceptual metaphor, "numbers are lengths" can be said to have introduced—in that it has allowed sense to be made of—the fraction. The fraction was always "there" however, in that it was a delitescent property of the confluence of numbers as objects with another metaphor, "sharing is division".

This last sentence might sound like a reversion to Platonism, but it is not. It is, rather, an application of John Searle's aforementioned argument that abstract entities do not affect our ontological commitments: "The introduction of ... abstract entities ... does not introduce a new ontological realm but is just a manner of speaking" (Searle, 2007, p. 24). Our recognising that one half is a number makes no difference in the world, no more than our recognition that the grass is green. What conceptual metaphors bring to this mix is the potential to discover new properties of familiar mathematical objects in virtue of the preservation of logic that inheres in spatially familiar experiences. The crossing of metaphors enhances the prospect of new discoveries.

A QUESTION OF HATS

As I prepared to unlock the door to our room one morning, one of my Year Seven mathematics enrichment girls made a reference to the word "literally". I picked it up to see if the girls knew what "literally" meant. They did. It is an interesting word because its popular usage as an emphatic, as in "They were literally rooted to the spot!" is contra to its actual meaning. That is, common usage indicates a degree of semantic reversal, almost to the point where the word is a contronym, such as sanction, and temper, for which two opposing meanings coexist in the language.

This impromptu moment led to an elaboration of logic and meaning in text, and the tensions therein. I recalled that I knew a teacher who on the occasion of observing a group of schoolgirls on excursion had made the remark, "All the girls are not wearing hats!" I invited the girls to speculate as to what she had meant. After some discussion, possibilities B, C, D and E were the shared results.

A. The teacher said, "All the girls are not wearing hats."

B. Most of the girls ~~probably~~ are not wearing hats.

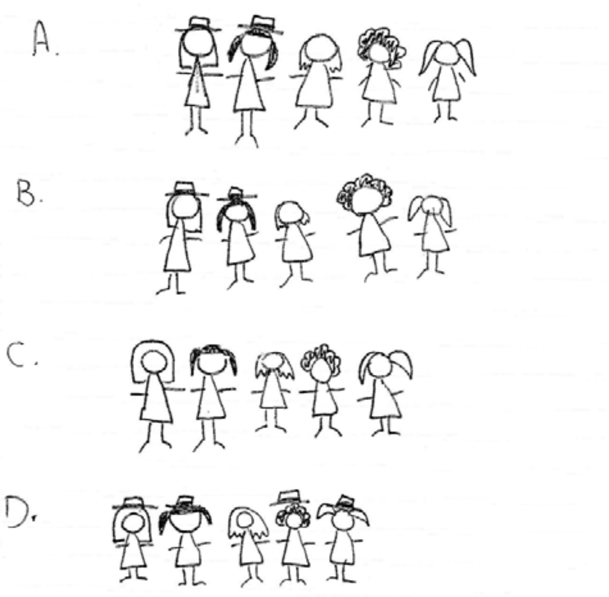
C. None of the girls are wearing hats.

D. Some of the girls are wearing hats.

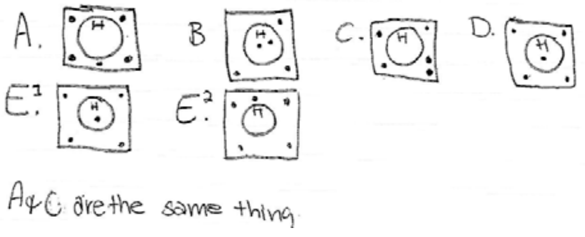
E. Did she mean, "Not all the girls are wearing hats"?

Gesine had suggested B, but withdrew the word "probably" after clarifying that the teacher had actually seen the students.

I asked the girls to draw diagrams to illustrate possible cases for each of A through E. Kelsey's diagrams for A to D are presented here, where A is already an interpretation of what Kelsey thinks that the teacher intended.



We then learned to use Venn diagrams to illustrate the situations. A and C were now understood to represent the same logical relationship between girls and hats.



By the end of the lesson we were able to analyse the standard type of problem that includes intersecting regions: in a class of twenty, eleven wear hats, five wear hats and blazers, four wear neither. How many wear blazers but not hats? Answer: five.

The inherent ambiguity in words like "some" came to the fore. The statement, "Not all the girls are wearing hats" seemed likewise ambiguous to the girls. Kelsey's E^1 and E^2 illustrate that we could interpret the statement to mean that it is not the case that *every* girl is wearing a hat, on which basis at least one girl is not wearing a hat; or that *all* the girls are *not* wearing hats, in which case each girl is hatless. The second interpretation, however, seems to derive its plausibility from an understanding that natural discourse does not follow strict rules of logic. That is, it is not only in the words but also in our interpretations of the intentions of the speaker that we find our uncertainty. This is our dialogical condition. I am confident that I knew what the teacher had intended by her words because I had overheard more of her speech than the single sentence I had shared with my students—and she had not intended C.

Thus we see that there is a tension between meaning as understood in everyday discourse and meaning in mathematical formulations. And yet, for all that, the deeper question may consist in acknowledging that our expectation of "meaning" is always contextualised by circumstances. Within mathematics, for example, we have seen that there is no univocal appreciation of what proof does, nor of what constitutes explanation. Lakoff and Núñez's (2000) attempt to trace mathematical meaning through inference preserving metaphors to a ground in our evolved spatial being is really an attempt *to disrupt a philosophical commitment to language as prior to being*, to ask us to reconsider what we expect when we ask, "What does this mean?" In the end, every question demands that we choose a level of description by which to frame our response.

Under our simple exploration of hats and intentions, then, was the injunction to remember that being logically correct is sometimes secondary to being interpreted in communication. What is the point, I asked the girls, in being logically precise if the very exercise of that precision encourages a less logically discerning interlocutor to misunderstand? Those with sufficient prescience have a responsibility to anticipate the interpretation.

GROUNDING NEGATIVES: STRETCHING METAPHORS

The embodied cognition approach of Lakoff and Núñez has the potential to provide the teacher of mathematics with a potentially useful way of making sense of how children learn, which is central to the development of my thesis. The ideas of inference preserving mappings taken together with a few fundamental grounding metaphors—metaphors that relate arithmetic directly to sense experience, such as "Arithmetic as Object Collection" (p. 54)—afford a reasonable and believable way to understand the growth of what we might call "natural mathematics" out of sense experience.

What Lakoff and Núñez (2000) dub the "fundamental metonymy of algebra" (p. 74) is a means of saying how we move from particular examples to general statements, as when, for example, we invoke induction to extend examples such as $3 \times 4 = 4 \times 3$ to a general statement of commutativity, $m \times n = n \times m$ for *all* numbers, *m* and *n*. That we have the capacity to make such statements at all is seen as a consequence of the way that metonymic behaviour—by which is meant the way in which we adopt a property or feature of something to stand for that thing, such as when we say "the crown" to represent a queen—is a feature of how our cognitive unconscious works. We move from specific examples (queens wear crowns) to a general representation (the crown for a queen). In mathematics we traditionally describe the movement from examples to general statements as inductive, but do not regard such movement as rigorous (or even reliable) until a proof of validity is provided. Lakoff and Núñez's purpose, however, is to describe the cognitive mechanism underlying induction, not the mathematical validation of induction in its own (mathematical) terms.

On the other hand, mathematics, even at the school level, is more complex than the mathematics easily described by "simple" cognitive features. Combinations of features may be required to deliver fully rounded meaning to mathematical structures. The introduction of negative numbers provides an early instance where what I have loosely referred to as natural mathematics encounters a serious challenge. An initial difficulty for students is the type of statement represented by $5 - {}^{-}3 = 5 + 3$. This is alternatively written as $5 - -3 = 5 + 3$. Even if this hurdle is successfully negotiated—or surrendered to a rote process of rule following—the second hurdle is even more serious: how does ${}^{-}3 \times {}^{-}2 = 6$? That is, how does it happen that the product of two negatives gives a positive result?

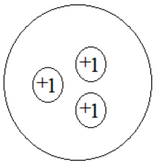
Some students will resign themselves at this point to not expecting mathematics to “make sense”—if they had not already reached that point of resignation—or a very few will see that it is a necessary result. Memorising that “two negatives give a positive” seems to aid students in their passage through the curriculum; it is consistent in a different way with the foregoing example of the type $5 - -3 = 5 + 3$; but it fails when applied as a verbal rule to equations of the type $-4 - 2 = -6$.

With the exception of the product of two negatives, the above difficulties can be managed with reference to the number line. Under the embodied cognition paradigm this would be a move that is in concert with the Measuring Stick metaphor. Under this metaphor, numbers are related to lengths (p. 68), and processes such as putting sticks of certain lengths together end to end, is metaphorically mapped to the mathematical concept of addition.

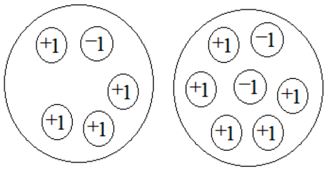
This metaphor (in effect) is employed when teachers use number lines to “explain” how to make calculations involving addition and subtraction of negative numbers. The most difficult, not to say cumbersome, analogy concerns subtracting negative numbers, which is to say operations of the type $x - {}^{-}y = x + y$. It is a relatively common procedure to associate movement with a direction on the number line, and “positive” or negative” with facing right or left, respectively. The operation $3 - {}^{-}2 = 3 + 2 = 5$ is explained, for example, as beginning at the position three units right of zero, taking two steps facing left (subtraction), but backwards (because the two is negative), thereby landing at five. This is essentially the process Lakoff and Núñez describe (p. 91).

An alternative approach exploits what arguably is a more fundamental feature of negative numbers whilst at the same time being conducive to ready interpretation as an application of the grounding metaphors Arithmetic is Object Collection and Zero is an Empty Collection.

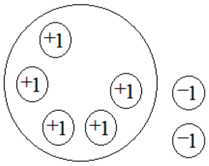
I can illustrate with $3 - {}^{-}2$. Collect three objects by physically placing them into a ring.



My students and I generally use coloured or labelled counters for this exercise. The objects might well be taken to be tokens of the form $^{+}1$. The presence of three of these tokens in a ring is, metaphorically speaking, the number three. Now, the subtraction sign (the first "–" in the expression $3-{}^{-}2$) is understood metaphorically as "remove a token from ring". The tokens to be removed are understood through the expression $^{-}2$ to be two tokens of the form $^{-}1$, since this is the understanding given to $^{-}2$. There are, however, none of these $^{-}1$ tokens in the ring waiting to be removed. The Zero metaphor is now exploited: under the idea that $^{+}1+{}^{-}1=0$, we can now insert as many parings of $^{-}1$ and $^{+}1$ tokens as we desire. Thus, for example, the following two collections also represent the number "three":



If we choose the second of the descriptions, we are now at liberty to "perform" the action dictated by the expression $3-{}^{-}2$: we remove (that is, subtract) two tokens of the form $^{-}1$ from the collection to obtain



This is a collection of objects (in the ring) which is understood to be the number five. Thus it is seen that $3-{}^{-}2=5$. Beyond this we still require algebraic metonymy to draw the inductive bow and so encompass the generalisation, $x-{}^{-}y=x+y$.

I have used both the measuring stick and the object collection-empty collection metaphors in the classroom to give meaning or facility—I am not sure which—to expressions of the form $x - {}^{-}y$. The reaction of students to the two presentations varies, but in general the number line formulation agrees with other presentations they have already encountered, and so it has a feeling of familiarity, while the object collection approach seems to be more explanatory or surprising than the number line. It is usually the case that the "Oooh!" reaction is stronger following the object collection version. It appears that the consequence is the more impressive for arising unexpectedly out of simple routine movements. In his *Apology*, G. H. Hardy (1992) also noted the aesthetic relevance of unexpectedness, along with inevitability and economy. My Year Sevens found the object collection explanation more appealing than the number line method, which they derided. The object collection approach is certainly more readily attached to algebraic formulations that introduce additive inverses.

I like the object collection approach for one other reason: it reveals that two apparently unrelated "facts", metaphors for addition and zero, can combine to produce something unexpected, something new (a point made earlier). I claim that this is a defining, constructive feature of mathematics.

Learning to multiply with negative numbers creates a new level of difficulty for the student. When one of the factors is positive, as in $3 \times {}^{-}2$, for example, the problem can be rendered sensible by application of basic metaphors. $3 \times {}^{-}2$ is three tokens of ${}^{-}2$, or ${}^{-}2$ collected three times. This can be translated as "two steps to the left" repeated three times, or thrice gathering two negative tokens. The result is ${}^{-}6$, readily pictured as a location on the number line or as six negative tokens. Even cases in which the order is reversed, ${}^{-}3 \times 2$, yield sense through an appeal to commutativity, a ready consequence of grounding metaphors. The product is reinterpreted as $2 \times {}^{-}3$ and the result obtained as above.

The product of two negative numbers, on the other hand, resists such a basic grounding. The appeal to commutativity, for example, is thwarted by a product of the type $^{-}2\times^{-}4$. One can commute this to $^{-}4\times^{-}2$ if one wishes, but the essential difficulty remains—how does one take a negative multiple of something? This is, of course, beyond "sense experience" (and so beyond a grounding metaphor). A teacher to whom I recently posed the problem responded by saying that this was where one had to "lift off the number line into another space."

That "other space" is very much the space in which advanced mathematics is done. This problem, indeed, represents a launching pad away from natural mathematics, the mathematics for which sense experience via grounding metaphors is sufficient to enable "understanding", to a deeper realm of the subject.

Appeals to sense-making in this scenario can proceed on several grounds. I have used a patterning procedure, for example, to suggest a degree of reasonableness for the statement "the product of two negatives is positive" in cases where I have felt that more (mathematically) sophisticated procedures would obfuscate meaning. That is, I have drawn on metonymy to establish plausibility. The pattern proceeds via an extension of the familiar multiplication table:

×	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
3				0			
2				0			
1				0			
0	0	0	0	0	0	0	0
$^{-}1$				0			
$^{-}2$				0			
$^{-}3$				0			

The zeros provide a ready means of sectioning the table into quadrants (note that multiplication by zero is immediately appealed to as requisite knowledge). The most familiar quadrant is readily constructed:

×	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
3				0	3	6	9
2				0	2	4	6
1				0	1	2	3
0	0	0	0	0	0	0	0
$^{-}1$				0			
$^{-}2$				0			
$^{-}3$				0			

Newly developed knowledge, still physically sensible, that the product of a positive and a negative yields a negative enables two more quadrants to be completed:

×	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
3	$^{-}9$	$^{-}6$	$^{-}3$	0	3	6	9
2	$^{-}6$	$^{-}4$	$^{-}2$	0	2	4	6
1	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
0	0	0	0	0	0	0	0
$^{-}1$				0	$^{-}1$	$^{-}2$	$^{-}3$
$^{-}2$				0	$^{-}2$	$^{-}4$	$^{-}6$
$^{-}3$				0	$^{-}3$	$^{-}6$	$^{-}9$

To this point the table has been completed by appeal to sense experience—the grounding metaphors. Now, however, comes the leap of faith: the decision to commit to something that is not immediately sensible. The table reveals simple counting patterns. The left column, for example, proceeds $^{-}9, ^{-}6, ^{-}3, 0$. If the student is prepared to acknowledge that the impetus of the pattern trumps the reluctance to depart familiar territory, then the patterns must be pursued to produce the inevitable result: the product of two negatives is positive. What is more, the table is (presumably) infinitely extendable and so able to account for all integers.

×	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
3	$^{-}9$	$^{-}6$	$^{-}3$	0	3	6	9
2	$^{-}6$	$^{-}4$	$^{-}2$	0	2	4	6
1	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
0	0	0	0	0	0	0	0
$^{-}1$	**3**	**2**	**1**	0	$^{-}1$	$^{-}2$	$^{-}3$
$^{-}2$	**6**	**4**	**2**	0	$^{-}2$	$^{-}4$	$^{-}6$
$^{-}3$	**9**	**6**	**3**	0	$^{-}3$	$^{-}6$	$^{-}9$

The key feature that I try to bring my students to appreciate is that the desire of the mathematician to be consistent with what he or she has previously agreed, to continue the pattern, exceeds the desire to “see” how each new discovery makes “real” or “physical” sense. There is, in fact, a sense of leaving the comfort of the familiar world behind and exploring a new terrain, a voyage of discovery, as it were. The voyage here is in the mind, as we say, and cherished notions may have to be sacrificed; chief among them, the notion that everything we see must clearly and immediately resonate with familiarity.

This patterning procedure, while generally effective if cries of excitement as girls discover patterns for themselves are any guide, nevertheless remains somewhat unsettling for me as a teacher of mathematics. There is something lacking in the derivation that is not immediately obvious. The appeal to patterns is seductive, but in this case we have no real answer to the question, “Must the pattern continue?” That is, while the process has pedagogic value, a judgement must be made as to whether this value outweighs its mathematical shortcomings. The question is, what are we trying to teach?[37]

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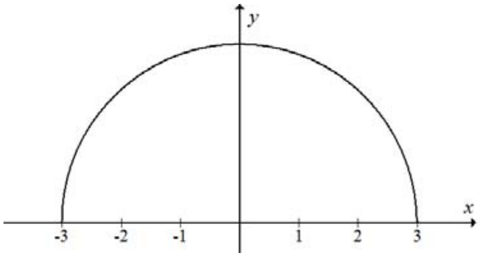
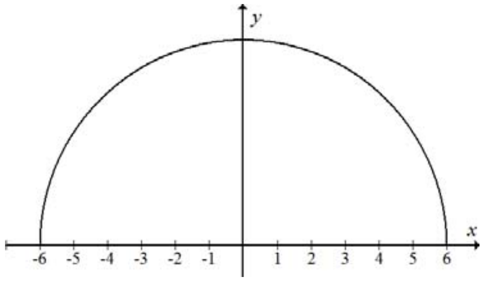
[37] Balanced, as ever, against, “Whose mathematics is it, anyway?”

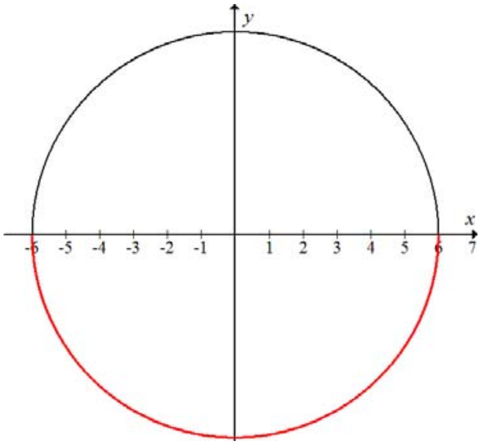
In order to accommodate multiplication of negative integers Lakoff and Núñez "stretch" (p. 89) the grounding metaphors by introducing a new metaphor, that of rotation. To multiply by $^{-}1$ is to rotate 180º anticlockwise about the zero point on the number line. Thus, for example, to multiply the number 3 by $^{-}1$ "means" to rotate from 3 to $^{-}3$.



To perform $^{-}2\times 3$ one must determine $^{-}1\times\left(2\times 3\right)$: since $2\times 3=6$ we find that rotation (multiplication by $^{-}1$) takes us to $^{-}6$. Note the natural way in which our language supports the motion aspect of the metaphor: we say, "*Takes us to….*"



In similar fashion, then, multiplying a negative by a negative, say $^{-}2\times{}^{-}3$, can be interpreted as $^{-}1\times\left(2\times{}^{-}3\right)$. Since $2\times{}^{-}3={}^{-}6$, we must rotate 180º from $^{-}6$ and thus arrive at 6:



Thus, under the rotation metaphor, ${}^{-}2 \times {}^{-}3 = 6$.

What is happening here? The process, say Lakoff and Núñez (2000), is a metaphorical blend:

> We form a blend in which we have both positive and negative numbers, addition for both, and multiplication for only positive numbers. To this conceptual blend we add the new metaphor for multiplication by negative numbers, *which is formulated in terms of the blend!* That is, to state the new metaphor, we must use
>
> - negative numbers as point-locations to the left of the origin [the zero point on the number line],
> - addition for positive and negative numbers in terms of movement, and
> - multiplication by positive numbers in terms of repeated addition a positive number of times, which results in a point-location. (pp. 91–92, authors' italics)

This particular approach to multiplication by negative numbers does have the potential to be pedagogically useful. We can see how it will help students successfully manipulate multiplications. We can see how it will afford a means of conceptualising multiplication by negative numbers. We might even recognise in it a useful, consistent, introduction to the idea that multiplication by the imaginary number i in the complex plane is interpreted as an anticlockwise rotation through

90º. It is not, however, easy to see that it will help students understand *why* the product of two negatives must give a positive. It will encourage students to produce the right result, but it has the flavour of a trick, of something new introduced after the fact to "make visible" what is known, but not to explain what is known. It also rather begs the question, "What is the *y* axis" in the diagrams I have shown. That is, what is the meaning or significance of the "space" through which one rotates?

To some extent, Lakoff and Núñez (2000) seem aware of these difficulties, for they state that,

> The metaphor for multiplication of negative numbers uses a natural cognitive mechanism. However, it does not arise from subitizing and counting as part of some natural activity. Instead, it is added to fit the needs of closure, which are given higher priority than consistency with subitizing or counting. (p. 92)

Closure is a fundamental concept in mathematics. Actions that break closure lead to expansions of structures with the possibility that closure will be achieved. Thus natural numbers, not closed under subtraction, yield the integers; which in turn, not closed under division, yield the rational numbers. We can ask for a psychological explanation of this, but in any event, it would appear to be basic mathematical behaviour to investigate closure, which really amounts to a thoroughgoing, not to say tautological, disentangling of consequences out of prepositional assumptions.

When Lakoff and Núñez introduce the rotation metaphor for multiplication by negative numbers, they break with their practice of demonstrating how physical interaction with our environment and the nature of our brains lead us to mathematics. Here it appears that the explanatory direction is reversed. That is, rather than showing how mathematics emerges from embodied cognition, it is shown how mathematics can be fitted into the embodied cognition paradigm. We must ask ourselves whether this approach, taken this far, achieves any better explanatory outcome than "traditional mathematics".

In the case of multiplying negative numbers, a traditional algebraic verification could proceed thus. Without introducing the rotation metaphor, we make do with three entailments of grounding metaphors:

> A number added to its negative gives zero, $a + {}^{-}a = 0$, the additive inverse;
>
> A number multiplied by zero gives zero, $a.0 = 0 = 0.a$;
>
> The distributive law, $a(b+c) = ab + ac$.

These statements "make sense" (statements arising out of grounding metaphors generally do) and they are sufficient to lead to the conclusion that the product of two negatives is positive, for, to take as an example ${}^{-}4 \times {}^{-}2$, we can note that $4 + {}^{-}4 = 0$. In this case, we can multiply this way of expressing zero by ${}^{-}2$ and obtain zero:

$$(4 + {}^{-}4) \times {}^{-}2 = 0.$$

The distributive law enables us to remove the brackets in the conventional way:

$$4 \times {}^{-}2 + {}^{-}4 \times {}^{-}2 = 0.$$

A consideration of these terms reveals that we have $4 \times {}^{-}2$ added to ${}^{-}4 \times {}^{-}2$ to give zero. But we know that the first of these terms is $4 \times {}^{-}2 = {}^{-}8$. Since the second term is added to ${}^{-}8$ to give zero, the second term must be 8. That is,

$${}^{-}4 \times {}^{-}2 = 8.$$

Thus we see in this example that the product of two negatives is necessarily positive. The argument can be generalised to any integers.

This derivation, however, borders on the abstract. It is, in my experience, somewhat challenging for students, and this is possibly because it is the one of the first times, if not the very first time, that consequences of accepted statements are pursued to produce an unexpected result, by which I mean a result that was not intuitively obvious.

It think it likely, moreover, that difficulties also stem from a lack of context—whence abstraction—and a related lack of familiarity. I am reminded of the "wax on, wax off" scene in the film *Karate Kid* (1984), where the young boy played by Ralph Macchio was set to work polishing. Tremendous persistence was demanded of him to apply a decontextualised rote procedure when he had hoped to learn a martial art. Only some time later did the physical significance of the training—the meaning of "wax on, wax off"—become clear, as the movements of waxing transferred to technical hand movements in the context of a fight. We teachers of mathematics are susceptible to criticism for asking our students to employ the equivalent of "wax on, wax off" where no fight scene ever eventuates.

The algebraic verification given above is hardly obvious. It does not usually occur to students of their own volition; certainly I have never encountered a student who had developed it without direction. Does it *explain* why two negatives multiply to give a positive? I think that it does, insofar as the three basic statements are accepted and it is agreed that the "usual" meanings of multiplication and addition are understandable. Does it give meaning? Well, yes and no. As I have been suggesting, it rather depends on what you demand of "meaning". I suspect that for some, meaning will be attributed only when immediate appeal can be made to the senses and to experience derived from those senses. The training of the mathematician on the other hand, as we have seen with Victoria, demands that we allow meaning to grow as one's hermeneutic domain expands to encompass deeper and more abstracted ideas. It is a form of failure to jettison a mathematical argument on the grounds that "it doesn't make sense" before one has learned to create the affective and cognitive space in which sense inheres. Creating the space takes time. One is required to become so familiar with concepts that they, themselves, become as manipulable as "real" objects. An algebraic argument, then, autopoietically determines the boundaries that determine its sense.

MANIPULABLE OBJECTS AND THE HERMENEUTICAL HELIX

A persistent difficulty in reading Lakoff and Núñez is the temptation to interpret their ideas as a more or less standard recapitulation of the structural relationships of dependent mathematical ideas. This is not, however, their intention. In laying out a series of conceptual metaphors, replete with entailments, they are attempting to generate an elucidation of how we achieve mathematical thought. A key component of their program, indeed, is the elucidation of *meaning*, but it is not necessarily the kind of meaning that a mathematician would recognise. They do not ask about the logical meaning of this or that mathematical concept, so much as how conceiving can be understood as meaningful to the thinker.

For the teacher this is significant. When my students demonstrate a typical case of overgeneralising linearity by equating $\log(x+y)$ with $\log x + \log y$, stating that $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$, or by writing $(x+y)^2 = x^2 + y^2$; or when an attempt to solve an equation founders on the failure to understand functional notation so that $\log 5x = 2$ leads to the curious equation $5x = \frac{2}{\log}$, we are left to ponder how it is that intelligent children can so misunderstand the structural characteristics of the symbols in play. Divorcing such puzzlement from a mathematical account and allowing a simpler, more grounded interpretation such as "objects can be moved" has the potential to yield insight into errors, if not into formal practice.

I was once analysing the equation-solving process with my Year Eight students, not merely to achieve a solution to a simple equation such as $6x = 17$, but to articulate in quite precise terms (Clarke, Xu, & Wang, 2010; Silverman & Thompson, 2008) the underlying subtleties: inverse operations, logical validity versus truth, maintenance of equivalence, and order of operations. I asked the girls how they *imagined* the transition from $6x = 17$ to $x = \frac{17}{6}$. They responded in terms of a formal step: "We divide both sides by 6!" None spoke of "moving the 6 across."

There is a sharp distinction to be made here. On the one hand, the girls expressed a logically valid and safe method. They were not convincing in their execution when pressed with other equations, however. Their method was secure only in that they had embraced the need to "do the same thing to both sides." When several potential steps presented themselves they struggled to know the order in which to complete them. Neither was their ability to appreciate the context of an equation robust. Answers were rarely compared against the equation for suitability, the preferred check being an inquisitive look at their teacher.

We can interpret this presentation as evidence of a lack of meaning in the equation solving process. The girls were relying on learned procedures. They were hesitant and error-prone.

An alternative method of solution, whereby 6 is "moved across", appeals to a physical reality—we can move objects. We can, moreover, readily understand that objects sometimes move along particular trajectories, so that $6x = 17 \Rightarrow x = \frac{17}{6}$ and $x + 6 = 17 \Rightarrow x = 17 - 6$, for example, need not be confused after initial practice. In terms akin to Lakoff and Núñez's (2000) metaphorical approach, the numbers (including *x*) serve as objects and the preserved inference is that objects can be moved in particular ways. The *meaning* inheres in the fact that we have an evolved capacity to understand movement. It *transfers* from the experiential to the mathematical: it is mapped from the source domain to the target domain, we might say. It is preserved with the metaphor of motion. It becomes *mathematically meaningful* when the target domain ceases to be envisioned as a target, when the equation solving process assumes its own particular quality in the practice of the learner.

John Mason (1989) finds the same effect. He discusses a process of abstraction by considering a case study in which a shift of attention is discernible, in which the foreground recedes and the particulars shift as the target of interest alters. In the initial phase, a mathematical sequence is of interest, but before long, the sequence is but an example of a class of sequences, and the attention shifts to the class of sequences as a whole. The class of sequences itself becomes an object as the initial sequence had itself begun as an object:

> When forms become objects or components of thought, and when with familiarity they become mentally manipulable, becoming, as it were, concrete, mathematics finds its greatest power. (p. 2)

In a tone that pre-empts Lakoff and Núñez somewhat, Mason theorises that

> the uses of the word *abstract* in mathematics by both novices and professionals refers to a common, root experience: an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression *as* an expression of generality, to seeing the expression *as* an object or property. Thus, abstracting lies between the expression of generality and the manipulation of that expression while, for example, constructing a convincing argument. (Mason, 1989, p. 2, author's italics)

The fleeting moment, the "delicate shift of attention", can be read as the "seeing" that occurs when "the penny drops", when a connection is made with some well understood concept that is rooted, ultimately, in grounding metaphors through a sequence of conceptual blends and metaphorical mappings across schemas. In dynamic systems terms, this signals the emergence of a relatively stable neuronal configuration, robust under minor perturbations.

I was once supervising a class of Year Seven girls. Several were given a set of Olympiad questions. The first question was:

> You have five-cent and thirteen-cent stamps. Their total value is exactly one dollar. How many five-cent stamps do you have?

A girl came to me and said, "I don't have any idea how to get started." I looked at her; she appeared blank, unsure, wanting. I suggested that a useful strategy can be to play with the problem, to get to know it. It is hard, looking at some problems, even to identify precisely where a difficulty lies. In this case I suspected that she did not appreciate that the word “exactly” in the question carried the key to seeing the way into the problem.

We imagined that we had but a few five-cent stamps in order to see what it would tell us about the number of thirteen-cent stamps required. We supposed we had eight five-cent stamps. She worked out that this would total to 40 cents, and it was obvious enough to her that 60 cents would be left over from the dollar. I asked her how many thirteen-cent stamps she would have. Her lack of familiarity with multiples of thirteen kept the key to the problem from her for a little longer, but as she worked with the numbers the moment came when she realised that she could not use thirteen-cent stamps to make 60 cents. Thus meaning in the question was revealed to her. It was not merely that one was required to find a way of constructing one dollar from an unknown number of stamps. The very possibility of constructing the dollar determined the combination of stamps.

From that moment she ventured on a search for multiples of thirteen. In fact, she continued to trial multiples of five and check the remainder from 100 for divisibility by thirteen, whereas she could have asked herself for a multiple of thirteen that ended in a five or zero: $5 \times 13 = 65$ is the case to consider, leaving her with seven five cent stamps (35 cents).

When she left me she was able to articulate what it was she was looking for: “Numbers so that thirteen goes into the remainder.”

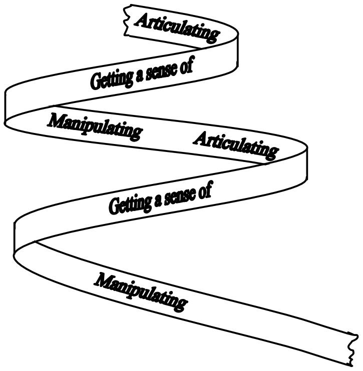
Of interest here is the thought that from her perspective there was no initial understanding of what even constituted the problem. She was unable to approach it. The context told her it was a problem—it was printed on a page of problems, after all—but she had no means of engaging with it. My role was to draw her into it. Mason’s mathematical sequence example refers to a similar situation, described by Brown (2001) as comprising

> a helix where the experience of a mathematical situation is seen as passing repeatedly through the “getting a sense of”, the “manipulating of” and the “articulating of” the problem. This has much in common with the notion of the

> hermeneutic circle which might be used here to describe the tension between interpreting a problem and making statements in respect of it, which in turn influence subsequent interpretations. (p. 52)

Mason proposes that the helix be

> used to describe the experience of moving from *manipulating* objects (physical, pictorial, symbolic, mental) to *getting-a-sense-of* some feature or property of those objects, to *articulating* that property as an expression of generality, to finding that expression becoming a confidence-inspiring entity which can be manipulated and used to seek out further properties. I suggest that the process of abstracting in mathematics lies in the momentary movement from articulating to manipulating. (Mason, 1989, p. 3, author's italics)



Mason’s purpose, in part at least, is to describe the process of abstraction, but this particular Olympiad question was not constructed to elicit an abstraction from our Year Seven student. The helix is, nevertheless, useful. As Mason says:

> it is intended to provide an image, linking the labels *manipulating, getting-a-sense-of, articulating* with personal experience, and so awakening a teacher to the value in noting such transitions explicitly and allowing students time to convert hard-won expressions of generality into confidently manipulable objects. (Mason, 1989, p. 8, author's italics)

Tall, Thomas, Davis, Gray and Simpson (2001), in similar mien, identify three ways in which mental objects can be constructed, in a hierarchy of increasing sophistication:

> - perceived objects, arising through empirical abstraction from objects in the environment (and later may be given successively subtler meaning through focusing also on verbal descriptions and definitions to construct platonic objects),
> - procepts, which first involve processes on real-world objects, using symbols which can then be manipulated as objects, upon which operations may be performed and symbolized in the same way,
>
> and
>
> - axiomatic objects, conceived by specifying criteria (axioms or definitions) from which properties are deduced by formal proof. (p. 19)

In all this we recognise the primordial significance of the body in the world. Abstraction signals a transitioning from what is grounded in sense experience, through metaphorical mappings, into mental objects and ultimately, as familiarity grows and permits, into purely mental constructs far removed from their antecedents. Meaning is preserved, but not necessarily consciously so. In the most formal cases, the thinker might resist assigning any kind of meaning, preferring instead to rely on strict procedural manipulation, in order that no methodological error can intrude, but even here we might speculate that the processes of formal operations are traceable to grounding metaphors. That is, while we can observe a shift in the nature of meaning as the level of sophistication of mathematical concepts increases from that which is nearly grounded in real world phenomena toward a more formal mathematical interpretation, we need not assume that the formal interpretations *supplant* the grounded cognitive unconscious. The fundamental processes of metaphorical movement continue to underlie even the most sophisticated mental imaginings.

CHAPTER SIX MATHEMATICAL EXPERIENCE

MATHEMATICIANS' COMMENTARIES

If we reflect on the phenomenology of our own thought, we will recognise that we do not always speak to ourselves with an "inner voice". Indeed, not only can we move and act without "saying" so, we can ponder and wonder with an unvoiced imagination. In like manner, formal symbolic language can fail to serve in cases of advanced mathematics. Symbols, often but not always voiced, may take on the role that words formerly fulfilled; but even they can fall short of what is required imaginatively to manipulate an idea. Even in situations of problem solving, the processes of mental manipulation sometimes exceed the capacity of the language to serve. Consider the following description given by Warwick Evers, an expert problem solver and former teaching colleague. I had asked him to reflect upon the question of problem solving. He presented me with an elaboration of Pick's Theorem—a theorem relating polygonal areas to points in a lattice—and included the following commentary on his attempts to make a particular type of proof:

> I thought about this for a good day, with little doodles on paper, seeking a way of ensuring that the division would always be possible in a non-intersecting polygon. Every argument that I made could, after a lapse of a few minutes or a few hours, be shot down with a counterexample. What I was doing was focusing on questions of convexity and concavity, but being distracted by the lattice setting. When I finally started to think polygons and convexity related to non-reflex and reflex angles, earlier arguments fell into place and no special cases were required (unlike my earlier considerations). In fact, the last piece fell into place as I watered the garden!
>
> Okay—a deal of experimentation, doodling, but nothing too pressured. The last piece falls into place when the mind is quite relaxed and quite divorced from mathematical thought. (Personal communication, 7 May, 2008)

The significance that Warwick attributes to being relaxed accords with my own experience of solving problems and with my observations of students over the years. A relaxed mind appears freer to think constructively than a mind under pressure, as we have already noted.

I asked Warwick to elaborate a little upon his thought processes. His reply came as follows:

> If we begin with visualisation, I draw the line at “hard outlines” such as occur in Euclidean geometry. If a really complex figure is involved I simply switch off, pick out one or several “fundamental” properties embedded in the figure and switch to trigonometry, vectors or complex numbers. Combinatorial problems I usually “see” as blurred or wavy lines connecting different objects or nodes, and the idea is to try to “unravel” at least some of the lines or to see how some invariant property is preserved through all the interactions. It is the invariance which is (generally) the thing which is important. (Personal communication, 14 May, 2008)

We see here the hallmark of an expert. The attention to invariants signals that Warwick, while not necessarily knowing precisely what to look for (remember Meno’s question), does have an expectation of what type of object will prove useful. He is able, moreover, to do without language that requires words:

> As to my conception of the objects of my thoughts—they are very simple. Points and lines, but not too many or they become a confusion—points on lines and intersections of lines are okay. Sometimes one object contained within another can imply that one appears to be implied by the other. (Personal communication, 14 May, 2008)

Not only that, but he is able to “switch off”, which suggests a degree of control over the broad parameterization of his conscious thought processes, while maintaining a faith in his ability to find helpful associations.

Language is most valuable at the beginning and end of the process, when one communicates with others—at the moments the problem is first encountered, that is to say, and at the time its solution is presented:

> Language comes into play in this sense—what does the question “mean”? By that, do I understand at least some of the implications of the problem, and how some of the conditions interact? This means that language is important at the “doodling” stage. Language is also important when at some stage you need to interpret what you have found—that is, put it into some meaningful context—and also to convince yourself that, say, what you see as a necessary condition is a necessary condition and that you haven't confused it with a sufficient condition. In other words, [language enables you to see] what has really been accomplished. (Personal communication, 14 May, 2008)

This aligns with Hadamard’s (1954) initial and final phases of preparation and verification.

If it were easy, we would all be expert puzzlers! Expertise demands payment, however, and in the case of mathematics it is often through accumulated hours that one learns to organise one’s mind and through which the possibility of thinking useful thoughts accrues:

> Experience is very important—it gives you more starting points and also suggests more likely strategies for success. The problem is, you have to solve some problems before you can build up a stock of useful experiences! Acquiring the experience takes time and intensive (not necessarily successful) effort. (Personal communication, 14 May, 2008)

Mason, too, identifies what an achievement it is to conceptualise a mathematical idea and use it fluently to enable oneself to learn still more:

> Articulation of a seeing of generality, first in words and pictures, and then in increasingly tight and economically succinct expressions, using symbols and perhaps diagrams, is a pinnacle of achievement, often achieved only after great struggle. It turns into a mere foothill as it becomes a staging post for further work with the expression as a manipulable object. (Mason, 1989, p. 3)

Warwick's description of his own thought process accords neatly with that given by Albert Einstein in a letter to Jacques Hadamard (1954). It is worth quoting Einstein at length so that the similarities with Warwick's experiences become clear:

> (A) The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined.
>
> There is, of course, a certain connection between those elements and relevant logical concepts. It is also clear that the desire to arrive finally at logically connected concepts is the emotional basis of this rather vague play with the above mentioned elements. But taken from a psychological viewpoint, this combinatory play seems to be the essential feature in productive thought before there is any connection with logical construction in words or other kinds of signs which can be communicated to others.
>
> (B) The above mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will.
>
> (C) According to what has been said, the play with the mentioned elements is aimed to be analogous to certain logical connections one is searching for. (pp. 142–143)

Both speak of a process of thinking that can be distinguished from language and seems, indeed, more closely related to the movement of figures or shapes. The transition to abstraction, it would appear, need not be to the exclusion of the "real world" so much as it requires the abandonment of formal language in deference to engaging with the world, where it is to be understood that "the world" here includes objects of the thinkers' own making. The conundrum Warwick identifies as the acquisition of experience meshes with this interpretation, because it is through deep and prolonged engagement that the world of thinkable objects is constructed.

Leone Burton (1998) surveyed practicing mathematicians in order to discern something of how they themselves determine when they come into knowledge, and found her respondents answered with a "lapse into lyricism and the use of metaphor" (p. 132), comparable to the prose of Warwick and of Einstein. Describing the responses of the mathematicians, she noted the importance of *feelings* in motivating research:

> You gain pleasure and satisfaction from the feelings which are associated with knowing. These feelings are exceptionally important since, often despite being unsure about the best path to take to reach your objective, because of your feelings you remain convinced that a path is there. Such conviction can feed enquiries that go on often over years before a resolution of the problem is completed. (p. 134)

Feelings were also important in determining *when* you know. As one male lecturer put it:

> When I think I know, I feel quite euphoric. So I go out and enjoy the happiness. Without going back and thinking about whether it was right or not but enjoy the happiness [*sic*]. When I discover something, I just enjoy the feeling. It is almost pointless to try and check it because you are so euphoric that you cannot possibly check. (p. 135)

Warwick had said it like this:

> You usually know that the pieces have fallen into place when a "simplicity" sweeps over everything—it is an intuitive feel. Of course, you then need to go over things again to make sure that you are not deceiving yourself—if an erroneous early step has been taken then the "simplicity" itself will be in error! (Personal communication, 14 May, 2008)

Burton summarises the experiences of her respondents:

> The rich sense of pleasure gained from achieving an Aha!, even one which is later followed by an Oops! permeates their descriptions as does the sense that every new journey is, indeed, a new journey and you face again all the hazards that have been faced before. You might accumulate experience which helps you on your way but,

> at root, you are an explorer reliant upon your own strategies, expectations and fallacies, and those of your fellow travellers. (p. 135)

In the end, then, we are describing a process of exploration. Mathematics arises as we learn to extend our engagement with real objects into thinkable mathematical objects. We think with these objects in ways that are metaphorically aligned with ways in which we engage with the world. As far as our unconscious, tacit thought processes are concerned, the thinkable mathematical objects *are* the world. The extent to which we are prepared to undergo this labour is determined in part by our motivation. Coupled with a propensity to pursue entailments of our thoughts, just as we are drawn to pursue horizons, this metaphorical entanglement, this bleeding of inner and outer worlds, produces a corpus consistent with the physical logicality of the very world it models.

In the following pages I describe a personal experience of wrestling with a mathematical problem, of feelings of uncertainty and the process of coming to know. Little will be lost in skipping over the mathematical details in order to garner the sense of the experience, if that is your wish.

AN ANATOMY OF PROBLEM SOLVING: A PERSONAL REFLECTION

Warwick asked me to solve this problem: given that $f: R \rightarrow R$ is continuous, that $f(x).f(f(x)) = 1$ and that $f(1000) = 999$, find $f(500)$.

My initial concern was to understand the problem. Rather, since the structure was clear enough, to become familiar with the particular features of the function *f*. I was, moreover, concerned to find a solution strategy. Indeed, this interest in *how* to solve the problem somewhat occluded my early perceptions of the function itself. This is suggestive of where I stood in relation to the problem. I am interested in how we think mathematically and among my students I have advocated that there is value in being aware of how one is thinking about a problem. Value there may be, but thinking about thinking is not the same thing as solving a problem.

So there I was, confronted with a curious function, armed with limited information and unclear of how to proceed. What I was confident of was that there was sufficient information in the presentation to ensure a solution could be constructed. This is because: (i) that is part of how the problem solving game is played in mathematical discourse—in contradistinction to the application of mathematics to practical problems, where the very formulation of the problem becomes a problem in itself, and where there is no guarantee of the possibility of progress, much less success, however that may be defined; and (ii) the problem, I was told, was sourced from an international competition. Moreover, and for the same reasons, not only could I be confident that the presentation contained *sufficient* information, I could be fairly confident that the problem contained only *necessary* information (I have mentioned this heuristic earlier, in the "Guess the Ages" problem). That is to say, it was likely that I would need to employ all the given information to construct a full solution. It is possible that a problem be posed with irrelevant, "spoiler" information, but that can increase the level of difficulty to a point where the pedagogical value of the exercise is rendered hostage to factors that threaten always to impede progress. These factors include anxiety, impatience and a lack of confidence.

My initial meanderings, then, were introspective. There was an early fear of failure. Every challenge requires one to find a way between the Charybdis of inertia and the Scylla of error, and the first imperative is therefore safely to overcome stasis whilst avoiding excessive commitment to a passage that will cause one to lose one's head! A heuristic often employed to escape this peril is to doodle, to experiment with the features of the problem, to test simple cases—as I had suggested to the Year Seven girl facing the 5-cent and 13-cent stamp problem. The purpose is twofold. The commencement of action displaces the stasis characteristic of anxiety; and one is able to become familiar with the features that characterise the problem. As one gains confidence the former consideration wanes.

In this case the initial investigations revealed little. It was known that $f(1000)=999$ and, since $f(x)\,f(f(x))=1$, I could quickly discern that

$$\begin{aligned} f(1000)\,f(f(1000)) &= 1 \\ 999\times f(999) &= 1 \\ f(999) &= \frac{1}{999} \end{aligned}$$

I could then use this result to establish that

$$\begin{aligned} f(999)\,f(f(999)) &= 1 \\ \frac{1}{999}\times f\left(\frac{1}{999}\right) &= 1 \\ f\left(\frac{1}{999}\right) &= 999 \end{aligned}$$

So there was a symmetry whereby $f(999)=\frac{1}{999}$ and $f\left(\frac{1}{999}\right)=999$.

The circularity revealed was at once encouraging and disappointing. It seemed that the path thus far trod had met its end where it began, and what had it revealed? It *felt* as though $f(x)=\frac{1}{x}$ was the "shape" of the function, in which case $f(500)=\frac{1}{500}$, but this was not the full story, for $f(1000)=999$. If I were to imagine another value, *a*, for which the function existed, the validity of which was guaranteed since the function was continuous over the real numbers, I could say that

$$\begin{aligned} f(a)\,f(f(a)) &= 1 \\ f(f(a)) &= \frac{1}{f(a)} \end{aligned}$$

If I allowed that $f(a)=b$ for some real *b*, then this would mean $f(b)=\frac{1}{b}$, but I also knew $f(1000)\neq\frac{1}{1000}$. Something was wrong—or misunderstood. What next?

Being immersed in solving a problem can involve protracted moments of forgetfulness. One attends deeply to the phenomenon at hand; so much so that one forgets oneself, becomes unaware that one is thinking. There can be times in which one's reflective awareness becomes suspended. On this occasion, however, I was quite conscious of my state of thinking. I had the sensation of watching myself address the task with more interest than I was giving to the task itself. My conscious deliberation on the problem was consequently stultified, halted. I could not simultaneously attend to both the way in which I was attending and to that to which I had set out to attend. When I became conscious of these moments I stopped, acknowledged the difficulty, and attempted to redirect my attention toward the problem by modifying my angle of attack.

What directions were there for me? I was aware that I had limited experience on which to draw, and that it was likely that there were approaches of which I was unaware that might prove fruitful, even essential. I recalled from the few similar problems that I had encountered how it had proved important to explore properties of the function in play by asking questions about its characteristics: is it injective, is it increasing, is it monotonic, where are its zeros? I knew this particular function was continuous and expected that to be important.

This suggested an avenue down which I could make my way, but progress proved fitful. I readily enough realised that the function had no zeros, and that, since f was continuous and $f\left(1000\right)$ was positive, the function must be positive for all x. But little else seemed apparent.

Having spent time on the problem, I would stop and do something else. I have learnt that releasing a problem, provided effort has been spent in becoming acquainted, does not necessarily spell an end to thought. On returning to the problem, one is able sometimes to see a new approach, or revise more astutely steps already undertaken. On occasions, an idea will present itself unbidden and unexpectedly, while attention is elsewhere. This phenomenon lends a certain romanticism to the discovery of solutions, understandable by anyone who has suddenly and with surprise recalled that for which they had sought in vain mere hours before. I suspect it is rarer than its notoriety suggests, for it is in itself so noteworthy that its occasional manifestation will displace from our collective memory the many other occasions in which hard work and attentiveness are the sole precursors of success.

In any event, cessation, diversion and recapitulation are features of a solution process necessitated by the interventions of daily life. Quiet hours in which to reflect and attend are rare enough that one cannot await their blessing. As it transpired, it was while supervising a class for an absent colleague that I was able to construct a solution. The class, once settled, was sufficiently on task that I could give time to the challenge of unravelling my discordant efforts and refining a more coherent path.

I summarised what I knew in a few brief notes. I reflected on the significance of continuity. I prepared a sketch graph in which the few points I had were plotted, and drew an undulating curve between the points, as continuity allowed. To this moment I had been distracted by the question of why 500 was the designated domain element. It had seemed worth noting that I was not actually required to determine f, only $f\left(500\right)$, and it was not without possibility that knowledge of $f\left(1000\right)$ would prove fundamental to determining $f\left(500\right)$ without knowing fully the nature of f. This thought, however, had so far earned me no reward.

In the end, though, it was a combination of the crude graph and one subtle shift of attention that enabled me to see the solution. The benefit of the graph was that it encouraged me to see that if $f\left(1000\right)=999$ and $f\left(999\right)=\frac{1}{999}$, then there must be some $x\in\left(999\,,1000\right)$ such that $\frac{1}{999}\leq f\left(x\right)\leq 999$. That is, I could employ the Intermediate Value Theorem. This observation pointed me quickly to its application on the more relevant interval: if $f\left(\frac{1}{999}\right)=999$ and $f(999)=\frac{1}{999}$, then there must be some $x\in\left(\frac{1}{999},999\right)$ such that $\frac{1}{999}\le f(x)\le 999$.

Then came the shift of attention. Rather than ask for y such that $f(500)=y$, I could ask for x such that $f(x)=500$. That there must be such an x was guaranteed by the Intermediate Value Theorem, so writing $f(x)=500$ for some x leads to

$$f(x)\,f\left(f(x)\right)=1\Rightarrow 500f(500)=1,$$

from which it emerges that $f(500)=\frac{1}{500}$.

This result had been suggested earlier in the piece, but it had lacked *validity*. Now I knew it was right, for I had employed the essential property of continuity and at the same time had not required an almost wishful appeal to $f(x)=\frac{1}{x}$. In fact, I could now see that for any $x\in\left[\frac{1}{999},999\right]$ the function would have the form $f(x)=\frac{1}{x}$, but if $x\notin\left[\frac{1}{999},999\right]$ then other rules would apply. For example, say $x>1000$ such that $f(x)=y,\ y\neq 0$. Then

$$f(x)\,f\left(f(x)\right)=1\Rightarrow y\times f(y)=1\Rightarrow f(y)=\frac{1}{y}.$$

From this it follows that

$$f(y)\,f\left(f(y)\right)=1\Rightarrow \frac{1}{y}\times f\left(\frac{1}{y}\right)=1\Rightarrow f\left(\frac{1}{y}\right)=y=f(x).$$

That is, $f(x)=y\Rightarrow f(y)=\frac{1}{y}\Rightarrow f\left(\frac{1}{y}\right)=f(x)$.

In other words, given some arbitrary value of the function, y, the entailment is that an interval is determined in which $f\left(x\right)=\frac{1}{x}$.

Let us reflect for a moment on the sense in which thoughts come to the conscious mind. During this solution process, as is so often the case, there was a genuine sense of interplay between the conscious and the unconscious, between the attending, directing, calculating mind and the blithe, spirited, freer part of the self that reveals only so much as can be made apparent in flashes, insights and sudden appearances.

I do not imply that the two are separate. I would rather imagine that in this interplay we see two facets of one being, two self-parts. On the one hand we have the language-imbued, directing self, that part of me who speaks in words. On the other, we have a less readily knowable self, for whom formal language appears not to be the mode of communication. When this self-part communicates with my linguistic self-part, it is revelatory, immediate and unspoken. I am learning to trust this self-part, to be at one with it, to allow it purchase on *my sense of self*. I begin to suspect it is this facet of my being that is able to recognise patterns, to recognise form in that which is presented to the being. This suspicion is framed by introspective, phenomenal apprehension, and further encouraged by interactions with my students. The moments in which forms and patterns are recognised, in which recognition erupts, have the feel and appearance of being gifts. When we suddenly comprehend, it is as though that part of ourselves which is given over to language is suddenly provided with the gift of having something about which it may speak.

SEMIOTICS: SIGNS AND SENSE

I now want to consider how signs and symbols can be understood within an embodied cognition epistemology. It was suggested earlier that there appears to be some temptation to misunderstand expressions such as $(x+y)^2$ and $\ln(x+y)$. Are such misunderstandings comprehensible within the framework of embodied cognition?

The function of symbols in mathematics, taken as a study in semiotics, should be of interest to the teacher concerned with how children learn. Duvall (2006) advances the observation that mathematics is distinguished by the paramount importance and large number of semiotic representations; and by the lack of access to real world objects to support semiotic representations. This last consideration leads to a cognitive paradox particular to mathematics: how can learners distinguish an object represented semiotically from the semiotic representation used if they have no access to the mathematical object apart from the semiotic representations? The paradox exists, for Duvall, because,

> in order to do any mathematical activity, semiotic representations must necessarily be used even if there is the choice of the kind of semiotic representation[;] but the mathematical objects must never be confused with the semiotic representations that are used. (p. 107)

It is the cognitive conflict between these requirements, says Duvall, that is a source of mathematical incomprehension for many learners. And yet, as we make the attempt to recast our thinking in light of the embodied cognition paradigm, we are forced to question whether there is such validity in drawing distinctions between signifiers and signified, at least in the case of mathematics. We have seen that Heike Weise (2003, 2007), for example, posits that number words *function as numbers*, in which case numbers are not objects per se, but tools serving purposes more or less arcane according to the cultural milieu of the thinker. Luis Radford (2001), writing within a semiotic framework, expresses this interpretative stance by noting that

> instead of seeing signs as the reflecting mirrors of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required

> by the contextual activities in which the individuals engage. As a result, there is a theoretical shift from what signs represent to what they enable us to do. (p. 241)

Font, Godino, Planas and Acevedo (2010) draw attention to the way in which we employ the mathematical-object-metaphor to approach the nature of mathematical entities: "The use of the object metaphor facilitates the transition from the ostensive representation of the object to an ideal and non-ostensive object" (p. 15). When contemplating the use of zero as a place-holder in our place-value numeration system, for example, the object "0" can be regarded as an ostensive object which functions to point to a non-ostensive mathematical object, the number zero. The non-ostensive object need not even exist, neither in the sense in which ostensive chairs and tables exist, nor even in the in the sense in which other mathematical objects, such as triangles, can be said to exist. The ostensive object $\frac{x}{0}$, for example, can be taken to point to a non-existent, non-ostensive mathematical object. "The existence of well-established ostensive objects that represent non-ostensive objects that do not exist facilitates the consideration of the non-ostensive object as different from the ostensive object that represents it" (Font, et al., 2010, p. 17). Thus, it is hoped, such awareness, both from teacher and student, can enhance understanding.

The foregoing discussion treats of mathematical objects, but this is not the same as according any particular ontological status to these putative objects: it is, rather, to indicate that via inference preserving mappings, one is able to think with mathematical objects metaphorically by applying the logical structure, which is to say the conceptual scheme, of a physical domain to the mathematical objects in question.

The cognitive paradox Duval draws our attention to is significant, nonetheless, because it highlights the question of why learners struggle to comprehend mathematics. Duval distinguishes between *treatments*, which are operations one can carry out whilst remaining within a semiotic system (applying algebraic operations, for example, to solve an equation) and *conversions*, by which one changes semiotic representation systems (from an algebraic system, for example, to a visual, or diagrammatic system). He argues that it is in the conversions, more so than in the treatments, that we will find clues to the cognitive complexity of mathematics:

> The first problem of comprehension in learning mathematics is a problem both of recognition and discrimination. When facing two representations from two different registers [semiotic systems], how can one recognize the same object represented within their representative content? In other words, how can a student discriminate in any semiotic representation what is mathematically relevant and what is not mathematically relevant? (Duval, 2006, p. 115)

We have previously seen an example in which the one mathematical idea, the sum of the first *n* natural numbers—the mathematical content—was expressed in several different forms (kinaesthetically, in using natural language; symbolically, and diagrammatically) for a group of Year Nine students. Whether in such situations a learner will recognize the unity of the mathematical content is an interesting question. If we ask ourselves whether the distance between your house and mine expressed alternatively in terms of the time taken to walk, the street blocks to cover, or the kilometres to traverse always represents the same content, we approach the point of Duval's semiotic concern. A pragmatic, empirical answer might be to analyse neurological states as these alternatives are considered and to see if there appear to be differences in activation responses. Given the nature of such inquiries, however, there would doubtless be substantial difficulties in interpreting any such responses.

At this time it is by no means clear to me, at least, that such a thing as unitary mathematical content exists. That is to say, when I present a mathematical lesson to a class of students, particularly when I open the presentation to several modes of thinking, I do not presume that one veridical mathematical idea is being learned. The attitude adopted is, rather, one of allowing the possibility that the varied life experiences of my students, including their formal educational experiences, will elicit somewhat individual responses. In an echo of the specificity-plasticity dialectic introduced earlier, it is likely that there will be broad agreement between students on what has been learnt, but that nuanced versions arise out of idiosyncratic considerations.

This is not to say that *any* mathematics will do. Mathematics has standards by which it permits judgements to be made, and it is these, ultimately, that bear upon a student's praxis:

> Most cases of warranted mathematical belief are cases in which an individual's belief is warranted in virtue of its being explicitly taught to him by a community authority or in virtue of his deriving it from explicitly taught beliefs using types of inference which were explicitly taught. This reduces the task of explaining most cases of warranted mathematical belief to that of accounting for the warranted belief of community authorities. (Kitcher, 1984, p. 225)

But we are more concerned here with the manner in which the student understands mathematical ideas, and in particular, with tacit knowledge. The embodied paradigm permits a certain license to each individual to frame learning by her own lights, according to her particular preferences. At the same time, the grounding metaphors function, as their name suggests, to ground conceptualisations against pragmatic reality. That said, the further from the ground ideas rise, the greater the effort required to ground them. But then, as the complexity of interrelated ideas mounts, so the sophistication of the thinker rises. In no small measure, it is the ability to maintain parity between the complexity of the mathematical ideas and the sophistication of the thinker that determines whether the ideas learned will commend themselves to the mathematical community.

FICTITIOUS PARTS

In keeping with Edmund Burke, whose work I introduced earlier, when I hear "cat" I do not imagine a furry animal with whiskers; not really. Rather, I let the sound of the word pass over me and I recognise that I know this word, that I could use it should I feel want or need. I have the potential to summon cat-knowledge: I could conjure recollections of cats I have known, synthesise cat features, cat colours, cat names, and all the cat-miscellanea that it has been my pleasure or pain to have experienced. That is what "cat" means to me. It is a token of potential action.

Other words are like this, too. "Number" can suggest particular examples, say three, or pi, each of which in turn betokens recollections of things past; it can summon itself as a thing to be contemplated, as in, "What is number?" It can bring to mind images of glyphs, of Babylonian cuneiform, of Roman inscriptions on high stones. In short, the word encourages me to drift on currents of associations.

This is what a thinkable object confers: the potential to think with a web of associated thoughts collectively related to one designation. What I do with these objects is much as I do with any object. I squeeze, hold, weigh, feel, throw, drop, balance, caress them. I manipulate them in my grasp. It is all I can do; I am only human. That all this takes place in my imagination, or endogenously, is no matter. I hold rocks, I hold convergent series; I peel onions, I solve equations. If at some time you ask me to explain what I am doing then I must wind my way back through my objects, unpacking tokens to find a language we share before I can convey some sense of my actions. My mind, like my hands, is indifferent to nouns; it only *acts*.

I find it interesting to note how ideas such as this have appeared now and then in history. Jeremy Bentham (1748–1832), for example, seems to have imagined the actions of the mind somewhat similarly in his *Essay on Logic*, in which, as part of his program of codifying thought suitable to utilitarian purposes, he ascribed the term *faculty* to the "fictitious part" of the mind implicated in thinking on abstract entities. The names given to the faculties—the perceptive, judicial, retentive, abstractive, imaginative, inventive, tactic, attentive, and so on—were intended as tokens of action, however, not so much as names of real objects, the faculties being, as Bentham describes them, "fictitious entities":

> But as to the several correspondent faculties, these belong to the class of purely fictitious entities, feigned in virtue of an irresistible demand, for the purposes of

> discourse. ... The word which is the name of ... [a faculty] is not indicative of anything but the *operation* which, when called into exercise, it performs, if it be an active faculty, or the impressions which, if it be a passive faculty, it receives. (Bentham, 1843b, pp. 223-224, author's italics)

Thus a half-real memory is construed within a fictitious retentive faculty. The significance of the faculty rests not in what it putatively *is*, for it is conceived as a linguistic device, but in what it *does*; and what it does is to perform an *action* of remembering. In terms of embodied cognition, I would suggest that the actions of the so-called faculties of the mind on "abstract" entities are of a kind with actions on real objects.

Elsewhere, in his *Essay on Language*, Bentham expressed what could be read as a view of language consonant with the embodied account:

> Throughout the whole field of language ... runs a line of what may be termed the immaterial language. Not that to every word that has a material import there belongs also an immaterial one; but that to every word that has an immaterial import there belongs, or at least did belong, a material one.
>
> In a word, our ideas coming, all of them, from our senses, from what other source can the signs of them—from what other source can our language come? (Bentham, 1843a, p. 329)

The thoughts with which we think, the thinkable objects, that is to say, find their provenance in the senses. It takes time to create a thinkable object, though, as we have seen; becoming acquainted with a new idea exposes us to the need to attach each thought to another. When I explore some unfamiliar mathematical theory it is like exploring new terrain. I must keep hold to the landmarks—I need to know how I can get back to where I began. I hear a word and must translate it through actions: "I apply this process and that definition," until, after long practice, I am able to think with this ensemble of actions as a betokened whole. The word becomes the marker of a thinkable object. I can explore more deeply now, I am more efficient. I am a miner delving, delving the better the deeper I go, the more words I betoken.

But I am too easily distracted. I hear a new word and I abandon my mine to another day: "Yet knowing how way leads on to way, I doubted if I should ever come back" (Frost, 2001, p. 105) .

Experts, on the other hand, are deep miners; they build extensive, yet narrow, disconnected pits in the landscape. Once in a rare while they discover diamonds. Generalists flit more widely across the surface, dipping into pits now and then, marvelling at the depths and mysteries therein, but dispersing their surveys further afield. Some others yet find life sufficiently worth the living in quiet enclosures, never wandering too far from old homes.

In the hills where I like to cycle there are coal mines, long abandoned, secret beneath quiet paddocks. The highland cattle do not know they are there. I know they are there, for I have seen the photographs. I have seen the grainy black and white images of men beside entrances, of the old tramway that carted the coal to ships waiting in the bay. The tramway has left its reminder in the landscape. You can still follow its course through the hills, like a thread in history it ties then with now.

All of our thoughts, all of our actions, are threads in history, ties in time, memories in the landscape. And what will the world allow? How will the world remember us?

PART FOUR VERIFICATION

Now to this final part in which I reflect upon what it means for me to work with an embodied epistemology. What does it afford my teaching? Is my understanding of how I should orient myself toward my students affected?

I draw on an evolutionary account of morality in part to situate my understanding, and to realign the question of what it means to be moral. I consider a felt, aesthetic response as significant to a determination of how I will act, and I move toward a view in which I feel less inclined to dissociate *my self* from my students.

In all this, the felt experience of my students transcends learning and the experience of the whole becomes the lesson learned.

CHAPTER SEVEN AN ETHICAL ORIENTATION

Therefore I am still
A lover of the meadows and the woods,
And mountains; and of all that we behold
From this green earth; of all the mighty world
Of eye, and ear, — both what they half create,
And what perceive; well pleased to recognise
In nature and the language of the sense
The anchor of my purest thoughts, the nurse,
The guide, the guardian of my heart, and soul
Of all my moral being.

William Wordsworth (1956),
Lines Composed a Few Miles above Tintern Abbey

EMBODIED COGNITION, ETHICS AND THE TEACHING OF MATHEMATICS

But beauty, real beauty, ends where an intellectual expression begins.

Oscar Wilde (2009, p. 6), *The Picture of Dorian Gray*

As I progress in my study of mathematical concepts as an embodied form of cognition, I am led, inevitably, to the practical question: what am I to do with this perspective? For educators this question informs an orientation toward our students, and as such it is essentially an ethical question. If I hold that mathematics, as we study it, is a consequence of our being human in the world, am I to behave any differently to the students in my care than if I believed in some other view of mathematics, or if I had no especial thoughts for the ontological status of mathematics at all? My response must be in accord with the principles of embodied cognition if it is to be coherent with the point of the question. My attempt to formulate a response, then, can begin with a consideration of evolutionary accounts of moral behaviour.

The problem for the evolutionist, and so for the embodied paradigm in general, is summarised thus by Charles Taylor (1989):

> It is a form of self-delusion to think that we do not speak from a moral orientation which we take to be right. This is a condition of the functioning self, not a metaphysical view we can put on or off. So the meta-construal of the neo-Nietzschean philosopher—"in holding my moral position, I am imposing (or collaborating in the imposition of) a regime of truth on the chaos, and so does everyone"—is just as impossible as the meta-construal of the empiricist—"in holding my moral position, I am projecting values on a neutral world of facts, and so does everyone". Both are incompatible with the way we cannot but understand ourselves in the actual practices which constitute holding that position: our deliberations, our serious assessments of ourselves and others. They are not construals you could actually make of your life while living it. (p. 99)

For me, however, the phrase "neutral world of facts" is bereft. No such world is compatible with the world envisioned in the embodied paradigm, but is rather an artefact of a way of seeing in which we disembody ourselves from our life's context. Taylor's empiricist is not an empiricist interested in embodied cognition. If there is a neutral world of facts, it is not a world consistent with the epistemological framework of this thesis.

In developing an evolutionary account we are obliged to render notions such as "being moral" with care. It is easy to question how moral behaviour could have arisen, but the asking is a form of assertion in disguise. What can we possibly mean when we assign the term "moral" to behaviour? If we imagine a species evolving from some simpler form, at what point in the evolutionary journey do we feel it sensible to speak of moral action? When we speak of morals we need to be careful to distinguish entailments that attach to the term—it is tempting to confuse morals with putative sources, for example, as when we claim some authority that tells us what it is moral to do; as it is to blur distinctions between descriptive and prescriptive accounts of behaviour.

A further confusion becomes apparent when we ask ourselves what *a* moral could possibly be. That is, when we allow the use of the term as a noun, rather than as an adjective. This is not an unfamiliar act: we do it repeatedly in mathematics and in discussions of concepts in general. It was noted earlier that Jeremy Bentham (1843b) was careful to recognise his fictitious faculties as aids to discourse, for example. We take what is in essence an action—*thinking* mathematically, for example, or *being* moral—and abstract the sense in which the adjective is applied to the status of a concept, itself an abstraction.

Our capacity to recognise and thence to categorise according to perceived similarities, where the similarities are themselves determined by our intentions toward those objects of perception—if we are hungry, for example, we see a plant differently than if we are looking for material to fashion a shelter—determines how we shape the world conceptually. The concept "cat" becomes an acknowledgement that this creature before me is as like those other creatures that we call "cat", and at the same time, a signal that I am in such a frame of mind as to label it "cat" and not "hunter", or "potential fur cap", or "source of disease pathogens" and so on. Juxtaposing Linda Smith's (2005) suggestion, cited earlier, that concepts are purely hypothetical constructs employed to denote emergent stabilities in the complex dynamical system proposed as our cognitive self against Bentham's fictitious faculties, encourages us to see concepts as fictitious aids to conversation: we *conceive*; we *think* with thinkable objects—but we do not *have* concepts in the way we have pens.

My expression of a concept, then, becomes a signal of my behaviour, my attitude to some entity—my autopoietic perspective. To say that there is a concept is to say that I bear a certain relationship to that which is conceived, that I understand some entity for what it is for me, what in Heideggerian (1996) terms is called "handiness" (p. 76), or readiness-to-hand. It implies recognition, which itself requires a spatial and temporal consonance, not to say physical apparatus with which to make the determination; and intention, in that my attitude to that recognised thing will determine how I see and interpret it. It is autopoietic in that the boundaries of this-and-that determine each other, and the quality of handiness is attenuated by the richness or paucity of experience.

A similar sense, derived from biological considerations, is captured in James J. Gibson's term "affordances" (Gibson, 1979; see also Scarantino, 2003, for a discussion):

> The *affordances* of the environment are what it *offers* the animal, what it *provides* or *furnishes,* either for good or ill. The verb *to afford* is found in the dictionary, but the noun *affordance* is not. I have made it up. I mean by it something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment. (p. 127, author's italics)

Reference to concepts as entities actually encourages an act of disembodiment, for as Mark Johnson (2007) notes,

> this exclusive attention to stable structures can entice us to succumb to the illusion of fixity, that is, the illusion that meanings are fixed, abstract entities that can float free of contexts and the ongoing flow of experience. Such a strategy of exclusion leaves out the body and our situated, embodied practices, along with all their intricate meaning. (p. 80)

Out of some putative fixity, then, we draw concepts including a multitude of mathematical forms, but then, as Johnson succinctly observes, "No sooner does *concept* (used as a noun) make concepts into things than we must find a *place* for concepts to exist or be" (Johnson, 2007, p. 90, author's italics), and we have created a duality. In terms of the fixity-change dialectic, one lesson of embodied cognition is that nouns can disguise verbs. If I feel warm towards you I am said to *have* affection for you, but if I am to understand this in embodied terms, I must break the solidity of the affection-entity into the transience of a relation. "Things fall apart," as Yeats (2003, p. 19) tells us—little is so fixed as we might imagine.

In like manner, it is doubtless convenient to speak of morals, as surely as it is to speak of triangles and integrals, but to seek habitation for such forms is to forget that they arise solely in the course of our living meaningful lives, that they are a part of what we call our *umwelt*, or, in autopoietic terms, our domain of effective behaviour, which is itself a qualitative acting-out of our existence in a real, physical world. Such forms are usefully manipulated in language and symbol, but coalesce as we draw near, like mercury puddling, until they are indistinguishable in the whole of our being, lost in the seeking through levels of description. This, indeed, can be read as a précis of the embodied cognition philosophy.

In his essay, *Explanation and Practical Reason*, Charles Taylor (1995) introduces us to strongly and weakly evaluated moral goals. Strong evaluations are such that to be denied their goal is to be less worthy; weak evaluations are such that to be denied (an ice cream, for example) is no big deal and does not diminish one:

> Something is a moral goal of ours not just in virtue of the fact that we are de facto committed to it. It must have a stronger status, that we see it as demanding, requiring, or calling for this commitment. While some goals would have no more claim on us if we ceased desiring them, such as my present aim to have a strawberry ice cream cone after lunch, a strongly evaluated goal is one such that, were we to cease desiring it, *we* would be shown up as insensitive or brutish or morally perverse. (p. 37, author’s italics)

We might readily admit that there is an abundance of evidence that we *feel impelled* by moral demands, that we *experience* morals as categorical imperatives; but this is not to say that the imperatives we feel constitute anything more than feelings. Feelings can be considered to be of two types. There are those that arise in response to direct engagement with phenomena, such as when we feel the heat of an iron, or the cool of a breeze; and there are those that are more distantly related to phenomena, such as when I feel sad at the news of a loss, or confused by an argument. If I am confused then there are antecedent causes of that confusion, which might include my previous knowledge of the issues at stake, the formulation of the argument that I perceive and so on, but there is no “confuser simpliciter” in the environment that has occasioned the confusion. The confusion felt is created within me in response to a circumstance in which I have become implicated—being in the argument, that is to say. To claim that the argument itself has caused the confusion, however, is not quite right, for others might hear the same argument and be not the least confused. *I* am implicated. In like manner, when I feel that I am bound to behave in some or other fashion, we need not presume that the feeling of being bound renders truthful any assertion that there is indeed, a binding force in the universe that causes me to feel so. There is a situation in which I am to be found which prompts the feeling of obligation, but the feeling arises in virtue of it being *my* feeling. There is, in short, reason to question whether morals as categorical imperatives (those that we would will to be universal laws: "Act only according to that maxim through which you can at the same time will that it become a universal law," in Immanuel Kant's (2012, p. 38) formulation) are some actual things we feel, or just the feelings themselves.

This stands before the observation that whereas I might be the only one confused by an argument, the draw of moral feelings is wide, and it is likely that a common core of moral feeling is universal among humans. This could include obligations such as "we ought to provide for our children". If the feeling is universal, then may it be likened to the feeling of the cool breeze? We *all* feel the cooling breeze because there is, in fact, a breeze to feel. But there is a logical flaw here. *If* there is a phenomenon to feel, then all normally constituted humans will feel that phenomenon directly. There being a simple environmental stimulus is *sufficient* to occasion universal feeling, that is to say. It is not, however, necessary. A universal feeling can arise in virtue of the universal constitution of humans. We can posit that we *all feel* that we are obliged to care for our children because that is how we have evolved, in contradistinction to the claim that it is *right* to care for our children.

This is no mere sophistry. If our account favours this possibility then the question of what we are to do is informed by the account itself. This is because such an account turns moral behaviour into an analysandum. We will be able to ask *why* we feel certain ways, as surely as we can validly ask why we feel hungry, or tired, or angry, and (in principle) maintain a realistic expectation of an answer.

This turn to a search for explanations is consistent with what Frans de Waal and Pier Ferrari (2010, p. 201) describe as a shift from a top-down to a bottom-up approach. We can look at altruistic behaviour, for example, and try to determine which species exhibit it; and yet this very endeavour is subject to our definitions of what altruism is. There is a temptation to frame meanings so that we wind up searching for behaviours or capacities that are, in virtue of our own biases, uniquely human. Recall that Descartes did as much in the seventeenth century and found cognition to be uniquely human, and only possible through the agency of God.

An alternative approach is to search for underlying mechanisms that enable behaviours and capacities in order to learn how such behaviours evolve and function—we do it for walking, flying and digesting; why not do likewise for empathy, sharing, and all forms of cognitive or putatively moral behaviour? After all, as de Waal and Ferrari put it, "*De novo* appearances of cognitive capacities are apparently as unlikely as *de novo* anatomical features" (de Waal & Ferrari, 2010, p. 205): advanced cognition is as unlikely to appear freshly in humans as are thumbs.

As to whether we *ought* to act in such and such a way, the basis on which we approach the question moves to new, better prepared ground: to the extent that we do seek to judge, we do not seek judgement against what is *right*, since no ontologically *right* action need be presumed to exist, so much as what *feels* right or what is judged to be *better* for some putative end—a form of hypothetical imperative, indeed (Kant, 2012). In the final analysis, since the account is built upon the *universal feeling* of what ought to be done, the discernment of the feeling will speak first and any rational calculus will stand or fall on pragmatic terms; that is, some form of social contract is called into play.

We can pass over the question of how to understand that social contract, save that we make certain observations. Within the literature there is a tendency to revere rationality at the expense of felt dispositions. Even where evolutionary accounts are invoked, it seems difficult to loosen a hold on “morals” as things to be explained in *rational* terms. Fritz Allhoff (2009) attempts to address what he identifies as a false dichotomy that emerges when morals are discussed: that of the realist-nihilist positions. The realist posits that morals instantiate some kind of *a priori* facts about the universe; whereas the nihilist, in brief, denies this and claims that there are no moral facts. And yet what we see as facts cannot be dissociated from the intentions with which we look. Allhoff agrees “that naturalistic accounts of the moral sentiments are not very congenial to Platonic or Moorean conceptions of the good” (p. 100), but adds that “there are conceptions of the good other than the Platonic or Moorean ones, and I argue that at least some of these conceptions are unchallenged by naturalistic accounts of the origin of moral sentiments” (p. 100). We might find elements of the account that Allhoff develops somewhat discordant with that which is outlined here, but let that rest—the observation that we are susceptible to being inveigled into accepting yet another dichotomy is salutary. We can choose to say morals have no ontological status or we can assign morals ontological status in virtue of our experiential bringing-into-being: the point turns merely on what we rationally intend by ontology, and that is precisely the kind of discussion that threatens to reduce the embodied understanding of our *being* moral.

Returning to an earlier theme, if we take it to be the case that processes of rational thought emerge out of our bodily engagement in the world, then since our emotional systems contribute significantly to the flow of that thought (see below), there is a deep sense in which the ground of our moral processing is to be found in *a recapitulation of our place in the world*. What we believe to be right is *right for what we are*, and were we to be different then it is to be anticipated that our sense of moral obligation would be altered.

Put differently, the construction of any social contract that fails adequately to acknowledge the best account of *brute facts* about our being cannot be sustained or defended as promoting a social good. All varieties of ethics are subject to the same test: how will people actually behave in the face of any ethical contract? This observation is not made to promote a laissez faire abandonment of obligations—we *feel* obligations, regardless—but rather to claim that the construction of any contract of oughts is an *a posteriori* act of knowing, where the prior knowing is embodied in our tacit selves, so that the practical test determines its application: if any ethical contract requires us to act in ways that do not accord with what we feel to be right, it will not pass the test. A social contract that will not be employed is no contract at all, merely a cultural decoration.

David Hume (2003) famously observed the tendency to drift from statements of what *is* to statements of what we *ought* to do:

> In every system of morality which I have hitherto met with, I have always remark'd, that the author proceeds for some time in the ordinary way of reasoning, and establishes the being of a God, or makes observations concerning human affairs; when of a sudden I am surpriz'd to find, that instead of the usual copulations of propositions, *is,* and *is not*, I meet with no proposition that is not connected with an *ought*, or an *ought not*. This change is imperceptible; but is, however, of the last consequence. For as this *ought*, or *ought not*, expresses some new relation of affirmation, 'tis necessary that it shou'd be observ'd and explain'd; and at the same time that a reason shou'd be given, for what seems altogether inconceivable, how this new relation can be a deduction from others, which are entirely different from it. (p. 334, author's italics)

Any proponent of embodied cognition needs take care, therefore, not to commit the same slip. And yet, I do not think that there is much danger. If embodied cognition offers anything to the formulation of moral principles, it is just this: the possibility that the basis on which principles are determined or evaluated be in accordance with the way in which we have evolved to be. For me this is no less evident than the simple observation that any meal prepared for human consumption had better take account of how we process food: a meal that appears beautiful and is yet unpalatable, or worse still, poisonous, is no meal at all.

The universal experience of the force of moral feeling offers no draw away from embodiment. Morals can be deemed real and universal in that we *all really do feel* obligations. Joyce (2000, 2009) and Ruse (1986), for example, speak of an error theory of morality, in which we are deemed wrong to believe in morals as if they were real in and of themselves, but wherein it is the very *belief* in morals and the concomitant feeling of being bound that makes morals what they are: effective means of regulating human behaviour to capitalise upon selection pressures.

Any rational calculus of how we are to act, moreover, will demand attention to the historicity of any situation. Whether the turn is to "constitutive goods" of the type Charles Taylor (1989) proposes, wherein exemplars and the evidence of history is corralled to the purpose of framing an ethical response, or to metaphysical beings, or to ancient texts or pragmatics or to the Enlightenment or televangelists is certain to remain a matter of debate. This much will not change. What does change however is the background against which the debate is carried out. No longer does one search for that which is *right*; rather, one seeks a passage that will accommodate one's innate disposition to feel discommoded by a situation *and yet act* meaningfully in pursuit of some hypothetical end. The embodied turn is to action over classification, to the *experience of being moral* over morality per se.

The significance of emotions in the discernment of moral behaviour was clear to Hume (2003): "Morality ... is more properly felt than judg'd of: tho' this feeling or sentiment is commonly so soft and gentle that we are apt to confound it with an idea" (p. 335). It was feelings, not rationality, that impelled moral behaviour. More recently, Antonio Damasio (2000) has described the importance of feelings and the *feeling of feelings* in moving to specific actions. His analysis draws a distinction between emotions and feelings. Emotions, more or less independent of consciousness, are background states against which more transient feelings have their play, whereas feelings are those bodily responses that we can know. Feelings, in essence, are our way of knowing our emotions:

> In practical terms this means that you cannot observe a feeling in someone else although you can observe a feeling in yourself when, as a conscious being, you perceive your own mental states. Likewise no one can observe your own feelings, but some aspects of the emotions that give rise to your feelings will be patently

> observable to others. ... The fabric of our minds and of our behavior is woven around continuous cycles of emotions followed by feelings that become known and beget new emotions, a running polyphony that underscores and punctuates specific thoughts in our minds and actions in our behavior. (pp. 42–43)

With this distinction in mind, the acknowledgement of moral feelings becomes a recognition that we are so constituted as to be affected emotionally by circumstances, which is to say that we respond not merely intellectually but through our bodies to situations that conjure a moral attitude. We respond in a visceral way to stimuli and become aware of our responses through our feelings.

Not only this, but the very comportment of our body acts to influence the nature of our emotional response to a stimulus and our modally-related emotional responses are recreated in acts of recall (see Niedenthal, 2007, for an overview). This is an active field of ongoing research, but the implications a teacher might draw include the possibility that by conspiring to have students adopt appropriate bodily states at the time of learning, the emotional valence of lessons and hence recall might be enhanced. This research has the potential to go quite some way to explaining the almost banal observation that engagement and interest contribute favourably to learning.

Given that feelings are our way of experiencing our emotional states, what other functions do they serve? That is, what benefit or cost obtains from knowing our emotional states? We might suspect that there are "lower" forms of life that have emotions without experiencing conscious awareness of those states and are yet functioning effectively in their domains. Damasio (2000) thinks that our awareness of our emotional states is an important step toward being able to make plans, particularly when it leads to an awareness of being aware:

> The simple process of feeling begins to give the organisms *incentive* to heed the results of emoting (suffering begins with feelings, although it is enhanced by knowing, and the same can be said for joy). The availability of feeling is also the stepping stone for the next development—*the feeling of knowing that we have feelings*. In turn, knowing is the stepping stone for the process of planning specific

> and nonstereotyped responses which can either complement an emotion or guarantee that the immediate gains brought by emotion can be maintained over time, or both. In other words, "feeling" feelings extends the reach of emotions by facilitating the planning of novel and customized forms of adaptive response. (pp. 284 – 285, author's italics)

It is this ability to "feel feelings" and to embrace the feeling as fundamental to what we are that offers to enlighten our considered ethical response. In an autopoietic turn, this capacity extends our effective domain of meaning. The recognition of feelings as the basis of our moral response opens the possibility of evenly evaluating options for action.

Thomas Huxley (2009), in his *Ethics and Evolution*, first published in 1894, was among the earliest to reflect upon the problem of linking evolution with moral behaviour. There was a genuine concern that the evolutionary process, or cosmic process, as he termed it, would stand as an analogy to inform our ethics, that we should organise our ethical response to emulate the cosmic process. But the analogy faltered on a confusion between the idea of "fitness" as an evolutionary term, and "best" as an evaluative term:

> There is another fallacy which appears to me to pervade the so-called "ethics of evolution." It is the notion that because, on the whole, animals and plants have advanced in perfection of organization by means of the struggle for existence and the consequent "survival of the fittest"; therefore men in society, men as ethical beings, must look to the same process to help them towards perfection. I suspect that this fallacy has arisen out of the unfortunate ambiguity of the phrase "survival of the fittest." "Fittest" has a connotation of "best"; and about "best" there hangs a moral flavour. In cosmic nature, however, what is "fittest" depends upon the conditions. (p. 80)

Against this, Huxley was of the view that it is our responsibility to actually *combat* the evolutionary process:

> Social progress means a checking of the cosmic process at every step and the substitution for it of another, which may be called the ethical process; the end of

> which is not the survival of those who may happen to be the fittest, in respect of the whole of the conditions which obtain, but of those which are ethically the best. ... The practice of that which is ethically best—what we call goodness or virtue—involves a course of conduct which, in all respects, is opposed to that which leads to success in the cosmic struggle for existence. (Huxley, 2009, pp. 81-82)

In this Huxley falls prey to the very forces he seeks to address: in opposing the evolutionary process to the ethical process, Huxley is taking as his model the garden, which in order to be maintained against the ravages of the environment demands attention and nurturing. Left to the forces of nature, it is overrun and lost. Our society is like the garden and, while it is the evolutionary process that has provided the plants, it is the gardeners who maintain the order:

> Laws and moral precepts are directed to the end of curbing the cosmic process and reminding the individual of his duty to the community, to the protection and influence of which he owes, if not existence itself, at least the life of something better than a brutal savage. (Huxley, 2009, p. 82)

Huxley is committed to an ethical teleology that stands in distinction against any modern evolutionary account. To suggest that virtue is opposed to evolutionary processes is to proffer a conundrum, indeed. About the best that an evolutionist can do with this is to allow that some of the gross selection pressures that have led to us being what we are have ameliorated in some relevant ways since prehistoric times, since we have assumed greater control over our environmental circumstances, so that we now evaluate our ancient inclinations in light of modern, pragmatic concerns. The conundrum bears attention, because, in truth, we sometimes do feel tensions between what we *feel* we ought to do and what we are encouraged or compelled by society to do. Should we allow abortion? Should we supply relief funds to strangers suffering far-off crises? Should we permit euthanasia? On a more mundane (and less technologically inspired) level, I can crave the taste of meat and yet cringe at the thought of killing an animal to obtain it. Should I accept the work of a butcher if I am unprepared to perform the function myself? But the conundrum as posed is no real puzzle: it is simply the recognition that tensions can exist between what we feel we ought to do and what we ultimately decide to do; and this is exacerbated when what we decide to do requires maintenance against what are regarded as “natural” feelings, as may sometimes be the case in complex, modern societies. If, however, following Huxley’s gardening analogy, one insists that we maintain our well-being through *battle* with evolution, one exposes a dualistic worldview in which ethical conduct will be seen as something beyond or other to what is deemed “natural”.

The source of the tension might lie, in part, in the complexity of society. In evolutionary terms, we expect that our behaviours are fitted to the maintenance of social groups of sizes that predominated in the early, formative ages of our story. Our sense of kinship will have been determined to a significant extent by the conditions prevalent in those times. In this modern era, however, when our populations are larger, feelings formerly reserved for closer associates are extended via a process of rational inclusion—what is right for my brother is right for my countryman, and so for all men. We may argue that it is *morally right* to do such and such, and yet struggle to *feel* it so—witness the variety of responses to asylum seekers attempting to reach our shores, for example. To some extent we evaluate each other according to the degree to which we are able to project our feelings of what we feel is right beyond those with whom we live in close alliance. Those who “fear the other” are deemed morally immature, while those who embrace and regard the other by extending the moral feelings they have for near-kin are deemed to be thinking, and possibly acting, morally.

When we endorse an embodied paradigm we no longer expect to find a strict isomorphism between the articulation of our moral concerns and the felt experience of our being moral. We understand that the articulation suffers the constraints of language; that language as a tool and arbiter of rational cognition does not so much show us the way to right behaviour, as it attempts to describe, categorise or justify what is already felt to be proper—less so the word-made-flesh, as the flesh-made-word. This is not to suggest that language cannot be turned to the purpose of articulating moral concerns, but rather that such discursive explorations might resonate more profoundly if they were to privilege some more "in tune" modes of expression before reifying disembodied elements of our felt experience. I therefore entertain the proposal that it is fruitful first to conceive of being moral as an aesthetic response.

THE AESTHETIC OF BEING

> My opinion is this: that deep thinking is attainable only by a man of deep feeling, and that all truth is a species of revelation.
>
> Samuel Taylor Coleridge, (1957, p. 126),
> *Letter to Thomas Poole*, March 23, 1801

Artists understand the possibility that morality could be an aesthetic response as more than a conceit. When the artist commits his body to the medium he finds a form of articulation no less esoteric than the written word, but less articulate. Not in words but in some closer, more direct engagement with the senses does the artist speak. The populist catch-cry that, "I don't know art, but I know what I like," conveys this sense of direct, personal appeal. Artists allow the voices of shape, space, time and tone to speak directly to experiential being. When the internationally renowned guitarist Tony McManus was asked on ABC radio (Lomax & Cooney, 19 March 2011) how he had selected which tune was right to play on which of various guitars at his disposal, he replied that the guitar had talked to him.

> I went ... with specific ideas of some of the pieces I wanted to record on particular instruments, and other ones. Paul Heumiller selected a bunch of guitars, some of which I had never heard of, and I would pick one up and just play and play and play and then bingo, something would happen. Like there's a melody of airs on the album—I can't remember, I think it's track two—and that literally came together an hour before it was recorded, and it was, you know, the guitar was talking to me.

When pressed to explain he said,

> What I was hearing were these beautiful layers of overtones. One of the things I like doing is hitting a harmonic and then finishing the chord somewhere else on the neck [demonstrates]. So I started noodling around with that, and then, just listening to the sound of the guitar, this version of *Donal Og*, a beautiful ballad ...—this is a Donegal version—suggested itself and then it led into a Robert Burns tune, called *The Lea Rig*. Other times I had very specific ideas of what I wanted to record, and which instruments [to play them on].

In this response McManus describes how the instrument becomes a part of his being at that time. His language attempts to convey the unboundedness between man and guitar, so that touching the instrument, hearing the sounds, *making* sounds in response, feeling his fingers on the neck, catching the tonal flux in his core, he conjures a dynamic feedback system looping through his musical repertoire, enabled by his technical mastery, constrained by mundane facts of space, time and the need to record, wherein there is no simple sense in which one can extricate man from guitar, memory or mastery from the ensemble. All together determine the autopoietic whole in an act of self-being-and-sustaining.

As observers we can identify the McManus-and-guitar system, but we rarely see it in its ontogenetic entirety. We will see him playing the guitar, but we will not see the life, the practice, and the learning that has brought him to this time and place. We will see the guitar, but we will not see the artistry that has gone into the production of that instrument, nor trace its ancestry through ages to antiquity. We will hear the notes, feel the music, but we will not recognise the folk traditions that have shaped the sounds through generations of playing, singing, living. If, however, we are so minded, we can become more learned about the elements of the system. This is the essence of *appreciation*, whereby we educate ourselves so that we become more able to understand the depth and complexity of the system before us.

As long as we are quiet, we will not overly perturb the system. While there is time, McManus will play on. Within the system, however, there are constant perturbations—it is the perturbations, the “noodling”, that help to define the system through its ability to respond, adapt and maintain its unitary identity. Neither is the system immune to our presence. If we cry “Stop!” the spell is broken and the system collapses—the music ends. If we merely dim the lights the system will suffer the perturbation and adjust itself to the altered conditions. We—as exemplars of that which is *not* the McManus-guitar system—are *structurally coupled* with it. We respond to it and it responds to us, and while ever we perturb each other within degrees that can be accommodated, the music plays on.

When McManus speaks of *Donal Og* suggesting itself, and of it leading into *The Lea Rig*, he uses passive language, and by it we understand that it was not he making the choices, not he ordering the tunes. Not the “he” that we would identify with “McManus”, that is to say, but rather, that it was something that had risen within the system through his being with the guitar, and his bringing to the instrument memories of a life immersed in traditional music.

The coherence of this unitary system, this McManus-guitar system, can be understood by us as observers, or from within the system itself. *We* can approach only from without, but within the system there is the capacity for a degree of conscious self-awareness to emerge—and yet this very act of becoming self-aware threatens to destabilize the system (see Maturana & Varela, 1992, pp. 97-98, on the domains specified by the structure of a unity). For the most part, we can imagine the player is in a state of “flow” (Csikszentmihalyi, 1997), but the turn to self-acknowledgement collapses this state, since a self-forgetting is characteristic of deep immersion in activity.

Allied with this turn to self-awareness is the drive to articulation. We imagine that speech begins when there is something to say—when we identify that we are in a position to exclaim, to describe, to utter forth our predicament—and when the tacit language of artistic expression or body-talk lacks the precision we seek. We might yet take care not to presume that speech-making derives from conscious awareness, however, for it is possible that proto-speech evolution recruited cross-modal neurological features, motor systems and mirror networks (Arbib, 2008, 2010;

Corballis, 2010; Rizzolatti & Arbib, 1998) and that language and action systems are intimately related today (Kühn & Brass, 2008). In either case, what is common is a structural coupling—as Shaun Gallagher (2005) has reminded us, we are always already coupled with the other—so that speech making in its most rudimentary characterisation is articulation as perturbation beyond the unity that is the self. As the sound of our own speech and of the response elicited in others is perceived, we become aware of our implication in the maintenance of the identities that constitute the self and the other—and since we are not the other, we hear ourselves *as* ourselves (Jardri et al., 2007).

This is not to be confused with the emergent identification of some *res cogitans*—such a disembodied being never did nor ever could exist. It is, rather, to acknowledge that within the embodied unity that is the self, a causally significant partial awareness persists, such that one is able to infer agency in speech. Speech, indeed, becomes a fulcrum on which the weight of ipseity is leveraged. Antonio Damasio (2006, p. 243) asserts that, "Language may not be the source of the self, but it certainly is the source of the 'I'."[38] But while speech encourages the sense that the self is lifted beyond the body, it is the ironic body that enables and maintains that belief.

FITTING AND FRAGMENTATION

> What is important in teaching is not the mechanical repetition
> of this or that gesture but a comprehension of the value of

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[38] Or perhaps we should say the speech encourages the sense that there *is* a self and that *it has a unitary body*. Morin (2009) provides a discussion of the intriguing first person, phenomenological account of Jill Bolte Taylor (2008, March; 2009), a neuroanatomist who suffered a stroke that left her with impaired language and inner speech abilities and concomitant self-awareness deficits. See also Mitchell (2009) and Schlinger (2009) for criticisms and clarifications.

> sentiments, emotions, and desires. Of the insecurity that can
> only be overcome by inspiring confidence. Of the fear that can
> only be abated to the degree that courage takes its place
>
> Paulo Freire, *Pedagogy of Freedom* (1998, p. 48).

I have so far sought to increase my apprehension of how we think mathematically, to order perceptions of number and speech. My simple purpose has been to render my teaching more tactful. This has led me to imagine our complex selves as implicated and coupled; as involved in living but concentrated within the deciphered world; turning to angels in our need for expression, yet tacitly alive to simple being. I have adopted the language of autopoiesis, flavoured by phenomenology, to express the way in which we construct our world even as our world informs us; and I have assayed aesthetical and moral feeling to frame an interpretation of how our embodiment is consistent with feelings of autonomy, agency and right behaviour—of what is felt to be good, in fact.

Throughout all, I have grounded my considerations by calling to mind my students. They have never been far from my pragmatic concerns, nor absent from my speculations. They challenge me still, as I continue to seek frequencies that induce resonance between our intermingled selves.

At the outset of my journey I borrowed from Thoreau to express the desire to go walking. So have I ventured through fields of study with a curious eye. All explorations are shaped by some binding principles, however, and no less this, my sojourn. My hope has been constrained—or enlivened—by my own selfish ambition to create an epistemology that will suffer my pursuit. For all its incompleteness, I have the germ of an idea here, an inkling of how mathematical thought is framed within us, by us, is a part of us—a germ with roots that arise in our pre-literate past, and run veiled and fractured through the articulations of Plato, Augustine and Descartes, to emerge in the post-enlightenment considerations of Hume, Bentham, and Burke, among others. A nascent idea that quickened in the age of phenomenology under considerations of philosophers such as Husserl, Heidegger, and Merleau-Ponty. Researchers including Maturana and Varela fostered the germinal sprout with naturalistic inquiries; philosophical reflections of cognitive science cultivated and nuanced its branches; and finally, revelations in neurological science are advancing the first suggestions of a flowering. Mathematical thought has been brought back to the body through the guidance of Dehaene, Lakoff, Núñez and many others, but our understanding of "the body" has grown so much to meet it.

It is nearly time to end this walk, but as I contemplate the paths I have wandered, I am led finally to a consideration of how I understand our very selves and our knowledge-worlds as *fitting*. How will the steps I have taken, the prospects on which I have mused, affect my teaching life? David Bohm (1996) offers guidance here, as he draws attention to the notions of fitting and fragmentation as foundational to the elaboration of an aesthetical response:

> The original meaning of this word [art] is "to fit." This meaning survives in articulate, article, artisan, artefact, and so on. Of course, in modern times the word "art" has come to mean mainly "to fit, in an aesthetic and emotional sense." However, the other words listed above show that art can also call attention to fitting in a functional sense. The fact that we are hardly aware of the syllable "art" in words such as articulate or artefact is an indication of an implicit but very deeply penetrating fragmentation in our thought between the aesthetic, emotional aspects of life and its practical functional aspects. (p. 99)

Science, mathematics, and philosophy are likewise notable for the ways in which they require a sense of fitting and are yet fragmentary ways of knowing:

> By now, philosophy has become one fragmentary field of specialization among many, but originally it meant a wholeness of understanding whose end was a kind of skill in seeing all knowledge as "fitting together." Thus, in its deepest and most comprehensive meaning, to know is an art, the impulse from which springs "love of wisdom." (Bohm, 1996, p. 100)

Bohm pursues the question of rational thought, and concludes that "reasoning is to be regarded as an art. And thus, in a deep sense, the artist, the scientist, and the mathematician are concerned with art in its most general significance, that is, with *fitting*" (p. 101). But fitting is a universal concern, and so,

> each human being *is* artist, scientist, and mathematician all in one, in the sense that he is most profoundly concerned with aesthetic and emotional fitting, with functional and practical fitting, with universal rational fitting, and, more generally, with fitting between his world view and his overall experience with the reality in which he lives. Even in his particular work he is always concerned with all these kinds of fitting, though, of course, with different sorts of emphasis. So one has to begin with a general feeling for the whole of human activity, both in society and in the individual. This is to be described as art: the action of fitting. (Bohm, 1996, p. 105)

Regarding our desire to fit, or to make things fit—to find coherence, we might say—as a primary concern enables us to recast our understanding of what it is to be moral, and even to do mathematics. This is the aesthetical response: to find ways to understand that make sense, first at an unconscious, emotional level, leading on to an awareness of the experience of emotions—to feelings, that is—thence to a conscious awareness of the feeling of feelings, and ultimately to representation in some form, be it sculptural, musical, visual; in speech, text or equation.

This allows Bohm to reflect that the "good"

> is that which fits, not only in practical function and in our feelings and aesthetic sensibilities, but also that which, by its action, leads to an ever-wider and deeper sort of fitting, in every phase of life, both for the individual and for society as a whole. (Bohm, 1996, p. 106)

This underlines the expansion of a hermeneutic horizon, or, in autopoietic terms, the broadening of an effective domain of being. Learning abstruse or arcane miscellanea can be deemed good if it opens us to greater possibilities. Knowing the divisibility test for three in base-five, to cite an example we have considered, can lead to a richer understanding of base-ten numbers. The difficulty, he says, consists in fragmentation, by which is meant the narrowing and splintering that ensues when men seek their own goods. The arterial narrowing that is the hallmark of ever-refined academic expertise might be understood as a case in point.

> To end this fragmentation is clearly of crucial importance if man is to cease to accomplish evil in the very act in which he pursues the good. So, what is needed is to pause and to inquire into the origin of this fragmentation in a mode of thought in which the separation of art, science, and mathematics from each other and from questions of morals and ethics plays a key role. (Bohm, 1996, p. 106)

The fragmentation Bohm identifies is not to be addressed through "moral exhortation, through compulsion, through being convinced of what is right, or through a new organization of society" (Bohm, 1996, p. 108), for that would be to

> try to solve the problem of fragmentation by engaging in more of the fragmentary mode of thought that produces the problem. Rather, what we have to do is to give serious and sustained attention to this mode of thought itself. And this is what we are beginning to do when we see how we have been conditioned to split art, science, and mathematics, and the desire for "the good" into separate compartments of life, so that we are not able to see the oneness of the deep impulse toward "fitting" that is behind all of these. (Bohm, 1996, p. 108)

What is required, says Bohm, is an understanding that this process of fragmentation is itself a form of "metaphysical art", a seeing of the world as it is in terms of convenient, constrained, delineated forms. For all that, and in contrast,

> It is especially important to realize that each man's thought arises in a cyclical movement in which he is exposed to the thought of other people and responds with a generally similar but somewhat different thought of his own. So it can be seen that, ultimately, all that man is, both physically and mentally, arises in his overall contacts with the whole world in which he lives. (Bohm, 1996, p. 122)

"No man," wrote John Donne in his 1624 *Meditations*, "is an island, entire of itself" (Donne, 1923, p. 98). No man, we might go further, is *a* man as such. We are implied in what we do, where and with whom we do it: "I am involved in mankind" (Donne, 1923, p. 98). Bohm's attention to the dangers of fragmentation alerts us to our collective responsibilities to each other.

Presaging a thesis of embodiment, and turning phenomenology to the arena of thought itself, Bohm concludes that what is needed is an awareness of *how thought becomes fragmented* into delineated forms together with an appreciation of the limits that pertain to such modes of thinking:

> Such attention discloses the *abstract character* of perception in terms of separate things, each with a fixed essential nature. When one sees this abstract character *as such* he is able to use this mode of thought within the limits in which it fits, without mistaking its general metaphysics for "an absolute truth about the whole of reality." And so the mind is free, at any moment, to give attention to new differences and new similarities, allowing for the perception of a new structure of "things." (Bohm, 1996, p. 124, author's italics)

This attention to "the abstract character of perception" with the express purpose of *seeing clearly* into the nature of things appears to be in concert with what Charles Taylor (1989) calls a best account in search of clairvoyance. The goal is not to make life fit some mode of reasoning, but to find some mode of reasoning that fits life:

> What we need to explain is people living their lives; the terms in which they cannot but avoid living them cannot be removed from the explanandum, unless we can propose other terms in which they could live them more clairvoyantly. We cannot just leap outside of these terms altogether, on the grounds that their logic doesn't fit some model of "science" and that we know a priori that human beings must be explicable in this "science". This begs the question. How can we ever know that humans can be explained by any scientific theory until we actually explain how they live their lives in its terms?
>
> This establishes what it means to "make sense" of our lives, in the meaning of my statement above. The terms we select have to make sense across the whole range of both explanatory and life uses. The terms indispensible for the latter are

> part of the story that makes best sense of us, unless and until we can replace them with more clairvoyant substitutes. The result of this search for clairvoyance yields the best account we can give at any time, and no epistemological or metaphysical considerations of a more general kind about science or nature can justify setting this aside. The best account in the above sense is trumps. (p. 58)

Narrow articulations can be fruitful, but their fruit can only be judged when tasted against the sensations of a full life. The embodied account seems to keep close to this in that it is fundamentally concerned with understanding meaning in terms of the experiences of lived lives; with fitting, I might now say.

Mark Johnson (2007) conveys the essence of the move away from fragmentation toward an awareness of a gestalt-based coherence in powerful prose as he recalls the effect of April light in a forested valley near his Oregon home:

> Once we are struck, caught up, seized, only then can we discriminate elements within our present situation. At this point, we may not always understand those April greens as the greens of spring oaks versus vine maples versus rhododendrons, though we understand that they are green leaves. Rather, we are simply able to differentiate colors, forms, and structures. When we see the oak-leaf green, as distinguished from the vine-maple green, we are *not* engaging in acts of synthesizing atomistic sense impressions into complex situations, or even objects. No! We are *discriminating* within a situation that was given to us whole. All of those qualities were potentially available in the situation together, and we selectively grasp some of them as salient, focal, differentiated. We are not *making* our world of objects, but we are instead *taking up* these objects in experience. In other words, objects are not so much *givens* as they are *takings*. (p. 75, author's italics)

This call for awareness and recognition is fundamental to the purpose I am pursuing here, learning how an embodied account of mathematical thinking fits within tactful teaching. Concentrated upon our fragmentary art of mathematics, I am especially conscious of potentially debilitating effects when one loses sight of the wholeness of the experience. Jerry King (1992) alerts us that, "What is needed is a real understanding of the mathematician's 'personal experience' with his subject. At the highest levels, there can be no doubt that this experience is largely aesthetic" (p. 143). Richard Hamming (1980) once proposed that a key driver in the development of mathematics is, indeed, a variety of aesthetic:

> Mathematics has been made by man and therefore is apt to be altered rather continuously by him. Perhaps the original sources of mathematics were forced on us, but ... we see that in the development of so simple a concept as number we have made choices for the extensions that were only partly controlled by necessity and often, it seems to me, more by aesthetics. We have tried to make mathematics a consistent, beautiful thing, and by so doing we have had an amazing number of successful applications to the real world. (pp. 86–87)

My considerations suggest that even at lower levels we experience *being mathematical* in aesthetic terms. Thus it is that we are susceptible to the joys of the "Aha!" and the perils of the "Oh, no!"

Mathematics has, indeed, been contemplated as an art. For Hans Freudenthal (1991) it is

> a mental art to be sure, which for most people will be closer to crafts than to sciences, a tool rather than an aim in itself, more relevant because it works than because it is certain. But why does it work? Because it is certain? Although many people trust mathematics more than it deserves, it works only when it is rightly applied. But what is right or wrong? Is there any way to verify it and if so, isn't such a verification again mathematics and—if it is—to which degree? Once one has admitted that mathematics is an art, one cannot shirk the responsibility of judging whether, in particular cases, it is being properly used or rather abused; while trying to decide, one behaves once again like a mathematician. (p. 2)

The question of judgement is central to mathematical deliberations, as it surely is to all thought, but judgement is susceptible of misattribution to logical, rational decision making. Even the use of the word "deliberation" betrays an attachment to some form of intentional choice making. Johnson (2007) encourages a remediation of this attitude, drawing on the works of Dewey:

> The crux of Dewey's entire argument is that what we call thinking, or reasoning, or logical inference, *could not even exist* without the felt qualities of situations.... *Insofar as logic pertains to real human inquiry, logic can't do anything without feeling.* Logic alone—pure formal logic—cannot circumscribe the phenomena under discussion. Logic alone cannot tell you what should count as relevant to your argument. Logic can only work because we take for granted the prior working of qualities in experienced situations. (p. 78, author's italics)

Neither will language alone suffice to guide the selection, for the deployment of language itself is now to be understood as folded within the embodied thought process; not a driving force deployed by some homunculus-agitator, but interwoven with and arising out of salience-detection within felt experience. As Hadamard (1954) wrote, "For those of us who do not think in words, the chief difficulty in understanding those who do lies in our inability to understand how they can be sure they are not misled by the words they use" (p. 93). Our discernment of the qualities of the whole situation cycled within and throughout our beings, incorporating all our histories, are the means by which the forms—words, symbols, texts—are recognised as apt and rise to bear, to embellish, embolden and perpetuate the self-making-work, the autopoietic phenomenon that is *thought*. Writes Johnson,

> The fateful error is to overlook much of what goes into making something meaningful to us. Then we are seduced into mistaking the forms for that which they inform, and we fool ourselves into thinking that it is the forms alone that make something meaningful, real, and knowable. We think that if we have succeeded in abstracting a form—conceptualizing some aspect of our experience—then we have captured the full meaning. (Johnson, 2007, p. 80)

It is on this basis that Freudenthal (1991) can recommend the value of "common sense" mathematics:

> In general I believe that in mathematics it would be more recommendable to start with common sense ideas rather than to reject them as outdated and better being suppressed. This belief is supported in any case by the more or less spontaneous development of mathematics. (p. 6)

This is a recognition that the experiential qualities of a young life, interpreted and reinterpreted through all that is innate to humanity, gives rise to folk mathematics. The duty of the teacher, then, is to see this for what it is and to understand how to become a working element in the education of each young person. Of forms that are forced on children as if they were in and of themselves meaningful—algorithms, rules and the like—Freudenthal says,

> Having been imposed they never had a real chance to develop into common sense of a higher order. Is it a privilege of old and wise people to doubt what looks like common sense, or can you not teach this behaviour to the young? (p. 8)

This attention to common sense mathematics highlights its contra-form: schooled or formal mathematics. It brings to mind the System 1-System 2 distinction of dual-process theory (Stanovich & West, 2000). Uri Leron and Orit Hazzan (2006) explain:

> According to this theory, our cognition and behavior operate in parallel in two quite different modes, called System 1 (S1) and System 2 (S2), roughly corresponding to our common sense notions of intuitive and analytical thinking. (p. 108)

System 1 thinking is fast, evolutionarily more ancient, and sits between perception and analytical cognition. System 2 is largely a product of cultural evolution, and probably what most people think of as “thinking”. These systems, let it be said, are another example of fictitious entities, useful in that they facilitate description, but not to be understood as simple constructs.

> Like perception, S1 processes are characterized as being fast, automatic, effortless, unconscious and inflexible (hard to change or overcome); unlike perception, S1 processes can be language-mediated and relate to events not in the here-and-now (i.e., events in far-away locations and in the past or future). In contrast, S2 processes are slow, conscious, effortful and relatively flexible. The two systems differ mainly on the dimension of accessibility: how fast and how easily things come to mind. In many situations, S1 and S2 work in concert, but there are situations ... in which S1 produces quick automatic non-normative responses, while

> S2 may or may not intervene in its role as monitor and critic. (Leron & Hazzan, 2006, p. 108)

Mirroring Freudenthal's question, Leron and Hazzan suggest that it might be possible to train students to monitor their own S1 and S2 thinking:

> We believe that having students monitor the S1/S2 interaction, and learn to avoid its most common pitfalls [e.g. a permissive S2 allowing S1 to operate on rough association], has a degree of specificity—hence trainability—that may be lacking in the older admonitions [to develop reflective habits when solving problems]. If analysing typical S1/S2 pitfalls became an inherent part of students' problem solving sessions, they might become more successful problem solvers and decision makers. (p. 124)

Such openness to the complications of learning becomes central to a serious pursuit of the teaching art—the exploration of how to fit, or accommodate, new with older ways of understanding. The enterprise falls within the range of activities captured under the tentative title *mathematicology*, offered by Willi Dörfler (2003):

> In mathematics education one has to study and investigate mathematics as it is currently practiced [*sic*], produced and used in all its forms. This is not a mathematical study but it is a meta-study of mathematics like musicology is the study of music.... Possibly, this field of research could be termed 'mathematicology'. Mathematicology should never be permitted to lose sight of the other pole of the relationship which mathematics education is about: the human being. It thus has to investigate mathematics as one of the two sides of mathematical activity yet not detached from the people who carry it out. (pp. 149–150)

Dörfler lays out what appears to be an incontrovertible statement and follows it with what could be taken as an imperative of the embodiment program:

> A pretext for a mathematicology is the acceptance and appreciation of the fact that all of mathematics and all about mathematics has been said and is to be said or written by humans. And therefore one has to look for the roots, the genesis, and

> the grounds and foundations of mathematics in this very human activity. (Dörfler, 2003, p. 157)

Teaching mathematics could be said to find its description and its challenge here: in relating the quality of lived experience to the expressions of mathematics. The point that emerges is that being aware of how we learn, what it means to think, *how the experience of the whole becomes the lesson learned*, offers the teacher a means of fostering and participating in educational experiences that are good in the deepest sense.

CONCEPTUAL CHANGE AND LITTLE LIES

There are occasions in a student's mathematical development when, if she is able to recognise it, crises of belief are imposed upon her mind. Often, it seems, these moments pass unacknowledged. An idea that threatens to cast dearly held anticipations of order awry is meekly accommodated within a flexible grasp of what constitutes mathematical truth. Let us pause a while and reflect upon the unheralded significance of learning to cope with a concept so fluid as "number": a paradigmatic case of turning and returning, of broadening a domain of effective being.

One day in a senior class, I commented to Grace and her fellow classmates that some, if not much, of what passes for education is a series of "little lies". I had heard the phrase years ago—I do not remember where—and it seemed to me to convey a significant truth. The impetus of our discussion was the need to agree on a value of the factorial of zero, written "0!". Factorials are simple enough concepts, it would seem. They arise in problems associated with combinatorics and probability, and also, as was our current circumstance, in considerations of Taylor series expansions of functions. In particular, we were considering the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ and } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots .$$

There is much that can be brought to a discussion when considering expressions such as these, including questions of convergence and the very idea that infinite expansions could be equal to a function expressed in what appears to be a closed, finite form. These topics were, indeed, discussed, but an interesting moment arose as we considered the factorial. We realised that these expressions could be written

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}; \text{ and}$$

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Well that was fine, but Grace was not quite prepared to concede that we should so readily allow 0!. She well knew that division by zero was “not allowed”, as was often said. Indeed, we had recently broached the question of *why* division by zero was not permitted. Division by the factorial of zero seemed little better, and possibly worse.

To be clear, the girls had encountered factorials before and regarded an expression of the type 5! as a rather unspectacular abbreviation of $5 \times 4 \times 3 \times 2 \times 1$. They were experienced in calculating large numbers—for factorials increase rapidly—and possibly thought the definition of a factorial

$$n! = n(n-1)(n-2)\ldots 3.2.1$$

sufficiently clear and unambiguous. So it is, if we restrict *n* to the natural numbers. Zero factorial (0!) had been encountered before now, and was “defined” as being equal to 1. A plausibility argument had even been given:

$$\begin{aligned} 4! &= 4 \times 3! \\ 3! &= 3 \times 2! \\ 2! &= 2 \times 1! \\ 1! &= 1 \times 0! \end{aligned}$$

This pattern seems to suggest that *if* we are to permit 0!, then it ought to be that number such that $1! = 1 \times 0!$, and since $1! = 1$ is manifestly clear, 0! itself must also be equal to 1, as 1 = 1×1.

Grace had seen this argument but was not satisfied. Something within her recoiled a little at 0!. "How," she asked, "can we multiply down to zero and not get zero?" She was supposing that since the definition of $n!$ as $n(n-1)(n-2)...3.2.1$ seemed to make little sense if *n* itself was less than zero, the "obvious" or "natural" extension was a redefinition as $n! = n(n-1)(n-2)...3.2.1.0$, where now *n* could be any non-negative integer. This would only make things worse, however, as it would mean that *all* factorials were equal to zero. I don't think Grace would have wanted that.

Grace's classmates, interestingly enough, were less concerned with the matter. They were prepared to accept the statement $0! = 1$ with equanimity. This is no small matter, and affords grounds for discussion, but enough for now—I will stay with Grace's concerns.

It was about this time that I suggested to Grace—and to her classmates—that they had encountered situations like this before: over and over, in fact. Was it not the case that once, when they were young, numbers were counting names, such as one, two, three, and so on? To speak of a number was to invoke what we now call a natural number. Even after rational numbers—fractions, as we called them—had been encountered, did we not yet hesitate to call $\frac{2}{3}$ a number? At some point fractions were accepted into the fold, however, and the word "number" had become more inclusive—we had altered our discernment of the gestalt.

Those curious negatives were eventually admitted, along with the mystery of pseudo-rules such as "two negatives make a positive"; as were oddities such as *e*, $\sqrt{2}$, and π. If it could be located on a real number line it was a number. Zero itself, it might be noted—and in some cases the number 1—was still a little suspect, since it seemed to have properties that set it aside from other, more routine numbers.

The number line, it transpired, became a kind of bench-test for candidacy into the fold of number-hood. It became the method par excellence of visualising numbers. Not even suggestions that the number line is infinitely dense and unbounded defeat our imaginations: we imagine we can see into its depths and believe in both infinitesimally small and vastly large numbers.

Grace and her classmates, however, had just been introduced to a new form of number, the complex number. This number does not permit itself to be ensnared within a single line. It demands a plane, and the erstwhile conception of numbers, those that inhabit the real number line, are now considered but a small subset of this expanded species.

At each stage in this development, the application of the term "number' had been restricted to but a part of its potential. We had spoken of numbers in limited ways. In this sense, we teachers had been parties to little lies, if only lies of omission, readily justifiable on grounds of educational expediency.

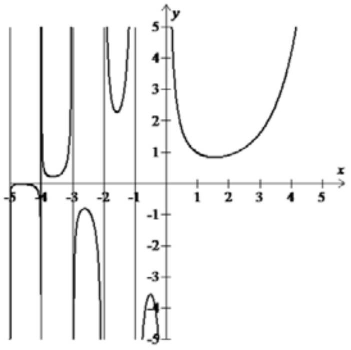
Questions arising from this somewhat vexed scenario are discussed at some length in a special issue of *Learning and Instruction*, "The conceptual change approach to mathematics learning and teaching". Kaarina Merenluoto and Erno Lehtinen (2004) for example, describe their finding that high-achieving students such as Grace undergo what they call an experience-of-conflict path, in which they exhibit a "metacognitive grasp of conflicting notions" (p. 526). In concert with Grace, students in their study show a reduced level of certainty in the face of conceptual change: "The reduced certainty of these students suggests the novelty of their thinking and the radical nature of the change experienced" (p. 526). We might say that these students know enough to realise that there is something happening that they do not quite understand, and they care enough to be discommoded! Merenluoto and Lehtinen advise that "in order for the students to be able to deal with the conflicting notions, it is mandatory to use teaching methods that support the development of meta-conceptual awareness and the use of metacognitive strategies in dealing with conflicting notions" (p. 531).

Grace and I had encountered little lies before, in another context: that of exponentiation. She had learned long ago that 5^2 meant "multiply 5 by itself twice over," so $5^2 = 5 \times 5 = 25$. This was readily expanded to expressions such as $3^4 = 3 \times 3 \times 3 \times 3$, or "multiply 3 by itself 4 times over." Even after students have met with negative and fractional exponents, for which the "multiply by itself" interpretation cannot apply, students cling to such statements. The newer interpretations, such as $3^{-2} = \frac{1}{3^2}$ and $3^{\frac{2}{5}} = \sqrt[5]{3^2}$ are resisted, and though correct statements may be produced upon demand as learned items, the newer interpretations are invested with less conviction than the former. The earlier interpretation was always a little lie, a narrow discernment; and could not be sustained against the expansion of one's outlook. It had, rather, to be restructured. Even statements such as $\sqrt{xy} = \sqrt{x}\sqrt{y}$ and $x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(x^m\right)^{\frac{1}{n}}$ need careful revision when visited in the realm of complex numbers.

Incorporation, more so than invalidation, becomes the means of acquiring new interpretations. Brian Greer (2004) notes that, "There has been a conceptual shift away from the view of mathematics learning as additive to the view that it is characterized by conceptual restructurings" (p. 546). At the revealing of each little lie, then, a greater truth has to be told, such that the earlier faith is not so much displaced as expanded. Early faiths are firmly grounded in rapid and readily available System 1 thinking, however, and too many revelations create a sense of suspicion. As time goes by, some students begin to regard mathematics with distrust, as the grounds for belief seem to shift at the whim of the teacher.

Factorials, then, were become the latest to succumb to the need for a broader interpretation. Against the background of Grace's doubt, and risking accusations of vague hand waving, I showed the girls an extension—the gamma function—that lent some additional structure to our erstwhile interpretation of factorials. This was not so much an elaboration of factorials per se, as a means of enabling one to find a factorial for a broadened set of numbers. In particular, it transpires that one can say

$$0! = \Gamma(1) = \int_0^\infty e^{-t} t^0 dt = \lim_{a \to \infty} \int_0^a e^{-t} dt = 1,$$



where $\Gamma(x)$ is the gamma function. The algebra required to understand the extension was beyond that previously experienced by the girls, so I was calling upon a level of faith—in me, as teacher, and in the mathematical institution at large—in order to communicate a broadened context for 0!. I had ensured that I could provide a graph of the relevant gamma function (actually an approximation, shown above), and it was this image that ultimately won the day. Faced with a picture that was given to them in faith they became believers.

Grace acknowledged that there was something there, and even if she still felt uneasy, even if the larger context into which factorials had been thrown was still somewhat mysterious to her, she was at least prepared to accept that there was a basis upon which one could say that 0! was indeed equal to 1, and that the "multiply down to one" interpretation was more of a narrowly delineated helpmate than a substantial and complete truth.

Of course, the corollary to this excursion is that Grace thereafter regarded all that was said in our classes as potential lies, even if little ones. And was she so wrong in that?

MATHEMATICS AS THREAT

In the end, embodied cognition demands attention to the ways in which we understand what it is *to fit*: to incorporate experience, to find coherence. In this we are always in thrall to our fundamental modes of fear and attraction. Consider the following reflection from Michelle, a student in my Year Twelve Mathematics Specialised class. She had recently sat a test paper on complex numbers, and had not performed as well as she would have liked. She checked over the questions afterwards and then sent me this email:

> I did most of the test questions again, and then checked them against the solutions— and apart from a few tiny algebraic mistakes that gave me the wrong sign or just a wrong answer (but right method), they were fine, which I don't understand. It's like I can do the questions, but as soon as it's a test I can't. How can I work on this for the next test? (Personal communication, 22 April, 2011)

I replied, offering a few thoughts and encouragements. She responded:

> It's just a bit frustrating because I really like maths, and I understand it when I'm just doing it at home or in class, but I'm just not going well this year in tests. I guess I'll just have to study harder then, and do a lot of practise [*sic*] questions. It's not that I really care about the points I'm going to get from specialised [*sic*], because I chose to do it because I loved [Mathematics] methods [*sic*] and thought I'd enjoy it, which I do, it's probably my favourite subject; but it's more that I want to be good at it because I enjoy it to the extent that I'd like to do something maths related as a career. And it's sort of making me think that I wouldn't actually be good enough to do it at uni [*sic*] next year. (Personal communication, 24 April, 2011)

Michelle's concern would not sound unfamiliar to many teachers. She expresses the frustration with the formal assessment process that asks students to solve problems under test conditions. Dismay threatens because for her mathematics is not just a means to an end so much as a potential end in itself—she is interested in pursuing mathematical studies at university. The assessment process has led her to question her fitness for that course of study, whereas her engagement with the actual mathematics of our subject has inspired her in that very direction.

Anna experienced something similar. She had prepared herself for a Year Eleven Mathematics Methods test by studying all weekend and even from 5.30 am on the day of the test. As she sat the test I could sense from her posture that something was wrong. At the end of the class she stayed back to see me, to express a mixture of apology and dismay at what she had perceived to be a poor performance. Stoicism gave way to tears as frustration and confusion—a form of grief—usurped her resolve to maintain dignity, which only amplified the intensity of my unwilled response. I was taken by the thought that she should even contemplate apologising to me for her inability, even as I was immediately captivated by her anguish: I felt her pain as surely as I have felt the pain of my own children at times in their childhood, and I wanted to comfort her.

There is research to indicate that such experiences of anxiety are not so unusual; the very fact that mathematics is important for Michelle and Anna increases the prospect that they will underperform in pressurised situations: "Ironically, those most likely to fail in demanding situations are those who, in the absence of pressure, have the greatest capacity for success" (Beilock, 2008, p. 342); and, what is worse, being aware that she has underperformed might increase the risk of underperforming in the future if she considers herself as one of a group of people who "struggle with tests," for, as Krendl, Richeson, Kelley and Heatherton (2008) have noted, "Reminding a group about negative stereotypes related to their abilities to perform a task has a deleterious impact on their ability to perform that task" (p. 174).

Sian Beilock (2010) summarises these findings in a discussion directed principally at gender stereotypes but informative for the implications of self-perceptions in general:

> When highly capable women are made aware of how they *should* perform, they recruit additional working-memory and emotional centers in the brain to deal with this information.
>
> These brain centers likely come into play to combat the negative thoughts and worries that arise from the idea that "girls can't do math" and, importantly, these same emotion-related brain areas are not as active when sex differences in math are not brought to the forefront of these women's awareness. When brainpower that could otherwise be devoted to math is instead redirected to controlling worrying, the test taker has fewer resources to support her math problem solving and, as a result, her performance suffers. (pp. 104–110, author's italics)

There is the very real danger of becoming ensnared in a self-perpetuating bind, here. Michelle believes she has underperformed; she thinks it is not her actual ability in mathematics that is at fault, so much as her ability to manage tests; it is likely this very awareness will recruit emotional centres and her working memory when she next sits a test, reducing her capacity to perform once again. Anna, too, believes she has underperformed, although ensuing dialogue uncovered mitigating factors that pointed in directions other than "test anxiety".

If the embodied cognition paradigm offers anything for the teacher of girls such as Michelle and Anna, it is perspective: Michelle is more than she seems, Anna lives a complicated life. They each made an appeal to me. At the level of the classroom teacher a manifestation of the embodiment of cognition paradigm becomes the acceptance that there are no sole-traders in learning. Michelle and Anna are multiply coupled in systems that weave, intersect and transform—we might say each girl *is* a multiply coupled system. To attend to Michelle's question is to become a part of her questioning. There is no choice in this: the fact that Michelle has asked my advice already implicates me in her search. I am involved, and while I can choose to withdraw or engage, I cannot choose disentanglement. I might say more—as a teacher immersed in thoughts of autopoiesis and embodiment I *feel* an obligation welling within our mutual situations, at one with Paulo Freire:

> As a teacher in an educational program, I cannot be satisfied simply with nice, theoretical elaborations regarding the ontological, political, and epistemological bases of educational practice. My theoretical explanation of such practice ought to be also a concrete and practical demonstration of what I am saying. A kind of incarnation joining theory and practice. In speaking of the construction of knowledge, I ought to be involved practically, incarnationally, in such construction and be involving the student in it also. (Freire, 1998, pp. 49-50)

What do I bring to Michelle's questioning and to Anna's distress? If I am a serious teacher then I make an effort to understand facts relevant to the discourse. In this case, I read the literature on mathematics anxiety, I orient myself to appreciate the teaching and learning process so that I can enact it more fully, I keep abreast of research in human development, I study mathematics. I cannot, however, become a specialist in any (much less *all*) of these—such an effort would leave no time to give classes. So I do what I can and where I lack knowledge or expertise I find others who can supply the deficit. I study the fragments but remain aware of the whole. I acknowledge the aesthetic that guides me toward fitting.

Michelle and Anna do not know this. They may maintain some nebulous notion that I will find something to say, but they do not know my personal angst, although they likely sense something in me. I am unable to solve their problems, nor even diagnose them. I suffer a pain of helplessness proportional to the depth of my desire to lend the very aid they beg. Here then is an irony—that which they seek to know of me I find in myself: the experience of underperforming when need of performance is greatest.

> I celebrate myself,
> And what I assume you shall assume,
> For every atom belonging to me as good belongs to you.
>
> Walt Whitman (2005), *Song of Myself*

Now I have something to reflect upon. Now I can feel *their* pain, for it is my pain, too. It is at this moment that the emotional response engenders a felt-feeling and transmogrifies into some form of articulation. But what will be the form of articulation? It will be something that arises out of me, out of my life's being-in-the-world. If authenticity is desirable, then I must articulate something of myself in order to maintain the dialogue that is emerging through Michelle's questioning, Anna's suffering. Inauthenticity shatters the unitary boundary. Each girl has sparked a moment in which we can partake; like guitarist-and-guitar in flow, she and I, in some interlude, become one unitary system, perturbed within by question and answer—voiced and tacit—as with all the subtle machinery of communication; framed in the histories of my reflections and her possible futures. I bring the past, she portends what is to be; we meet in the now.

That is what Michelle and Anna lack, what they desire of me, and what I, as a teacher, can provide. Youth has a future, age has a past. The joy of teaching lies in the possibility that in the interstices of our having-been and will-be we meet and permit inspirations to be shared. Backgrounds and foregrounds flitter and shake; "crabbèd age and youth" *do* live together[39], suspending identities, floating above articulations; now she is all youthful hopes and I am sage experience, now she is conservative fears and I am easy abandonment.

Not so much what *I* bring, then, nor what *she* asks, as what *we* become. The lesson of embodied cognition is that *we* supersedes *I*, that emotion transcends and imbues thought, that learning is not something you get from me nor I from you, but a blending of corporeal experiences articulated through a process of refined abstraction and ever replunged into the pleasures and hazards of living a life.

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[39] "Crabbed age and youth cannot live together, / Youth is full of pleasance, Age is full of care", from Shakespeare's *The Passionate Pilgrim*, XII (Hayward, 1956).

What we did as we climbed, and what we talked of
Matters not much, nor to what it led, –
Something that life will not be balked of
Without rude reason till hope is dead,
And feeling fled.
It filled but a minute. But was there ever
A time of such quality, since or before,
In that hill's story? To one mind never,
Though it has been climbed, foot-swift, foot-sore,
By thousands more.

Thomas Hardy, (1956), *At Castle Boterel*

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