

**Department of Mathematics and Statistics**

**Computational Studies of Some Fuzzy Mathematical Problems**

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**This thesis is presented for the Degree of  
Doctor of Philosophy  
of  
Curtin University**

**September 2012**

# Declaration

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

**Signature:** 

**Date: 5 September 2012**

# Acknowledgement

I am extremely grateful to my supervisor, Professor Yong Hong Wu, for his guidance and exceptional support during my study. In addition I sincerely thank all the students and staff in the Department of Mathematics and Statistics for their encouragement during my study at Curtin. During my PhD candidacy I have been engaged in the preparation of this thesis for some time whilst working full time, managing a laboratory. It has been a difficult and worthwhile task that would have been much more difficult without the support of all those around me.

# Abstract

In modelling and optimizing real world systems and processes, one usually ends up with a linear or nonlinear programming problem, namely maximizing one or more objective functions subject to a set of constraint equations or inequalities. For many cases, the constraints do not need to be satisfied exactly, and the coefficients involved in the model are imprecise in nature and have to be described by fuzzy numbers to reflect the real world nature. The resulting mathematical programming problem is referred to as a fuzzy mathematical programming problem.

Over the past decades, a great deal of work has been conducted to study fuzzy mathematical programming problems and a large volume of results have been obtained. However, many issues have not been resolved. This research is thus undertaken to study two types of fuzzy mathematical programming problems. The first type of problems is fuzzy linear programming in which the objective function contains fuzzy numbers. To solve this type of problems, we firstly introduce the concept of fuzzy max order and non-dominated optimal solution to fuzzy mathematical programming problems within the framework of fuzzy mathematics. Then, based on the new concept introduced, various theorems are developed, which involve converting the fuzzy linear programming problem to a four objective linear programming problem of non-fuzzy members. The theoretical results and methods developed are then validated and their applications for solving fuzzy linear problems are demonstrated through examples.

The second type of problems which we tackle in this research is fuzzy linear programming in which the constraint equations or inequalities contain fuzzy numbers. For this work, we first introduce a new concept, the  $\alpha$ -fuzzy max order. Based on this concept, the general framework of an  $\alpha$ -fuzzy max order method is developed for solving fuzzy linear programming problems with fuzzy parameters in the constraints. For the special cases in which the constraints consist of inequalities containing fuzzy numbers with isosceles triangle or trapezoidal membership functions, we prove that the feasible solution space can be determined by the respective  $3n$  or  $4n$  non-fuzzy inequalities. For the general cases in which the

constraints contain fuzzy numbers with any other form of membership functions, robust numerical algorithms have been developed for the determination of the feasible solution space and the optimal solution to the fuzzy linear programming problem in which the constraints contain fuzzy parameters. Further, by using the results for both the first and second types of problems, general algorithms have also been developed for the general fuzzy linear programming problems in which both the objective function and the constraint inequalities contain fuzzy numbers with any forms of membership functions. Some examples are then presented to validate the theoretical results and the algorithms developed, and to demonstrate their applications.

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# Chapter 1

## Introduction

### 1.1 Background

In optimizing real world systems, one usually ends up with a linear and nonlinear programming problem. For many cases, the coefficients involved in the objective and constraint functions are imprecise in nature and have to be interpreted as fuzzy numbers to reflect the real world situation. The resulting mathematical programming problem is therefore referred to as a fuzzy mathematical programming problem (Furukawa (1994), Sakawa (1993), Sakawa et al (1991)).

Applications of fuzzy linear and nonlinear programming problems in practical applications are widespread. Economic and financial systems are particularly prone to the utilization of vague or fuzzy data. Ponsard (1988) conducted a survey of fuzzy mathematical models in economics and concluded that the use of fuzzy analysis led to results that could not be obtained by classical methods. Gutierrez and Carmona (1988) applied a fuzzy set approach to financial ratio analysis. Sanches, Pamplona and Montevechi (2005) investigated capital budgeting using triangle fuzzy numbers, because of the uncertainty of future cash flows. They found that the visualization of the various membership functions contributed to improving the decision making resources. Ostermark (1989) investigated the application of fuzzy methodology in the Capital Asset Pricing Model and the management of financial portfolios. Tanaka, Guo and Turksen (2000) examined the selection of portfolios based on fuzzy probabilities and possibility distributions, and attempted to incorporate the authors' judgement. Following on his earlier work, Ostermark (1996) developed a fuzzy control model for dynamic portfolio management, using risk measurement and future expected prices as fuzzy values. Vercher, Bermudez and Segura (2007) presented two fuzzy portfolio selection models under downside risk measures (i.e. aiming at

minimizing the risk of negative financial outcomes on the portfolio), using trapezoidal membership functions. Many other application areas for fuzzy programming problems exist in finance (Gupta, Mehlawat and Saxena, (2008); Huang (2007); Tiryaki and Ahlatcioglu (2009); Chrysafis, Papadopoulos and Papaschinopoulos (2008)) as well as areas as diverse as pharmacology (Lilic, Sproule, Turksen and Naranjo (2002)) and water management (Slowinski, (1986)).

In recent years, various attempts have been made to study the solution of fuzzy mathematical programming problems with objective functions involving fuzzy numbers, either from a theoretical or computational point of view. Hannan (1981) examined linear programming equations with multiple fuzzy goals. Luhandjula (1984) formulated fuzzy linear programming (FLP) problems as semi-infinite linear programming problems with infinitely many objective functions. Buckley (1988) considered linear programming problems where all the parameters may be triangle fuzzy numbers. Delgado and Verdegay (1989) considered the general structure of the FLP problem and examined solutions to it. Buckley (1990) investigated two solutions to multi-objective linear programs where all the parameters are fuzzy variables. Buckley and Qu (1990) explored the use of  $\alpha$ -cuts to solve fuzzy equations. They showed that one can obtain an incorrect result by evaluating a fuzzy equation by this method. Buckley and Qu (1991) introduced new solutions for fuzzy equations based on the unified extension and possibility theories. They showed that a fuzzy quadratic equation with real fuzzy number coefficients always has a solution. Buckley (1992) conducted a review of the classical methods (extension principle,  $\alpha$ -cuts) developed for solving fuzzy equations. He then proposed a number of areas for further research. Buckley (1992) applied his earlier developed solutions to linear, nonlinear and differential fuzzy equations. Tanaka, Ichihashi and Asai (1991), formulated the fuzzy linear programming (FLP) problem as a parametric linear programming problem, while Luhandjura (1987) formulated the (FLP) problem as a semi-finite linear programming problem with infinitely many objective functions. More recently, Maeda (2001) formulated the (FLP) problem as a two-objective linear programming problem. However, Maeda's work is only applicable to fuzzy numbers with triangle membership functions. Przybylski, Gandibleux and Ehrgott (2010) investigated a two phase solution to multi-objective integer programming problems with three objective functions. Nasser and Gholami (2011) examined linear systems

of equations with trapezoidal fuzzy numbers. Kuwano (1996) used the goal programming approach to solve fuzzy multi-objective linear programming problems. Li, Xu and Gen (2006) investigated multi-objective linear programming problems with random coefficients and developed a generic algorithm designed to solve multi-objective non-linear programming problems. Lu, Ruan, Wu and Zhang (2007) developed an  $\alpha$ -fuzzy goal approximate algorithm to assist in obtaining satisfactory solutions for fuzzy multi-objective linear programming problems.

Although previous work has led to a theoretical basis for the solution of fuzzy mathematical programming problems, many issues have not been resolved, and further development is required. In this thesis, we will study two fuzzy mathematical problems.

## 1.2 Objectives

The objective of the thesis is to develop new robust methods for solving fuzzy linear programming problems. Based on existing theories and previous work in this field, the specific objectives of the thesis are as follows:

- (1) Develop a robust theory and method to solve fuzzy linear programming problems with objective functions involving fuzzy numbers, and demonstrate its applications.
- (2) Develop a robust method and algorithms to solve fuzzy linear programming problems in which the constraints involve fuzzy numbers and demonstrate its application.
- (3) Develop a robust method and algorithms to solve fuzzy linear programming problems in which both the constraints and objective function involve fuzzy coefficients for various forms of membership functions.

### **1.3 Outline of the thesis**

This thesis focuses on the study of two fuzzy mathematics problems including the fuzzy linear programming problem with fuzzy parameters in the objective function, and the fuzzy linear programming problem involving fuzzy parameters in the constraints.

The thesis consists of five chapters. Chapter 1 introduces the background of the research and the objective of the research.

Chapter 2 reviews the basic theories of fuzzy mathematics, including basic concepts, definitions and operations. Particular focus is on the existing theories and literature on fuzzy linear programming and fuzzy nonlinear programming.

Chapter 3 presents a new method for solving fuzzy linear programming problems with fuzzy parameters in the objective function. Various theorems are established to convert the fuzzy linear programming problem to a four objective linear programming problem. Two examples are then given to illustrate the procedure for solving this type of fuzzy linear programming problem by the developed method.

Chapter 4 introduces a new concept, the  $\alpha$ -fuzzy max order, for the study of fuzzy linear programming problems with fuzzy parameters in the constraints. Then, by utilizing the results in Chapter 3, the general case of linear programming problems involving fuzzy parameters in both the objective function and the constraints is investigated. Analytical results corresponding to two special cases of fuzzy number membership functions are derived. For the general case of fuzzy number membership functions, three numerical algorithms are developed. They are the determination of the feasible solution space, the fuzzy optimization problem with fuzzy parameters in the constraints and the fuzzy optimization problem with fuzzy parameters in both the objective function and the constraints.

Chapter 5 presents a summary of the thesis together with areas for further research.

# Chapter 2

## Literature Review

### 2.1 General

This thesis is focussed on the study of fuzzy linear programming which involves fuzzy parameters. The field of study involves the complex interaction of various subjects including fuzzy set theories, mathematical programming, multi-objective mathematical programming and fuzzy mathematical programming. Hence, in this chapter, we review the existing theories and methods in these fields relevant to this research.

The rest of this chapter is organised as follows. Section 2.2 presents the basic concepts, operations and theories of fuzzy sets. Section 2.3 introduces the standard form of linear programming followed by the formulation of fuzzy linear programming and the method of solution. Section 2.4 gives the general form of multi-objective linear programming and various existing methods for the solution including the weighting method and the weighted minmax method. Section 2.5 presents the general form of fuzzy multi-objective linear programming and a solution method. Section 2.6 reviews some basic concepts, theories and methods for nonlinear programming and fuzzy nonlinear programming, followed by brief concluding remarks in section 2.7.

### 2.2 Fuzzy sets, fuzzy numbers and fuzzy operations

In this section, we will review the basic definitions and results on fuzzy sets and fuzzy numbers.

(a) **Fuzzy Sets**

**Definition 2.1.** (Fuzzy Sets): Let  $X$  be a universal set. Then a fuzzy set  $\tilde{A}$  is defined as the set of elements  $x$  equipped with a membership function  $\mu_{\tilde{A}}(x)$ , namely:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \quad (2.1)$$

where  $\mu_{\tilde{A}}(x) : X \rightarrow [0,1]$  is the grade of membership of  $x$  in  $\tilde{A}$  with the larger  $\mu_{\tilde{A}}(x)$  value representing a higher grade of membership of  $x$  in  $\tilde{A}$ .

**Remark 2.1.**

If  $X$  is a countable or a finite set with elements  $x_1, x_2, \dots, x_n$ , namely:

$$X = \{x_1, x_2, \dots, x_n\},$$

then a fuzzy set  $\tilde{A}$  on  $X$  can also be expressed by:

$$\tilde{A} = \{(x_1, \mu_{\tilde{A}}(x_1)), (x_2, \mu_{\tilde{A}}(x_2)), \dots, (x_n, \mu_{\tilde{A}}(x_n))\} \quad (2.2)$$

or

$$\tilde{A} = \sum_{i=1}^n \mu_{\tilde{A}}(x_i) / x_i. \quad (2.3)$$

If  $X$  is infinite or not countable, a fuzzy set  $\tilde{A}$  on  $X$  may be expressed as:

$$\tilde{A} = \int_X \mu_{\tilde{A}}(x) / x. \quad (2.4)$$

**Example 2.1.** Let  $\tilde{A}$  be a fuzzy set representing integers approximately equal to 10. Then  $\tilde{A}$  can be subjectively defined as:

$$\tilde{A} = \{(8, 0.5), (9, 0.8), (10, 1), (11, 0.8), (12, 0.5)\}$$

**Example 2.2.** Consider the fuzzy set  $\tilde{A}$  for old houses.

$$\text{Let } S(x; a, b) = \begin{cases} 0, & x < a \\ 2 \left( \frac{x-a}{b-a} \right)^2, & a \leq x < \frac{a+b}{2} \\ 1 - 2 \left( \frac{x-a}{b-a} \right)^2, & \frac{a+b}{2} \leq x < b \\ 1, & b \leq x \end{cases} \quad (2.5)$$

which is shown in Figure 2.1.

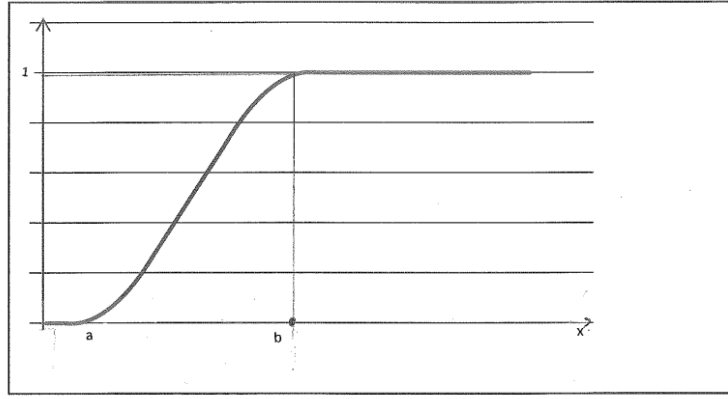


Figure 2.1 : The  $S(x; a, b)$  membership function

Then the fuzzy set  $\tilde{A}$  for old houses can be expressed by:

$$\int_{\mathcal{R}} S(x; 30, 50)/x ,$$

that is, a house that is more than 50 years old is known as an old house, while those houses in the range 30 – 49 years are called old to some degree.

**Remark 2.2.** When the membership function has only two values 0 and 1. i.e.:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

then  $\tilde{A}$  becomes an ordinary set instead of a fuzzy set, i.e.:

$$A = \{x, \mu_A(x) | x \in X\} = \{x \in [a, b]\}$$

**Remark 2.3.** Different membership functions can be used to represent different types of fuzzy concepts/quantities, such as low temperature, comfortable temperature and high temperature. Except for the  $S$ -type membership function as given in (2.5), some other typical types of membership functions are listed below:

(i)  $Z$  function :

$$Z = 1 - S(x; a, b) \tag{2.6}$$

where  $S$  is as defined in (2.5)

(ii)  $\pi$  function :

$$\pi(x; a, b) = \begin{cases} S(x; b - a, b), & x \leq b \\ Z(x; b, b + a), & x > b \end{cases} \tag{2.7}$$

(iii) Triangle function :

$$T(x; a, b) = \begin{cases} 0, & x < a \\ \frac{2(x-a)}{b-a}, & a \leq x < \frac{a+b}{2} \\ \frac{2(b-x)}{b-a}, & \frac{a+b}{2} \leq x < b \\ 0, & x \geq b \end{cases} \quad (2.8)$$

(iv) Trapezoidal function :

$$\text{Tr}(x; a, b, \ell) = \begin{cases} 0, & x < a \\ \frac{2(x-a)}{b-a-\ell}, & a \leq x < \frac{a+b-\ell}{2} \\ 1, & \frac{a+b-\ell}{2} \leq x < \frac{a+b+\ell}{2} \\ \frac{2(b-x)}{b-a-\ell}, & \frac{a+b+\ell}{2} \leq x < b \\ 0, & x \geq b \end{cases} \quad (2.9)$$

The above functions are shown graphically in figures 2.2 – 2.5.

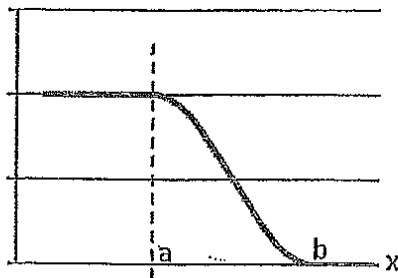


Figure 2.2 : Z Function

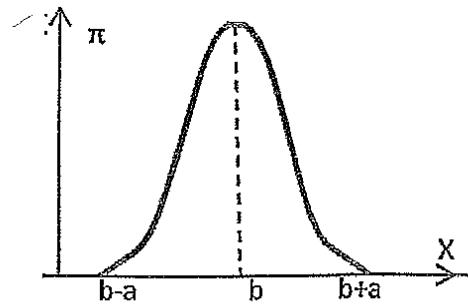


Figure 2.3 :  $\pi$  Function

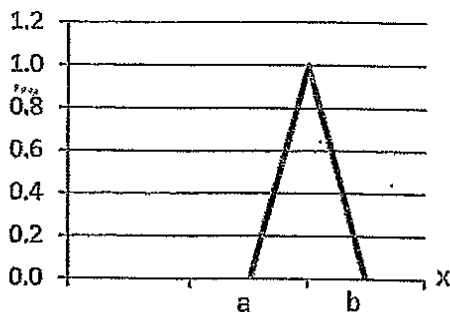


Figure 2.4 : Triangle function

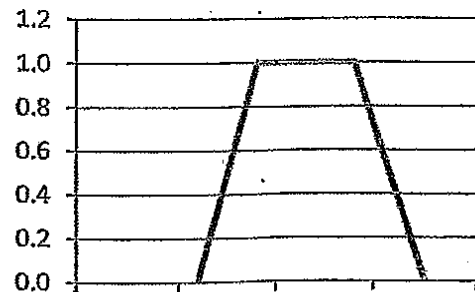


Figure 2.5 : Trapezoidal function



**Definition 2.2.** (Support of a Fuzzy Set) [Sakawa, 1993]

Let  $\tilde{A}$  be a fuzzy set of  $X$ . The support of  $\tilde{A}$  is the set of points in  $X$  at which  $\mu_{\tilde{A}}(x) > 0$ :

$$\text{supp}(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}.$$

**Definition 2.3.** (Height of a Fuzzy Set) [Sakawa, 1993]

The height of a fuzzy set  $\tilde{A}$  on  $X$  is the least upper bound of  $\mu_{\tilde{A}}(x)$ , that is:

$$h(\tilde{A}) = \text{Sup}_{x \in X} \mu_{\tilde{A}}(x).$$

**Definition 2.4.** (Normal Fuzzy Set) [Sakawa, 1993]

A fuzzy set is said to be normal if its height has unity. If it is not normal, it can be normalized by redefining the membership function as  $\mu_{\tilde{A}}(x)/h(\tilde{A})$ .

In the following, some basic set theoretic operations for fuzzy set are presented.

- (i) **Equality:** The fuzzy sets  $A$  and  $B$  are equal if and only if their membership functions are identically equal, namely:

$$A = B \text{ if and only if } \mu_A(x) = \mu_B(x) \quad \forall x \in X.$$

- (ii) **Containment:** The fuzzy set  $A$  is contained in  $B$  if and only if its membership function is less than or equal to that of  $B$  everywhere on  $X$ , namely:

$$A \subseteq B \text{ if and only if } \mu_A(x) \leq \mu_B(x) \quad \forall x \in X.$$

- (iii) **Complementation:** The complement of  $A$  on  $X$  is given by  $\bar{A}$  with membership function:

$$\mu_{\bar{A}} = 1 - \mu_A(x) \quad \forall x \in X.$$

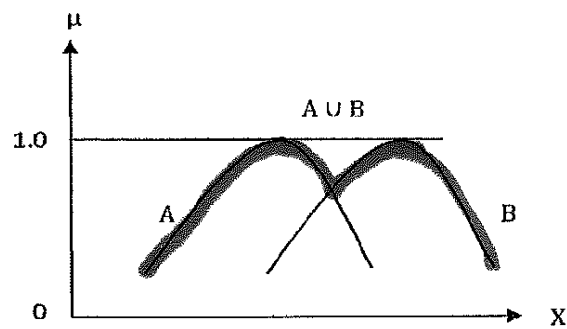
- (iv) **Union:** The union of two fuzzy sets  $\tilde{A}$  and  $\tilde{B}$  is a fuzzy set  $\tilde{A} \cup \tilde{B}$  with membership function:

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.$$

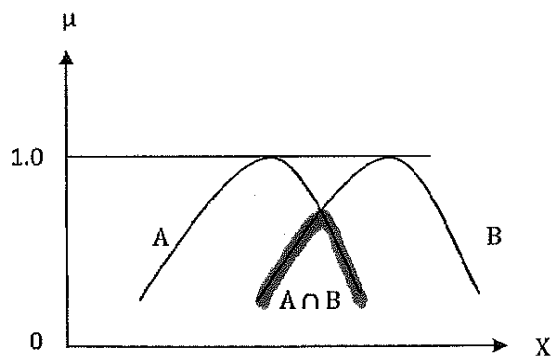
- (v) **Intersection:** The intersection of two fuzzy sets  $\tilde{A}$  and  $\tilde{B}$  on  $X$  is a fuzzy set  $\tilde{A} \cap \tilde{B}$  with membership function:

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \quad \forall x \in X.$$

**Remark 2.4.** Obviously, the union  $A \cup B$  is the smallest fuzzy set containing both  $A$  and  $B$ , as shown in Figure 2.6(a), while the intersection  $A \cap B$  is the largest fuzzy set contained in both  $A$  and  $B$  as shown in Figure 2.6(b).



(a)



(b)

Figure 2.6 : Diagrams showing (a)  $A \cup B$ ; (b)  $A \cap B$

It has also been shown that many properties/identities for ordinary (crisp) sets are valid for fuzzy sets, particularly the following properties:

**(i) Community law**

$$A \cup B = B \cup A.$$

**(ii) Associativity law**

$$(A \cup B) \cup C = A \cup (B \cup C).$$

**(iii) De Morgan's laws**

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B},$$

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}.$$

**(iv) Distributivity laws**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Some algebraic operations for fuzzy sets are given below:

**(i) Algebraic product**

$$AB : \mu_{AB}(x) = \mu_A(x)\mu_B(x).$$

**(ii) Algebraic Sum**

$$A + B : \mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x).$$

**Definition 2.5.** The  $\alpha$ -cut of a fuzzy set  $A$  is a crisp set  $A_\alpha$  given by:

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}, \quad \alpha \in (0,1).$$

From the definition, it is clear that:

$$A_{\alpha_1} \subseteq A_{\alpha_2} \text{ if } \alpha_1 \geq \alpha_2,$$

which is illustrated in Figure 2.7.

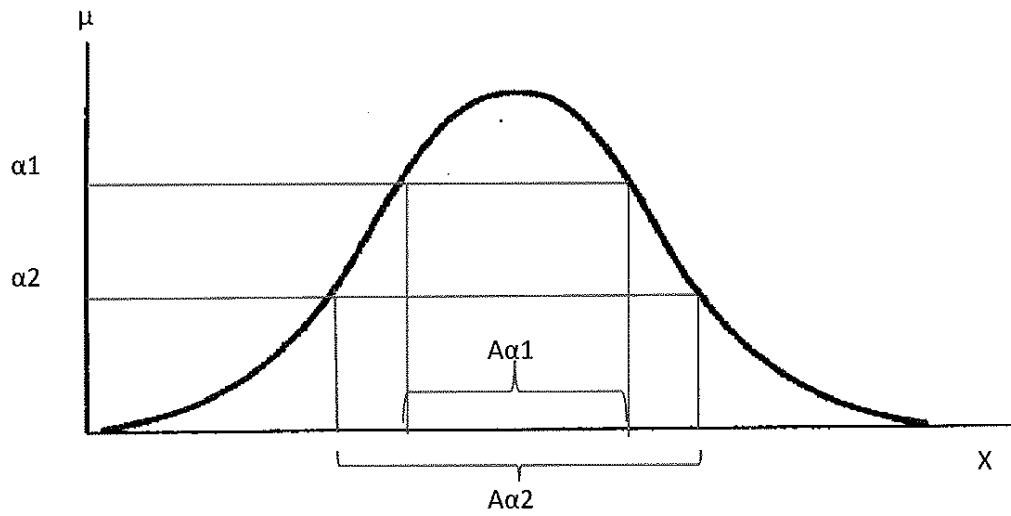


Figure 2.7: Examples of  $\alpha$ -cuts

Now with the concept of  $\alpha$ -cut, a fuzzy set  $A$  can be decomposed into a series of  $\alpha$ -cuts, as stated by the following theorem.

**Theorem 2.1.** (Decomposition Theorem) [Sakawa, 1993]

A fuzzy set  $A$  can be represented by:

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_{\alpha}$$

where  $\alpha A_{\alpha}$  is the algebraic product of a scalar  $\alpha$  with the  $\alpha$ -cut,  $A_{\alpha}$ .

**Example 2.3.** Let  $A = 0.2/2 + 0.4/1 + 0.6/7 + 0.8/6 + 1/8$ . Then we can represent  $A$  by:

$$A = 0.2 A_{0.2} \cup 0.4 A_{0.4} \cup 0.6 A_{0.6} \cup 0.8 A_{0.8} \cup 1 A_1$$

where:

$$A_{0.2} = (2, 1, 7, 6, 8)$$

$$A_{0.4} = (1, 7, 6, 8)$$

$$A_{0.6} = (7, 6, 8)$$

$$A_{0.8} = (6, 8)$$

$$A_1 = (8)$$

**(b) Fuzzy numbers and fuzzy operations**

A fuzzy set is said to be a convex set if its  $\alpha$ -level sets are convex, i.e.:

$$\mu_A(\beta x_1 + (1 - \beta)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \quad \forall x_1, x_2 \in X \text{ and } \beta \in (0,1)$$

Figure 2.8 shows a convex fuzzy set  $A$ .

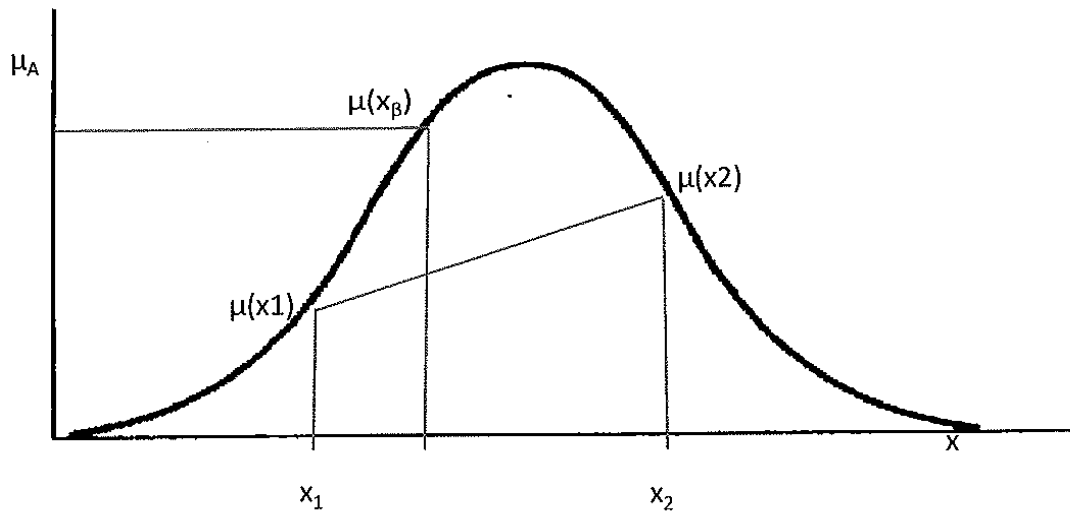


Figure 2.8 : Diagram showing a convex fuzzy set

**Definition 2.6.** (Fuzzy numbers)

A fuzzy number  $\tilde{A}$  is a convex normalized fuzzy set whose membership function is piecewise continuous.

**Remark 2.5.** From the definition of fuzzy number, the  $\alpha$ -cut can be represented by the closed interval:

$$A_\alpha = \{x \in R | \mu_A(x) \geq \alpha\} = [A_\alpha^L, A_\alpha^R].$$

Two different approaches may be used for the arithmetic calculation of fuzzy numbers, including the approach based on the interval arithmetic on the  $\alpha$ -cuts of the fuzzy numbers and the approach based on the extension principle of Zadeh (1965, 1975). In the following, we will review the first approach.

Let  $A$  and  $B$  be two fuzzy numbers respectively with the following  $\alpha$ -cuts:

$$A_\alpha = [a_\alpha^L, a_\alpha^R], \quad B_\alpha = [b_\alpha^L, b_\alpha^R] \quad \alpha \in (0,1].$$

Let  $*$  denote any of the arithmetic operations (+), (-), ( $\cdot$ ), ( $/$ ). The operation of fuzzy numbers is defined by:

$$A * B = \bigcup_{\alpha} \alpha(A * B)_\alpha$$

with

$$(A * B)_\alpha = A_\alpha * B_\alpha, \quad \alpha(0,1],$$

where  $(A * B)_\alpha, A_\alpha$  and  $B_\alpha$  are closed intervals. Obviously  $(A + B)_\alpha$  can be obtained by the interval arithmetic applying on  $A_\alpha$  and  $B_\alpha$ . More specifically:

$$A_\alpha(+ )B_\alpha = [a_\alpha^L + b_\alpha^L, a_\alpha^R + b_\alpha^R],$$

$$A_\alpha(- )B_\alpha = [a_\alpha^L - b_\alpha^R, a_\alpha^R - b_\alpha^L],$$

$$A_\alpha(\cdot )B_\alpha = [a_\alpha^L b_\alpha^L, a_\alpha^R b_\alpha^R] \quad \text{for } A \text{ and } B \text{ in } \mathbb{R}_+,$$

$$A_\alpha(/ )B_\alpha = \left[ \frac{a_\alpha^L}{b_\alpha^R}, \frac{a_\alpha^R}{b_\alpha^L} \right] \quad \text{for } A \text{ and } B \text{ in } \mathbb{R}_+,$$

$$(k \cdot A)_\alpha = k \cdot A_\alpha = [ka_\alpha^L, ka_\alpha^R], \quad k > 0.$$

**Example 2.4.** (Arithmetic for fuzzy numbers with triangle membership function  $\mu_{Tr}(x)$  given by (2.8) ). From (2.8), the  $\alpha$ -cut of the triangle fuzzy number  $A = [a_L, a_R]$  is the closed interval:

$$A_\alpha = [a_\alpha^L, a_\alpha^R] = \left[ \frac{(\alpha_R - \alpha_L)\alpha}{2} + a_L, \alpha_R - \frac{(\alpha_R - \alpha_L)\alpha}{2} \right], \quad \alpha \in (0,1]$$

Let  $B=[b_l, b_R]$  be another fuzzy number, then by using the  $\alpha$ -cuts of  $A$  and  $B$ , one can calculate  $A*B$  where  $*$  refers to (+), (-), ( $\cdot$ ), ( $/$ ),  $\vee$  and  $\wedge$  operations. For example, we have:

$$A(+ )B = \bigcup_{\alpha} \alpha(A + B)_{\alpha},$$

$$A(- )B = \bigcup_{\alpha} \alpha(A - B)_{\alpha},$$

$$-A = -\bigcup_{\alpha} \alpha A_{\alpha},$$

$$kA = k\bigcup_{\alpha} \alpha A_{\alpha}, \quad k > 0,$$

where:

$$\begin{aligned} (A + B)_{\alpha} &= A_{\alpha} + B_{\alpha} \\ &= [a_L + b_L + \frac{1}{2}(a_R - a_L + b_R - b_L)\alpha, \\ &\quad a_R + b_R - \frac{1}{2}(a_R - a_L + b_R - b_L)\alpha], \end{aligned}$$

$$\begin{aligned} (A - B)_{\alpha} &= A_{\alpha} - B_{\alpha} \\ &= [a_L - b_R + \frac{1}{2}(a_R - a_L + b_R - b_L)\alpha, \\ &\quad a_R - b_L - \frac{1}{2}(a_R - a_L - b_R + b_L)\alpha]. \end{aligned}$$

## 2.3 Linear programming and fuzzy linear programming

$$\begin{aligned} \text{Let} \quad \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)^T, \\ \mathbf{c} &= (c_1, c_2, \dots, c_n), \\ \mathbf{b} &= (b_1, b_2, \dots, b_n)^T, \\ \mathbf{A} &= [a_{ij}], i = 1, m; \quad j = 1, n. \end{aligned}$$

Then a typical linear programming problem can be written as:

$$\begin{aligned} \min \quad & y = \mathbf{c}\mathbf{x} && (2.10) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

To soften the rigid requirements of strictly minimizing the objective function and strictly satisfying the constraints, Zimmerman (1978) proposed the fuzzy linear programming problem in the form of:

$$\begin{aligned} & cx \preceq z_0 \\ \text{subject to} & \quad Ax \preceq b, \\ & x \geq 0, \end{aligned} \tag{2.11}$$

where the first formula states that the objective  $cx$  should be essentially smaller than or equal to an aspiration level  $z_0$  of the  $z$ , while the second formula says that  $Ax$  should be essentially smaller than or equal to  $b$ . Combining the fuzzy goal and the fuzzy constraints together, the fuzzy linear programming problem (2.11) was expressed by Zimmerman (1978) as:

$$\begin{aligned} Bx & \preceq b, \\ x & > 0, \end{aligned} \tag{2.12}$$

where:

$$\begin{aligned} B & = (c, A)^T, \\ b' & = (z_0, b)^T. \end{aligned}$$

The author then proposed the following membership function for the  $i$ th fuzzy inequality  $(Bx)_i \preceq b'_i$ , namely:

$$\mu_i((Bx)_i) = \begin{cases} 1 & \text{if } (Bx)_i \leq b'_i \\ 1 - \frac{(Bx)_i - b'_i}{d_i} & b'_i \leq (Bx)_i \leq b'_i + d_i \\ 0 & (Bx)_i \geq b'_i + d_i \end{cases}$$

where  $d_i$  denotes the limit of the admissible violation of the inequality. It is obvious that the membership function takes the value 1 if the constraint is completely satisfied and the value 0 if the constraint is violated.

From the above membership function, it is clear that maximizing the degree of satisfaction of the constraints is equivalent to maximizing the membership function



$\mu_i((Bx)_i)$ . That is, the problem of finding the solution of (2.12) is to determine  $x^*$  that maximizes the minimum membership function value, namely:

$$\mu_D(x^*) = \max_{x \geq 0} \min_{i=0,1,\dots,n} \{\mu_i((Bx)_i)\}$$

or

$$\mu_D(x^*) = \max_{x \geq 0} \min_{i=0,1,\dots,n} \left\{ 1 + \frac{b'_i}{d_i} - \frac{1}{d_i} (Bx)_i \right\}$$

which can also be formulated as:

$$\begin{aligned} & \max && Q && (2.13) \\ \text{subject to} && Q &\leq 1 + \frac{b_i}{d_i} - \frac{1}{d_i} (Bx)_i && i = 0, 1, \dots, n \\ && x &\geq 0. \end{aligned}$$

## 2.4 Multi-objective linear programming

Let

$$\begin{aligned} c &= (c_{ij}) && \text{with } i = 1, \dots, k; \quad j = 1, 2, \dots, n; \\ x &= (x_1, x_2, \dots, x_n)^T, \\ z &= (z_1(x), z_2(x), \dots, z_k(x))^T, \\ A &= [a_{ij}] && \text{with } i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n; \\ b &= (b_1, b_2, \dots, b_m)^T. \end{aligned}$$

Then, a typical multi-objective linear programming problem is expressed by the vector optimization problem:

$$\begin{aligned} & \min && z(x) = cx \\ \text{subject to} && x &\in X \\ \text{where} && X &= \{x \in R^n | Ax \leq b, x \geq 0\}. \end{aligned}$$

Solutions to the above problem may be defined in different senses, as detailed in the following:

**Definition 2.7.** [Complete Optimal Solution] (Sakawa, 1993)

$x^*$  is said to be a complete optimal solution, if and only if there exists  $x^* \in X$  such that  $z_i(x^*) \leq z_i(x), i = 1, 2, \dots, k$  for all  $x \in X$ .

**Definition 2.8.** [Pareto Optimal Solution] (Sakawa, 1993)

$x^*$  is said to be a Pareto optimal solution, if and only if there does not exist another  $x \in X$  such that,  $z_i(x) \leq z_i(x^*)$  for all  $x$  and  $z_j(x) < z_j(x^*)$  for at least one  $j$ .

**Definition 2.9.** [Weak Pareto Optimal Solution] (Sakawa, 1993)

$x^*$  is said to be a Weak Pareto optimal solution, if and only if there does not exist another  $x \in X$  such that  $z_i(x) < z_i(x^*), i = 1, 2, \dots, k$ .

Various methods have been established to solve the multi-objective linear programming problem, including the weighting method, the constraint method and the weighted minmax method.

In the weighting method, the MOLP problem is formulated as:

$$\begin{aligned} \min \quad & wZ(x) = \sum_{i=1}^k w_i z_i(x) \\ \text{subject to} \quad & x \in X, \end{aligned}$$

where  $w = (w_1, w_2 \dots w_k)$  is the vector weighting coefficients. If  $x^*$  is an optimal solution of the weighting problem for some  $w > 0$ , then  $x^*$  is a Pareto optimal solution of the MOLP problem.

In the constraint method, only one objective function is kept as an objective function, all others are taken as irregularity constraints (Haimes 1971, 1974). The constraint problem is defined as:

$$\begin{array}{ll} \min & z_j(x) \\ \text{subject to} & x_i(x) \leq \varepsilon_i \quad i = 1, 2, \dots, k; i \neq j. \end{array}$$

It has been proposed that if  $x^*$  is a unique optimal solution to the constraint problem for some  $\varepsilon_i, i = 1, 2, \dots, k; i \neq j$ , then  $x^*$  is a Pareto optimal solution to the MOLP problem.

In the weighted minmax method, the linear MOLP problem is formulated by:

$$\begin{array}{ll} \min & \max_{i=1,k} w_i z_i(x) \\ \text{subject to} & x \in X, \end{array}$$

or by using an auxiliary variable, namely:

$$\begin{array}{ll} \min & v \\ \text{subject to} & w_i z_i(x) \leq v \quad i = 1, 2, \dots, k \\ & x \in X. \end{array}$$

It has been proved that if  $x^*$  is a unique optimal solution of the weighted problem for some  $w = (w_1, w_2, \dots, w_k) \geq 0$ , then  $x^*$  is a Pareto optimal solution of the MOLP problem.

## 2.5 Fuzzy multi-objective linear programming

Let

$$\begin{aligned} c &= (c_{i1}, c_{i2}, \dots, c_{in}) \\ x &= (x_1, x_2, \dots, x_n)^T \\ b &= (b_1, b_2, \dots, b_n)^T \\ z_i(x) &= c_i x \\ A &= [a_{ij}] , \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n. \end{aligned}$$

Then the typical fuzzy multi-objective linear programming problem can be written as:

$$\begin{aligned} \min \quad & (z_1(x), z_2(x), \dots, z_k(x))^T \\ & \text{in the sense that each } z_i(x) \text{ is substantially less} \\ & \text{than or equal to some value} \\ \text{subject to} \quad & Ax \leq b, x \geq 0 \end{aligned}$$

It has been established that the corresponding linear membership function for the above fuzzy goal can be defined by:

$$\mu_i^L(z_i(x)) = \begin{cases} 0, & \text{if } z_i(x) \geq z_i^0 \\ \frac{z_i(x) - z_i^0}{z_i^1 - z_i^0}, & z_i^0 > z_i(x) \geq z_i^1 \\ 1, & z_i(x) < z_i^1 \end{cases}$$

where  $z_i^0$  and  $z_i^1$  are respectively the objective function  $z_i(x)$  such that the degree of membership is 0 and 1.

Using the above membership function, the fuzzy multi-objective programming problem can be formulated as:

$$\begin{aligned} \max \quad & \min\{\mu_i^L(z_i(x))\} \\ \text{subject to} \quad & Ax \leq b, x \geq 0 \end{aligned} \tag{2.14}$$

which is equivalent to:

$$\begin{aligned} \max \quad & Q \\ \text{subject to} \quad & Q \leq \mu_i^L(z_i(x)) \quad i = 1, 2, \dots, k \\ & Ax \leq b, \\ & x \geq 0. \end{aligned} \tag{2.15}$$

## 2.6 Nonlinear programming and Fuzzy nonlinear programming

### (a) Nonlinear programming

Let  $x = (x_1, x_2, \dots, x_n)$  be an  $n$ -dimensional vector of decision variable,  
 $f(x_1, x_2, \dots, x_n)$  be a nonlinear function,  
 $g_i(x_1, x_2, \dots, x_n) \leq 0, (i = 1, m)$ , be nonlinear inequality constraints,  
then the nonlinear programming problem can be written as:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & x \in X := \{x \in R^n \mid g_i(x) \leq 0, i = \\ & 1, \dots, n\} \end{aligned}$$

Various notations and definitions used in nonlinear programming are listed below.

**Local minimum.** A point  $x^*$  is a local minimum if there exists a real  $\delta > 0$  such that  $f(x) \geq f(x^*)$  for all  $x \in X$  satisfying  $\|x - x^*\| < \delta$ .

**Global minimum.** A point  $x^*$  is a global minimum if and only if  $f(x) \geq f(x^*)$  for all  $x \in X$ .

**Convex function.** A function  $f(x)$ , defined on a non empty convex set, is convex if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad \forall x_1, x_2 \in S \text{ and } \forall \lambda \in [0,1]$$

**Active constraint.** An inequality constraint  $g_i(x) \leq 0$  is called an active constraint at  $x^*$  if  $g_i(x^*) = 0$ .

The Kuhn-Tucker conditions have been established for the determination of the local minimum, namely:

$$\begin{aligned} \frac{\partial f}{\partial x_i} + \sum_{j=1}^n \lambda_j \frac{\partial g_j}{\partial x_i} &= 0, \quad i = 1, 2, \dots, n. & (2.16) \\ \lambda_j g_j &= 0, \quad j = 1, 2, \dots, m, \\ g_j &\leq 0, \quad j = 1, 2, \dots, m, \\ \lambda_j &\geq 0, \quad j = 1, 2, \dots, m. \end{aligned}$$

When the nonlinear programming problem involves equality constraints, namely:

$$\begin{aligned}
& \min && f(x) && (2.17) \\
& \text{subject to} && g_i(x) \leq 0, \quad i = 1, 2, \dots, p, \\
& && h_j(x) = 0, \quad j = 1, 2, \dots, q,
\end{aligned}$$

then the Kuhn-Tucker conditions become:

$$\begin{aligned}
\nabla f + \sum_{i=1}^p \lambda_i \nabla g_i - \sum_{j=1}^q \beta_j \nabla h_j &= 0 && (2.18) \\
\lambda_i g_i &= 0, \quad i = 1, 2, \dots, p, \\
g_i &\leq 0, \quad i = 1, 2, \dots, p, \\
h_j &= 0, \quad j = 1, 2, \dots, q, \\
\lambda_i &\geq 0, \quad i = 1, 2, \dots, p.
\end{aligned}$$

**(a) Fuzzy nonlinear programming**

Zimmerman (1978) proposed the fuzzy version of the nonlinear programming problem, namely:

$$\begin{aligned}
& \widetilde{\min} && f(x) && (2.19) \\
& \text{subject to} && g_i(x) \preceq 0, \quad i = 1, 2, \dots, n.
\end{aligned}$$

where  $\widetilde{\min}$  means that the objective function should be minimized as much as possible, while  $\preceq$  states that the constraints should be satisfied as much as possible.

As for the linear programming, the objective function can be replaced by the following fuzzy constraint:

$$f(x) \preceq f'$$

or for notation convenience as:

$$g_o(x) \preceq g'_o.$$

As for the fuzzy linear programming, the membership function for the typical  $i$ th fuzzy constraints  $g_i(x) \leq 0$  is:

$$\mu_i(g_i(x)) = \begin{cases} 1 & \text{if } g_i(x) \leq g_i' \\ d_i(x), & g_i' \leq g_i(x) \leq g_i^0 \\ 0, & g_i(x) \geq g_i^0 \end{cases}$$

where  $d_i(x)$  is a strictly monotone decreasing function with respect to  $g$ .

Using the above membership functions, based on Bellman and Zadeh (1970), finding the solution of the fuzzy nonlinear programming problem is to determine  $x^*$  such that:

$$\mu_D(x^*) = \max \min_{i=0,1,\dots,n} \{\mu_i(g_i(x))\} ,$$

or

$$\begin{aligned} \min \quad & q & (2.20) \\ \text{such that} \quad & q \leq \mu_i(g_i(x)), \quad i = 0, 1, 2, \dots, m . \end{aligned}$$

Hence, existing numerical algorithms of nonlinear programming can be used to solve the above problem.

## 2.7 Concluding Remarks

Fuzzy mathematical programming problems involving fuzzy parameters have not yet been fully investigated and thus are the focus of this PhD research. The basic concepts, definitions, theories and solution methods relevant to the PhD research have been reviewed and presented, and will form the basis for the PhD research. The main research development will be presented in Chapters 3 and 4.

# Chapter 3

## Solution of fuzzy linear programming as constrained optimisation problems

### 3.1 General

Attempts to optimize real world systems usually end up with a linear or nonlinear programming problem. In many cases, because of their imprecise nature, the coefficients involved in the objective and constraint functions have to be interpreted as fuzzy numbers to reflect the real world situation. This mathematical problem is known as a fuzzy mathematical programming problem.

In recent years efforts have been made to study the solution to fuzzy mathematical programming problems from both a theoretical and computational point of view. The fuzzy linear programming (FLP) problem was formulated by Tanaka et al (1991) as a parametric linear programming problem. The FLP problem was formulated as a semi-finite linear programming problem with infinitely many objective functions by Luhandjura (1984). Maeda (2001) recently formulated the FLP problem as a two-objective linear programming problem but this work is only applicable to problems which involve fuzzy numbers with triangle membership functions.

In this chapter we formulated the FLP problem as a multi-objective linear programming (MOLP) problem with four-objective functions. This work is in line with Maeda's (2001) development. In comparison, our work is applicable to problems involving fuzzy numbers with any form of membership functions. The rest of the thesis is organized as follows. In Section 3.2, we give some basic definitions and theorems fundamental to the development are described in Section 3.3. In Section 3.3, we firstly give the concepts of the optimal solution to a FLP problem. Then a number of theorems are developed to convert the FLP problem to a MOLP problem with four-objective functions. In Section 3.4, two illustrative examples are given to demonstrate the procedure for solving FLP problems.



## 3.2 Preliminary

In this section, we describe and develop some fundamental theorems and definitions required in the thesis.

Let  $R$  be the set of all real numbers,  $R^n$  be an  $n$ -dimensional Euclidean space and  $x = (x_1, x_2, \dots, x_n)^T$ ,  $y = (y_1, y_2, \dots, y_n)^T \in R^n$  be any two vectors, where  $x_i, y_i \in R$ ,  $i = 1, 2, \dots, n$ , and  $T$  denotes the transpose of the vector. Then we denote the inner product of  $x$  and  $y$  by  $\langle x, y \rangle$ . For any two vectors  $x, y \in R^n$ , we write  $x \cong y$  if and only if  $x_i \cong y_i, \forall i = 1, 2, \dots, n$ ;  $x \geq y$  if and only if  $x \geq y$  and  $x \neq y$ ;  $x > y$  if and only if  $x_i > y_i, \forall i = 1, 2, \dots, n$ .

**Definition 3.1.** As defined in Chapter 2, a fuzzy number  $\tilde{a}$  is defined as a fuzzy set on  $R$ , whose membership function  $\mu_{\tilde{a}}$  satisfies the following conditions:

- (i)  $\mu_{\tilde{a}}$  is a mapping from  $R$  to the closed interval  $[0, 1]$ ;
- (ii) it is normal. i.e., there exists  $x \in R$  such that  $\mu_{\tilde{a}}(x) = 1$ ;
- (iii) whenever  $\lambda \in (0, 1]$ ,  $a_\lambda = \{x; \mu_{\tilde{a}}(x) \geq \lambda\}$  is a closed interval, denoted by  $[a_\lambda^L, a_\lambda^R]$ .

Let  $F(R)$  be the set of all fuzzy numbers. By the decomposition theorem of fuzzy set, we have:

$$\tilde{a} = \bigcup_{\lambda \in (0, 1]} \lambda [a_\lambda^L, a_\lambda^R]$$

$$\tilde{a} = \bigcup_{\lambda \in R_0} \lambda [a_\lambda^L, a_\lambda^R]$$

for every  $\tilde{a} \in F(R)$ , where  $R_0$  is all rational numbers in  $(0, 1]$ .

For any real number  $\lambda \in R$ , we define  $\mu_\lambda(x)$  by:

$$\mu_\lambda(x) = \begin{cases} 1 & \text{if and only if } x = \lambda, \\ 0 & \text{if and only if } x \neq \lambda. \end{cases}$$

Then  $\lambda \in F(R)$ .

**Definition 3.2.** If for every positive real number  $M$ , there exists  $\lambda_0 \in (0,1]$  such that  $M \leq a_{\lambda_0}^R$  or  $a_{\lambda_0}^L \leq -M$ , then the fuzzy number  $\tilde{a}$  is said to belong to fuzzy infinity, written as  $\tilde{a} \in \tilde{\infty}$ . If  $\tilde{a} \notin \tilde{\infty}$ , then the fuzzy number  $\tilde{a}$  is said to be a finite fuzzy number.

Let  $F^*(R)$  be the set of all finite fuzzy numbers in  $R$ .

**Theorem 3.1.**

Let  $\tilde{a}$  be a fuzzy set on  $R$ , then  $\tilde{a} \in F(R)$  if and only if  $\mu_{\tilde{a}}$  satisfies:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & x \in [m, n] \\ L(x), & x < m \\ R(x), & x > n \end{cases}$$

where  $L(x)$  is the right continuous monotone increasing function,  $0 \leq L(x) < 1$  and  $\lim_{x \rightarrow -\infty} L(x) = 0$ ,  $R(x)$  is the left continuous monotone decreasing function,  $0 \leq R(x) < 1$  and  $\lim_{x \rightarrow \infty} R(x) = 0$ .

**Corollary 3.1.**

If  $\tilde{a} \in F^*(R)$ , then there exists  $|x_i| < \infty, i = 1,2$  such that  $L(x_1) = R(x_2) = 0$ . i.e., the support of  $\tilde{a}$  is a bounded set.

**Corollary 3.2.** For every  $\tilde{a} \in F(R)$  and  $\lambda_1, \lambda_2 \in (0,1]$ , if  $\lambda_1 \leq \lambda_2$ , then  $a_{\lambda_2} \subset a_{\lambda_1}$ .

**Definition 3.3.**

For any  $\tilde{a}, \tilde{b} \in F(R)$  and  $0 \leq \lambda \in R$ , the sum of two fuzzy numbers  $\tilde{a} + \tilde{b}$  and the scalar product of  $\lambda$  and  $\tilde{a}$  are defined by the membership functions:

$$\mu_{\tilde{a}+\tilde{b}}(t) = \sup_{t=u+v} \min \{ \mu_{\tilde{a}}(u), \mu_{\tilde{b}}(v) \},$$

$$\mu_{\lambda\tilde{a}}(t) = \max \{ 0, \sup_{t=\lambda u} \mu_{\tilde{a}}(u) \},$$

where we set  $\sup \{ \emptyset \} = -\infty$ .

**Theorem 3.2.** For any  $\tilde{a}, \tilde{b} \in F(R)$  and  $0 \leq \alpha \in R$ ,

$$\tilde{a} + \tilde{b} = \bigcup_{\lambda \in (0,1]} \lambda [a_\lambda^L + b_\lambda^L, a_\lambda^R + b_\lambda^R],$$

$$\alpha \tilde{a} = \bigcup_{\lambda \in (0,1]} \lambda [\alpha a_\lambda^L, \alpha a_\lambda^R].$$

**Definition 3.4.** Let  $\tilde{a} \in F(R), i = 1, 2, \dots, n$ . We define  $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$

$$\mu_{\tilde{a}} : R^n \rightarrow [0,1],$$

$$x \rightarrow \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(x_i),$$

where  $x = (x_1, x_2, \dots, x_n)^T \in R^n$ .

Then  $\tilde{a}$  is called an n-dimensional fuzzy number on  $R^n$ . If  $\tilde{a}_i \in F^*(R), i = 1, 2, \dots, n$ ,  $\tilde{a}$  is called an n-dimensional finite fuzzy number on  $R^n$ .

Let  $F(R^n)$  and  $F^*(R^n)$  be the set of all n-dimensional fuzzy numbers and the set of all n-dimensional finite fuzzy numbers on  $R^n$  respectively.

**Proposition 3.1.** For every  $\tilde{a} \in F(R), \tilde{a}$  is normal.

**Proof.** Since  $\tilde{a} \in F(R^n)$ , there exist  $\tilde{a}_i \in F(R), i = 1, 2, \dots, n$  such that  $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$ . As  $\tilde{a}_i (i = 1, 2, \dots, n)$  is normal, it follows that there exists  $x_i \in R (i = 1, 2, \dots, n)$  such that  $\mu_{\tilde{a}_i}(x_i) = 1 (i = 1, 2, \dots, n)$ . Let  $x = (x_1, x_2, \dots, x_n)^T \in R^n$ , then:

$$\mu_{\tilde{a}}(x) = \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(x_i) = 1.$$

which implies that  $\tilde{a}$  is normal.

**Proposition 3.2.** For every  $\tilde{a} \in F(R)$ , the  $\lambda$ -section of  $\tilde{a}$  is an n-dimensional closed rectangular region for any  $\lambda \in (0,1]$ .

**Proof.** Since  $\tilde{a} \in F(R^n)$ , there exist  $\tilde{a}_i \in F(R), i = 1, 2, \dots, n$  such that  $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$ . As the  $\lambda$ -section of  $\tilde{a}_i$  ( $i = 1, 2, \dots, n$ ) is a closed interval  $[a_{i\lambda}^L, a_{i\lambda}^R]$  ( $i = 1, 2, \dots, n$ ), we have:

$$\begin{aligned}
a_i &= \{x; \mu_{\tilde{a}}(x) \geq \lambda\} \\
&= \{x; \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(x_i) \geq \lambda\} \\
&= \{x; \mu_{\tilde{a}_i}(x_i) \geq \lambda, \quad i = 1, 2, \dots, n\} \\
&= \{x = (x_1, x_2, \dots, x_n)^T; \quad x_i \in a_{i\lambda}, \quad i = 1, 2, \dots, n\} \\
&= \{x = (x_1, x_2, \dots, x_n)^T; \quad x_i \in [a_{i\lambda}^L, a_{i\lambda}^R], \quad i = 1, 2, \dots, n\}
\end{aligned}$$

for any  $\lambda \in (0, 1]$ . This implies that the  $\lambda$ -section of  $\tilde{a}$  is an  $n$ -dimensional closed rectangular region for any  $\lambda \in (0, 1]$ .

**Proposition 3.3.** For every  $\tilde{a} \in F(R^n)$ ,  $\tilde{a}$  is a complex fuzzy set. i.e.:

$$\mu_{\tilde{a}}(\lambda x + (1 - \lambda)y) \geq \mu_{\tilde{a}}(x) \wedge \mu_{\tilde{a}}(y),$$

whenever  $\lambda \in [0, 1], x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in R^n$ .

**Proof.** For every  $\lambda \in [0, 1], x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in R^n$ , from  $\tilde{a} \in F(R^n)$ , there exist  $\tilde{a}_i \in F(R^n), i = 1, 2, \dots, n$  such that  $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$ . It follows, by the fact that  $\tilde{a}_i \in F(R), i = 1, 2, \dots, n$ , are convex fuzzy sets, that:

$$\mu_{\tilde{a}_i}(\lambda x_i + (1 - \lambda)y_i) \geq \mu_{\tilde{a}_i}(x_i) \wedge \mu_{\tilde{a}_i}(y_i), \quad i = 1, 2, \dots, n.$$

Therefore:

$$\begin{aligned}
\mu_{\tilde{a}}(\lambda x + (1 - \lambda)y) &= \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(x)(\lambda x_i + (1 - \lambda)y_i) \\
&\geq \bigwedge_{i=1}^n (\mu_{\tilde{a}_i}(x) \wedge \mu_{\tilde{a}_i}(y)) \\
&= \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(x_i) \wedge \bigwedge_{i=1}^n \mu_{\tilde{a}_i}(y_i) \\
&= \mu_{\tilde{a}}(x) \wedge \mu_{\tilde{a}}(y),
\end{aligned}$$

which implies that  $\tilde{a}$  is a convex fuzzy set.

**Proposition 3.4.** For every  $\tilde{a} \in F(R^n)$  and  $\lambda_1, \lambda_2 \in (0,1]$ , if  $\lambda_1 \leq \lambda_2$  then  $\alpha_{\lambda_2} \subset \alpha_{\lambda_1}$ .

**Proof.** Obvious.

**Definition 3.5.** For any n-dimensional fuzzy numbers  $\tilde{a}, \tilde{b} \in F(R^n)$ , we define:

1.  $\tilde{a} \cong \tilde{b}$  iff  $a_{i\lambda}^L \cong b_{i\lambda}^L$  and  $a_{i\lambda}^R \cong b_{i\lambda}^R$ ,  $i = 1, 2, \dots, n$ ,  $\lambda \in (0,1]$ ,
2.  $\tilde{a} \succeq \tilde{b}$  iff  $a_{i\lambda}^L \geq b_{i\lambda}^L$  and  $a_{i\lambda}^R \geq b_{i\lambda}^R$ ,  $i = 1, 2, \dots, n$ ,  $\lambda \in (0,1]$ ,
3.  $\tilde{a} \succ \tilde{b}$  iff  $a_{i\lambda}^L > b_{i\lambda}^L$  and  $a_{i\lambda}^R > b_{i\lambda}^R$ ,  $i = 1, 2, \dots, n$ ,  $\lambda \in (0,1]$ .

We call the binary relations  $\cong$ ,  $\succeq$  and  $\succ$  a fuzzy max order, a strict fuzzy max order and a strong fuzzy max order, respectively.

### 3.3 Fuzzy linear programming with fuzzy max order

In this section, we study the solutions to fuzzy linear programming problems. Firstly, we define the concepts of optimal solutions to fuzzy linear programming problems and then investigate their properties.

Let us consider the following problem:

$$(FLP) \quad \left\{ \begin{array}{l} \text{maximize} \quad \langle \tilde{c}, x \rangle_F = \sum_{i=1}^n \tilde{c}_i x_i \\ \text{subject to} \quad Ax \leq b, x \geq 0, \end{array} \right. \quad (3.1)$$

Where  $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)^T \in F^*(R^n)$  and  $A$  is an  $m \times n$  matrix and  $b \in R^m$  whose elements are given by  $a_{ij}$  and  $b_j$ , respectively.

For the sake of simplicity, we set  $X = \{x \in R^n; Ax \leq b, x \geq 0\}$  and assume that  $X$  is compact. In a fuzzy linear programming problem, for each  $x \in X$ , the value of the objective function  $\langle \tilde{c}, x \rangle$  is a fuzzy number. Thus, we shall introduce the following concepts of optimal solutions to fuzzy linear programming problems.

**Definition 3.6.** A point  $x^* \in X$  is said to be an optimal solution to the FLP problem if it holds that  $\langle \tilde{c}, x^* \rangle_F \cong \langle \tilde{c}, x \rangle_F$  for all  $x \in X$ .

**Definition 3.7.** A point  $x^* \in X$  is said to be a non-dominated solution to the FLP problem if there does not exist  $x \in X$  such that  $\langle \tilde{c}, x \rangle_F \succcurlyeq \langle \tilde{c}, x^* \rangle_F$  holds.

**Definition 3.8.** A point  $x^* \in X$  is said to be a weak non-dominated solution to the FLP problem if there is no  $x \in X$  such that  $\langle \tilde{c}, x \rangle_F > \langle \tilde{c}, x^* \rangle_F$  holds.

We denote the sets of all non-dominated solutions and all weak non-dominated solutions to the fuzzy linear programming problem by  $X^F$  and  $X^{wF}$ , respectively. Then, by definition, it holds that  $X^F \subset X^{wF}$ .

Associated with the fuzzy linear programming (FLP) problem, we consider the following multi-objective linear programming (MOLP) problem:

$$(MOLP) \begin{cases} \text{maximize} & (\langle c_0^L, x \rangle, \langle c_1^L, x \rangle, \langle c_0^R, x \rangle, \langle c_1^R, x \rangle)^T, \\ \text{subject to} & Ax \leq b, x \geq 0, \end{cases} \quad (3.2)$$

where  $c_i^L = (c_{i1}^L, c_{i2}^L, \dots, c_{in}^L)^T$ ,  $c_i^R = (c_{i1}^R, c_{i2}^R, \dots, c_{in}^R)^T \in R^n$ ,  $i = 0, 1$ .

In the following, we introduce the concepts of optimal solutions of the multi-objective linear programming (MOLP) problem.

**Definition 3.9.** A point  $x^* \in X$  is said to be a complete optimal solution to the MOLP problem if it holds that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \cong (\langle c_0^L, x \rangle, \langle c_1^L, x \rangle, \langle c_0^R, x \rangle, \langle c_1^R, x \rangle)^T$$

for all  $x \in X$ .

**Definition 3.10.** A point  $x^* \in X$  is said to be a Pareto optimal solution to the MOLP problem if there is no  $x \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \preceq (\langle c_0^L, x \rangle, \langle c_1^L, x \rangle, \langle c_0^R, x \rangle, \langle c_1^R, x \rangle)^T$$

holds.

**Definition 3.11.** A point  $x^* \in X$  is said to be a weak Pareto optimal solution to the MOLP problem if there is no  $x \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T > (\langle c_0^L, x \rangle, \langle c_1^L, x \rangle, \langle c_0^R, x \rangle, \langle c_1^R, x \rangle)^T$$

holds.

**Lemma 3.1.** For any  $a, b, c \in R^n$  and  $a \preceq b \preceq c$  if there exist  $x^*, x \in R^n$  such that:

$$\langle a, x \rangle \preceq \langle a, x^* \rangle \text{ and } \langle c, x \rangle \preceq \langle c, x^* \rangle, \text{ then } \langle b, x \rangle \preceq \langle b, x^* \rangle.$$

**Proof.**

If:

$$\langle a, x \rangle \preceq \langle a, x^* \rangle \text{ and } \langle c, x \rangle \preceq \langle c, x^* \rangle$$

we have:

$$0 \preceq \langle a, (x^* - x) \rangle \text{ and } 0 \preceq \langle c, (x^* - x) \rangle.$$

Further, from  $a \preceq b \preceq c$ , we have:

$$\begin{aligned} \langle c, (x^* - x) \rangle - \langle a, (x^* - x) \rangle &= \langle (c - a), (x^* - x) \rangle \\ &\succeq \langle (c - b), (x^* - x) \rangle \\ &= \langle c, (x^* - x) \rangle - \langle b, (x^* - x) \rangle \end{aligned}$$

Therefore:

$$\langle b, (x^* - x) \rangle \succeq \langle a, (x^* - x) \rangle \succeq 0$$

and hence:

$$\langle b, x \rangle \preceq \langle b, x^* \rangle.$$

**Theorem 3.3.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. Then  $x^*$  is an optimal solution to the problem if and only if  $x^*$  is a complete optimal solution to the MOLP problem.

**Proof.** If  $x^*$  is an optimal solution to the FLP problem, then for any  $x \in X$ , we have:

$$\langle \tilde{c}, x^* \rangle_F \geq \langle \tilde{c}, x \rangle_F.$$

Therefore, for any  $\lambda \in [0, 1]$ , we have:

$$\left( \sum_{i=1}^n \tilde{c}_i x_i^* \right)_\lambda^L \cong \left( \sum_{i=1}^n \tilde{c}_i x_i \right)_\lambda^L,$$

$$\left( \sum_{i=1}^n \tilde{c}_i x_i^* \right)_\lambda^R \cong \left( \sum_{i=1}^n \tilde{c}_i x_i \right)_\lambda^R,$$

that is:

$$\sum_{i=1}^n c_{i\lambda}^L x_i^* \cong \sum_{i=1}^n c_{i\lambda}^L x_i,$$

$$\sum_{i=1}^n c_{i\lambda}^R x_i^* \cong \sum_{i=1}^n c_{i\lambda}^R x_i$$

Hence  $x^*$  is a complete optimal solution to the MOLP problem by Definition 3.4.

If  $x^*$  is a complete optimal solution to the MOLP problem, then for all  $x \in X$ , we have:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \cong (\langle c_0^L, x \rangle, \langle c_1^L, x \rangle, \langle c_0^R, x \rangle, \langle c_1^R, x \rangle)^T$$

that is:

$$\sum_{i=1}^n c_{i0}^L x_i^* \cong \sum_{i=1}^n c_{i0}^L x_i, \quad \sum_{i=1}^n c_{i1}^L x_i^* \cong \sum_{i=1}^n c_{i1}^L x_i,$$



$$\sum_{i=1}^n c_{i1}^R x_i^* \cong \sum_{i=1}^n c_{i1}^R x_i, \quad \sum_{i=1}^n c_{i1}^R x_i^* \cong \sum_{i=1}^n c_{i1}^R x_i$$

As  $\tilde{c} \in F^*(R^n)$ , for any  $\lambda \in [0, 1]$ , we have:

$$c_0^L \cong c_\lambda^L \cong c_1^L, \quad c_1^R \cong c_\lambda^R \cong c_0^R$$

From Lemma 3.1, we have:

$$\sum_{i=1}^n c_{i\lambda}^L x_i^* \cong \sum_{i=1}^n c_{i\lambda}^L x_i, \quad \sum_{i=1}^n c_{i\lambda}^R x_i^* \cong \sum_{i=1}^n c_{i\lambda}^R x_i$$

for any  $\lambda \in [0, 1]$ . Therefore  $x^*$  is an optimal solution to the FLP problem.

**Theorem 3.4.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. Then  $x^*$  is a non-dominated solution to the problem if and only if  $x^*$  is a Pareto optimal solution to the MOLP problem.

**Proof.** Let  $x^* \in X$  be a non-dominated solution to the FLP problem. On the contrary, we suppose that there exists a  $\bar{x} \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \preceq (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T. \quad (3.3)$$

Therefore:

$$0 \preceq (\langle c_0^L, \bar{x} \rangle - \langle c_0^L, x^* \rangle, \langle c_1^L, \bar{x} \rangle - \langle c_1^L, x^* \rangle, \langle c_0^R, \bar{x} \rangle - \langle c_0^R, x^* \rangle, \langle c_1^R, \bar{x} \rangle - \langle c_1^R, x^* \rangle)^T. \quad (3.4)$$

Hence:

$$\begin{aligned} 0 &\preceq \langle c_0^L, \bar{x} \rangle - \langle c_0^L, x^* \rangle, & 0 &\preceq \langle c_1^L, \bar{x} \rangle - \langle c_1^L, x^* \rangle, \\ 0 &\preceq \langle c_0^R, \bar{x} \rangle - \langle c_0^R, x^* \rangle, & 0 &\preceq \langle c_1^R, \bar{x} \rangle - \langle c_1^R, x^* \rangle. \end{aligned}$$

That is:

$$\langle c_0^L, \bar{x} \rangle \geq \langle c_0^L, x^* \rangle, \langle c_1^L, \bar{x} \rangle \geq \langle c_1^L, x^* \rangle,$$

$$\langle c_0^R, \bar{x} \rangle \geq \langle c_0^R, x^* \rangle, \langle c_1^R, \bar{x} \rangle \geq \langle c_1^R, x^* \rangle.$$

By using Lemma 3.1, for any  $\lambda \in [0, 1]$ , we have:

$$\langle c_\lambda^L, x^* \rangle \leq \langle c_\lambda^L, \bar{x} \rangle, \quad \langle c_\lambda^R, x^* \rangle \leq \langle c_\lambda^R, \bar{x} \rangle$$

that is,  $\langle \tilde{c}, \bar{x} \rangle_F \succcurlyeq \langle \tilde{c}, x^* \rangle_F$ . However, this contradicts the assumption that  $x^* \in X$  is a non-dominated solution to the FLP problem.

Let  $x^* \in X$  be a Pareto optimal solution to the MOLP problem. If  $x^*$  is not a non-dominated solution to the problem, then there exists  $\bar{x} \in X$  such that  $\langle \tilde{c}, \bar{x} \rangle_F \succcurlyeq \langle \tilde{c}, x^* \rangle_F$ . Therefore, for any  $\lambda \in [0, 1]$ , we have:

$$\left( \sum_{i=1}^n \tilde{c}_i x_i^* \right)_\lambda^L \leq \left( \sum_{i=1}^n \tilde{c}_i \bar{x}_i \right)_\lambda^L, \quad \left( \sum_{i=1}^n \tilde{c}_i x_i^* \right)_\lambda^R \leq \left( \sum_{i=1}^n \tilde{c}_i \bar{x}_i \right)_\lambda^R,$$

that is:

$$\langle c_\lambda^L, x^* \rangle \leq \langle c_\lambda^L, \bar{x} \rangle, \quad \langle c_\lambda^R, x^* \rangle \leq \langle c_\lambda^R, \bar{x} \rangle.$$

Hence, for  $\lambda = 0$  and  $\lambda = 1$ , we have:

$$\langle c_0^L, \bar{x} \rangle \geq \langle c_0^L, x^* \rangle, \quad \langle c_1^L, \bar{x} \rangle \geq \langle c_1^L, x^* \rangle,$$

$$\langle c_0^R, \bar{x} \rangle \geq \langle c_0^R, x^* \rangle, \quad \langle c_1^R, \bar{x} \rangle \geq \langle c_1^R, x^* \rangle,$$

Which contradicts the assumption that  $x^* \in X$  is a Pareto optimal solution to the MOLP problem.

**Theorem 3.5.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. Then  $x^*$  is a weak non-dominated solution to the problem if and only if  $x^*$  is a weak Pareto optimal solution to the MOLP problem.

**Proof.** Similar to that of Theorem 3.4.

From Theorems 3.3, 3.4 and 3.5, in order to find all optimal or non-dominated or all weak non-dominated solutions to the FLP problem, it suffices to find all complete or Pareto or weak Pareto optimal solutions to the MOLP problem. Now, associated with the MOLP problem, we consider the following weighted linear programming problem defined by Kuhn and Tucker (1951) and Zadeh (1963):

$$(MOLP_w) \begin{cases} \text{maximize} & \langle w, \tilde{c}, x \rangle = w_0^L \langle c_0^L, x \rangle + w_1^L \langle c_1^L, x \rangle \\ & + w_0^R \langle c_0^R, x \rangle + w_1^R \langle c_1^R, x \rangle \\ \text{subject to} & Ax \leq b, \quad x \geq 0 \end{cases}$$

where:

$$c_i^L = (c_{i1}^L, c_{i2}^L, \dots, c_{in}^L)^T, \quad c_i^R = (c_{i1}^R, c_{i2}^R, \dots, c_{in}^R)^T \in R^n, i = 0, 1, \\ w = (w_0^L, w_1^L, w_0^R, w_1^R) \geq 0.$$

**Theorem 3.6.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. If it is an optimal solution of the  $MOLP_w$  problem for some  $w > 0$ , then it is a non-dominated solution to the FLP problem.

**Proof.** If an optimal solution  $x^*$  to the  $MOLP_w$  problem is not a non-dominated solution to the FLP problem, from Theorem 3.4, it is not a Pareto optimal solution to the MOLP problem, thus there exists a  $\bar{x} \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \leq (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T. \quad (3.5)$$

Hence, there exists at least a  $c_i^L$  or  $c_i^R$ ,  $i = 1, 2$  such that “ $<$ ” holds. Noting that:

$$w = (w_0^L, w_1^L, w_0^R, w_1^R) > 0,$$

this implies:

$$\begin{aligned} \langle w, \tilde{c}, x^* \rangle &= w_0^L \langle c_0^L, x^* \rangle + w_1^L \langle c_1^L, x^* \rangle + w_0^R \langle c_0^R, x^* \rangle + w_1^R \langle c_1^R, x^* \rangle \\ &< \langle c_0^L, \bar{x} \rangle + w_1^L \langle c_1^L, \bar{x} \rangle + w_0^R \langle c_0^R, \bar{x} \rangle + w_1^R \langle c_1^R, \bar{x} \rangle \\ &= \langle w, \tilde{c}, \bar{x} \rangle. \end{aligned}$$

However, this contradicts the assumption that  $x^*$  is an optimal solution to the MOLP<sub>w</sub> problem for some  $w > 0$ .

**Theorem 3.7.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. If it is a non-dominated solution to the problem, then it is an optimal solution to the MOLP<sub>w</sub> problem for some  $w \geq 0$ .

**Proof.** If  $x^*$  is a non-dominated solution to the FLP problem, then it is a Pareto optimal solution to the MOLP problem from Theorem 3.4. By using Theorem 4.2 of Maeda (2001), it is an optimal solution to the MOLP<sub>w</sub> problem for some  $w \geq 0$ .

**Theorem 3.8.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem, then it is an optimal solution of the MOLP<sub>w</sub> problem for some  $w \geq 0$  if and only if it is a weak non-dominated solution to the FLP problem.

**Proof.** Similar to the proofs for Theorems 3.6 and 3.7.

Now, associated with the MOLP problem, we consider the following constrained multi-objective linear programming (CMOLP) problem defined by Haimes et al. (1971) and Haimes and Hall (1974):

$$(CMOLP) \quad \begin{cases} \text{maximize} & \langle c_i, x \rangle \\ \text{subject to} & \langle c_j, x \rangle \geq \varepsilon_j, j = 1, 2, 3, 4; \quad j \neq i \\ & Ax \leq b, x \geq 0 \end{cases} \quad (3.6)$$

where  $c_1 = (c_{01}^L, c_{02}^L, \dots, c_{0n}^L)^T$ ,  $c_2 = (c_{11}^L, c_{12}^L, \dots, c_{1n}^L)^T$ ,  $c_3 = (c_{01}^R, c_{02}^R, \dots, c_{0n}^R)^T$ ,  $c_4 = (c_{11}^R, c_{12}^R, \dots, c_{1n}^R)^T \in R^n$ , and  $\varepsilon_j$  is the minimum acceptable values for objectives corresponding to  $j \neq i$ .

**Theorem 3.9.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. If it is a unique optimal solution of the CMOLP problem for some  $\varepsilon_j$ ,  $j = 1, 2, 3, 4$  and  $j \neq i$ , then it is a non-dominated solution to the FLP problem.

**Proof.** If a unique optimal solution  $x^*$  to the CMOLP problem is not a non-dominated solution to the FLP problem, then it is not a Pareto optimal solution to the MOLP problem from Theorem 3.4, therefore there exists a  $\bar{x} \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \leq (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T. \quad (3.7)$$

This means:

$$\varepsilon_j \leq \langle c_j, x^* \rangle < \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4; \quad j \neq i, \quad \langle c_i, x^* \rangle < \langle c_i, \bar{x} \rangle,$$

which contradicts the assumption that  $x^*$  is a unique optimal solution of the CMOLP problem for some  $\varepsilon_j$ ,  $j = 1, 2, 3, 4; j \neq i$ .

**Theorem 3.10.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. If it is a non-dominated solution to the problem, then it is an optimal solution of the CMOLP problem for some  $\varepsilon_j$ ,  $j = 1, 2, 3, 4$  and  $j \neq i$ .

**Proof.** If  $x^*$  is a non-dominated solution to the FLP problem, then it is a Pareto optimal solution to the MOLP problem from Theorem 3.4. Suppose  $x^*$  is not an optimal solution of the CMOLP problem for some  $\varepsilon_j$ ,  $j = 1, 2, 3, 4; j \neq i$ , then there exists a  $x^* \in X$  such that:

$$\langle c_j, x^* \rangle = \varepsilon_j \leq \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4; \quad j \neq i, \quad \langle c_i, x^* \rangle < \langle c_i, \bar{x} \rangle,$$

which contradicts the fact that  $x^*$  is a Pareto optimal solution to the MOLP problem.

**Theorem 3.11.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. If it is an optimal solution of the CMOLP problem for some  $\varepsilon_j, j = 1, 2, 3, 4; j \neq i$ , then it is a weak non-dominated solution to the FLP problem.

**Proof.** If an optimal solution  $x^*$  to the CMOLP problem is not a weak non-dominated solution to the FLP problem, then it is not a weak Pareto optimal solution to the MOLP problem from Theorem 3.5. Therefore, there exists a  $x^* \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T < (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T . \quad (3.8)$$

This means:

$$\varepsilon_j \leq \langle c_j, x^* \rangle < \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4; j \neq i, \quad \langle c_i, x^* \rangle < \langle c_i, \bar{x} \rangle,$$

which contradicts the assumption that  $x^*$  is an optimal solution of the CMOLP problem for some  $\varepsilon_j, j = 1, 2, 3, 4; j \neq i$ .

Now, associated with the MOLP problem, we consider the following weighted maximum linear programming problem defined by (Bowman 1976):

$$(\text{MOLP}_{\text{wm}}) = \begin{cases} \text{maximize} & \min_{i=1,2,3,4} w_i \langle c_i, x \rangle \\ \text{subject to} & Ax \leq b, x \geq 0 \end{cases} \quad (3.9)$$

Where:

$$\begin{aligned} c_1 &= (c_{01}^L, c_{02}^L, \dots, c_{0n}^L)^T, c_2 = (c_{11}^L, c_{12}^L, \dots, c_{1n}^L)^T, \\ c_3 &= (c_{01}^R, c_{02}^R, \dots, c_{0n}^R)^T, c_4 = (c_{11}^R, c_{12}^R, \dots, c_{1n}^R)^T \in R^n, \\ w &= (w_1, w_2, w_3, w_4) \geq 0. \end{aligned}$$

**Theorem 3.12.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. If it is a unique optimal solution of the MOLP<sub>wm</sub> problem for some  $w \geq 0$ , then it is a non-dominated solution to the FLP problem.

**Proof.** If a unique optimal solution  $x^*$  to the MOLP<sub>wm</sub> problem for some  $w \geq 0$  is not a non-dominated solution to the FLP problem, then it is not a Pareto optimal solution to the MOLP problem from Theorem 3.4. Therefore there exists a  $\bar{x} \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T \leq (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T. \quad (3.10)$$

In view of  $w = (w_1, w_2, w_3, w_4) \geq 0$ , it follows:

$$w_j \langle c_j, x^* \rangle \leq w_j \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4.$$

Hence:

$$\min_{j=1,2,3,4} w_j \langle c_j, x^* \rangle \leq \min_{j=1,2,3,4} w_j \langle c_j, \bar{x} \rangle$$

which contradicts the assumption that  $x^*$  is a unique optimal solution of the MOLP<sub>wm</sub> problem for some  $w = (w_1, w_2, w_3, w_4) \geq 0$ .

**Theorem 3.13.** Let a point  $x^* \in X$  be any feasible solution to the FLP problem. If it is a non-dominated solution to the problem, then it is an optimal solution of the MOLP<sub>wm</sub> problem for some  $w = (w_1, w_2, w_3, w_4) > 0$ .

**Proof.** If  $x^*$  is a non-dominated solution to the FLP problem then it is a Pareto optimal solution to the MOLP problem from Theorem 3.4. Here, without loss of generality, we assume that  $\langle c_j, x \rangle > 0, j = 1, 2, 3, 4$  for all  $x \in X$  and choose  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$  such that  $w_j^* \langle c_j, x^* \rangle = v, j = 1, 2, 3, 4$ . Now, we assume

that  $x^*$  is not an optimal solution of the  $\text{MOLP}_{wm}$  problem for  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$ , then there exists a  $\bar{x} \in X$  such that:

$$w_j \langle c_j, x^* \rangle < w_j^* \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4.$$

Noting  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$ , this implies:

$$\langle c_j, x^* \rangle < \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4.$$

which contradicts the fact that  $x^*$  is a Pareto optimal solution to the MOLP problem.

**Theorem 3.14.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. If it is an optimal solution of the  $\text{MOLP}_{wm}$  problem for some  $w \geq 0$ , then it is a weak non-dominated solution to the FLP problem.

**Proof.** If an optimal solution  $x^*$  to the  $\text{MOLP}_{wm}$  problem for some  $w \geq 0$  is not a weak non-dominated solution to the FLP problem, then it is not a weak Pareto optimal solution to the MOLP problem from Theorem 3.5. Therefore, there exists a  $x^* \in X$  such that:

$$(\langle c_0^L, x^* \rangle, \langle c_1^L, x^* \rangle, \langle c_0^R, x^* \rangle, \langle c_1^R, x^* \rangle)^T < (\langle c_0^L, \bar{x} \rangle, \langle c_1^L, \bar{x} \rangle, \langle c_0^R, \bar{x} \rangle, \langle c_1^R, \bar{x} \rangle)^T. \quad (3.11)$$

In view of  $w = (w_1, w_2, w_3, w_4) \geq 0$ , it follows:

$$w_j \langle c_j, x^* \rangle \leq w_j \langle c_j, \bar{x} \rangle, \quad j = 1, 2, 3, 4.$$

Hence:

$$\min_{j=1,2,3,4} w_i \langle c_i, x^* \rangle < \min_{j=1,2,3,4} w_i \langle c_i, \bar{x} \rangle$$



which contradicts the assumption that  $x^*$  is a unique optimal solution of the MOLP<sub>wm</sub> problem for some  $w = (w_1, w_2, w_3, w_4) \geq 0$ .

**Theorem 3.15.** Let a point  $x^* \in X$  be a feasible solution to the FLP problem. If it is a weak non-dominated solution to the problem, then it is an optimal solution of the MOLP<sub>wm</sub> problem for some  $w = (w_1, w_2, w_3, w_4) \geq 0$ .

**Proof.** If  $x^*$  is a weak non-dominated solution to the FLP problem then it is a weak Pareto optimal solution to the MOLP problem from Theorem 3.5. Here, without loss of generality, we can assume that  $\langle c_j, x \rangle > 0, j = 1, 2, 3, 4$  for all  $x \in X$  and choose  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$  such that  $w_j^* \langle c_j, x^* \rangle = v, j = 1, 2, 3, 4$ . Now, we assume  $x^*$  is not an optimal solution of the MOLP<sub>wm</sub> problem for  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$ , then there exists a  $\bar{x} \in X$  such that:

$$w_j^* \langle c_j, x^* \rangle < w_j^* \langle c_j, \bar{x} \rangle, j = 1, 2, 3, 4.$$

Noting  $w = (w_1^*, w_2^*, w_3^*, w_4^*) > 0$ , this implies:

$$\langle c_j, x^* \rangle < \langle c_j, \bar{x} \rangle, j = 1, 2, 3, 4.$$

which contradicts the fact that  $x^*$  is a Pareto optimal solution to the MOLP problem.

### 3.4 Examples

To conclude this section, we give two numerical examples in this section.

#### Example 3.1.

$$\begin{cases} \text{maximize} & f((x, y)) = \tilde{c}_1 x + \tilde{c}_2 y & (3.12) \\ \text{subject to} & x + 4y \leq 14 & (3.13) \\ & 4x + 10y \leq 38 \\ & 28x - 5y \leq 56 \\ & x \geq 0, y \geq 0 \end{cases}$$

where the membership functions of  $\tilde{c}_1$  and  $\tilde{c}_2$  are:

$$\mu_{\tilde{c}_1}(x) = \begin{cases} 0, & x < 5 \\ x - 5, & 5 \leq x < 6 \\ 1, & 6 \leq x \leq 7 \\ \frac{20 - x}{13}, & 7 < x \leq 20 \\ 0, & 20 < x \end{cases}$$

$$\mu_{\tilde{c}_2}(x) = \begin{cases} 0, & x < 16 \\ x - 16, & 16 \leq x < 17 \\ 1, & 17 \leq x \leq 18 \\ \frac{40 - x}{22}, & 18 < x \leq 40 \\ 0, & 40 < x \end{cases}$$

Associated with the (FLP) problem, consider the following multi-objective linear programming problem:

$$(MOLP) \begin{cases} \text{maximize} & \{5x + 16y, 6x + 17y, 7x + 18y, 20x + 40y\} \\ \text{subject to} & (3.13) \end{cases}.$$

According to Theorem 3.3, the optimal solution to the (MOLP) problem is an optimal solution to the (FLP) problem. To solve the (MOLP) problem, we consider the following weighted maximization problem:

$$(MOLP_{w_{\max}}) \begin{cases} \text{maximize} & w_1(5x + 16y) + w_2(6x + 17y) + w_3(7x + 18y) \\ & + w_4(20x + 40y) \\ \text{subject to} & (3.13) \end{cases}.$$

From Theorem 3.12, if  $(x^*, y^*)$  is a unique optimal non-dominated solution to the  $(MOLP_{w_{\max}})$  problem for some  $w \geq 0$ , then it is a non-dominated solution to the (FLP) problem. Standard optimization techniques can then be used to solve the problem. Obviously, the solution of the problem depends on the choice of the weighting for the objective functions. For example, for  $w_1 = w_4 = 0$ ,  $w_2 = w_3 = 1$ , the solution is  $(x^*, y^*) = (2, 3)$  and then the membership function of:

$$f((x^*, y^*)) = f((2,3)) = 2\tilde{c}_1 + 3\tilde{c}_2$$

is:

$$\mu_{f((2,3))}(x) = \begin{cases} 0, & x < 58 \\ \frac{x - 58}{5}, & 58 \leq x < 63 \\ 1, & 63 \leq x \leq 68 \\ \frac{160 - x}{92}, & 68 < x \leq 160 \\ 0, & 160 < x \end{cases}$$

While, for  $w_1 = 0$ ,  $w_2 = w_3 = 1$ ,  $w_4 = 1/2$ , the solution is  $(x^*, y^*) = (2.5, 2.8)$  and then the membership function of:

$$f((x^*, y^*)) = f((2.5, 2.8)) = 2.5\tilde{c}_1 + 2.8\tilde{c}_2$$

is:

$$\mu_{f((2.5, 2.8))}(x) = \begin{cases} 0, & x < 57.3 \\ \frac{x - 57.3}{5.3}, & 57.3 \leq x < 62.6 \\ 1, & 62.6 \leq x \leq 68.9 \\ \frac{162 - x}{92.1}, & 69.9 < x \leq 162 \\ 0, & 162 < x \end{cases}$$

### Example 3.2.

In this example, we consider the same (FLP) problem as given in Maeda (2001) and we will show that our method will include Maeda's method as a special case. The following is the (FLP) problem under consideration:

$$\begin{aligned}
& \left. \begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \right\} \begin{array}{l} f = \widetilde{10}x_1 + \widetilde{34}x_2 \\ 3x_1 + 10x_2 \leq 1200 \\ 7x_1 + 25x_2 \leq 2950 \\ 8x_1 + 25x_2 \leq 3075 \\ 0 \leq x_1 \leq 180 \\ 0 \leq x_2 \leq 100 \end{array} \quad \begin{array}{l} (3.14) \\ (3.15) \end{array}
\end{aligned}$$

where the membership functions of  $\widetilde{10}$  and  $\widetilde{34}$  are:

$$\mu_{\widetilde{10}_1}(x) = \begin{cases} 0, & x \leq 7 \\ \frac{x-7}{3}, & 7 < x \leq 10 \\ 1, & x = 10 \\ \frac{13-x}{3}, & 10 < x < 13 \\ 0, & 13 < x \end{cases} ,$$

$$\mu_{\widetilde{34}}(x) = \begin{cases} 0, & x \leq 27 \\ \frac{x-27}{7}, & 27 < x \leq 34 \\ 1, & x = 34 \\ \frac{41-x}{7}, & 34 < x < 41 \\ 0, & 41 < x \end{cases} .$$

Following the same procedure as used in Example 3.1, the problem of finding the solution of the (FLP) problem becomes the problem of finding the solution of the following (MOLP<sub>wmax</sub>) problem:

$$\begin{aligned}
(MOLP_{wmax}) \left\{ \begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \right. & \begin{array}{l} (w_1 + w_2 + w_3)(10x_1 + 34x_2) + (w_3 - w_1)(3x_1 + 7x_2) \\ (3.15) \end{array} \\
& (3.16)
\end{aligned}$$

By choosing  $w_1 + w_2 + w_3 = 1$  and  $w_3 - w_1 = \lambda$  with  $\lambda$  varying from  $-1$  to  $1$ , the (MOLP<sub>wmax</sub>) problem becomes:

$$\begin{cases} \text{maximize} & (10x_1 + 34x_2) + \lambda(3x_1 + 7x_2), \\ \text{subject to} & \end{cases} \quad (3.17)$$

which is exactly the same as the parametric linear programming derived by Maeda (2001) using his bi-criteria method. An application of a standard optimization technique to the above problem will then lead to the same solution as that given by Maeda (2001). This example clearly indicates that our solution method includes Maeda's method as a special case.

A further analysis shows that if all the fuzzy coefficients  $\tilde{c}_i$  in the objective function have triangle membership functions and the weightings for  $c_i^L$  and  $c_i^R$  are chosen to be equal (i.e.,  $w_1 = w_3$ ), then the solution of the fuzzy linear programming problem is the same as the solution to the normal non-fuzzy linear programming problem with the coefficients  $\tilde{c}_i$  replaced by a non-fuzzy value  $c_i$  satisfying  $\mu_{\tilde{c}_i}(c_i) = 1$ .

### 3.5 Concluding Remarks

In this chapter, we consider the solution to fuzzy linear programming problems which involve fuzzy numbers in the coefficients of objective functions. Firstly we introduce and investigate various concepts of optimal solutions to fuzzy linear programming problems. Consequently a number of theorems have been developed to convert the fuzzy linear programming problem to a multi-objective optimization problem with four objective functions. Finally, we successfully demonstrated the validity of our methods through two examples. We have shown in Example 3.2 that our method of solution has included an existing method as a special case.

# Chapter 4

## Solution of fuzzy linear optimisation problems by the $\alpha$ -fuzzy max order

### 4.1 General

In Chapter 3, we study the solution to fuzzy mathematical programming problems with objective functions involving fuzzy numbers, and various theorems and methods have been developed to deal with the problems. In this chapter, we focus on another aspect of fuzzy mathematical programming problems, namely dealing with constraints involving fuzzy numbers.

The rest of this chapter is organized as follows. In Section 4.2, we firstly introduce the concept of  $\alpha$ -fuzzy max order and use the concept to define a fuzzy optimization problem with constraints involving fuzzy numbers. Then, two important theorems are developed concerning the determination of the feasible solution spaces defined by the constraint inequalities involving only fuzzy numbers respectively with isosceles triangle membership functions and trapezoidal membership functions. In Section 4.3, by utilizing the results in Chapter 3 and Section 4.2, we develop the  $\alpha$ -fuzzy max order method for solving the mathematical programming problems where both the objective function and the constraint equations/inequalities contain fuzzy coefficients, and some theoretical results are obtained for problems involving fuzzy coefficients with trapezoidal membership functions. In Section 4.4, various numerical algorithms are developed for the solution to various fuzzy mathematical programming problems involving fuzzy coefficients with any form of membership functions. Finally in Section 4.5, three illustrative examples are given to demonstrate the validity of the methods and the algorithms developed.

## 4.2 The $\alpha$ -fuzzy max order for problems with fuzzy constraints

In this section, we consider the following fuzzy linear programming (FLP) problem in which the constraint equation and/or inequalities involve fuzzy coefficients:

$$(\text{FLP}) \begin{cases} \text{maximize} & \langle c, x \rangle = \sum_{i=1}^n c_i x_i \\ \text{subject to} & \tilde{A}x \preceq \tilde{b}, x \geq 0, \end{cases} \quad (4.1)$$

where:

$c = (c_1, c_2, \dots, c_n)^T \in R^n$ ,  $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m)^T \in F^*(R^m)$ ,  $\tilde{A} = (\tilde{a}_{ij})$ ,  
in which  $\tilde{a}_{ij} \in F^*(R^m)$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$ .

Associated with the (FLP) problem, consider the following  $(\text{FLP}_\lambda)$  problem:

$$(\text{FLP}_\lambda) \begin{cases} \text{maximize} & \langle c, x \rangle = \sum_{i=1}^n c_i x_i \\ \text{subject to} & A_\lambda^L x \leq b_\lambda^L, A_\lambda^R x \leq b_\lambda^R, x \geq 0, \forall \lambda \in [0, 1]. \end{cases} \quad (4.2)$$

**Theorem 4.1.** If  $x^*$  is the solution of the  $(\text{FLP}_\lambda)$  problem, it is also the solution to the (FLP) problem.

**Proof.** The proof is obvious from Definition 3.5.

Obviously, a feasible solution must satisfy the constraints for all  $\lambda \in [0, 1]$ . However, in general, this requirement is too strong. Now consider a typical coefficient  $c_i$  represented by a fuzzy number  $\tilde{c}_i$ . The possibility of such a parameter  $c_i$  taking a value in the range  $[c_{i\lambda}^L, c_{i\lambda}^R]$  is  $\lambda$  or above, while the possibility of  $c_i$  taking a value beyond  $[c_{i\lambda}^L, c_{i\lambda}^R]$  is less than  $\lambda$ . Thus, one would generally be more interested in solutions obtained using coefficients  $c_i$  taking values in  $[c_{i\lambda}^L, c_{i\lambda}^R]$  with  $\lambda \geq \alpha > 0$ . As a special case, if the coefficients involved are either real numbers or fuzzy numbers with triangle membership functions, then, we will have the usual non-fuzzy

optimization problem, supposing we choose  $\alpha = 1$ . To formulate this idea, we introduce the following definitions.

**Definition 4.1.** For any  $n$ -dimensional fuzzy numbers  $\tilde{a}, \tilde{b} \in F(R^n)$ , we define:

$$\tilde{a} \underset{=}{\succ}_{\alpha} \tilde{b} \text{ if and only if } a_{i\lambda}^L \geq b_{i\lambda}^L \text{ and } a_{i\lambda}^R \geq b_{i\lambda}^R, i = 1, 2, \dots, n, \forall \lambda \in [\alpha, 1];$$

$$\tilde{a} \underset{>}{\succ}_{\alpha} \tilde{b} \text{ if and only if } a_{i\lambda}^L \geq b_{i\lambda}^L \text{ and } a_{i\lambda}^R \geq b_{i\lambda}^R, i = 1, 2, \dots, n, \forall \lambda \in [\alpha, 1];$$

$$\tilde{a} \succ_{\alpha} \tilde{b} \text{ if and only if } a_{i\lambda}^L > b_{i\lambda}^L \text{ and } a_{i\lambda}^R > b_{i\lambda}^R, i = 1, 2, \dots, n, \forall \lambda \in [\alpha, 1].$$

We call the binary relations  $\underset{=}{\succ}_{\alpha}, \underset{>}{\succ}_{\alpha}$  and  $\succ_{\alpha}$  an  $\alpha$ -fuzzy max order, a strict  $\alpha$ -fuzzy max order and a strong  $\alpha$ -fuzzy max order, respectively.

With Definition 4.1, we turn our interest to the solution of the following problem:

$$(FLP_{\alpha}) \begin{cases} \text{maximize} & \langle c, x \rangle = \sum_{i=1}^n c_i x_i \\ \text{subject to} & \tilde{A}x \underset{=}{\prec}_{\alpha} \tilde{b}, x \geq 0. \end{cases} \quad (4.3)$$

Associated with the  $(FLP_{\alpha})$  problem, we now consider the following problem:

$$(FLP_{\alpha\lambda}) \begin{cases} \text{maximize} & \langle c, x \rangle = \sum_{i=1}^n c_i x_i \\ \text{subject to} & A_{\lambda}^L x \leq b_{\lambda}^L, A_{\lambda}^R x \leq b_{\lambda}^R, x \geq 0, \forall \lambda \in [0, 1]. \end{cases} \quad (4.4)$$

Where:

$$c = (c_i)_{1 \times n},$$

$$A_{\lambda}^L = (a_{ij\lambda}^L)_{m \times n}, A_{\lambda}^R = (a_{ij\lambda}^R)_{m \times n}, b_{\lambda}^L = (b_{i\lambda}^L)_{m \times 1} \text{ and } b_{\lambda}^R = (b_{i\lambda}^R)_{m \times 1}.$$

**Theorem 4.2.** Let  $x^*$  be the solution of the  $(FLP_{\alpha\lambda})$  problem (4.4). Then it is also a solution of the  $(FLP_{\alpha})$  problem defined by (4.3).



**Proof.** The proof is obvious from Definition 4.1.

**Theorem 4.3.** If all the fuzzy coefficients  $\tilde{a}_{ij}$  and  $\tilde{b}_i$  have isosceles triangle membership functions:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z - h_{\tilde{z}} \\ \frac{t - z + h_{\tilde{z}}}{h_{\tilde{z}}} & z - h_{\tilde{z}} \leq t < z \\ \frac{-t + z + h_{\tilde{z}}}{h_{\tilde{z}}} & z \leq t < z + h_{\tilde{z}} \\ 0 & z + h_{\tilde{z}} \leq t \end{cases}, \quad (4.5)$$

where  $\tilde{z}$  denotes  $\tilde{a}_{ij}$  or  $\tilde{b}_i$ ,  $z$  and  $h_{\tilde{z}}$  are the centre and the deviation parameter of  $\tilde{z}$  respectively, then, the space of feasible solutions  $X$  is defined by the set of  $x \in R^n$  with  $x_i$ , for  $i = 1, 2, \dots, n$ , satisfying:

$$\left\{ \begin{array}{l} \sum_{j=1}^n a_{ij} x_j \leq b_i \\ \sum_{j=1}^n [a_{ij} - (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_i - (1 - \alpha)h_{\tilde{b}_i} \\ \sum_{j=1}^n [a_{ij} + (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_i - (1 - \alpha)h_{\tilde{b}_i} \\ x_i \geq 0 \end{array} \right. \quad (4.6)$$

**Proof.**

From Theorem 4.1,  $X$  is defined by:

$$X = \left\{ x \in R^n \left| \sum_{j=1}^n a_{ij\lambda}^L x_j \leq b_{i\lambda}^L, \sum_{j=1}^n a_{ij\lambda}^R x_j \leq b_{i\lambda}^R, x \geq 0, \forall \lambda \in [\alpha, 1], \text{ and } i = 1, 2, \dots, m \right. \right\} \quad (4.7)$$

That is,  $X$  is the set of  $x \in R^n$  with  $x \geq 0$  and satisfying:

$$I_{i\lambda} = \sum_{j=1}^n a_{ij\lambda}^L x_j - b_{i\lambda}^L \leq 0,$$

$$J_{i\lambda} = \sum_{j=1}^n a_{ij\lambda}^R x_j - b_{i\lambda}^R \leq 0, \quad \forall \lambda \in [\alpha, 1], \text{ and } i = 1, 2, \dots, m. \quad (4.8)$$

For fuzzy numbers with isosceles triangle membership functions, we have:

$$a_{ij\lambda}^L = a_{ij} - h_{\tilde{a}_{ij}}(1 - \lambda),$$

$$a_{ij\lambda}^R = a_{ij} + h_{\tilde{a}_{ij}}(1 - \lambda), \quad (4.9)$$

$$b_{i\lambda}^L = b_i - h_{\tilde{b}_i}(1 - \lambda),$$

$$b_{i\lambda}^R = b_i + h_{\tilde{b}_i}(1 - \lambda). \quad (4.10)$$

Substituting (4.9) and (4.10) into (4.8), we have:

$$I_{i\lambda} = \sum_{j=1}^n [a_{ij} - h_{\tilde{a}_{ij}}(1 - \lambda)] x_j - [b_i - h_{\tilde{b}_i}(1 - \lambda)], \quad (4.11)$$

$$J_{i\lambda} = \sum_{j=1}^n [a_{ij} + h_{\tilde{a}_{ij}}(1 - \lambda)] x_j - [b_i + h_{\tilde{b}_i}(1 - \lambda)]. \quad (4.12)$$

Now, our problem is to show that  $I_{i\lambda} \leq 0$ ,  $J_{i\lambda} \leq 0$ ,  $\forall \lambda \in [\alpha, 1]$ , and  $i = 1, 2, \dots, m$  if (4.6) is satisfied. From (4.6)<sub>1</sub>, we have:

$$\sum_{j=1}^n a_{ij} x_j - b_i \leq 0. \quad (4.13)$$

From (4.6)<sub>2,3</sub>, we obtain:

$$\sum_{j=1}^n (1 - \alpha) (h_{\tilde{a}_{ij}} x_j - h_{\tilde{b}_i}) \geq \sum_{j=1}^n a_{ij} x_j - b_i, \quad (4.14)$$

$$\sum_{j=1}^n (1 - \alpha) (h_{\tilde{a}_{ij}} x_j - h_{\tilde{b}_i}) \leq - \sum_{j=1}^n a_{ij} x_j - b_i, \quad (4.15)$$

Thus, from (4.11) and (4.12) and using (4.13) - (4.15), we have, for any  $\lambda \in [\alpha, 1]$  and  $i = 1, 2, \dots, m$ :

$$\begin{aligned}
I_{i\lambda} &= \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) - \left( \sum_{j=1}^n a_{ij}x_j - h_{\tilde{b}_i} \right) (1 - \lambda) \\
&< \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) - \left( \sum_{j=1}^n a_{ij}x_j - h_{\tilde{b}_i} \right) \frac{(1 - \lambda)}{(1 - \alpha)} \\
&= \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) \frac{(\lambda - \alpha)}{(1 - \alpha)} \\
&\leq 0
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
J_{i\lambda} &= \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) + \left( \sum_{j=1}^n a_{ij}x_j - h_{\tilde{b}_i} \right) (1 - \lambda) \\
&< \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) - \left( \sum_{j=1}^n a_{ij}x_j - h_{\tilde{b}_i} \right) \frac{(1 - \lambda)}{(1 - \alpha)} \\
&= \left( \sum_{j=1}^n a_{ij}x_j - b_i \right) \frac{(\lambda - \alpha)}{(1 - \alpha)} \\
&\leq 0
\end{aligned} \tag{4.17}$$

The proof is complete.

**Theorem 4.4.** If all the fuzzy coefficients  $\tilde{a}_{ij}$  and  $\tilde{b}_i$  have trapezoidal triangle membership functions:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & , & t \leq z_L - h \\ \frac{t - z_L + h}{h} & , & z_L - h \leq t < z_L \\ 1 & , & z_L \leq t < z_R \\ \frac{z_R + h - t}{h} & , & z_R < t < z_R + h \\ 0 & , & t \geq z_R + h \end{cases} \tag{4.18}$$

where  $\tilde{z}$  denotes  $\tilde{a}_{ij}$  or  $\tilde{b}_i$ ,  $z_R, z_L$  and  $h$  are the parameters of the  $\tilde{z}$  membership function, then, the space of feasible solutions  $X$  is defined by the set of  $x \in R^n$  with  $x_i$ , for  $i = 1, 2, \dots, n$ , satisfying:

$$\left\{ \begin{array}{l} \sum_{j=1}^n a_{ijL} x_j \leq b_{iL}, \quad \sum_{j=1}^n a_{ijR} x_j \leq b_{iR} \\ \sum_{j=1}^n [a_{ijL} - (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_{iL} - (1 - \alpha)h_{\tilde{b}_i} \\ \sum_{j=1}^n [a_{ijR} + (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_{iR} + (1 - \alpha)h_{\tilde{b}_i} \\ x_i \geq 0 \end{array} \right. \quad (4.19)$$

**Proof.**

From Theorem 4.2,  $X$  is defined by:

$$X = \left\{ x \in R^n \left| \sum_{j=1}^n a_{ij\lambda}^L x_j - b_{i\lambda}^L, a_{ij\lambda}^R x_j - b_{i\lambda}^R, \lambda \in [\alpha, 1], \text{ and } i = 1, 2, \dots, m \right. \right\}. \quad (4.20)$$

That is,  $X$  is the set of  $x \in R^n$  with  $x \geq 0$  satisfying:

$$\begin{aligned} I_{i\lambda} &= \sum_{j=1}^n a_{ij\lambda}^L x_j - b_{i\lambda}^L \leq 0, \\ J_{i\lambda} &= \sum_{j=1}^n a_{ij\lambda}^R x_j - b_{i\lambda}^R \leq 0, \quad \forall \lambda \in [\alpha, 1], \text{ and } i = 1, 2, \dots, m. \end{aligned} \quad (4.21)$$

For fuzzy numbers with trapezoidal membership functions, we have:

$$a_{ij\lambda}^L = a_{ijL} - h_{\tilde{a}_{ij}}(1 - \lambda), \quad a_{ij\lambda}^R = a_{ijR} + h_{\tilde{a}_{ij}}(1 + \lambda), \quad (4.22)$$

$$b_{i\lambda}^L = b_{iL} - h_{\tilde{b}_i}(1 - \lambda), \quad b_{i\lambda}^R = b_{iR} + h_{\tilde{b}_i}(1 - \lambda). \quad (4.23)$$

Substituting (4.22) and (4.23) into (4.21), we have:

$$I_{i\lambda} = \sum_{j=1}^n [a_{ijL} - h_{\tilde{a}_{ij}}(1-\lambda)]x_j - [b_{iL} - h_{\tilde{b}_i}(1-\lambda)], \quad (4.24)$$

$$J_{i\lambda} = \sum_{j=1}^n [a_{ijR} + h_{\tilde{a}_{ij}}(1-\lambda)]x_j - [b_{iR} + h_{\tilde{b}_i}(1-\lambda)]. \quad (4.25)$$

Now, our problem is to show that  $I_{i\lambda} \leq 0$ ,  $J_{i\lambda} \leq 0$ ,  $\forall \lambda \in [a, 1]$  and  $i = 1, 2, \dots, m$  if (4.19) is satisfied. From (4.19)<sub>1</sub>, we have:

$$\sum_{j=1}^n a_{ijL}x_j - b_{iL} \leq 0, \quad (4.26)$$

$$\sum_{j=1}^n a_{ijR}x_j - b_{iR} \leq 0. \quad (4.27)$$

From (4.19)<sub>2,3</sub>, we obtain:

$$\sum_{j=1}^n (1-\alpha)(h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i}) \geq \sum_{j=1}^n a_{ijL}x_j - b_{iL}, \quad (4.28)$$

$$\sum_{j=1}^n (1-\alpha)(h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i}) \leq -\left(\sum_{j=1}^n a_{ijR}x_j - b_{iR}\right), \quad (4.29)$$

Thus, from (4.24) and (4.25) and using (4.28)-(4.29), we have, for any  $\lambda \in [a, 1]$  and  $i = 1, 2, \dots, m$ :

$$\begin{aligned} I_{i\lambda} &= \left(\sum_{j=1}^n a_{ijL}x_j - b_{iL}\right) - \left(\sum_{j=1}^n h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i}\right)(1-\lambda) \\ &< \left(\sum_{j=1}^n a_{ijL}x_j - b_{iL}\right) - \left(\sum_{j=1}^n a_{ijL}x_j - b_{iL}\right)\frac{(1-\lambda)}{(1-\alpha)} \\ &= \left(\sum_{j=1}^n a_{ijL}x_j - b_{iL}\right)\frac{(\lambda-\alpha)}{(1-\alpha)} \\ &\leq 0 \end{aligned} \quad (4.30)$$

$$\begin{aligned}
J_{i\lambda} &= \left( \sum_{j=1}^n a_{ijR}x_j - b_{iR} \right) + \left( \sum_{j=1}^n h_{\tilde{a}_{ij}}x_j - h_{\tilde{b}_i} \right) (1 - \lambda) \\
&< \left( \sum_{j=1}^n a_{ijR}x_j - b_{iR} \right) - \left( \sum_{j=1}^n a_{ijR}x_j - b_{iR} \right) \frac{(1 - \lambda)}{(1 - \alpha)} \\
&= \left( \sum_{j=1}^n a_{ijR}x_j - b_{iR} \right) \frac{(\lambda - \alpha)}{(1 - \alpha)} \\
&\leq 0
\end{aligned} \tag{4.31}$$

The proof is complete.

### 4.3 The $\alpha$ -fuzzy max order for problems with fuzzy coefficients in both objective functions and constraints

In this section, we are concerned with the following fuzzy linear programming problem:

$$(\text{FLP}) \begin{cases} \max & \langle \tilde{c}, x \rangle = \sum_{i=1}^n \tilde{c}_i x_i \\ \text{subject to} & \tilde{A}x \leq \tilde{b}, \quad x \geq 0 \end{cases}, \tag{4.32}$$

where  $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)^T \in F^*(R^n)$ ,  $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)^T \in F^*(R^n)$ ,

$$\tilde{A} = (\tilde{a}_{ij}) \quad i = 1, n; j = 1, m,$$

in which  $\tilde{a}_{ij} \in F^*(R^n)$ .

Associated with the FLP problem, we consider the following multi-objective linear programming (MOLP $_{\alpha\lambda}$ ) problem:

$$(\text{MOLP}_{\alpha\lambda}) \begin{cases} \max & (\langle c_{\alpha}^L, x \rangle, \langle c_{\alpha}^R, x \rangle, \langle c_1^L, x \rangle, \langle c_1^R, x \rangle) \\ \text{subject to} & A_{\lambda}^L \leq b_{\lambda}^L, A_{\lambda}^R \leq b_{\lambda}^R, \quad x \geq 0 \quad \forall \lambda \in [\alpha, 1] \end{cases} \tag{4.33}$$

$c_{\alpha}^L = (c_{1\alpha}^L, c_{2\alpha}^L, \dots, c_{n\alpha}^L)$  and  $c_{\alpha}^R = (c_{1\alpha}^R, c_{2\alpha}^R, \dots, c_{n\alpha}^R)$  are the left and right limits of the  $\alpha$ -cut of  $\tilde{c}$ ;  $c_1^L = (c_{11}^L, c_{21}^L, \dots, c_{n1}^L)$  and  $c_1^R = (c_{11}^R, c_{21}^R, \dots, c_{n1}^R)$  are the left and right limits of the fuzzy number  $\tilde{c}$  corresponding to the membership function value one.

**Theorem 4.5.** Let  $x^*$  be the solution of the MOLP $_{\alpha\lambda}$  with  $\alpha=0$ , then it is also a solution to the (FLP) problem.

**Proof.** The proof is obvious from Theorem 3.3 and Theorem 4.1

As stated in Section 4.2, a feasible solution must satisfy the constraints for all  $\lambda \in [0,1]$ . Also, for the solution to FLP, the  $\alpha$  value in the coefficients of the objective function must be equal to 0. However, as discussed in Section 4.2, one usually is more interested in the solutions obtained using coefficients taking values in  $[c_{i\lambda}^L, c_{i\lambda}^R]$  where  $\lambda \geq \alpha > 0$ . Combining the results in Chapter 3 and Section 4.2, we can obtain the following two theorems.

**Theorem 4.6.** If all the fuzzy coefficients  $\tilde{A}$  and  $\tilde{b}$  have isosceles triangle membership functions:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & t < z - h_{\tilde{z}} \\ \frac{t - z + h_{\tilde{z}}}{h_{\tilde{z}}} & z - h_{\tilde{z}} \leq t < z \\ \frac{-t + z + h_{\tilde{z}}}{h_{\tilde{z}}} & z \leq t < z + h_{\tilde{z}} \\ 0 & z + h_{\tilde{z}} \leq t \end{cases},$$

where  $\tilde{z}$  denotes  $\tilde{a}_{ij}$  or  $\tilde{b}_i$ ,  $z$  and  $h_{\tilde{z}}$  are the centre and the deviation parameters of  $\tilde{z}$  respectively, then the solutions to the MOLP $_{\alpha\lambda}$  problem can be obtained by solving the following problem:

$$\begin{array}{ll} \text{maximize} & (\langle c_{\alpha}^L, x \rangle, \langle c_{\alpha}^R, x \rangle, \langle c_1^L, x \rangle, \langle c_1^R, x \rangle) \\ \text{subject to} & \begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i \\ \sum_{j=1}^n [a_{ij} - (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_i - (1 - \alpha)h_{\tilde{b}_i} \\ \sum_{j=1}^n [a_{ij} + (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_i - (1 - \alpha)h_{\tilde{b}_i} \\ x_i \geq 0. \end{cases} \end{array} \quad (4.34)$$

**Proof.** From Theorem 4.3.

**Theorem 4.7.** If all the fuzzy coefficients  $\tilde{A}$  and  $\tilde{b}$  have trapezoidal membership functions in the form of:

$$\mu_{\tilde{z}}(t) = \begin{cases} 0 & , & t \leq z_L - h \\ \frac{t - z_L + h}{h} & , & z_L - h \leq t < z_L \\ 1 & , & z_L \leq t < z_R \\ \frac{z_R + h - t}{h} & , & z_L \leq t < z_R + h \\ 0 & , & t \geq z_R + h \end{cases}$$

then the solution to the MOLP $_{\alpha\lambda}$  problem can be obtained by solving the following problem:

$$\text{maximize } (\langle c_{\alpha}^L, x \rangle, \langle c_{\alpha}^R, x \rangle, \langle c_1^L, x \rangle, \langle c_1^R, x \rangle)$$

$$\text{subject to } \begin{cases} \sum_{j=1}^n a_{ijL} x_j \leq b_{iL} \\ \sum_{j=1}^n a_{ijR} x_j \leq b_{iR} \\ \sum_{j=1}^n [a_{ijL} - (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_{iL} - (1 - \alpha)h_{\tilde{b}_i} \\ \sum_{j=1}^n [a_{ijR} + (1 - \alpha)h_{\tilde{a}_{ij}}] x_j \leq b_{iR} + (1 - \alpha)h_{\tilde{b}_i} \\ x_i \geq 0 \end{cases} \quad (4.35)$$

**Proof.** From Theorem 4.4.

## 4.4 Numerical algorithms

It should be emphasized that for problems involving fuzzy numbers with nonlinear membership functions, the results in Section 4.3 will not be applicable. However, based on Theorem 4.2, we can derive a numerical algorithm for the determination of



the space of feasible solutions, an algorithm for the numerical solution to the (FLP<sub>αλ</sub>) problem defined by (4.4), and an algorithm for the numerical solution to the (MOLP<sub>αλ</sub>) problem defined by (4.33). For simplicity in presentation, we define:

$$X_\lambda = \{x \in R^n | A_\lambda^L x \leq b_\lambda^L, A_\lambda^R x \leq b_\lambda^R, x \geq 0 \forall \lambda \in [\alpha, 1]\}.$$

Algorithm for the space of feasible solutions X:

Let the interval  $[\alpha, 1]$  be divided into  $m$  sub-intervals with  $(m+1)$  nodes  $\lambda_i$  ( $i = 0, m$ ) arranged in the order  $\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_m = 1$ .

Step 1: Set  $m = 2$ , then determine

$$X^m = \bigcap_{j=0}^m X_{\lambda_j};$$

Step 2: Determine

$$X^{2m} = \bigcap_{j=0}^{2m} X_{\lambda_j};$$

Step 3: If  $(X^{2m} \sim X^m)$ , then  $X \approx X^{2m}$ . Otherwise, set  $m$  to  $2m$  and go to Step 2 where  $X^{2m} \sim X^m$  means that the space  $X^{2m}$  is close to  $X^m$ , namely

$$[x_i^L, x_i^R]^{2m} \approx [x_i^L, x_i^R]^m, \quad \forall i = 1, 2, \dots, n,$$

in which  $[x_i^L, x_i^R]^{2m}$  represents the interval of  $x_i$  obtained by using  $2m$  sub-intervals.

Algorithm for the (FLP<sub>αλ</sub>) problem defined by (4.4):

Let the interval  $[\alpha, 1]$  be divided into  $m$  sub-intervals with  $(m+1)$  nodes  $\lambda_i$  ( $i = 0, m$ ) arranged in the order  $\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_m = 1$  and denoted by:

$$(FLP_{\alpha\lambda})_m : \begin{cases} \text{maximize} & \langle c, x \rangle \\ \text{subject to} & x \in X^m \end{cases} \quad (4.36)$$

Step 1: Set  $m = 2$ , then solve the  $(FLP_{\alpha\lambda})_m$  problem for  $(x)_m$ , where  $(x)_m = (x_1, x_2, \dots, x_n)_m$  and the subscript  $m$  indicates that the result is obtained subject to constraint  $x \in X^m$ ;

Step 2: Solve the  $(FLP_{\alpha\lambda})_{2m}$  problem for  $(x)_{2m}$ ;

Step 3: If  $\|(x)_{2m} - (x)_m\| < \text{Tol}$  the solution of the  $(FLP_{\alpha\lambda})$  problem is  $x^* = (x)_{2m}$ . Otherwise, update  $m$  to  $2m$  and go to Step 2.

Algorithm for the  $(MOLP_{\alpha\lambda})$  problem defined by (4.33):

Let the interval  $[\alpha, 1]$  be divided into  $m$  sub-intervals with  $(m+1)$  nodes  $\lambda_i$  ( $i = 0, 1, 2, \dots, m$ ) arranged in the order  $\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_m = 1$  and denoted by:

$$(MOLP_{\alpha\lambda})_m : \begin{cases} \text{maximize} & \{\langle c_{\alpha}^L, x \rangle, \langle c_{\alpha}^R, x \rangle, \langle c_1^L, x \rangle, \langle c_1^R, x \rangle\} \\ \text{subject to} & x \in X^m \end{cases} \quad (4.37)$$

Step 1: Set  $m = 2$ , then solve the  $(MOLP_{\alpha\lambda})_m$  problem for  $(x)_m$ , where  $(x)_m = (x_1, x_2, \dots, x_n)_m$  and the subscript  $m$  indicates that the result is obtained subject to constraint  $x \in X^m$ ;

Step 2: Solve the  $(MOLP_{\alpha\lambda})_{2m}$  problem for  $(x)_{2m}$ ;

Step 3: If  $\|(x)_{2m} - (x)_m\| < \text{Tol}$ , the solution of the (MOLP $_{\alpha\lambda}$ ) problem defined by (4.33) is  $x^* = (x)_{2m}$ ; or otherwise, update  $m = 2m$  and go to Step 2.

## 4.5 Examples

To conclude this paper, we give several examples in this section.

### Example 4.1. [Fuzzy constraints with triangle membership function]

$$(FLP) \begin{cases} \text{maximize} & f(x, y) = 19x + 7y \\ \text{subject to} & \tilde{7}x + \tilde{6}y \underset{=_{\alpha}}{<} \tilde{42} \\ & \tilde{5}x + \tilde{9}y \underset{=_{\alpha}}{<} \tilde{45} \\ & x - y \underset{=}{<} 4 \\ & x \geq 0, y \geq 0 \end{cases} \quad (4.38)$$

where  $\alpha = \tilde{0.5}, \tilde{5}, \tilde{6}, \tilde{7}$  and  $\tilde{9}$  are fuzzy numbers with membership functions given by:

$$\mu_{\tilde{c}}(x) = \begin{cases} 0, & x < c - 1 \\ x - c + 1, & c - 1 \leq x < c \\ 1 + c - x, & c \leq x < c + 1 \\ 0, & c + 1 \leq x \end{cases} \quad (4.39)$$

where  $c$  denotes a non-fuzzy value satisfying  $\mu_{\tilde{c}}(c) = 1$ , while,  $\tilde{42}$  and  $\tilde{45}$  are fuzzy numbers with membership functions given by:

$$\mu_{\tilde{c}}(x) = \begin{cases} 0, & x < c - 2 \\ \frac{x - c + 2}{2}, & c - 2 \leq x < c \\ \frac{2 + c - x}{2}, & c \leq x < c + 2 \\ 0, & c + 2 \leq x \end{cases} \quad (4.40)$$

If all fuzzy numbers  $\tilde{c}_i$  are replaced by non-fuzzy values  $c_i$  satisfying  $\mu_{\tilde{c}_i}(c_i) = 1$ , then the (FLP) problem becomes the normal linear mathematical programming problem. The solution in this case is  $(x^*, y^*) = \left(5\frac{1}{13}, 1\frac{1}{13}\right)$  with objective function value  $-104.000$ . If fuzziness has to be considered, by Theorem 4.2, the problem becomes a usual linear programming problem subject to eleven constraint inequalities, namely:

$$\begin{aligned}
 \max \quad & f(x, y) = 19x + 7y \\
 \text{Subject to} \quad & 7x + 6y \leq 42 \\
 & 5x + 9y \leq 45 \\
 & x - y \leq 4 \\
 & 6.5x + 5.5y \leq 41 \\
 & 4.5x + 8.5y \leq 44 \\
 & 0.5x - 1.5y \leq 3 \\
 & 7.5x + 9.5y \leq 43 \\
 & 5.5x + 9.5y \leq 46 \\
 & 1.5x - 0.5y \leq 5 \\
 & x \geq 0 \\
 & y \geq 0.
 \end{aligned}$$

The solution in this case is  $(x^*, y^*) = (4.9286, 0.9286)$  with objective function value  $-100.143$ . As a validation of the algorithms presented in Section 4.4, the feasible solution space  $X$  and the  $(FLP)_{\alpha\lambda}$  problem are also solved by the algorithm for  $X$  and the algorithm for the  $(FLP)_{\alpha\lambda}$  problem respectively. The solutions obtained are the same as those from Theorem 4.2.

**Example 4.2. [fuzzy constraints with trapezoidal membership function]**

$$(FLP) \begin{cases} \text{maximize} & f(x, y) = 15x + 8y \\ \text{subject to} & \tilde{5}x + \tilde{8}y \underset{=_{\alpha}}{<} \tilde{35} \\ & \tilde{9}x + \tilde{3}y \underset{=_{\alpha}}{<} \tilde{46} \\ & x \geq 0, y \geq 0 \end{cases} \quad (4.41)$$

where  $\alpha = \widetilde{0.6}, \widetilde{3}, \widetilde{5}, \widetilde{8}$  and  $\widetilde{9}$  are fuzzy numbers with membership functions given by:

$$\mu_{\widetilde{z}}(t) = \begin{cases} 0 & , \quad t \leq c_L - 2 \\ \frac{t - c_L + 2}{2} & , \quad c_L - 2 \leq t < c_L \\ 1 & , \quad c_L \leq t < c_R \\ \frac{c_R + 2 - t}{2} & , \quad c_R \leq t < c_R + 2 \\ 0 & , \quad t \geq c_R + 2 \end{cases}$$

where  $(c_L, c_R)$  for  $\widetilde{3}, \widetilde{5}, \widetilde{8}$  and  $\widetilde{9}$  are respectively  $[2.5, 3.5], [4.5, 5.5], [7.5, 8.5]$  and  $[8.5, 9.5]$ , while,  $\widetilde{35}$  and  $\widetilde{46}$  are fuzzy numbers with membership functions given by:

$$\mu_{\widetilde{z}}(t) = \begin{cases} 0, & t \leq z_L - 4 \\ \frac{t - z_L + 4}{4}, & z_L - 4 \leq t < z_L \\ 1, & z_L \leq t < z_R \\ \frac{z_R + 4 - t}{4}, & z_R \leq t < z_R + 4 \\ 0, & t \geq z_R + 4 \end{cases}, \quad (4.42)$$

where the  $[z_L, z_R]$  for  $\widetilde{35}$  and  $\widetilde{46}$  are respectively  $[34, 36]$  and  $[45, 47]$ .

If all fuzzy numbers  $\widetilde{c}_i$  are replaced by non-fuzzy values  $c_i = \frac{c_L + c_R}{2}$  satisfying  $\mu_{\widetilde{c}_i}(c_i) = 1$ , then the (FLP) problem becomes the normal linear mathematical programming problem and the solution in this case is  $x = (4.614, 1.491)$  with objective function value 81.1404. If fuzziness has to be considered, by Theorem 4.4, the problem becomes a usual linear programming problem subject to ten constraint inequalities, namely:

$$\begin{aligned} \max \quad & f(x,y)=15x + 8y \\ \text{subject to} \quad & 4.5x + 7.5y \leq 34 \\ & 8.5x + 2.5y \leq 45 \\ & 5.5x + 8.5y \leq 36 \\ & 9.5x + 3.5y \leq 47 \end{aligned}$$

$$3.7x + 6.7y \leq 32.4$$

$$7.7x + 1.7y \leq 43.4$$

$$6.3x + 9.3y \leq 37.6$$

$$10.3x + 4.3y \leq 48.6$$

$$x \geq 0$$

$$y \geq 0.$$

The solution to this is  $x = (4.2256, 1.1805)$  with objective function value 72.8282.

**Example 4.3. [Both objective function and constraints involve fuzzy numbers]**

$$\left\{ \begin{array}{l} \text{maximize } f(x, y) = x_1 + \tilde{2}x_2 \\ \text{subject to } \tilde{2}x_1 + \tilde{6}x_2 \leq \tilde{27} \\ \tilde{8}x_1 + \tilde{6}x_2 \leq \tilde{45} \\ \tilde{3}x_1 + \tilde{1}x_2 \leq \tilde{15} \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right.$$

where the membership functions for the fuzzy numbers  $\tilde{2}, \tilde{3}, \tilde{6}, \tilde{8}, \tilde{15}, \tilde{27}$  and  $\tilde{45}$  are respectively:

$$\mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & 1 \leq t < 3 \\ 4 - t & 3 \leq t \leq 4 \\ 0 & 4 \leq t \end{cases}$$

$$\mu_{\tilde{1}}(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & 2 \leq t \end{cases}$$

$$\mu_{\tilde{2}}(t) = \begin{cases} 0 & t < 1 \\ t - 1 & 1 \leq t < 2 \\ 3 - t & 2 \leq t < 3 \\ 0 & 3 \leq t \end{cases}$$

$$\mu_{\tilde{3}}(t) = \begin{cases} 0 & t < 2 \\ t - 2 & 2 \leq t < 3 \\ 4 - t & 3 \leq t < 4 \\ 0 & 4 \leq t \end{cases}$$

$$\mu_{\tilde{6}}(t) = \begin{cases} 0 & t < 5 \\ t - 5 & 5 \leq t < 6 \\ 7 - t & 6 \leq t < 7 \\ 0 & 7 \leq t \end{cases}$$

$$\mu_{\tilde{8}}(t) = \begin{cases} 0 & t < 7 \\ t - 7 & 7 \leq t < 8 \\ 9 - t & 8 \leq t < 9 \\ 0 & 9 \leq t \end{cases}$$

$$\mu_{\tilde{15}}(t) = \begin{cases} 0 & t < 12 \\ \frac{t - 12}{3} & 12 \leq t < 15 \\ \frac{18 - t}{3} & 15 \leq t < 18 \\ 0 & 18 \leq t \end{cases}$$

$$\mu_{\tilde{27}}(t) = \begin{cases} 0 & t < 23 \\ \frac{t - 23}{4} & 23 \leq t < 27 \\ \frac{31 - t}{4} & 27 \leq t < 31 \\ 0 & 31 \leq t \end{cases}$$

$$\mu_{\tilde{45}}(t) = \begin{cases} 0 & t < 39 \\ \frac{t - 39}{6} & 39 \leq t < 45 \\ \frac{51 - t}{6} & 45 \leq t < 51 \\ 0 & 51 \leq t \end{cases}$$

For this problem, if the fuzzy numbers  $\tilde{c}_i$  are replaced by non-fuzzy values  $c_i = \frac{1}{2}(c_{i1}^L + c_{i1}^R)$  where  $c_{i1}^L$  and  $c_{i1}^R$  are the left and right values of  $\tilde{c}_i$  and where the membership function is equal to one, then the problem has the solution  $x_1 = 3, x_2 = 3.5$  and the number value of  $f$  is 10. If fuzziness is to be considered with  $\alpha = 0.5$ , by Theorems 4.3 and 4.4, the problem becomes a multi-objective linear programming problem subject to eleven constraints, namely:

$$\begin{aligned}
& \max \quad \{x_1 + 0.5x_2, x_1 + 3.5x_2, x_1 + x_2, x_1 + 3x_2\} \\
& \text{subject to} \quad 2x_1 + 6x_2 \leq 27 \\
& \quad \quad \quad 8x_1 + 6x_2 \leq 45 \\
& \quad \quad \quad 3x_1 + x_2 \leq 15 \\
& \quad \quad \quad x_1 + 5x_2 \leq 25 \\
& \quad \quad \quad 7x_1 + 5x_2 \leq 42 \\
& \quad \quad \quad 2x_1 + 0 \leq 13.5 \\
& \quad \quad \quad 3x_1 + 7x_2 \leq 29 \\
& \quad \quad \quad 9x_1 + 7x_2 \leq 48 \\
& \quad \quad \quad 4x_1 + 2x_2 \leq 16.5 \\
& \quad \quad \quad x_1 \geq 0 \\
& \quad \quad \quad x_2 \geq 0
\end{aligned}$$

which is a multi-objective linear programming problem. Various methods can be used to solve the problem, such as the weighting method and the weighted minmax method. Here the weighting method is used and thus the new objective function to maximize is:

$$\begin{aligned}
f_w(x, y) &= w_1(x_1 + 0.5x_2) + w_2(x_1 + 3.5x_2) + w_3(x_1 + x_2) \\
&\quad + w_4(x_1 + 3x_2) \\
&= (w_1 + w_2 + w_3 + w_4)x_1 + (0.5w_1 + 3.5w_2 + w_3 + 3w_4)x_2
\end{aligned}$$

Choose  $w_1 = w_2 = 0.5$ ,  $w_3 = w_4 = 1.0$ . Then, by maximizing  $f_w(x, y)$  subject to the constraints, we obtain the solution  $x_1 = 2.6, x_2 = 3.0$ . The membership function of the corresponding objective function  $f(x^*, y^*) = f(2.6, 3.0) = 2.6 + 3 * \tilde{2}$  is:

$$\mu_{f(2.6,3.0)}(x) = \begin{cases} 0 & x < 2.6 \\ \frac{x - 2.6}{3} & 2.6 \leq x < 5.6 \\ 1 & 5.6 \leq x < 11.6 \\ \frac{14.6 - x}{3} & 11.6 \leq x < 14.6 \\ 0 & 14.6 \leq x \end{cases}$$



## 4.6 Concluding Remarks

In this chapter, we introduce a new concept, the  $\alpha$ -fuzzy max order, and then apply the concept in the study of fuzzy linear constrained optimization problems. For optimization problems with constraints given by  $n$  inequalities involving fuzzy numbers with isosceles triangle membership functions, we have successfully proved that the feasible solution space is determined by  $3n$  non fuzzy inequalities. For optimization problems with constraints involving fuzzy numbers with other forms of membership functions, we develop three numerical algorithms respectively for the determination of the feasible solution space and the solutions to two types of fuzzy optimization problems. Through three illustrative examples, we have successfully demonstrated the validity of the methods and the numerical algorithms developed in the work.

# Chapter 5

## Summary and further research

### 5.1 Summary

In this thesis, we study the theoretical and computational aspects of fuzzy mathematical programming problems involving fuzzy parameters. The research consists mainly of two parts. The first part focuses on fuzzy linear programming problems with fuzzy parameters in the objective function, namely:

$$(\text{FLP}) \begin{cases} \text{maximize} & \langle \tilde{c}, x \rangle_F = \sum_{i=1}^n \tilde{c}_i x_i \\ \text{subject to} & Ax \leq b, x \geq 0, \end{cases}$$

where  $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n)^T \in F^*(R^n)$ ,  $A$  is an  $m \times n$  matrix and  $b \in R^m$  whose elements are given by  $a_{ij}$  and  $b_j$ , respectively. The key results achieved for this part include various aspects:

- (i) The concepts of fuzzy max order, non-dominated optimal solution to fuzzy linear programming, complete optimal solution and Pareto optimal solution are introduced within the framework for fuzzy mathematical programming.
- (ii) Based on the concepts introduced in (i), various theorems have been developed addressing various theoretical and computational aspects of the problem, including the relationship between the optimal solution to the fuzzy linear programming problem with the complete optimal solution to a normal multi-objective linear programming problem (Theorem 3.3), the relationship between the non-dominated solution to the FLP with the Pareto optimal solution to a MOLP problem (Theorem 3.4), the relation between the non-dominated solution to the FLP problem with the optimal solution of a weighted multi-objective linear programming problem

(Theorems 3.6 – 3.8), the relation between the non-dominated solution to the FLP problem with the constrained linear programming problem (Theorems 3.9 – 3.11), and the relation between the non-dominated solution to the FLP problems with the optimal solution of a weighted maximum linear programming problem (Theorems 3.12 – 3.15).

- (iii) The results achieved in (ii) have formed a theoretical basis for solving fuzzy linear programming problems with objective functions involving fuzzy parameters by converting the problems into multi-objective linear mathematical programming problems of real numbers. The validation and application of the results achieved in (ii) for solving FLP problems have been demonstrated successfully through two examples.

The second part studies fuzzy linear programming problems in which the constraints involve fuzzy parameters. For this part, the key results achieved include the following aspects:

- (i) A new concept, the  $\alpha$ -fuzzy max order, has been developed for the study of fuzzy mathematical programming problems in which both the objective function and the constraint equations/inequalities contain fuzzy coefficients.
- (ii) Based on the  $\alpha$ -fuzzy max order, an  $\alpha$ -fuzzy max order method has been developed for solving fuzzy mathematical problems with fuzzy parameters in both the objective function and the constraints.
- (iii) For constraints given by inequalities involving fuzzy numbers with isosceles triangle membership functions, we prove that the feasible solution space can be defined by  $3n$  non-fuzzy inequalities; while for constraints consisting of  $n$  inequalities with fuzzy numbers of trapezoidal

membership functions, we proved that the feasible solution space can be determined by  $4n$  non-fuzzy inequalities.

- (iv) For constraints involving fuzzy numbers with any nonlinear forms of membership functions, numerical algorithms have been developed for the determination of the feasible solution space and the optimal solution of the general fuzzy linear programming problem where both the objective function and the constraints contain fuzzy numbers.
  
- (v) The methods developed in (iii) and (iv) have been validated and demonstrated successfully through three examples.

Two research papers have been produced from the research during my PhD enrolment including:

- (1) Zhang, G.Q., Wu, Y.H., Remias, M. and Lu, J. An  $\alpha$ -Fuzzy Max Order and Solution of Linear Constrained Fuzzy Optimization Problems. *East-West J. of Mathematics, Special Volume*.
  
- (2) Zhang, G.Q., Wu, Y.H., Remias, M. and Lu, J. Formulation of Fuzzy Linear Programming Problems as Four-objective Constrained Optimization Problems, *Applied Mathematics and Computation*, 139, 383-399.

## **5.2 Further Research**

Further research work includes extending the theories and methods developed to more complex fuzzy mathematical programming problems, such as fuzzy quadratic programming and fuzzy nonlinear programming where both the objective function and the constraint equations or inequalities contain fuzzy numbers with nonlinear membership functions. Other further research is to apply the theories and methods to real world problems, such as economics, finance, facility location design, logistics planning and portfolio selection.

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