Aspects of non-geometry in string theory

Dissertation

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Zusammenfassung

Diese Arbeit befasst sich mit den Erscheinungsformen der sogenannten Nicht-Geometrie in der Stringtheorie. Sie verwendet verschiedene Rahmenwerke um zu erkunden wie Nicht-Geometrie im Zielraum in Erscheinung tritt, wie Nicht-Geometrie und nicht-geometrische Flüsse zusammenhängen, wie Nicht-Geometrie in effektiven Feldtheorien erfaßt werden kann und wie eine mögliche Erweiterung der gewöhnlichen Stringtheorie Weltflächen-Wirkung nicht-geometrischen Konfigurationen Rechnung tragen kann.

Der erste Teil bietet ein Beispiel dafür, daß Nicht-Geometrie nicht-kommutierende Koordinatenfelder für geschlossene Strings implizieren kann. Drei T-duale Hintergründe werden untersucht, der Drei-Torus mit konstantem H-Fluß, der verdrehte Torus und der Torus mit nicht-geometrischem Fluß Q. Unter der Annahme verdünnter Flüsse werden sowohl eine klassische Moden-Entwicklung als auch die kanonische Quantisierung je zur linearen Ordnung im Flußparameter für den zweiten Fall durchgeführt. T-Dualität wird dann verwendet, um die Kommutatoren der einzelnen Moden mit dem Koordinatenfeld-Kommutator im nicht-geometrischen dritten Hintergrund in Beziehung zu setzen. Nicht-Kommutativität tritt tatsächlich auf und kann mit dem nicht-geometrischen Fluß Q und der Windungszahl des Strings in Verbindung gebracht werden. Sie erscheint als intrinsisch stringtheoretische Eigenschaft.

Der zweite Teil untersucht Nicht-Geometrie im Kontext zweier zehndimensionaler effektiver Feldtheorien: Doppelfeldtheorie und Supergravitation. Eine Feld-Redefinition in der Form einer T-Dualitäts-Transformation wird implementiert. Sie offeriert einen Satz alternativer Feldvariablen, der es erlaubt, höherdimensionale nicht-geometrische Flüsse Q und R zu definieren. Die Perspektive der Doppelfeldtheorie bietet eine geometrische Interpretation dieser Flüsse, indem sie einen neuen Typ kovarianter Windungs-Ableitungen in Betracht zieht. Die Perspektive der Supergravitation erlaubt es dagegen, das Zusammenspiel von nicht-geometrischen Konfigurationen und nicht-geometrischen Flüssen zu untersuchen. Für das Beispiel des Drei-Torus kann eine wohldefinierte Wirkung formuliert werden, eine einfache dimensionale Reduktion stellt die Verbindung zum bekannten vierdimensionalen Potential her. Dies bestätigt die korrekte Hebung von Q und R zu höheren Dimensionen. Wie erwartet sind die beiden Perspektiven durch Lösungen der starken Bedingung in Doppelfeldtheorie zueinander äquivalent.

Im dritten Teil wird ein Weltflächen-Modell mit verdoppelten Koordinatenfeldern vorgeschlagen. Es ist manifest T-dualitäts-kovariant und erhält Lorentz-Invarianz. Die Hälfte der Weltflächen-Freiheitsgrade ist redundant durch die Fixierung einer zugrundeliegenden Eichsymmetrie der Koordinatenfelder. Die Weltflächen-Wirkung kann durch eine Verschiebung der entsprechenden Lagrange-Multiplikatoren in verschiedene Formen gebracht werden, insbesondere die der Standard-Weltflächen-Wirkung. Nicht-geometrische Flüsse sind durch einen verdoppelten anti-symmetrischen Tensor implementiert, die zugehörigen Transformationen der Flüsse unter T-Dualität werden dadurch reproduziert. Die Renormierung der Theorie wird untersucht, indem eine nicht-lineare Hintergrund- / Quanten-Aufteilung und eine angepaßte Normalkoordinaten-Entwicklung durchgeführt wird. Der Propagator der verdoppelten Koordinatenfelder enthält dann einen Projektor, welcher die Hälfte der propagierenden Freiheitsgrade unterdrückt. Die Bewegungsgleichungen im verdoppelten Zielraum werden ermitteln, indem Weyl-Invarianz auf Ein-Schleifen-Niveau gefordert wird. Eine dieser Gleichungen ähnelt der starken Bedingung in Doppelfeldtheorie, die anderen scheinen neuartig zu sein.

Abstract

This thesis investigates various manifestations of non-geometry in string theory. It utilises different frameworks to study how non-geometry appears in the target space, how non-geometry and non-geometric fluxes are interconnected, how non-geometry can be captured in effective field theories and how a possible extension of the standard string worldsheet model can accommodate non-geometric setups.

The first part provides an example that non-geometry can imply non-commutativity of the closed string coordinate fields. Three T-dual frames are investigated, the three-torus with constant H-flux, the twisted torus and the torus with non-geometric flux Q. Under the assumption of dilute flux, a mode expansion and the canonical quantisation are carried out in the second case up to linear order in the flux parameter. T-duality is then used to relate the commutators of the string expansion modes to the coordinate field commutator in the non-geometric third frame. Non-commutativity is found and related to the non-geometric flux Q and the string winding, it therefore appears as an intrinsically string theoretic feature.

The second part investigates non-geometry in the context of ten-dimensional effective field theories, i.e. double field theory and supergravity. A field redefinition is implemented that takes the form of a T-duality transformation, it reveals an alternative set of field variables allowing to define non-geometric fluxes Q and R in higher dimensions. The perspective of double field theory provides a geometric interpretation of those by taking into account a new type of covariant winding derivative. The perspective of the ten-dimensional supergravity allows to investigate the interplay between non-geometric field configurations and non-geometric fluxes. For the three-torus example, a well-defined action can be found, and a simple dimensional reduction makes contact to the known four-dimensional potential. It thus proves the correct uplift of Q and R to higher dimensions. As expected, the two perspectives are related to each other by solutions of the strong constraint.

In the third part, a worldsheet model with doubled coordinate fields is suggested. It is manifestly T-duality covariant and preserves worldsheet Lorentz invariance. Half of the worldsheet degrees of freedom are made redundant by fixing an underlying gauge symmetry of the coordinate fields. Shifting the respective Lagrange multiplier casts the action into various guises, including the standard worldsheet model. Non-geometric fluxes are incorporated by a doubled anti-symmetric tensor, and the appropriate transformations of fluxes under T-duality operations are reproduced. The renormalisation of the theory is investigated by using a non-linear background / quantum split and an adapted normal coordinate expansion. The propagator of the doubled coordinate fields contains a projection that removes half of the propagating degrees of freedom. The doubled target space equations of motion are determined by requiring one-loop Weyl invariance. One of them resembles the strong constraint of double field theory, while others seem to be novel.

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Chapter 1

Introduction

Looking through a microscope reveals small things that are not visible to the naked eye. How small can these things be? It depends on the wavelength of the used light. Smaller things need smaller wavelengths and thus higher energies. The microscopes of particle physics are accelerator experiments, and instead of light, particles like protons are used to probe phenomenological data. They can carry huge amounts of energy and thus allow for a huge magnification.

Some surprises were revealed. Matter does not consist of impartible atoms, but rather of much smaller entities that come in groups and arrange into atoms as composite objects. Fundamental forces were shown to be mediated by other small entities, in contrast to the long-standing description by force fields. The view on how matter is composed and by which mechanism its dynamics are determined has changed dramatically by simply using finer probes.

Surprisingly, a similar story can be told for the framework that was believed to be fundamental: geometry. For decades in the history of natural sciences, geometry had been considered as a stage for all the plays that were performed by the various physical theories. But the stage turned out to be part of the play, and a new structure to be viewed in the microscope appeared - spacetime itself. By refining the probes, which means to use more massive objects in this case, geometry was revealed to be a dynamic part of nature. General relativity showed how matter deforms the spacetime it lives in, and a very subtle interrelation between probe and object of study began to loom.

Can the story end like in the case of atoms? Could it be that another refinement of the probes reveals completely new structures that underlie our common notion of geometry?

String theory offers a new type of microscope as it changes the set of probes. Not zero-dimensional point particles but one-dimensional strings now make the fundamental entities. And indeed, it seems that this change gives rise to a completely different view on what geometry is.

Not only does string theory claim to supplement our four-dimensional world with tiny compact extra dimensions, but also can these dimensions be constructed in a way that they, intuitively, look non-geometric. Schematically, such a situation could be as in fig. 1.1. A torus was constructed by taking the fibration of a circle over a circle, but there is a mismatch when closing the figure. On one end the fibre circle has radius R, on the other end it has radius $1/R$. These ends, in the framework of ordinary Riemannian geometry, cannot be glued together.

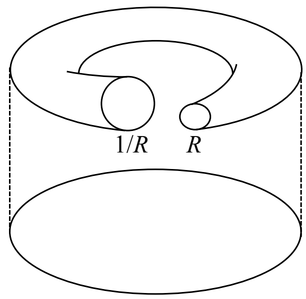


Figure 1.1: A non-geometric torus

Surprisingly, in string theory they can! It turns out that for a string winding around the fibre circle, it does not matter whether the radius is R or $1/R$. A special symmetry of string theory allows to identify the different lengths. The strange torus is an allowed configuration in string theory, and it is one of the simplest examples of non-geometry.

This thesis investigates the features of non-geometry and how they can be captured in the framework of string theory.

1.1 String theory

1.1.1 Worldsheet model

What is string theory?

String theory[1] is a quantum theory of one-dimensional objects. It thereby offers a radical generalisation of ordinary quantum field theory, whose fundamental entities are zero-dimensional point particles.

String theory brings a different perspective on elementary particle physics. Taking seriously the idea that nature is best described by tiny one-dimensional objects reduces all known particles and forces to the dynamics of those strings. Different particles with different masses are then simply vibrational modes with different energies. And these excitations are not the only modes that strings can have. When propagating in topologically non-trivial spaces, they can wind and gain energy from the corresponding winding-modes. Strings probe geometry differently than point particles.

String theory contains general relativity. Eintein's equations are part of its consistency conditions, so it also explains what geometry is. And being a quantum theory, it offers the possibility to formulate a quantum version of the gravitational force. This is not possible in the framework of quantum field theory with point particles.

[1] The most prominent introductory textbooks are [9], [10], [11], [12], [13], [14].

The two perspectives on string theory

String theory offers two perspectives on what spacetime really is. On the one hand, strings are supposed to be objects that propagate in spacetime, which itself then can be considered as the fixed stage for string dynamics. On the other hand, spacetime is determined by the dynamics of the contained strings, and is in a sense emergent from their properties. It is useful to introduce the concept of a worldsheet, in order to get a clearer understanding of the two different points of view.

A worldsheet is the two-dimensional surface that a moving string sweeps out in spacetime, similar to the worldline of a relativistic point particle. Such a two-dimensional surface is parametrised by two worldsheet coordinates τ and σ, which are embedded into spacetime by coordinate fields $X^\mu(\tau, \sigma)$. One can now differentiate between the worldsheet perspective on string theory, and the target space perspective.

The worldsheet perspective views string theory as a theory of two-dimensional surfaces, given by a two-dimensional quantum field theory. In particular, the worldsheet will be equipped with a metric and can have very distinct topologies, according to the two different types of strings. For open strings with free endpoints it would be a sheet with boundaries, whereas for closed strings it could, for example, be a cylinder. Taking into account interactions would allow even more complicated structures: the creation and annihilation of a closed string will result in a toroidal worldsheet, the splitting of one closed string into two will result in a Y-shaped tube, similar to pants.

The target space perspective on string theory determines the dynamics of the coordinate fields X^μ with the aim of interpreting them as actual coordinates of the spacetime manifold itself. Properties of the moving strings then allow conclusions on, for example, its topological or geometric features. For example, the symmetric coupling matrix of the coordinate fields will be interpreted as the metric of the target space. This is how one could say that spacetime is emergent from the description of the string dynamics.

The target space perspective then allows to assume the existence of effective field theories, that capture particular features of the full string theory in the form of quantum field theories on the respective spacetime manifold. Different vibrational excitations of the string can then be seen as different particles with different masses, and the effective theory is about their dynamics and interactions. This allows to connect string theory to the standard model of particle physics.

An action principle for string theory

When it comes to the mathematical formulation of string theory, the worldsheet perspective is remarkably helpful. The dynamics of a string can be captured by a very simple idea: like the surface of a soap bubble tends to minimise its energy and therefore its area, the string worldsheet is conjectured to do the same. The only necessary ingredient is that the string has tension, like the bubble has surface tension.

Such an idea can easily be embedded in the framework of variational calculus by formulating an action. This goes in perfect analogy to the action principle of a relativistic point particle, that claims the minimisation of the worldline length. The corresponding string theory formulation goes by the name Nambu-Goto action and is obtained from the following generalisation scheme,

$$S_0 = -m \int \mathrm{d}s \quad \longrightarrow \quad S_1 = -T \int \mathrm{d}\mu_1 \ , \tag{1.1}$$

where T is the string tension, and $\mathrm{d}\mu_1$ the (1+1) dimensional volume element.

Quantisation and conformal invariance

Although its underlying idea is simple, the Nambu-Goto action is difficult to quantise due to a square root it contains. Luckily, there is an equivalent formulation called the Polyakov action. It has the form of a general non-linear sigma model with fields $X^\mu(\tau, \sigma)$, and as such can be dealt with much easier. Furthermore, the Polyakov formulation makes manifest many important mathematical features of string theory.

The two-fold perspective on string theory is mirrored in the sigma model itself:

$$S \sim \int \mathrm{d}^2\sigma \sqrt{|\eta|}\, G_{\mu\nu}(X)\, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{1.2}$$

On the one hand, it is a field theory in two dimensions τ and σ that describes the worldsheet with metric $\eta^{\alpha\beta}(\tau, \sigma)$. On the other hand, it describes a D-dimensional spacetime with metric $G_{\mu\nu}(X)$ and coordinates X^μ.

One of the most important features of the Polyakov action is that it reveals the Weyl invariance of the worldsheet. A Weyl transformation rescales the worldsheet metric η with a conformal factor, and the far reaching implication of this symmetry is that string theory can be viewed as conformal field theory. This allows to apply the whole set of tools for the investigation of conformal field theories on string theory. The worldsheet perspective then would summarise this as: string theory is the study of worldsheet dynamics with the tools of conformal field theories.

When it comes to quantising, it is in general not guaranteed that classical symmetries remain unbroken. For consistent quantum theories such gauge anomalies must not occur. The claim of Weyl invariance being anomaly free imposes very strong constraints on the quantisation of string theory, and is often employed to find the correct renormalisation scheme.

Geometry and string theory

By adopting the target space perspective, one tries to investigate properties of the spacetime manifold with the help of particular features in string theory. One simple example is the interpretation of the coupling matrix $G_{\mu\nu}$ as target space metric. In order to do so consistently, one has to prove the tensorial nature of this object. And indeed, the Polyakov action allows to show that a field redefinition of the coordinate fields X^μ induces a certain transformation of $G_{\mu\nu}$:

$$X^\mu \to X'^\mu(X) \quad \Rightarrow \quad G_{\mu\nu}(X) \to G'_{\rho\sigma}(X') \frac{\partial X'^\rho}{\partial X^\mu} \frac{\partial X'^\sigma}{\partial X^\nu} \ . \tag{1.3}$$

Interpreting X^μ as coordinates on a manifold, the field redefinition is nothing else than a diffeomorphism, and the transformation of $G_{\mu\nu}$ is exactly as expected for a tensor under such a diffeomorphism.

An perhaps even simpler example shows how symmetries of the Polyakov sigma model can be interpreted as symmetries of the spacetime manifold. In the case of a constant field $G_{\mu\nu}$, the following symmetry of the two-dimensional field theory appears,

$$X^{\mu}(\tau,\sigma) \;\rightarrow\; \Lambda^{\mu}{}_{\nu}X^{\nu}(\tau,\sigma) + a^{\mu} \ , \tag{1.4}$$

and it is nothing else than Poincaré invariance for the target space.

A more far reaching consequence of the target space perspective is that string theory, under certain condition, predicts the number D of spacetime dimensions. Although physics appears to happen in a four-dimensional framework, for example the fermionic extension of string theory dictates a ten-dimensional framework when Lorentz invariance is claimed to be preserved. In the purely bosonic case the statement is similarly astonishing, as there it has to be $D = 26$. A huge part of string theory research is concerned with solving this discrepancy.

Other more sophisticated examples of the interplay between string theoretic features and geometry show up when considering properties of the spacetime manifold like its structure group or holonomy. Actually, the additional dimensions have to form very special types of manifolds like Calabi-Yau spaces, which challenges the naive assumption that physics takes place in a rather trivial space. String theory offers the instrumentarium to detect the subtle geometric structure which the world actually possesses.

Eventually, this research work is dedicated to the study of string theoretic features that prevent an interpretation of spacetime as a manifold at all. This, accordingly, has been collected under the name of "non-geometry".

1.1.2 Dualities

Many string theories

Both the Nambu-Goto action and the Polyakov formulation do not take into account fermionic degrees of freedom. The existence of fermions thus requires an extension of the so far bosonic string theory, which offers a whole variety of possibilities. It had cost a huge effort (a 'superstring revolution') to explore the constraints on such constructions.

Apart from the already mentioned condition that the conformal invariance has to be anomaly free, consistent formulations of string theory are restricted by many other requirements. In particular, the stability of the vacuum demands the introduction of fermions together with supersymmetry, which relates bosons and fermions to each other. Supersymmetry itself does only allow for very particular realisations. Altogether, it turns out that there are only three different consistent and stable formulations of string theory: type I, type II and heterotic, where the last two split into IIA and IIB, or $SO(32)$ and $E_8 \times E_8$, respectively.

Dualities

At first sight, it seemed confusing to have many different string theories, as it was expected to have only one. But step by step it was shown that there are relations between the five string theories. These so-called dualities tied a net of equivalent formulations, as for example T-duality related type IIA and type IIB string theory, or S-duality showed equivalence of type I and $SO(32)$ heterotic string theory. In the end, it was conjectured that there is an underlying theory, M-theory, that unifies the five string theories. It is supposed to contain them as limiting cases.

In general, the notion of a duality employs a very simple idea: The path integral allows to deduce empirically relevant quantities and could therefore be taken as the entity that represents all predictions of a theory. There might be different formulations of the same path integral, and they can come, in particular, with different actions. In this sense, different actions are representing the same theory. For string theory, the change of its action is eminently important when taking the target space perspective. Different formulations may correspond to completely different notions of spacetime.

T-duality and geometry

To illustrate the idea of a duality in string theory, it is useful to introduce T-duality in a little more detail[2]. It will play an important role in this thesis.

In the path integral formalism, T-duality relates different actions that describe different geometries. Roughly speaking, small becomes large and vice versa. In more detail, one can find that two T-dual actions contain target space metrics $G_{\mu\nu}$ and $G'_{\mu\nu}$ which are related by very specific rules. In particular, certain elements of the metrics become inverse to each other. This will generally not only change radii, but rather the whole manifold.

The differences between two T-dual models will become even more dramatic when one considers a possible extension of the Polyakov action. It can be complemented by another term that introduces an antisymmetric coupling matrix $B_{\mu\nu}(X)$. This matrix can be interpreted as an antisymmetric tensor field on the target space manifold. It is the higher dimensional analogue to the coupling of a point particle to a magnetic field via the vector potential A_μ. The full bosonic string action then reads

$$S \sim \int \mathrm{d}^2\sigma \ \left(G_{\mu\nu}(X)\, \eta^{\alpha\beta} + B_{\mu\nu}(X)\, \epsilon^{\alpha\beta} \right)\ \partial_\alpha X^\mu \partial_\beta X^\nu \ . \tag{1.5}$$

It can then be shown that a solution with both metric and B-field in general is dual to another background where components of the former B-field are mixed into the new $G'_{\mu\nu}$. This will complicate the geometry of the target space manifold when interpreting the latter as its metric.

T-duality can only be found for specific solutions of the equations of motion. Such solutions determine the target space metric $G_{\mu\nu}$, and it has been shown that T-duality does only work for field configurations that have isometries, i.e. that are invariant under a shift of the coordinate fields in a particular direction. In the case that a solution has more than one isometry, all possible T-dualities can be mixed with each other, which mathematically means that they form a group[3].

Different views on T-duality are feasible[4]. Apart from the path integral argument that was presented here, it is, for instance, possible to understand T-duality a canonical transformation of the field variables. A very enlightening perspective on T-duality arises when calculating the allowed vibrational energies of a bosonic closed string. For the simple example of a circular coordinate (which accordingly must be one of the extra dimensions) with radius R, the string can have winding N and its momentum is quantised, $p = n/R$, $n \in \mathbb{Z}$. A sketchy version[5] of

[2] Technical details are worked out in appendix A.

[3] In the case of d isometries, this group is $O(d,d)$, cf. the discussion in appendix A.

[4] For an overview, cf. [15].

[5] In particular, no units of α' are displayed.

the resulting mass formula then reads

$$m^2 = \frac{n^2}{R^2} + N^2 R^2 + (\dots) \; . \tag{1.6}$$

It comes from the energy dispersion relation that relates the square of the total energy to the square of the momentum. For a string, winding adds additional energy that is roughly proportional to the string length, which makes the second term on the right-hand side. It is immediately clear that

$$n \leftrightarrow N \; , \quad R \to \frac{1}{R} \tag{1.7}$$

leaves the above energy spectrum invariant. The radial inversion matches the expected effect of a T-duality transformation in that direction, and it can indeed be shown that the presented exchange rule is nothing else than a T-duality transformation. A refined investigation reveals T-duality as a symmetry of the full conformal field theory underlying string theory.

1.1.3 Phenomenology

String theory as a theory of everything

String theory challenges many established notions in elementary particle physics, first of all the notion of point particles itself. That physicists have spent a lot of effort on its development does not come without substantial reasons. String theory is considered as the one unifying theory in theoretical physics. It promises the successful marriage of quantum field theory and general relativity. By doing so, it will answer the many questions that arise when large distances and quantum effects meet, as for example: What is the quantum nature of black holes? or: What happened in the beginning of the universe? In addition, string theory promises to find answers to open problems in the standard model of particle physics: Why are there so many free parameters tuned very accurately to their actual values?, How can the hierarchy problem be solved? How can supersymmetry introduce the gauge coupling unification? or even: What is the origin of the particle species? Also cosmological problems can be addressed in string theory, like: How can inflation be explained? or: What is dark matter? Not few researches have pursued their work on string theory with the assumption that it will eventually make a 'theory of everything'.

Energy scales

Despite the motivating promise of solving problems in the regime of ordinary particle physics, it is not that simple to make contact with this scale from the perspective of strings. A simple estimate arises as follows: string theory is considered to be a quantum description of gravity and therefore has to include the three fundamental constants $\hbar$, c and G. They can uniquely be combined into a length scale and a mass scale:

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{3/2} = 1.6 \times 10^{-35} \text{ m} \; , \quad m_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2 \; . \tag{1.8}$$

These are called Planck length and Planck mass, and consequently make the characteristic scales for string theory. In conclusion, strings are so tiny that they will look like particles for current accelerator experiments and thus can never be detected directly.

There are two scales: the string scale[6] around the Planck mass and the energy scale set by the range of particle accelerators like the LHC experiment. They are separated by 16 orders of magnitude, but a connection between them can be made by a so-called effective theory that reproduces the 'low-energy' behaviour of strings. One very important area of research in string theory is the investigation of such low-energy effective actions.

Low-energy effective theories

Finding the appropriate low-energy effective theory of string theory is not straightforward. But there are certain indications of what theory to look for. The target space perspective dictates, for a string theory with bosons and fermions, a ten-dimensional spacetime. In addition, for consistency there has to be supersymmetry. Together with the fact that it is a local quantum field theory which is looked for, the evidence becomes almost compulsory that the low-energy effective theory of string theory is a ten-dimensional supergravity.

And indeed, the connection can be made more precise by investigating the conformal symmetry of string theory. In a perturbative approach to a path integral quantisation, this symmetry becomes anomalous. When performing a renormalisation of the theory, it is necessary to fulfill certain conditions, and these can be rephrased as making the anomaly of the Weyl invariance vanish. By doing so at leading order in the perturbative expansion, one obtains equations that include the target space fields G and B. These can be interpreted as equations of motion for an effective theory of the massless string excitations. The corresponding action can be constructed and it turns out to be the bosonic part of a ten-dimensional supergravity theory. Eventually, all five string theories can at low energies be effectively described by supergravity theories in ten dimensions.

The famous claim that string theory constitutes a quantum theory of gravity is based on one of the equations of motion obtained in this way. In a special case, it is equivalent to the Einstein equation, which means that clasically and at low energies, string theory contains general relativity.

Another approach is possible, and it leads to the same result: By employing conformal field theoretic methods, one can compute scattering amplitudes of string modes. It is then possible to guess an effective action that reproduces exactly these amplitudes, and that turns out to be the ten-dimensional supergravity again.

In these constructions, the low-energy effective theory only takes into account very light modes of the string. This usually excludes modes that come from a winding around topologically non-trivial directions. It is then particularly difficult to implement T-duality, which exchanges the potentially light momentum modes with the heavy winding modes. Recently, a completely different approach to the low-energy effective dynamics of string theory has been suggested: Double field theory [16, 17, 18]. It takes into account both momentum and winding modes at the costs of doubling the number of target space dimensions to $D = 20$. The construction then rests on the additional assumption of a reduction condition, called the "strong constraint", that brings the target space back to $D = 10$. Imposing a solution of the strong constraint has been shown to make double field theory equivalent to the conventional supergravity effective theory.

[6] When considering compactifications, the string scale does not necessary have the same order as the Planck scale but rather depends on the size of the compact dimensions.

Compactification

Going to an effective theory has not solved the problem of dimensionality. Still, the predicted number of dimensions from string theory is ten, which heavily disagrees with the observable four spacetime dimensions. The solution of this problem goes by the name of compactification, and follows the idea that the six extra dimension are made undetectable. This is achieved by making them extremely small, such that the energy necessary to probe them is by far higher than current accelerator experiments can achieve. As an immediate consequence from being small, the extra dimensions have to be compact, hence the term compactification.

Formally, a compactification can be performed by integrating out the extra dimension in the effective ten-dimensional action. This is possible when a specific ansatz is made for the structure of the spacetime manifold. The simplest setup is a product of two manifolds $M = M_4 \times M_6$, where an external spacetime manifold M_4 with Minkowskian signature and an internal manifold M_6 together make the full ten-dimensional spacetime that is needed. The result is a four-dimensional effective field theory in M_4. It will turn out that the geometry of the internal manifold deforms the properties of this four-dimensional theory. More sophisticated ansatzes use the so-called warping, where the size of the external spacetime manifold depends on the position in the internal manifold.

In terms of scales, the integration of the internal directions inserts a volume factor in front of the respective action. The four-dimensional Planck scale is then given as the product of this volume and the ten-dimensional string scale. Depending on the actual size of the extra dimensions, one can find different hierarchies of scales, where both a high but also a rather low string scale are possible.

In order to find solutions to the compactified theory, it is necessary to apply a product ansatz to the fields as well. In practice, this procedure is complicated and can be simplified by using a technical coincidence: on the side of the effective four-dimensional theory, a certain amount of supersymmetry is wanted for various reasons. It has to be broken dynamically later on. The ten-dimensional conditions that assert this supersymmetry in four-dimensions imply, under particular circumstances, the equations of motion. Solving the supersymmetry conditions with the corresponding product ansatz is technically easier and in such cases solves the equations of motion automatically[7]. Furthermore, following this approach allows to find strong conditions on the internal geometry such that the manifold M_6 can be characterised explicitly. In the simplest case it is, for instance, restricted to be a Calabi-Yau space.

For some cases, the detour to an effective ten-dimensional theory is not necessary, and one can determine the low-energy effective action in four-dimensions directly from string theory. Orbifold compactifications make an example of this procedure. By applying conformal field theoretic methods, the light modes are identified as well as certain scattering amplitudes of those. It is then possible to conclude on the effective four-dimensional theory by claiming that it has to reproduce these modes and amplitudes.

Flux compactifications

In the field of compactifications from the ten-dimensional effective theory, it has turned out that Calabi-Yau manifolds are not satisfactory. They necessarily imply external spacetime manifolds M_4 that are Minkowski and, in particular, do not have a cosmological constant.

[7]For details on the solution of supersymmetry conditions see [8], which will not be considered any further in this work.

Furthermore, the effective four-dimensional theory contains fields, the so-called moduli, that stem from geometric data of the internal manifold and do not have determined vacuum expectation values. They often determine the value of four-dimensional physical quantities such as coupling constants and therefore should be fixed in order for the theory to be able to make valuable predictions.

Luckily, at this stage not all possibilities of string theory have been employed. The massless spectrum of type II theories, for example, not only contains the already mentioned tensor fields $G_{\mu\nu}$ and $B_{\mu\nu}$ (which, together with the scalar dilaton field Φ make the so-called NSNS sector). It furthermore involves a set of p-forms: a one-form and a three-form in type IIA, a two-form and a four-form in type IIB. These make the so-called RR sector. In addition, there are fields in the fermionic sector, as for example the supersymmetry partner to the dilaton.

It is possible to compactify a ten-dimensional theory with some of the p-form fluxes from the RR sector having a nonzero vacuum expectation value. They then have to wrap cycles in the internal manifold M_6 only, in order not to preserve maximal symmetry in M_4. This will change the energy density of M_4 in a very particular way, such that deforming it, roughly speaking, would result in working against a pressure. In this way, moduli that come from such geometric deformations can obtain a vacuum expectation value. They get stabilised and no longer disturb the predictive power of the resulting four-dimensional theory.

In terms of solving the supersymmetry conditions, compactification with fluxes can easily be formulated. The new flux terms simply appear in a specific way in the equations that claim a certain degree of remaining supersymmetry, and this results in a more complex condition on the internal geometry of the internal manifold that then is connected to the values of the fluxes. In a simple case, nonzero RR fluxes induce torsion of the internal manifold, which will in turn affect the properties of the four-dimensional effective theory after integration.

1.2 Non-geometry and non-geometric fluxes

1.2.1 Non-geometry

It is not always possible to interpret the string coordinates as coordinates on a manifold. String theory offers the possibility of having valid solutions that exceed Riemannian geometry. Such setups are called non-geometric, and their existence once more show that the worldsheet perspective does not always have a target space counterpart.

Structure group and non-geometry

Two important examples have appeared in the literature, and they illustrate how polymorphic the features of non-geometry are. The asymmetric orbifold construction[8] presents a setup that offers not even the slightest possibility for a geometric interpretation. It comes with string coordinate fields that transform differently in the left- and right-moving part, and thus cannot be a composed as coordinates of a manifold.

The other class of examples for non-geometry is given by configurations that involve fields with non-geometric monodromies. Such monodromies arise when the compactification manifold is a fibration, and one considers how fields behave when following a closed loop in one of the fibre directions. In the simplest case [22, 23] the fibration is toroidal, but such that, for example, the corresponding metric is not single-valued under a closed loop. This

[8] Starting with [19] and [20], in this context recently refined in [21].

would be allowed as long as the different values could be connected by a transformation in the structure group, as for example by a diffeomorphism. But it turns out that there are string solutions where one has to include transformations that are not part of the geometric structure group.

Pictorially, this can be understood from fig. 1.1: The internal manifold is shown as a fibration of S^1 over a base circle S^1. For a geometric configuration this would result in a torus, but in the case of non-geometry the metric cannot be patched by diffeomorphisms as it has to turn a radius R into a radius $1/R$.

Interestingly, the additional transformations have to be $O(D,D)$ transformations [24, 25], or T-duality transformations in other words. Therefore, one characterisation of non-geometric string setups is that they necessitate the inclusion of T-duality transformations in the structure group of the target space manifold. Strictly speaking, the latter then ceases to be a manifold, and there have been many proposals of constructions that overcome this issue.

One of such proposals is the T-fold construction [26, 27] that introduces doubled coordinate fields in the worldsheet theory. It accommodates $O(D,D)$ transformations in a straightforward way, similar to the target space construction of double field theory. Furthermore, the T-fold has to be completed by a section condition, that halves the number of coordinate fields. Non-geometry then appears in the form of preventing a global solution to the section condition.

Features of non-geometry

Not only the metric of the potential target space might be affected by non-geometric monodromies, but also the Kalb-Ramond field B. For a geometric setup it is allowed to be patched around closed loops by diffeomorphisms, as any tensor, and gauge transformations that make a particular symmetry in string theory. Nevertheless, there are string solutions that involve the B-field in a more sophisticated way, as it can be equipped with non-geometric monodromies as well. Actually, an enhancement of the structure group by T-duality transformations implies that changing the coordinate patch may include a mixing of metric and B-field.

Many cases of non-geometric setups are constructed by applying a T-duality transformation on a known geometric setup. It is then straightforward, how the inclusion of T-dualities in the structure group helps to remedy the ill-definedness of the target space fields, as will be discussed in chapter 3. But not all non-geometric setups are of this type, as for example the asymmetric orbifold construction, and there is still no conclusive classification that contains all possible types.

Such an inconclusiveness not only appears for the construction but also for the characterisation of non-geometry, as non-trivial monodromies of the fields might not be the only feature of non-geometric backgrounds. Other characteristics have been suggested: [28] considers the appearance of non-commuting coordinate fields, which opens the possibility of applying an uncertainty principle to spacetime itself[9]. In [30, 31], even the appearance of non-associativity of the coordinates was argued for.

It is, in particular, important to carefully distinguish between local and global characterisations of non-geometry. Non-trivial monodromies are only detectable when following a closed loop around a compact direction, they are global characteristics of the spacetime

[9] Early speculations about quantum features of spacetime appeared in [29].

structure under investigation. Point particles would not suffice as probes for such features, but strings can have winding around compact dimensions and thus are sensitive to those. Setups that appear geometric locally but have non-geometric properties globally are often differentiated from setups that do not even have a local description, such as backgrounds with non-associative coordinate fields.

One important conjecture in this context is that non-geometric setups constructed as T-duals of geometric backgrounds can only be of the first type, i.e. they have locally geometric descriptions but fail to be geometric globally. It was nevertheless suggested to formally apply the T-duality rules in cases where there is no isometry, which is then conjectured to lead to backgrounds that even lack a locally geometric description[10].

The three-torus with flux

The most prominent example of a non-geometric setup is the three-torus with H-flux and its T-duals [23]. It is a toy model in the sense that it has to be supplemented by other ingredients to make a full string theory setup, as it only gives three of the six internal dimensions. Nevertheless, it illustrates many of the general ideas about non-geometry discussed so far.

A simple realisation of this toy model is to take a three-torus with unit radii and a Kalb-Ramond field B that only depends on one direction, implying a constant H-flux. The fields could be defined as

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & X^3 & 0 \\ -X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.9}$$

and it is obvious that this background has two isometries, namely in the remaining two coordinate directions along X^1 and X^2. That will allow to obtain two T-dual models.

The torus construction becomes visible when taking into account periodic identifications of the coordinates, expressed as $X^i \sim X^i + 1$. As presented here, the fields are well-defined under these monodromies: G remains invariant under $X^i \to X^i + 1$ and the B-field undergoes a simple gauge transformation.

This will change dramatically when investigating the model after having performed the two allowed T-duality transformations at once. The fields are then given by[11]

$$G' = \frac{1}{1+(X^3)^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+(X^3)^2 \end{pmatrix}, \quad B' = \frac{1}{1+(X^3)^2} \begin{pmatrix} 0 & -X^3 & 0 \\ X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.10}$$

and because of the denominator it is not possible to relate $G(X^3)$ and $B(X^3)$ to $G(X^3+1)$ and $B(X^3+1)$ by diffeomorphisms or gauge transformations any more. These fields are ill-defined in the above sense, as they have non-geometric monodromies. Still, they fall into the first class of non-geometric constructions because they obviously have a local description.

1.2.2 Non-geometric fluxes

There is a completely different point of view on non-geometry that has its origins in the four-dimensional effective theory. Based on the observation that non-geometric setups can be created by applying T-duality transformations on geometric backgrounds, it shall here be discussed how T-duality manifests itself in four-dimensional structures. The basic idea is that

[10] See for example [32], p. 4f. and [33], p. 14f.

[11] See appendix A for the technical details.

in the standard procedure of flux compactifications something is missing to allow for a proper four-dimensional action of T-duality. The missing objects have been denoted “non-geometric fluxes”, and they indicate that non-geometry can be revealed even in lower dimensions.

Effects of fluxes in four dimensions

Turning on fluxes will have an effect on the geometry of the internal manifold. This will enter the compactification procedure such that the effective four-dimensional theory depends on the choice of fluxes. In other words, when reducing supergravity as the ten-dimensional effective field theory of string theory to four dimensions, the background fluxes decide on the properties of the lower-dimensional theory, such as the spectrum of particles, their couplings and the cosmological constant.

For example, a simple torus compactification of type II supergravity from ten to four dimensions results in a supergravity with too much supersymmetry. It would not allow for chiral matter, and as this is phenomenologically unsatisfactory, a more sophisticated construction has to be found. Compactifications with fluxes [34, 35, 36] offer a broad variety of constructions that feature more realistic properties.

The four-dimensional effective theory changes dramatically when turning on background fluxes, as for example the H-flux that comes from the Kalb-Ramond field B, the Ramond Ramond fluxes, or so-called geometric flux f that comes from torsion in the internal manifold. The resulting theory is a gauged supergravity with non-Abelian gauge symmetries. It, most importantly, obtains a potential from the fluxes such that some of the four-dimensional fields acquire masses. This is known as moduli stabilisation, because it reduces the number of fields with no vacuum expectation value. Furthermore, there can be a nonzero cosmological constant and also a mechanism for spontaneous supersymmetry breaking - both phenomenologically highly important features.

For these reasons, flux compactifications of string theory have become a very active area of research over the last decade. Here, the point of interest is how a new type of flux has been discovered in that context, the so-called non-geometric flux. Two main arguments have been made in favour of an extension of the flux spectrum [33]: one that uses the superpotential, and one that employs the gauge algebra of the gauged supergravity.

Non-geometric fluxes from the superpotential

There are many different gauged supergravities and one possibility to classify them is to find a structure that appears in all of them. Such a structure is the superpotential, that together with the so-called Kähler potential is enough to specify the particular theory [12]. Other important quantities, like the potential that determines masses and couplings of the fields, can be calculated from the superpotential.

In principle, the superpotential for many gauged supergravities that come from a string theory compactification can be determined by using the so-called Gukov-Vafa-Witten formula [38]. It determines exactly how the various fluxes enter. The idea here is that this superpotential might not be complete: By definition, T-duality transformations in the string theory origin must not change the four-dimensional theory, but, in fact, the GVW superpotential

[12] Strictly speaking, one also needs the so-called gauge kinetic function [35], but it will play no role in the following. Another structure that allows to classify all gauged supergravity theories is the so-called embedding tensor [37].

does change. There are T-dual string models with different superpotential and, accordingly, different effective four-dimensional theories. From this perspective, it was proposed [33, 32, 39] to add new terms, i.e. new types of fluxes, such that T-duality invariance in four-dimensions is retained.

A very basic setup to study these new types of fluxes is the comparison between two particular string theory solutions, namely between a torus compactification of type IIB string theory and a twisted torus compactification of type IIA string theory. These two models are T-dual to each other and therefore should deliver the same superpotential. Instead, the GVW formula gives a superpotential of the form

$$W = P_1(\tau) + SP_2(\tau) \tag{1.11}$$

for the first case of compactification, whereas in the second case it has the form

$$W = P_1(\tau) + SP_2(\tau) + UP_3(\tau)\ . \tag{1.12}$$

Obviously, the two superpotentials mismatch, in particular all P_3 terms are not present in the first case. It was therefore suggested [33] to take a general superpotential for both cases, that looks like the second one. Internally, it introduces new coefficients to make the matching complete. These new coefficients cannot be related to the known types of fluxes, which are exhausted by the original form of the superpotential. They have been associated to a new type of fluxes, the so-called non-geometric fluxes. Eventually, such an extension has to be understood in the sense that the GVW formula does not capture all necessary modes.

This construction is completely generic and based on arguments in the four-dimensional theory only[13]. It does not provide any explanation of the higher-dimensional origin of the new fluxes, such that it is unclear which features of string theory are responsible for the appearance of non-geometric fluxes in four dimensions.

In order to arrange the findings in a more systematic way, one can order all types of fluxes[14], geometric and non-geometric, in a chain whose links are given by T-duality transformations. This works as follows: To start with, a configuration with H-flux in type IIB torus compactifications is considered. Then, its T-dual is the above mentioned type IIA compactification on a twisted torus [41]. The twisting can be characterised by a distinct type of flux denoted geometric flux f. It is the T-dual object to H, that itself vanishes for the twisted torus.

As discussed, the claim of a T-duality invariant superpotential implies the existence of new types of fluxes. Bases on the necessary index structure and on how T-duality transformations interchange upper and lower indices in four dimensions, one can conclude that they come as two different objects Q and R. The chain of fluxes then reads:

$$H_{abc} \quad \longrightarrow \quad f^a{}_{bc} \quad \longrightarrow \quad Q_c{}^{ab} \quad \longrightarrow \quad R^{abc}\ . \tag{1.13}$$

Each arrow stands for a T-duality transformation, and the first step is clear from the examples presented here. Furthermore, in order to obtain the correct index structure, two or three T-duality transformations on the original background are necessary to find non-geometric fluxes Q or R, respectively.

[13]There has been an independent argument for the proposed superpotential in [40].

[14]Focusing on the NSNS sector only.

So far, it is only a matter of wording that the new objects Q and R have been named 'non-geometric' fluxes. The actual connection between them and non-geometry was established for the particular case of the three-torus with H-flux, that already served as a toy model above [39]. One T-duality transformation on the original setup results in a twisted torus configuration with geometric flux f only. It enters the metric as off-diagonal component, and necessitates a twisted periodic identification of the coordinates, thus the name twisted torus. More precisely, the metric

$$ds^2 = (dX^1 - f^1{}_{23}X^3 dX^2)^2 + (dX^2)^2 + (dX^2)^2 \ , \tag{1.14}$$

is only well-defined under the torus transformation $X^3 \to X^3 + 1$ when identifying

$$(X^1, X^2, X^3) \sim (X^1 + f^1{}_{23}X^2, X^2, X^3 + 1) \ . \tag{1.15}$$

This transformation establishes the connection $H_{abc} \to f^a{}_{bc}$. A further T-duality transformation results in a non-geometric background. When running around the base circle in X^3 direction, the non-geometric transition function of the metric can be shown to depend again on a parameter that was given by $f^1{}_{23}$ in the previous frame. It has to have a different index structure, though, and therefore expands the T-duality chain of fluxes by $f^a{}_{bc} \to Q_c{}^{ab}$.

The existence of the last type of flux cannot be shown by such an argument, but formally there has to exist an object with index structure R^{abc} that adds the correct coefficients to the superpotential. In this sense, that last part of the T-duality chain is often stated as a "formal" extension[15]. There are arguments [32] stating that backgrounds with R-flux do not admit locally geometric descriptions, as it inhibits the existence of zero-dimensional objects. This has been refined in [30, 31], where a nonzero R-flux was conjectured to imply non-associative coordinate fields.

In summary, it should be stated that non-geometric fluxes have been loosely connected to non-geometry for very particular setups, whereas a general statement is missing.

Non-geometric fluxes from gauge algebras

It is possible to motivate the existence of non-geometric fluxes from another perspective: one can examine the gauge algebra of the effective four-dimensional gauged supergravity theory that is obtained by compactification. Again, it does not meet the expectation of being invariant under T-duality transformations, but can be complemented by additional coefficients that exactly meet the structures appearing in the superpotential extension.

In the simplest case with no fluxes, the reduction of the ten-dimensional effective string theory on a six-torus T^6 leads to an Abelian gauge group $U(1)^{12}$. In the deformed case, that has H-flux and geometric flux f from a twist of the tori, the gauge group becomes non-Abelian. Six of the corresponding generators descend from diffeomorphisms in the internal space, denoted by Z_a, and the other six generators X^a descend from shifts of the B-fields. It can be shown that the fluxes H and f make the structure constants of the gauge algebra [42],

$$\begin{aligned} [Z_a, Z_b] &= H_{abc}X^c + f^c{}_{ab}Z_c \\ [Z_a, X^b] &= -f^b{}_{ac}X^c \\ [X^a, X^b] &= 0 \ . \end{aligned} \tag{1.16}$$

[15] Cf. [33], p. 20, or [39], p. 14.

It can easily be checked that the duality group $O(6,6,\mathbb{Z})$, making the four-dimensional analogue of T-duality transformations, does not leave the algebra closed. Roughly, it interchanges the upper and lower indices, i.e. $Z_a \overset{T_a}{\leftrightarrow} X^a$, and transforms the flux coefficients along the T-duality chain. Then, for example, a transformation in a-direction of the second row would necessitate a nonzero right-hand side in the third row, having a term $Q_c{}^{ab}X^c$. The general extension along these lines reads [43]

$$\begin{aligned} [Z_a, Z_b] &= H_{abc}X^c + f^c{}_{ab}Z_c \\ [Z_a, X^b] &= -f^b{}_{ac}X^c + Q_a{}^{bc}Z_c \\ [X^a, X^b] &= Q_c{}^{ab}X^c + R^{abc}Z_c \ , \end{aligned} \tag{1.17}$$

and exactly reproduces the new types of fluxes Q and R which were found for the T-duality invariant superpotential.

Benefit from non-geometric fluxes

It seems, that the introduction of non-geometric fluxes is motivated by a rather technical argument only. But there is also a highly important impact on the phenomenological predictions of the corresponding four-dimensional theories.

First, non-geometric fluxes help to stabilise the moduli: Any string theory compactification usually comes with massless fields that are physically unwanted. They descend, for example, from geometric data of the compactification manifold or from fluxes. The extended superpotential now contains all such moduli of the model, and as the ordinary potential can be derived from it, they are all equipped with a vacuum expectation, i.e. they are stabilised. In [32] this was shown to hold at least at tree level but to be inaccessible with geometric fluxes only. Further evidence is given in [44, 45, 46, 47] for more refined models, where also constraints from one-loop corrections are considered. A numerical analysis was performed in [48, 49].

Second, non-geometric fluxes help to fulfill cosmological requirements: Ordinary flux compactifications usually allow for anti-de Sitter solutions only, in some cases for Minkowski solutions as well. To have a positive cosmological constant turned out to be a notoriously hard problem, as such solutions are usually metastable at most [34]. Recently, indications were found that non-geometric fluxes allow solutions with de Sitter vacua in four-dimensions, as they add positively signed terms to the potential [47, 48, 49, 50, 51].

Third, non-geometric fluxes allow for new supersymmetry breaking solutions: Phenomenologically, the four-dimensional gauged supergravity should have an $N = 1$ supersymmetry that, eventually, is broken either spontaneously or dynamically. Ordinary flux compactifications allow for solutions with such features, but these are often plagued by the huge difference between the typical energy scales. On the phenomenological side, supersymmetry should be broken at low scales, which is opposed to the high string scale of such solutions [36]. It has been shown that in some cases non-geometric fluxes help to implement supersymmetry breaking at low energy scales [45, 51].

These observations motivate the study of non-geometric fluxes from the phenomenological side and show that non-geometry goes beyond a mere technical interest in string theory.

1.3 This work

Non-geometry has been introduced as the idea that string theory offers a more general perspective on geometry than point particles do. It seems to exceed the framework of Riemannian manifolds when dualities like T-duality are taken into account. The literature provides a set of particular examples and some constructions that try to introduce more general notions of geometry. Nevertheless, the topic, as a young and active field, tends to remain fragmented, and some questions still have not found a definite answer. This work presents an investigation of three selected main topics in the field of non-geometry. They shall be sketched here, a more detailed discussion can be found in the respective chapters.

Non-commutativity

Leaving the framework of ordinary Riemannian geometry could imply many new features of spacetime, where non-commutativity of the coordinates is a very prominent one [28, 30, 31]. Although indications have been given, that the commutator of coordinate fields for closed strings could be nonzero for non-geometric setups, the connection has not been fully established. In particular, it was conjectured that non-geometric fluxes could be the source of non-commuting coordinates [28]. This was presented as an analogy to findings for open strings, where a nonzero geometric H-flux leads to non-commuting string endpoint coordinates [52]. Here, it shall be asked:

- When can a non-geometric setup be characterised by non-commuting coordinates?
- What determines the non-commutativity? Can it be connected to non-geometric fluxes?

Effective field theories

These questions immediately lead to the problem that so far it is not clear how to implement non-geometric fluxes in higher dimensions. They have been discovered in the context of four-dimensional compactifications of string theory, but they seem to be unrelated to the known constituents of ten-dimensional supergravity as low-energy effective theory. The latter is completely determined by the NSNS flux H, its geometric counterpart f that enters the metric as torsion, and the various RR fluxes. As non-geometric fluxes seem to be strongly related to T-duality, one might suspect that they make a truly string theoretic feature that so far has not been considered.

A related open question is, whether non-geometric fluxes are in general related to non-geometry. It has been shown that this is the case for certain examples, but generally one should be careful not to be misled by the similar naming. Only when the characterisation of non-geometry and the definition of non-geometric fluxes are available for the *same* framework, it can be checked how far-reaching the joint appearance of these two notions is.

Furthermore, the appearance of ill-defined fields threatens the applicability of supergravity theories as effective string models. The example of the three-torus with H-flux has shown, that the string worldsheet model can lead to a metric that has lost its tensorial character. Taking over such a field to the supergravity model leads to an ill-defined integral in the action and it seems necessary to remedy this deficiency before applying a compactification procedure.

This work investigates the topic and tries to find answers to the following questions:

- How can a higher dimensional origin of non-geometric fluxes in four dimensions be defined?
- What is the connection between non-geometry and non-geometric fluxes?
- How does non-geometry appear in effective field theories and how can these be kept consistent?

There is another issue that appears when asking about the higher dimensional origin of non-geometric fluxes: So far, the investigation of phenomenological implications has assumed that all types of fluxes can be turned on in arbitrary combinations, only restricted by certain differential conditions that assure the consistency of the four-dimensional structures (the so-called Bianchi identities). Whether string theory allows to have all four types of fluxes at the same time is not clear, but can be undeceived by finding a higher dimensional origin for the four-dimensional objects. Therefore, the following question shall be added to the research plan:

- Can the four types of fluxes be turned on at will?

Worldsheet model

T-duality has shown to be important in two ways: Either it makes the necessary extension of the structure group in the case of a non-geometric configuration, or it acts as a generating tool for such configurations. On the other hand, it is not a symmetry of string theory in the strict sense. The original derivation of T-duality rather relates two different worldsheet theories. It, therefore, seems desirable to find a manifest implementation of T-duality in string theory, and doubling the number of coordinates has shown to be effective along these lines.

On the level of effective theories, for example, double field theory provides manifest T-duality invariance by introducing a doubled spacetime manifold. It will be heavily used for the investigation of the preceding topic, but one can wonder how reliable any effective theory could be when considering non-geometry as an actual string theory appearance. Therefore, the study shall be pursued a little further, and it will be aimed at a worldsheet characterisation of non-geometry.

Indeed, there are worldsheet constructions that provide manifest T-duality invariance, most importantly the T-fold construction mentioned above. There, the doubling of the coordinate fields has to be undone by imposing a constraint. As this is implemented 'by hand', it remains unclear how the theory can be quantised. In other words, it is not fully clarified in what sense the T-fold construction is a reformulation of string theory.

When it comes to non-geometric fluxes, there are also attempts to implement them on the worldsheet in the same way as the H-flux [53, 54]. Unfortunately, they are incompatible with the T-fold construction and thus are not the best framework to investigate non-geometry.

Here, a new worldsheet model is proposed that aims at solving these problems, and in particular is used to find answers to the following general questions:

- Is there a formulation of string theory that implements T-duality manifestly?
- Does such a formulation help to characterise non-geometry?
- Can it provide direct access to non-geometric fluxes?

Common theme

To conclude the introduction of the research topics of this thesis, it should be noted that the above questions turn out to have a common theme. It shall make the most important question to be studied in the following, and will appear in many places under different circumstances: What is the relation between non-geometry and non-geometric fluxes?

$$\boxed{\text{Non-geometry} \longleftrightarrow \text{Non-geometric fluxes}}$$

1.3.1 Method of investigation

In the following chapters, a whole variety of different frameworks shall be employed to address the research topics developed here. Non-geometry and its relation to non-geometric fluxes will be investigated from as many angles as possible with the hope of finding a clear underlying structure. This is particularly promising as all frameworks are closely connected to each other, and it seems plausible that findings in one direction will be mirrored at other places as well.

Figure 1.2 summarises the web of frameworks that will be used. It shows three stages: On the top, the worldsheet perspective comes in two models, the standard sigma model of string theory and its suggested extension to a sigma model with doubled coordinate fields. The double-ended arrow indicates that the latter is constructed such that it can be connected to the former.

The middle stage gives two effective field theories of string theory: the ten-dimensional supergravity and its T-duality invariant analogue, double field theory. A horizontal arrow indicates that solutions of the strong constraint reduce the latter to the former. A vertical arrow on the right emphasises that the supergravity is considered as the effective field theory of the standard worldsheet theory. The possible connection between the doubled worldsheet model and double field theory is indicated by a vertical arrow on the left.

Eventually, both higher-dimensional effective theories can be compactified to the effective four-dimensional supergravity theory that makes the basis for potential phenomenological conclusions.

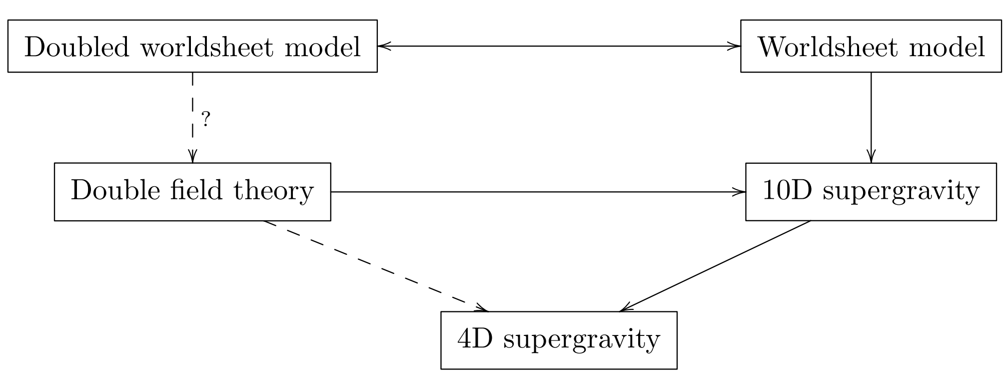


Figure 1.2: Possible frameworks to investigate non-geometry and non-geometric fluxes, and their interrelations

The partition of the various lines of research into different chapters is as follows:

- In chapter 2, the appearance of non-commutativity in the target space coordinates for non-geometric setups will be investigated. A direct canonical quantisation of the coordinates is employed, such that only the string theory worldsheet model itself has to be used. Still, a connection between the non-commutativity and non-geometric fluxes is conjectured, and therefore a close link to the effective field theories, in particular to ten-dimensional supergravity, is made.

- Chapter 3 heavily uses the connection, that starts in double field theory, passes supergravity in ten-dimensions and finally ends in a four-dimensional supergravity. It tries to reveal non-geometric fluxes in the higher-dimensional effective field theories, and shows that such theories can deal with non-geometric configurations.
  The use of double field theory is motivated by the fact that T-duality appears as a global symmetry there, which facilitates the investigation of non-geometric fluxes considerably. Ten-dimensional supergravity turns out to be the appropriate framework to study non-geometric backgrounds and how non-geometry appears in the effective field theory. In addition, it will be investigated how the non-geometric fluxes eventually reduce to the corresponding objects in four-dimensions. This will be done using a simple dimensional reduction.

- Chapter 4 finally considers a possible reformulation of the standard string theory worldsheet model as a manifestly T-duality covariant sigma model with doubled coordinate fields. Again, it shall be studied how non-geometric fluxes can be implemented and what their relation to non-geometry is. Having doubled coordinates, the new worldsheet model will be suspected to provide a direct origin of double field theory, and this connection shall be investigated by the above mentioned method of obtaining effective equations of motion from claiming conformal invariance.

Each chapter ends with its own summary and discussion, where also important connections to recent developments in the literature are drawn. The thesis then closes with a short conclusion, and two appendices provide a technical introduction to T-duality and a summary of conventions and technicalities.

Chapter 2

The torus with H-flux

This chapter, presenting the results of [2], investigates the conjecture that non-geometry, in certain cases at least, implies non-commutativity of the coordinate fields. In order to do so, a closed string setup with three toroidal directions is considered. The target space fields are assumed to have two isometries, such that in total there are three possible T-dual frames: the torus with H-flux, the twisted torus and a non-geometric frame with Q-flux. They correspond to the first three steps of the chain $H \to f \to Q \to R$. It is not only shown that a string specific property, namely the winding around one of the toroidal directions, is responsible for the coordinates to be non-commuting, but also that the 'amount' of non-commutativity can be expressed by integrating the non-geometric flux. The strategy is to perform a canonical quantisation of the geometric frame, and translate the commutators to the non-geometric frame via T-duality.

The structure of this chapter is as follows:

2.1 provides an introduction to the topic and the relevant ideas in the literature.

2.2 performs the canonical quantisation of the twisted torus as a geometric frame.

2.3 constructs solutions in the non-geometric Q-flux frame and applies the results of the former quantisation to compute the coordinate commutators.

2.4 discusses the problem of undetermined commutation relations and how non-commutativity can be shown.

2.5 checks the consistency of the whole procedure by independently determining a classical solution for the torus with H-flux and relating it to the previous findings.

2.6 provides a summary of the results and discusses their implications.

2.1 Introduction

The idea that strings may show non-commutative behaviour has been followed for more than fifteen years, even before the appearance of non-geometry. It was first investigated in the case of open strings, where it turned out that the coordinates of the endpoints on D-branes can have nonzero commutators. Important steps in this field of research have been taken for backgrounds with constant B-fields, i.e. vanishing H-flux, that in the presence of D-branes cannot be gauged to zero. It alters the geometry of a flux-free background in both a non-trivial and, nevertheless, technically tractable way. Results of particular interest are given in the following publications:

- [52] shows that the end-points of open strings on D-branes in a background with constant B-field have non-commutative coordinates. The result is obtained from a slightly modified canonical quantisation. In particular, it was found that the non-commutativity is related to the B-field itself,

  $$[X^i(\tau,\sigma), X^j(\tau,\sigma')]_{\sigma,\sigma'=0,\pi} \sim \mathrm{i}B^{ij} \ , \tag{2.1}$$

  for coordinates on the D-brane and up to first order in a small B expansion.

- [55] employed a direct canonical quantisation by imposing particular boundary conditions, but found conflicting results to [52]. In a follow-up publication [56] the authors then corrected their findings and agreed for a particular gauge choice by using the quantisation method of Dirac.

- In [57], the authors investigated D-branes in a background with constant B-field. They calculate operator products of open string vertex operators and find a non-commutative multiplication in the world-volume algebra. In particular, again, the non-commutativity is proportional to the B-field for small B.

- The famous [58] examines open strings in a constant B-field background. Amongst other results, like the Seiberg-Witten map, it shows that the effective action for such a setup can be described by a gauge theory on a non-commutative spacetime. In particular, the authors compute an operator product expansion and interpret it as a commutator, finding non-commuting coordinates $[X^i, X^j] \neq 0$.

For the present work, these ideas provide motivation for the following theses: a nonzero B-field can be the source of non-commutativity; canonical quantisation might provide the right framework to capture this effect; it might be hoped that similar investigations in a framework of conformal field theory bring concordant results.

Unfortunately, in order to connect non-geometry and non-commutativity, a dramatic change in the setup is necessary. As non-geometry here is taken to be generated by T-duality the investigation to follow has to be about closed strings. T-duality of open strings (see for example [59] or [60]) shall not be considered.

Non-commutativity of closed strings has been investigated only very recently. This is in particular due to the complication that backgrounds with constant B-field no longer provide valuable examples as in this case such a field is gauge equivalent to no field at all. The simplest non-trivial example thus is given by a constant H-flux, but that turns the underlying sigma model into an interacting one which is technically much more involved. Nevertheless, there are important contributions that either deal with particular exactly solvable models, or with approximations for small flux densities:

- In [30] the three-bracket structure for a particular WZW model with non-vanishing H-flux has been investigated. The string coordinates for closed strings are shown to be non-commutative as well as non-associative. The authors conjectured that the source of the non-associativity is the H-flux itself,

$$[X^i, X^j, X^k] \sim H^{ijk} \ . \tag{2.2}$$

  The results follow from a computation in conformal field theory with no particular embedding into string theory.

- [28] provides indications of non-commutativity for the closed string coordinates in backgrounds that are T-dual to backgrounds with non-trivial geometric fluxes f and H. The investigation rests on an analysis of the possible monodromies and uses the mode expansion of the free string. This is argued to be valid as it makes the lowest order of an expansion in the H-flux, that for small H fulfills the target space equations of motion.

- [31] investigates non-associativity for general closed string backgrounds with constant three-form flux by computing three-point functions. The authors conjecture that the relevant flux background is given by constant H-flux. As the back-reaction to the geometry is argued to appear at second order in the flux only, the approximation is considered valid for a conformal field theory at linear order in H. Furthermore, it is argued that, by dimensional analysis, higher order α' corrections cannot obstruct this linear order approximation. The investigation deals with all four T-dual setups appearing in the flux chain $H \to f \to Q \to R$, and it is conjectured that the non-commutativity of the string coordinates in the Q-flux frame is proportional to the winding of the string.

- [21] provides an example of flux backgrounds that is exact to all orders in α' and has a non-commutative structure for the closed string coordinates. It is constructed as a freely acting asymmetric orbifolds. The non-commutativity is related to Q-flux by an asymmetric Scherk-Schwarz mechanism.

For this chapter, in particular, the idea of an expansion in the flux parameter and the restriction of the analysis to linear order has been inspiring. It was put on solid grounds from at least two different perspectives, namely for the mode expansion of the string coordinates in [28] and from a conformal field theory point of view in [31]. Here it shall be taken to be valid also for the procedure of canonical quantisation.

There are also other approaches to non-commutativity in string theory, which only play a minor role in the research presented here, but in some sense add credibility to the idea that T-duality, non-geometry and non-commutativity are closely intertwined notions.

- Non-commutative tori: T-duals of a two-torus with nonzero H-flux are not necessarily torus fibrations anymore. The “missing” duals can be modeled by non-commutative tori; see e.g. [61, 62]. This is a more mathematical direction of research.

- Matrix models: [63] draws a connection between fluxes from the T-duality chain and non-commutativity or non-associativity, respectively. It rests on the BFSS matrix model [64] that hypothetically serves as a non-perturbative formulation of M-theory, the proposed unifying meta-framework for all five string theories.

Method

The most basic example for non-geometry is the three-torus with H-flux and its T-duals [23]. It shall here serve as the object of study, with the following conventions: The first frame has string coordinates X^μ, with $\mu = 1, 2, 3$, and a nonzero H-flux, i.e. a B-field that depends linearly on the coordinates. It is denoted as the 'torus with H-flux frame'. The first T-dual of it has coordinates Y^μ and a vanishing B-field, as dictated by the T-duality rules. It is denoted as 'the twisted torus frame' and still remains in the range of geometric string configurations. The last T-dual, in contrast, is non-geometric. It comes with coordinate fields Z^μ and has ill-defined target space fields G and B. The three frames are in accordance with the first three parts of the non-geometric flux chain $H \to f \to Q$.

The main goal of this chapter is to find the commutator

$$\left[Z^1(\tau,\sigma), Z^2(\tau,\sigma')\right] \ , \tag{2.3}$$

in the sense of canonically quantised string coordinates Z^μ. The usual procedure of canonical quantisation, in this context explained in great detail for example in [14], consists of three steps:

1. The classical solutions to the worldsheet equations of motion and the boundary conditions are obtained as mode expansions Z^μ in the worldsheet coordinates τ and σ.

2. These coordinate fields Z^μ are turned into operators by promoting the particular expansion coefficients to operators.

3. By employing canonical equal time, i.e. equal τ, commutators to the coordinate operators consistent commutation relations for the expansion coefficients can be read off.

Being a well-established procedure in quantum field theory, canonical quantisation follows the analogy to quantum mechanics, where position and momentum operators do not commute. It simply claims that the field operator and its corresponding canonical momentum, obtained as the functional derivative of the Lagrangian, do not commute. String theory is, in this sense, simply taken to be a two-dimensional field theory on the worldsheet, having a set of bosonic fields Z^μ. Conclusively, the standard way of quantising these coordinate fields Z^μ would be to *impose* a right-hand side for (2.3).

Here, it is suspected that non-geometry undermines the validity of such a procedure. Instead, the proposal is to leave (2.3) undetermined for the time being and gain information about it by relating commutators from the twisted torus frame via T-duality. In that frame, the fields Y^μ are suspected to be quantisable by the canonical procedure:

$$\left[Y^\mu(\tau,\sigma), Y^\nu(\tau,\sigma')\right] = 0 \ . \tag{2.4}$$

In detail, the applied strategy is suggested to be the following:

1. Solve the equations of motion for the Y^μ under appropriate boundary conditions. A mode expansion is implied.

2. Promote an appropriate set of expansion coefficients to operators. Employ canonical equal-time commutators and read off a consistent algebra of commutation relations for the expansion coefficients.

3. Use T-duality to construct a solution to the equations of motion for the Q-flux frame Z^μ. The expansion coefficients are given by expansion coefficients from the twisted torus frame.

4. Use the algebra of expansion coefficients from step 2 to compute the commutator (2.3).

Although this recipe might be considered straightforward, there are a couple of subtleties that shall be commented on.

In step 3 the construction rests on T-duality formulated as rules that involve derivatives of string coordinates in different T-duality frames. These are developed in appendix A, (A.10), and are in a sense complementary to the T-duality rules ('Buscher rules') for the target space fields (A.8).

To obtain a classical solution one has to involve integrations and thus not all constituents of the coordinates Z^μ can be related to expansion coefficients of the coordinates Y^μ. In particular, there will arise new position zero modes that enter step 4 in undetermined commutation relations. These can in principle be fixed at will but there are physical arguments and analogies that indicate a favouritism of particular values. A whole subsection 2.4 is devoted to the discussion of possible sets of such commutation relations, where, eventually, one set is argued to uniquely meet all physical requirements.

Inspired from the literature, the whole analysis shall be performed as an expansion in the flux parameter H. Although in the twisted torus frame it does not appear directly, the metric contains components related to the original H-flux and thus can be expanded. Not only the solution to the worldsheet equations of motion shall be expressed in this manner, but also it is assumed that the canonical quantisation can be reasonably applied order by order in such an expansion.

All results shall be restricted to first order in the flux parameter as it turns out that the target space equations of motion are then satisfied automatically. Because the flux parameter is taken to be infinitesimally small the whole approach is referred to as "dilute flux" approximation. Concerns that topological constrictions render the flux parameter integer can be rebutted as the actual parameter contains the inverse radii of the torus, see equation (2.27) and the discussion there. Thus, a dilute flux can be reached by having a large fibre volume.

Not all commutation relations in the twisted torus frame can be determined. This is basically due to applying the quantisation procedure order by order. Whereas the zeroth order allows to find all commutation relations which then coincide with the free string relations, the first order calculation is only able to solve for particular combinations of commutators. Furthermore, as in step 4 only one particular direction of the coordinate commutator shall be computed - the focus of this chapter lies in whether non-commutativity can be found at all, and not in how it quantitatively reveals itself in all possible directions - the canonical quantisation of the twisted torus is as well only performed in this $(\mu, \nu) = (1, 2)$ direction. In a sense, the result should better be called a "partial" quantisation.

From the perspective of the target space fields G and B, the H-flux frame and the non-geometric Q-flux frame look similar when restricted to first order in the flux parameter expansion. One could thus think that it is not necessary to determine the solutions Z^μ by T-duality from the beforehand obtained solutions Y^μ, but rather solve the worldsheet equations of motion directly. What prevents this idea from working is the rather involved boundary conditions for the Z^μ that have to be imposed. These do not coincide with the ones in the H-flux frame, not even in the dilute flux approximation. In the following, it is assumed that T-duality provides the only systematic way to conclude on these boundary conditions.

It could also be questioned to start from the twisted torus frame, that has quite more involved equations of motion than the H-flux frame. Two reasons speak against starting from the latter frame directly. First, one would have to apply two T-dualities and thus would encounter more integration constants from integrating the T-duality rules. Second, these T-duality rules themselves are roughly as complicated as the equations of motion for the twisted torus. In total, such a procedure seems to be at a disadvantage.

In the following, only the torus fibration $T^2 \times T$ shall be considered. To turn this into a full string theory background, there have to be three more internal directions in the case of a superstring theory. In addition, more ingredients like RR fluxes and orientifold sources have to be considered[1]. In this sense, the quantisation presented here is only about a part of the internal space and especially, non-commutativity is not to be expected in the four-dimensional spacetime.

2.2 The geometric frame: twisted torus

In this section, the twisted torus background shall be investigated on a classical as well as on a quantised level. Starting from a three-torus with constant H-flux, a T-duality is performed along one of the two fibre directions to find the target space fields G and B for the twisted torus frame. A rescaling of the coordinates hides all three torus radii and brings the worldsheet equations of motion into a convenient form. Nevertheless, they consist of a highly intertwined set of partial differential equations that involve the flux parameter H. The suggested method for solving these equations will therefore be an expansion up to first order in this parameter, which by an analysis of the target space equations of motion can be consistently taken to be small. Additionally, due to the non-diagonal form of the metric, there is a non-trivial boundary condition imposed on the first fibre coordinate that mixes it with the second one. This, as well, can be handled with the flux parameter expansion.

The quantisation procedure follows the standard method of canonical quantisation and can be parted into different orders in the flux parameter expansion, too. This allows for determining all necessary commutation relations amongst the introduced expansion coefficient operators. Nevertheless, the procedure does not allow for obtaining all possible commutation relations and thus it is, strictly speaking, not possible to prove consistency of the quantisation. Up to first order the fulfillment of the canonical commutators can be guaranteed, though.

2.2.1 Classical solutions

The standard worldsheet action of string theory in the form of Polyakov[2] shall be taken as a starting point,

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left(G_{\mu\nu}(\mathcal{X})\ \eta^{\alpha\beta} + B_{\mu\nu}(\mathcal{X})\ \varepsilon^{\alpha\beta} \right) \partial_\alpha \mathcal{X}^\mu \partial_\beta \mathcal{X}^\nu \ , \tag{2.5}$$

with $\eta_{\tau\tau} = -\eta_{\sigma\sigma} = -1$, $\varepsilon_{\tau\sigma} = -\varepsilon_{\sigma\tau} = 1$ and all other components zero. By convention it is $\alpha' = 1/2$. The dilaton term shall not be considered in the following investigation, as a constant dilaton is assumed later on. The coordinates $\mathcal{X}^\mu$ are here taken to be generic, but

[1] For a detailed account on consistent ten-dimensional constructions, refer to [65] or [66].

[2] Here as it can be found in [10], eq. (3.7.6). Cf. also with the form presented in the introduction, where no B-field was considered.

in the following will be taken to specialise to one of the coordinate sets X^μ, Y^μ or Z^μ that were introduced above.

The first frame to be considered here, with coordinates X^μ, is taken to be a three-torus with radii R_μ that is characterised by the torus identifications

$$X^\mu \sim X^\mu + 2\pi R_\mu \ . \tag{2.6}$$

This background shall be equipped with a constant H-flux

$$H_3 = H \, \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \ , \tag{2.7}$$

quantified by a constant H. This constant will make the expansion parameter for the classical solutions and the commutation relations in the quantisation procedure after being taken infinitesimally small later on. The target space metric has a diagonal form and contains the three torus radii R_μ, whereas gauge invariance for the B-field is used to set the latter to a particular form,

$$G = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} \ , \quad B = \begin{pmatrix} 0 & HX^3 & 0 \\ -HX^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

As can easily be seen from this setup, there is no dependence of the target space fields on the coordinate fields $X^{1,2}$. This shows that there are two isometries in the defined geometry and thus the sufficient condition for the existence of two T-dual configurations is satisfied.

As was explained above, the classical solutions to the worldsheet equations of motion in the given T-duality frame are of minor interest only - they will, for a check of consistency, be determined later on, see section 2.5. The procedure here is to perform a T-duality in the X^1-direction by applying the T-duality rules[3] for the target space fields (A.8). In this case they read

$$\begin{aligned} G_{11} &\to \frac{1}{G_{11}} = \frac{1}{R_1^2} \\ G_{22} &\to G_{22} - \frac{B_{21}B_{12}}{G_{11}} = R_2^2 + (HX^3)^2 \frac{1}{R_1^2} \end{aligned} \tag{2.9}$$

and

$$\left.\begin{aligned} (G+B)_{12} &\to -\frac{(G+B)_{12}}{G_{11}} = -\frac{1}{R_1^2} HX^3 \\ (G+B)_{21} &\to +\frac{(G+B)_{21}}{G_{11}} = -\frac{1}{R_1^2} HX^3 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} & B_{12} = B_{21} = 0 \\ & G_{12} = G_{21} = -\frac{HX^3}{R_1^2} \end{aligned}\right. \ , \tag{2.10}$$

where all other components are mapped into themselves. As the new frame obtained by these mappings shall be denoted by coordinates Y^μ it is required to set $X^3 = Y^3$. This is physically justified as T-duality leaves invariant the $\mu = 3$ direction, that makes the base fibre of the original configuration. For the quantisation of the third frame, to be discussed in section 2.3, this gives rise to subtleties with the position zero modes. Throughout the whole analysis, the coordinates thus shall be fixed to be identical in all T-duality frames, $X^3 = Y^3 = Z^3$.

[3] There is a change of signs due to different conventions in this chapter, see the remarks in appendix A.

Eventually, the target space fields after performing a T-duality in the $\mu = 1$ direction read

$$G \to G = \begin{pmatrix} \frac{1}{R_1^2} & -\frac{HY^3}{R_1^2} & 0 \\ -\frac{HY^3}{R_1^2} & R_2^2 + \left(\frac{HY^3}{R_1}\right)^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} , \quad B = 0 \; . \tag{2.11}$$

Given the intricate structure of the metric one can check that in order to have a well-defined manifold the torus identifications (2.6) have to be modified. It is sufficient to declare

$$\begin{aligned} (Y^1, Y^2, Y^3) &\sim (Y^1 + 2\pi, Y^2, Y^3) \\ &\sim (Y^1, Y^2 + 2\pi, Y^3) \\ &\sim (Y^1 + 2\pi H Y^2, Y^2, Y^3 + 2\pi) \; , \end{aligned} \tag{2.12}$$

which can be verified by recognising the invariance of the line element

$$\mathrm{d}s^2 = \frac{1}{R_1^2}(\mathrm{d}Y^1)^2 + R_3^2(\mathrm{d}Y^3)^2 + \left(R_2^2 + \left(\frac{HY^3}{R_1}\right)^2\right)(\mathrm{d}Y^2)^2 - \frac{2HY^3}{R_1^2}\mathrm{d}Y^1\mathrm{d}Y^2 \; . \tag{2.13}$$

In fact, the manifold (2.11) can be regarded as a three-dimensional nilmanifold, generated by a particular Heisenberg algebra, whose Maurer-Cartan one-forms are rendered globally well-defined by imposing (2.12).

It turns out that the so far explicitly shown radii R_μ can be conveniently hidden by a rescaling

$$Y^1 \to R_1 Y^1 \; , \; Y^{2,3} \to \frac{1}{R_{2,3}} Y^{2,3} \; , \tag{2.14}$$

and its generalisation to any tensor,

$$\begin{aligned} T^{\mu\cdots} &\to \begin{cases} \frac{1}{R_\mu} \cdots T^{\mu\cdots} & \text{for } \mu \neq 1 \\ R_1 \cdots T^{\mu\cdots} & \text{for } \mu = 1 \end{cases} \\ T_{\mu\ldots} &\to \begin{cases} R_\mu \cdots T^{\mu\cdots} & \text{for } \mu \neq 1 \\ \frac{1}{R_1} \cdots T^{\mu\cdots} & \text{for } \mu = 1 \end{cases} \; . \end{aligned} \tag{2.15}$$

Furthermore, the H-flux parameter has to be rescaled as

$$H \to H R_1 R_2 R_3 \; , \tag{2.16}$$

which can be motivated by a rescaling scheme similar to (2.15) that is adapted to the situation in the original torus with H-flux frame.

The rescaling (2.15) leaves invariant the line element (2.13) as well as the worldsheet equations of motion, which will be determined below. Moreover, it does not change the T-duality relations (2.124), nor the canonical commutation relations, that will be imposed at a later stage. For the target space fields, the only effect of (2.15) is that all radii are hidden in G,

$$G = \begin{pmatrix} 1 & -HY^3 & 0 \\ -HY^3 & 1 + (HY^3)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad B = 0 \; . \tag{2.17}$$

Having set up the target space fields for the twisted torus that shall be investigated here, it is now necessary to specify equations of motion as well as the boundary conditions. The former can be read off from (2.5), and for the rescaled fields (2.17) read

$$\partial_\alpha \partial^\alpha Y^1 = H \left(Y^3 \partial_\alpha \partial^\alpha Y^2 + \partial_\alpha Y^2 \partial^\alpha Y^3 \right) \tag{2.18}$$

$$\partial_\alpha \partial^\alpha Y^2 = H \left(\partial_\alpha Y^1 \partial^\alpha Y^3 - H Y^3 \partial_\alpha Y^2 \partial^\alpha Y^3 \right) \tag{2.19}$$

$$\partial_\alpha \partial^\alpha Y^3 = H \left(-\partial_\alpha Y^1 \partial^\alpha Y^2 + H Y^3 \partial_\alpha Y^2 \partial^\alpha Y^2 \right) \ . \tag{2.20}$$

The boundary conditions are chosen to be

$$Y^1(\tau, \sigma + 2\pi) = Y^1(\tau, \sigma) + 2\pi N^1 + 2\pi N^3 H Y^2(\tau, \sigma) \tag{2.21}$$

$$Y^2(\tau, \sigma + 2\pi) = Y^2(\tau, \sigma) + 2\pi N^2 \tag{2.22}$$

$$Y^3(\tau, \sigma + 2\pi) = Y^3(\tau, \sigma) + 2\pi N^3 \ , \tag{2.23}$$

in accordance with the above identifications (2.12). In particular, the condition for Y^1 can be traced back to the third line of (2.12). Furthermore, the introduction of a winding number N^μ for all three coordinates is a slight generalisation of (2.12) that still preserves the line element or the Maurer-Cartan one-forms, respectively.

The task of the next section will be to determine the most general solutions to this set of equations (2.18)-(2.20) and boundary conditions (2.21)-(2.23).

Solutions to the worldsheet equations of motion

Before a solution to the worldsheet equations of motion is constructed, it is necessary to make sure the consistency of the construction. In particular, it has to be ensured that Weyl invariance is preserved at least to the one-loop level, as mentioned in the introduction. The corresponding β-functional conditions can be written as target space equations of motion, here with zero B-field,

$$\mathcal{R} + 4(\nabla^2 \phi - (\partial\phi)^2) = 0 \tag{2.24}$$

$$\mathcal{R}_{\mu\nu} - \frac{G_{\mu\nu}}{2}\mathcal{R} + 2\nabla_\mu \partial_\nu \phi - 2G_{\mu\nu}(\nabla^2\phi - (\partial\phi)^2) = 0 \ .$$

Assuming a constant dilaton reduces these equations to

$$\mathcal{R}_{\mu\nu} = 0 \ , \tag{2.25}$$

where the components of the Ricci tensor without the above defined rescaling read

$$\mathcal{R}_{11} = \frac{1}{2R_1^2}\left(\frac{H}{R_1R_2R_3}\right)^2 \ , \ \mathcal{R}_{22/33} = -\frac{R_{2/3}^2}{2}\left(\frac{H}{R_1R_2R_3}\right)^2 \ , \ \mathcal{R} = -\frac{1}{2}\left(\frac{H}{R_1R_2R_3}\right)^2 \ . \tag{2.26}$$

Equation (2.25) can be solved conveniently by assuming the so-called dilute flux approximation,

$$\frac{H}{R_1R_2R_3} \ll 1 \ . \tag{2.27}$$

In case of the twisted torus, the dilute flux approximation is also called weak curvature approximation, and has for example been considered in [67, 68], [28] or [31]. In a sense, such an approximation avoids to add further ingredients like RR flux and sources as in [65].

The rescaling (2.16) turns the dilute flux approximation into

$$H \ll 1 \ . \tag{2.28}$$

Accordingly, any reasoning during the solving or quantising procedure is taken to make sense up to the first order $\mathcal{O}(H^1)$ only. This has an enormous impact on all steps that follow.

To begin with, the worldsheet equations of motion (2.18)-(2.20) simplify to

$$\partial_\alpha \partial^\alpha Y^\mu = H\theta^\mu{}_{\nu\rho} \partial_\alpha Y^\nu \partial^\alpha Y^\rho \ , \tag{2.29}$$

where the symbol θ is an abbreviation for

$$\theta^1{}_{23} = \theta^2{}_{13} = -\theta^3{}_{12} = 1 \ , \tag{2.30}$$

with all other components being zero. This corresponds to an expansion of the target space fields to

$$G = \begin{pmatrix} 1 & -HY^3 & 0 \\ -HY^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(H^2) \ , \ B = 0 \ . \tag{2.31}$$

To find solutions to these equations, the following strategy is applied: first, the boundary conditions (2.21)-(2.23) are used to find σ-periodic combinations of coordinate fields. Second, those combinations are Fourier expanded with τ-dependent expansion coefficients. Third, the coefficients can be solved for by applying the equations of motion (2.29).

Combinations that are 2π-periodic in σ are given by

$$Y^{2,3} - N^{2,3}\sigma \ , \tag{2.32}$$

$$Y^1 - N^1\sigma - N^3 H\sigma(Y^2 - N^2\sigma) + \frac{1}{2} N^3 H N^2 \sigma(2\pi - \sigma) \ . \tag{2.33}$$

Their Fourier expansion reads

$$\begin{aligned} Y^1(\tau,\sigma) = {} & N^1\sigma + \sum_{n\in\mathbb{Z}} b^1_n(\tau) e^{-in\sigma} \\ & + H\Big(N^3\sigma(Y^2 - N^2\sigma) - \frac{1}{2} N^3 N^2 \sigma(2\pi - \sigma) + \sum_{n\in\mathbb{Z}} c^1_n(\tau) e^{-in\sigma} \Big) \end{aligned} \tag{2.34}$$

$$Y^{2,3}(\tau,\sigma) = N^{2,3}\sigma + \sum_{n\in\mathbb{Z}} b^{2,3}_n(\tau) e^{-in\sigma} + H\Big(\sum_{n\in\mathbb{Z}} c^{2,3}_n(\tau) e^{-in\sigma} \Big) \ , \tag{2.35}$$

where a split of the expansion coefficients into different orders in H was introduced. It is even possible to separate the full solution,

$$Y^\mu(\tau,\sigma) = Y^\mu_0(\tau,\sigma) + HY^\mu_H(\tau,\sigma) + \mathcal{O}(H^2) \ . \tag{2.36}$$

That turns the non-linear equations (2.29) into a set of two sets of linear equations, a homogenous one for Y^μ_0 and an inhomogenous one for Y^μ_H,

$$\partial_\alpha \partial^\alpha Y^\mu_0 = 0 \tag{2.37}$$

$$\partial_\alpha \partial^\alpha Y^\mu_H = \theta^\mu{}_{\nu\rho} \, \partial_\alpha Y^\nu_0 \partial^\alpha Y^\rho_0 \ . \tag{2.38}$$

At zeroth order, the solution is nothing else than the free string solution with winding given by

$$b_0^\mu(\tau) = y^\mu + p^\mu\tau \ , \ b_{n\neq 0}^\mu(\tau) = \frac{\mathrm{i}}{2n}\left(\tilde{\alpha}_n^\mu e^{-\mathrm{i}n\tau} - \alpha_{-n}^\mu e^{\mathrm{i}n\tau}\right) \tag{2.39}$$

or, when composed to the coordinate solution,

$$Y_0^\mu = y^\mu + p^\mu\tau + N^\mu\sigma + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_n^\mu e^{-\mathrm{i}n\sigma_+} + \alpha_n^\mu e^{-\mathrm{i}n\sigma_-}\right) \ , \tag{2.40}$$

where the abbreviation $\sigma_\pm = \tau \pm \sigma$ is used. As it is usually done, a decomposition into left-moving and right-moving parts is used in the following sections,

$$Y_0^\mu = Y_{0L}^\mu + Y_{0R}^\mu \ , \ \tilde{Y}_0^\mu = Y_{0L}^\mu - Y_{0R}^\mu \ , \tag{2.41}$$

with

$$\begin{aligned} Y_{0L}^\mu &= y_L^\mu + p_L^\mu\sigma_+ + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\mathrm{i}n\sigma_+} \ , \\ Y_{0R}^\mu &= y_R^\mu + p_R^\mu\sigma_- + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-\mathrm{i}n\sigma_-} \ , \end{aligned} \tag{2.42}$$

and the natural splitting,

$$\begin{aligned} y^\mu &= y_L^\mu + y_R^\mu \ , \ \tilde{y}^\mu = y_L^\mu - y_R^\mu \ , \\ p^\mu &= p_L^\mu + p_R^\mu \ , \ N^\mu = p_L^\mu - p_R^\mu \ . \end{aligned} \tag{2.43}$$

At order $\mathcal{O}(H^1)$, the Fourier expansion (2.34) and (2.35) is plugged into the corresponding equation of motion (2.38),

$$\begin{aligned} \partial_\alpha\partial^\alpha\Big(\sum_{n\in\mathbb{Z}} c_n^\mu(\tau)e^{-\mathrm{i}n\sigma}\Big) &= \theta^\mu{}_{\nu\rho}\ \partial_\alpha Y_0^\nu \partial^\alpha Y_0^\rho \\ &\quad - \lambda^\mu{}_{23}\ \partial_\alpha\partial^\alpha\left(N^3\sigma(Y_0^2 - N^2\sigma) - \frac{1}{2}N^3N^2\sigma(2\pi-\sigma)\right) \ . \end{aligned} \tag{2.44}$$

The symbol $\lambda^\mu{}_{\nu\rho}$ is an abbreviation for $\lambda^1{}_{23} = 1$ and all other components zero. Inserting the zeroth order solution (2.40) gives

$$\begin{aligned} -\sum_{n\in\mathbb{Z}} e^{-\mathrm{i}n\sigma}(n^2c_n^\mu + \partial_\tau^2 c_n^\mu) &= \theta^\mu{}_{\nu\rho}\ (N^\rho N^\nu - p^\rho p^\nu) \\ &\quad - \frac{\theta^\mu{}_{\nu\rho}}{2}\sum_{n\neq 0} e^{-\mathrm{i}n\sigma}\Big(e^{-\mathrm{i}n\tau}\ \left(\tilde{\alpha}_n^\nu(p^\rho - N^\rho) + \tilde{\alpha}_n^\rho(p^\nu - N^\nu)\right) \\ &\qquad\qquad + e^{\mathrm{i}n\tau}\ \left(\alpha_{-n}^\nu(p^\rho + N^\rho) + \alpha_{-n}^\rho(p^\nu + N^\nu)\right)\Big) \\ &\quad - \frac{\theta^\mu{}_{\nu\rho}}{2}\sum_{m,p\neq 0} e^{-\mathrm{i}\sigma(p-m)}\Big(e^{-\mathrm{i}\tau(p+m)}\ \tilde{\alpha}_p^\rho\alpha_m^\nu + e^{\mathrm{i}\tau(p+m)}\ \alpha_{-p}^\rho\tilde{\alpha}_{-m}^\nu\Big) \\ &\quad - \lambda^\mu{}_{23}\ N^3N^2 - \lambda^\mu{}_{23}\ N^3\sum_{n\neq 0} e^{-\mathrm{i}n\sigma}\Big(e^{-\mathrm{i}n\tau}\ \tilde{\alpha}_n^2 - e^{\mathrm{i}n\tau}\ \alpha_{-n}^2\Big) \ . \end{aligned} \tag{2.45}$$

Decomposed into single Fourier modes, this results in differential equations for each coefficient with respect to its τ-dependence,

$$
\begin{aligned}
\partial_\tau^2 c_0^\mu &= \theta^\mu{}_{\nu\rho}\ (p^\rho p^\nu - N^\rho N^\nu) + \lambda^\mu{}_{23}\ N^3 N^2 \\
&\quad + \frac{\theta^\mu{}_{\nu\rho}}{2} \sum_{n\neq 0} \Big(e^{-\mathrm{i}\tau 2n}\ \tilde{\alpha}_n^\rho \alpha_n^\nu + e^{\mathrm{i}\tau 2n}\ \alpha_{-n}^\rho \tilde{\alpha}_{-n}^\nu \Big)\ , \qquad (2.46) \\
n^2 c_n^\mu + \partial_\tau^2 c_n^\mu &= \lambda^\mu{}_{23}\ N^3 \Big(e^{-\mathrm{i}n\tau}\ \tilde{\alpha}_n^2 - e^{\mathrm{i}n\tau}\ \alpha_{-n}^2 \Big) \qquad (2.47) \\
&\quad + \frac{\theta^\mu{}_{\nu\rho}}{2} \Big(e^{-\mathrm{i}n\tau}\ \big(\tilde{\alpha}_n^\nu (p^\rho - N^\rho) + \tilde{\alpha}_n^\rho (p^\nu - N^\nu)\big) \\
&\qquad + e^{\mathrm{i}n\tau}\ \big(\alpha_{-n}^\nu (p^\rho + N^\rho) + \alpha_{-n}^\rho (p^\nu + N^\nu)\big)\Big) \\
&\quad + \frac{\theta^\mu{}_{\nu\rho}}{2} \sum_{p\neq 0,n} \Big(e^{-\mathrm{i}\tau(2p-n)}\ \tilde{\alpha}_p^\rho \alpha_{p-n}^\nu + e^{\mathrm{i}\tau(2p-n)}\ \alpha_{-p}^\rho \tilde{\alpha}_{n-p}^\nu \Big) \\
&\quad \text{for } n \neq 0\ .
\end{aligned}
$$

The last line was obtained by an index shift $n = p - m$. A general solution to these equations is given by

$$
\begin{aligned}
c_0^\mu(\tau) &= y_H^\mu + p_H^\mu\ \tau + \Big(\theta^\mu{}_{\nu\rho}\ (p^\rho p^\nu - N^\rho N^\nu) + \lambda^\mu{}_{23}\ N^3 N^2 \Big)\ \frac{\tau^2}{2} \qquad (2.48) \\
&\quad - \frac{\theta^\mu{}_{\nu\rho}}{8} \sum_{n\neq 0} \frac{1}{n^2} \Big(e^{-\mathrm{i}\tau 2n}\ \tilde{\alpha}_n^\rho \alpha_n^\nu + e^{\mathrm{i}\tau 2n}\ \alpha_{-n}^\rho \tilde{\alpha}_{-n}^\nu \Big)\ , \\
c_n^\mu(\tau) &= \frac{\mathrm{i}}{2n} \Big(\tilde{g}_n^\mu e^{-\mathrm{i}n\tau} - \gamma_{-n}^\mu e^{\mathrm{i}n\tau} \Big) \qquad (2.49) \\
&\quad + \lambda^\mu{}_{23}\ N^3 \frac{\mathrm{i}}{2n}\ \tau\ \Big(e^{-\mathrm{i}n\tau}\ \tilde{\alpha}_n^2 + e^{\mathrm{i}n\tau}\ \alpha_{-n}^2 \Big) \\
&\quad + \theta^\mu{}_{\nu\rho} \frac{\mathrm{i}}{4n}\ \tau\ \Big(e^{-\mathrm{i}n\tau}\ \big(\tilde{\alpha}_n^\nu (p^\rho - N^\rho) + \tilde{\alpha}_n^\rho (p^\nu - N^\nu)\big) \\
&\qquad - e^{\mathrm{i}n\tau}\ \big(\alpha_{-n}^\nu (p^\rho + N^\rho) + \alpha_{-n}^\rho (p^\nu + N^\nu)\big)\Big) \\
&\quad + \theta^\mu{}_{\nu\rho} \frac{1}{2} \sum_{p\neq 0,n} \frac{1}{n^2 - (2p-n)^2} \Big(e^{-\mathrm{i}\tau(2p-n)}\ \tilde{\alpha}_p^\rho \alpha_{p-n}^\nu + e^{\mathrm{i}\tau(2p-n)}\ \alpha_{-p}^\rho \tilde{\alpha}_{n-p}^\nu \Big) \\
&\quad \text{for } n \neq 0\ .
\end{aligned}
$$

There are four new coefficients, a first order position y_H^μ, a first order momentum p_H^μ, and two first order oscillators $\tilde{\gamma}_n^\mu$, γ_n^μ. Having the expansion coefficients c_n^μ at hand, the full solution at order $\mathcal{O}(H^1)$ can be given by employing (2.34), (2.35) and (2.36),

$$
\begin{aligned}
Y_H^\mu(\tau,\sigma) = {} & y_H^\mu + p_H^\mu\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^\mu e^{-\mathrm{i}n\sigma_+} + \gamma_n^\mu e^{-\mathrm{i}n\sigma_-}\right) \\
& + \theta^\mu{}_{\nu\rho}\ (p^\rho p^\nu - N^\rho N^\nu)\ \frac{\tau^2}{2} \\
& + \theta^\mu{}_{\nu\rho}\ \frac{1}{2}\ \tau\ \left(p^\rho Y_0^\nu|_\Sigma - N^\rho \tilde{Y}_0^\nu|_\Sigma + p^\nu Y_0^\rho|_\Sigma - N^\nu \tilde{Y}_0^\rho|_\Sigma\right) \\
& - \theta^\mu{}_{\nu\rho}\ \frac{1}{4}\left(\tilde{Y}_0^\nu|_\Sigma \tilde{Y}_0^\rho|_\Sigma - Y_0^\nu|_\Sigma Y_0^\rho|_\Sigma\right) \\
& + \lambda^\mu{}_{23}\ N^3\left(N^2\ \frac{\tau^2}{2} + \tau\ \tilde{Y}_0^2|_\Sigma + \sigma(Y_0^2 - N^2\sigma) - \frac{1}{2}N^2\sigma(2\pi-\sigma)\right)\ ,
\end{aligned}
\tag{2.50}
$$

with the abbreviation

$$
Y_0^\mu|_\Sigma = \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_n^\mu e^{-\mathrm{i}n\sigma_+} + \alpha_n^\mu e^{-\mathrm{i}n\sigma_-}\right) \tag{2.51}
$$

$$
\tilde{Y}_0^\mu|_\Sigma = \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_n^\mu e^{-\mathrm{i}n\sigma_+} - \alpha_n^\mu e^{-\mathrm{i}n\sigma_-}\right)\ . \tag{2.52}
$$

For obtaining the last but one line, it might be helpful to make use of the following relation,

$$
-\frac{1}{2}\left(\tilde{Y}_0^\nu|_\Sigma \tilde{Y}_0^\rho|_\Sigma - Y_0^\nu|_\Sigma Y_0^\rho|_\Sigma\right) = Y_{0L}^\nu|_\Sigma\ . \tag{2.53}
$$

Although there are many possible reformulations of (2.50), the form shown here makes clear that the solution fulfils the boundary conditions. All terms but the last line are 2π-periodic in σ, whereas the latter is exactly reproducing the last term of (2.21). Nevertheless, it is possible to rewrite (2.50) as

$$
Y_H^\mu(\tau,\sigma) = -\frac{\theta^\mu{}_{\nu\rho}}{4}\left(\tilde{Y}_0^\nu \tilde{Y}_0^\rho - Y_0^\nu Y_0^\rho\right) + f_L^\mu(\sigma_+) + f_R^\mu(\sigma_-)\ , \tag{2.54}
$$

which, together with the relation

$$
-\frac{1}{4}\partial_\alpha\partial^\alpha(\tilde{Y}_0^\nu \tilde{Y}_0^\rho - Y_0^\nu Y_0^\rho) = \partial_\alpha Y_0^\nu \partial^\alpha Y_0^\rho\ , \tag{2.55}
$$

makes clear that the equation of motion (2.38) is solved.

For later use, another rewriting of (2.50) shall be given, namely

$$\begin{aligned}
Y^\mu_H(\tau,\sigma) = {} & y^\mu_H + p^\mu_H\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}^\mu_n e^{-\mathrm{i}n\sigma_+} + \gamma^\mu_n e^{-\mathrm{i}n\sigma_-}\right) \\
& - \theta^\mu{}_{\nu\rho}\ \frac{1}{4}\left(\tilde{Y}^\nu_0\tilde{Y}^\rho_0 - Y^\nu_0 Y^\rho_0\right) - \theta^\mu{}_{\nu\rho}\ \frac{1}{4}(\tilde{y}^\nu\tilde{y}^\rho - y^\nu y^\rho) \\
& + \theta^\mu{}_{\nu\rho}\ \frac{1}{4}\left(\tilde{Y}^\nu_0\tilde{y}^\rho - Y^\nu_0 y^\rho + \tilde{Y}^\rho_0\tilde{y}^\nu - Y^\rho_0 y^\nu\right) \\
& + \theta^\mu{}_{\nu\rho}\ \frac{1}{4}\tau\left(p^\nu(Y^\rho_0 - y^\rho) + p^\rho(Y^\nu_0 - y^\nu) - N^\nu(\tilde{Y}^\rho_0 - \tilde{y}^\rho) - N^\rho(\tilde{Y}^\nu_0 - \tilde{y}^\nu)\right) \\
& + \theta^\mu{}_{\nu\rho}\ \frac{1}{4}\sigma\left(p^\nu(\tilde{Y}^\rho_0 - \tilde{y}^\rho) + p^\rho(\tilde{Y}^\nu_0 - \tilde{y}^\nu) - N^\nu(Y^\rho_0 - y^\rho) - N^\rho(Y^\nu_0 - y^\nu)\right) \\
& - \theta^\mu{}_{\nu\rho}\ \frac{1}{4}(\tau^2+\sigma^2)(p^\nu p^\rho - N^\nu N^\rho) \\
& + \lambda^\mu{}_{23}\ N^3\left(\tau(\tilde{Y}^2_0 - \tilde{y}^2) + \sigma(Y^2_0 - \pi N^2) - \frac{1}{2}(\tau^2+\sigma^2)N^2 - \tau\sigma p^2\right)\ .
\end{aligned} \tag{2.56}$$

2.2.2 Quantisation

Carrying out the canonical quantisation procedure, the following expansion coefficients are promoted to operators,

$$y^\mu, p^\mu, N^\mu, \tilde{\alpha}^\mu_n, \alpha^\mu_n, y^\mu_H, p^\mu_H, \tilde{\gamma}^\mu_n, \gamma^\mu_n\ . \tag{2.57}$$

They have to be arranged into an algebra of commutators such that the canonical equal-τ commutation relations are fulfilled,

$$[Y^\mu(\tau,\sigma), Y^\nu(\tau,\sigma')] = 0 \tag{2.58}$$

$$[\mathcal{P}_\mu(\tau,\sigma), \mathcal{P}_\nu(\tau,\sigma')] = 0 \tag{2.59}$$

$$[Y^\mu(\tau,\sigma), \mathcal{P}_\nu(\tau,\sigma')] = \mathrm{i}\ \delta^\mu_\nu\ \delta(\sigma-\sigma')\ . \tag{2.60}$$

The canonical momentum is given by

$$\begin{aligned}
\mathcal{P}_\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\tau\mathcal{X}^\mu} &= \frac{1}{\pi}\left(G_{\mu\nu}(\mathcal{X})\partial_\tau\mathcal{X}^\nu + B_{\mu\nu}(\mathcal{X})\partial_\sigma\mathcal{X}^\nu\right) \\
&= \frac{1}{\pi}G_{\mu\nu}(Y)\partial_\tau Y^\nu\ ,
\end{aligned} \tag{2.61}$$

with (2.5) and (2.17).

By construction, it is possible to split the analysis of the commutators for the expansion coefficients into two separate orders in the flux parameter H.

Zeroth order - the free string

The metric (2.17) reduces to $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1,1,1)$ at order $\mathcal{O}(H^0)$. Accordingly, the canonical commutation relations become

$$[Y^\mu_0(\tau,\sigma), Y^\nu_0(\tau,\sigma')] = 0 \tag{2.62}$$

$$[\partial_\tau Y^\mu_0(\tau,\sigma), \partial_\tau Y^\nu_0(\tau,\sigma')] = 0 \tag{2.63}$$

$$[Y^\mu_0(\tau,\sigma), \partial_\tau Y^\nu_0(\tau,\sigma')] = \mathrm{i}\pi\ \eta^{\mu\nu}\ \delta(\sigma-\sigma')\ . \tag{2.64}$$

Taking into account the solutions (2.40) for Y_0, one finds the standard [4] commutation algebra

$$\begin{aligned}
&[\tilde{\alpha}^\mu_m, \alpha^\nu_n] = 0 \ , \\
&[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = [\alpha^\mu_m, \alpha^\nu_n] = m\ \delta_{m,-n}\ \eta^{\mu\nu} \ , \\
&[p^\mu, p^\nu] = [N^\mu, N^\nu] = [p^\mu, N^\nu] = [y^\mu, y^\nu] = [y^\mu, N^\nu] = 0 \ , \\
&[y^\mu, p^\nu] = \tfrac{\mathrm{i}}{2}\eta^{\mu\nu} \ , \\
&[\tilde{\alpha}^\mu_m, p^\nu] = [\tilde{\alpha}^\mu_m, N^\nu] = [\alpha^\mu_m, p^\nu] = [\alpha^\mu_m, N^\nu] = [\tilde{\alpha}^\mu_m, y^\nu] = [\alpha^\mu_m, y^\nu] = 0 \ ,
\end{aligned} \tag{2.65}$$

valid for arbitrary $m, n \in \mathbb{Z}^*$. To illustrate the procedure that is applied at linear order $\mathcal{O}(H^1)$, it is now shown how to obtain (2.65). First, a simplifying notation has to be introduced,

$$\epsilon = \begin{cases} +1 \mathrel{\hat{=}} L \\ -1 \mathrel{\hat{=}} R \end{cases} \ , \tag{2.66}$$

and

$$\alpha^\mu_{n\epsilon} = \begin{cases} \tilde{\alpha}^\mu_n & \text{for } \epsilon = +1 \\ \alpha^\mu_n & \text{for } \epsilon = -1 \end{cases} \ . \tag{2.67}$$

This helps to define

$$\partial_{\sigma_\epsilon} = \frac{1}{2}(\partial_\tau + \epsilon\ \partial_\sigma) \ , \tag{2.68}$$

such that, for example,

$$\partial_{\sigma_\epsilon} Y^\mu_0 = p^\mu_\epsilon + \frac{1}{2}\sum_{n\neq 0} \alpha^\mu_{n\epsilon}\ e^{-in\sigma_\epsilon} \ . \tag{2.69}$$

This is a particularly useful quantity, as it contains only two operators. In combination with the derived commutator

$$[\partial_{\sigma_{\epsilon_1}} Y^\mu_0(\tau,\sigma), \partial_{\sigma'_{\epsilon_2}} Y^\nu_0(\tau,\sigma')] = \frac{\mathrm{i}\pi}{4}(\epsilon_1 + \epsilon_2)\eta^{\mu\nu}\partial_\sigma\delta(\sigma - \sigma') \ , \tag{2.70}$$

where all three commutators (2.62)-(2.64) and derivatives of those have been used, one can conclude on commutators of p^μ_ϵ and $\alpha^\mu_{n\epsilon}$. To this end, one inserts (2.69) and (B.1), and identifies the various Fourier coefficients. The results have to be inserted into the commutation relations (2.62)-(2.64) to find all missing relation of (2.65). In particular, it has to be assumed that the position zero mode can be split into a left- and a right-moving part, or more concretely,

$$[y^\mu_{\epsilon_1}, y^\nu_{\epsilon_2}] = [y^\mu_{\epsilon_1}, \alpha^\nu_{n\epsilon_2}] = 0 \ , \ \forall n \neq 0 \ , \tag{2.71}$$

$$[y^\mu_{\epsilon_1}, p^\nu_{\epsilon_2}] = \delta_{\epsilon_1,\epsilon_2}\ \frac{\mathrm{i}}{4}\eta^{\mu\nu} \ . \tag{2.72}$$

[4] cf. (1.4.6) of [10] or (2.2.7)ff. of [9]

First order

To proceed in the same manner as was done for the zeroth order the introduction of a more convenient notation is indicated. For any expression, the addition of $|_H$ singles out only terms of order $\mathcal{O}(H^1)$. The vanishing commutator (2.58), for example, contains two terms at linear order,

$$0 = [Y^\mu(\tau,\sigma), Y^\nu(\tau,\sigma')]|_H = H\ [Y_0^\mu(\tau,\sigma), Y_H^\nu(\tau,\sigma')] + H\ [Y_H^\mu(\tau,\sigma), Y_0^\nu(\tau,\sigma')]\ . \tag{2.73}$$

One has to be careful, not to conclude too quickly on something like

$$0 = [Y^\mu(\tau,\sigma), Y^\nu(\tau,\sigma')]|_H \overset{?}{\sim} [Y_H^\mu(\tau,\sigma), Y_H^\nu(\tau,\sigma)] \tag{2.74}$$

which is of order $\mathcal{O}(H^2)$ and not well-defined in the approximation used here.

To find out the linear order terms of the commutator between two canonical momenta, (2.59), a little more work is required. This is mainly due to the non-trivial off-diagonal terms in the twisted torus metric, as can be seen in

$$[\partial_\tau Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')] = \pi^2\ [G^{\mu\rho}(Y)\mathcal{P}_\rho(\tau,\sigma), G^{\nu\lambda}(Y)\mathcal{P}_\lambda(\tau,\sigma')]\ . \tag{2.75}$$

To expand the right-hand side, one uses the general property of commutators that products behave as

$$[AB, C] = A[B,C] + [A,C]B\ , \tag{2.76}$$

the form of the inverse metric,

$$G^{-1} = \begin{pmatrix} 1 & HY^3 & 0 \\ HY^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(H^2)\ , \tag{2.77}$$

and the last commutator (2.60). Eventually, two directions have a non-trivial right-hand side,

$$[\partial_\tau Y^3(\tau,\sigma), \partial_\tau Y^1(\tau,\sigma')]|_H = -\mathrm{i}\pi H\ \delta(\sigma-\sigma')\ \partial_\tau Y_0^2(\tau,\sigma')\ , \tag{2.78}$$

$$[\partial_\tau Y^3(\tau,\sigma), \partial_\tau Y^2(\tau,\sigma')]|_H = -\mathrm{i}\pi H\ \delta(\sigma-\sigma')\ \partial_\tau Y_0^1(\tau,\sigma')\ , \tag{2.79}$$

$$[\partial_\tau Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')]|_H = 0 \quad \text{for all other } (\mu,\nu)\ . \tag{2.80}$$

The commutator of a coordinate with its τ-derivative can be obtained from (2.60) by multiplying with the inverse metric from the left. As coordinates commute, (2.58), the metric in the canonical momentum (2.61) can be cancelled and as result one has

$$[Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')] = \mathrm{i}\pi\ \delta(\sigma-\sigma')\ G^{\mu\nu}(Y)(\tau,\sigma')\ , \tag{2.81}$$

or

$$[Y^1(\tau,\sigma), \partial_\tau Y^2(\tau,\sigma')]|_H = \mathrm{i}\pi H\ \delta(\sigma-\sigma')\ Y_0^3(\tau,\sigma')\ , \tag{2.82}$$

$$[Y^2(\tau,\sigma), \partial_\tau Y^1(\tau,\sigma')]|_H = \mathrm{i}\pi H\ \delta(\sigma-\sigma')\ Y_0^3(\tau,\sigma')\ , \tag{2.83}$$

$$[Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')]|_H = 0 \quad \text{for all other } (\mu,\nu)\ , \tag{2.84}$$

when written out. Acting with a σ-derivative on these, one finds

$$[\partial_\sigma Y^1(\tau,\sigma), \partial_\tau Y^2(\tau,\sigma')]|_H = [\partial_\sigma Y^2(\tau,\sigma), \partial_\tau Y^1(\tau,\sigma')]|_H = \mathrm{i}\pi H\ \partial_\sigma\delta(\sigma-\sigma')\ Y_0^3(\tau,\sigma')\ . \tag{2.85}$$

Eventually, using (2.73), (2.80) and (2.85), a very useful commutator can be derived,

$$\begin{aligned}[\partial_{\sigma_{\epsilon_1}} Y^1(\tau,\sigma), \partial_{\sigma'_{\epsilon_2}} Y^2(\tau,\sigma')]|_H &= [\partial_{\sigma_{\epsilon_1}} Y^2(\tau,\sigma), \partial_{\sigma'_{\epsilon_2}} Y^1(\tau,\sigma')]|_H \\ &= \frac{\mathrm{i}\pi}{4} H\ \partial_\sigma \delta(\sigma-\sigma')\ \Big(\epsilon_1\ Y_0^3(\tau,\sigma') + \epsilon_2\ Y_0^3(\tau,\sigma)\Big)\ ,\end{aligned} \tag{2.86}$$

which is the linear order analogue to (2.70). The symbol ϵ is used in the same way.

The next step in the quantisation procedure at linear order is to identify expressions, analogous to (2.69) at zeroth order, that extract as few expansion coefficients as possible. When choosing these expressions, one also has to take into account that only combinations of commutators, as in (2.73), are determined. As it becomes clear after some steps in the derivation, such a useful expression is given by

$$\begin{aligned}\Pi^\mu_\epsilon := \partial_{\sigma_\epsilon} Y^\mu_H &+ \frac{1}{2}\partial_{\sigma_\epsilon} Y^\nu_{0\epsilon}\Big(-\theta^\mu{}_{\nu\rho} Y^\rho_{0(-\epsilon)} + \theta^\mu{}_{\nu\rho}(y^\rho_{-\epsilon} - \sigma_\epsilon p^\rho_{-\epsilon}) - 2\lambda^\mu{}_{\nu\rho}\sigma_\epsilon(p^\rho_\epsilon - p^\rho_{-\epsilon})\Big) \\ &+ \frac{1}{2}\partial_{\sigma_\epsilon} Y^\rho_{0\epsilon}\Big(-\theta^\mu{}_{\nu\rho} Y^\nu_{0(-\epsilon)} + \theta^\mu{}_{\nu\rho}(y^\nu_{-\epsilon} - \sigma_\epsilon p^\nu_{-\epsilon})\Big)\ .\end{aligned} \tag{2.87}$$

It shows the following dependence on the particular expansion coefficients,

$$\begin{aligned}\Pi^\mu_\epsilon = \frac{p^\mu_H}{2} &+ \lambda^\mu{}_{23}\frac{N^3\epsilon}{2}\ (y^2 - \pi N^2) \\ &+ \frac{1}{2}\sum_{n\neq 0} e^{-\mathrm{i}n\sigma_\epsilon}\left(\gamma^\mu_{n\epsilon} + \theta^\mu{}_{\nu\rho}\frac{\mathrm{i}}{n}\ p^{(\nu}_{-\epsilon}\alpha^{\rho)}_{n\epsilon} + \lambda^\mu{}_{\nu\rho}N^\rho\epsilon\frac{\mathrm{i}}{n}\ \alpha^\nu_{n\epsilon}\right)\ .\end{aligned} \tag{2.88}$$

To obtain this result, it was used that the symbol ϵ allows for the following manipulations,

$$\tilde{y}^2 = -\epsilon y^2 + 2\epsilon y^2_\epsilon\ , \tag{2.89}$$

and

$$N^3 = \epsilon(p^3_\epsilon - p^3_{-\epsilon})\ . \tag{2.90}$$

Furthermore, by definition of the symbol λ, it is,

$$\lambda^\mu{}_{23} N^3 Y^2_{0\epsilon} = \lambda^\mu{}_{\nu\rho} N^\rho Y^\nu_{0\epsilon}\ . \tag{2.91}$$

The structure of (2.73) is taken into account by considering the combination

$$[H\ \Pi^\mu_{\epsilon_1}(\tau,\sigma), \partial_{\sigma'_{\epsilon_2}} Y^\nu_0(\tau,\sigma')] + [\partial_{\sigma_{\epsilon_1}} Y^\mu_0(\tau,\sigma), H\ \Pi^\nu_{\epsilon_2}(\tau,\sigma')]\ . \tag{2.92}$$

It will be evaluated in two ways, similar to the procedure at zeroth order. First, the result (2.88) can be plugged in. That will result in a mode expansion with particular commutators as coefficients. Second, the definition (2.87) and all derived canonical commutators, in particular (2.86), are plugged in. The result, again, is a mode expansion with zeroth order operators as coefficients. Eventually, a matching between these two results gives the desired commutators.

From here on, all commutators are only evaluated in the fibre directions, i.e. all variables μ and ν become placeholders for the particular values $(\mu,\nu) = (1,2)$ or $(2,1)$. As it was mentioned in the introduction of this chapter, such a restriction is possible, as the following investigation focuses on finding non-commutativity at least in one particular direction and does not aim at classifying all possible commutators.

To start with the first evaluation, (2.88), (2.69) and also the zeroth order commutators (2.65) are plugged in,

$$
\begin{aligned}
&[H\ \Pi^\mu_{\epsilon_1}(\tau,\sigma),\partial_{\sigma'_{\epsilon_2}}Y^\nu_0(\tau,\sigma')]+[\partial_{\sigma_{\epsilon_1}}Y^\mu_0(\tau,\sigma),H\ \Pi^\nu_{\epsilon_2}(\tau,\sigma')] \qquad (2.93)\\
&=\frac{H}{2}\Bigg([p^\mu_H,p^\nu_{\epsilon_2}]-[p^\nu_H,p^\mu_{\epsilon_1}]+\frac{\mathrm{i}N^3}{4}(\lambda^\mu{}_{23}\epsilon_1-\lambda^\nu{}_{23}\epsilon_2)+\frac{\mathrm{i}N^3\epsilon_1}{2}\ \delta_{\epsilon_1,\epsilon_2}\ (\lambda^\mu{}_{23}-\lambda^\nu{}_{23})\sum_{m\neq 0}e^{-\mathrm{i}m\epsilon_1(\sigma-\sigma')}\\
&\quad+\sum_{m\neq 0}e^{-\mathrm{i}m\sigma_{\epsilon_1}}\left([\gamma^\mu_{m\epsilon_1},p^\nu_{\epsilon_2}]-\frac{1}{2}[p^\nu_H,\alpha^\mu_{m\epsilon_1}]\right)-\sum_{n\neq 0}e^{-\mathrm{i}n\sigma'_{\epsilon_2}}\left([\gamma^\nu_{n\epsilon_2},p^\mu_{\epsilon_1}]-\frac{1}{2}[p^\mu_H,\alpha^\nu_{n\epsilon_2}]\right)\\
&\quad+\frac{1}{2}\sum_{m,n\neq 0}e^{-\mathrm{i}((m+n)\tau+m\epsilon_1\sigma+n\epsilon_2\sigma')}\Big([\gamma^\mu_{m\epsilon_1},\alpha^\nu_{n\epsilon_2}]-[\gamma^\nu_{n\epsilon_2},\alpha^\mu_{m\epsilon_1}]\Big)\Bigg)\ .
\end{aligned}
$$

The second evaluation uses the definition (2.87) and many of the results so far obtained, and gives

$$
\begin{aligned}
&[H\ \Pi^\mu_{\epsilon_1}(\tau,\sigma),\partial_{\sigma'_{\epsilon_2}}Y^\nu_0(\tau,\sigma')]+[\partial_{\sigma_{\epsilon_1}}Y^\mu_0(\tau,\sigma),H\ \Pi^\nu_{\epsilon_2}(\tau,\sigma')] \qquad (2.94)\\
&\qquad\qquad-\frac{\mathrm{i}\pi}{4}HN^3\ (\sigma'-\sigma)\ \partial_\sigma\delta(\sigma-\sigma')\ (\lambda^\mu{}_{23}\epsilon_1-\lambda^\nu{}_{23}\epsilon_2)\\
&=\ [H\ \Pi^1_{\epsilon_1}(\tau,\sigma),\partial_{\sigma'_{\epsilon_2}}Y^2_0(\tau,\sigma')]+[\partial_{\sigma_{\epsilon_1}}Y^1_0(\tau,\sigma),H\ \Pi^2_{\epsilon_2}(\tau,\sigma')]-\frac{\mathrm{i}\pi}{4}HN^3\ (\sigma'-\sigma)\ \partial_\sigma\delta(\sigma-\sigma')\epsilon_1\\
&=\ [H\ \Pi^2_{\epsilon_1}(\tau,\sigma),\partial_{\sigma'_{\epsilon_2}}Y^1_0(\tau,\sigma')]+[\partial_{\sigma_{\epsilon_1}}Y^2_0(\tau,\sigma),H\ \Pi^1_{\epsilon_2}(\tau,\sigma')]+\frac{\mathrm{i}\pi}{4}HN^3\ (\sigma'-\sigma)\ \partial_\sigma\delta(\sigma-\sigma')\epsilon_2\\
&=\ \frac{\mathrm{i}H}{8}\delta_{\epsilon_1,-\epsilon_2}\Big(1-2\pi\ \delta(\sigma-\sigma')\Big)\Big(\partial_{\sigma_{\epsilon_1}}Y^3_0(\tau,\sigma)-\partial_{\sigma'_{-\epsilon_1}}Y^3_0(\tau,\sigma')\Big)\\
&+\frac{\mathrm{i}H}{4}\pi\Big(\delta_{\epsilon_1,\epsilon_2}\ \epsilon_1\ \big(2y^3-Y^3_{0-\epsilon_1}|_\Sigma(\tau,\sigma)-Y^3_{0-\epsilon_1}|_\Sigma(\tau,\sigma')\big)+\epsilon_1Y^3_0|_\Sigma(\tau,\sigma')+\epsilon_2Y^3_0|_\Sigma(\tau,\sigma)\Big)\partial_\sigma\delta(\sigma-\sigma')\ .
\end{aligned}
$$

Using the property (B.3) discussed in the appendix, the commutator can be brought to the useful form

$$
\begin{aligned}
&[H\ \Pi^\mu_{\epsilon_1}(\tau,\sigma),\partial_{\sigma'_{\epsilon_2}}Y^\nu_0(\tau,\sigma')]+[\partial_{\sigma_{\epsilon_1}}Y^\mu_0(\tau,\sigma),H\ \Pi^\nu_{\epsilon_2}(\tau,\sigma')] \qquad (2.95)\\
&=\frac{H}{2}\Bigg(\frac{\mathrm{i}N^3}{4}\ (\lambda^\mu{}_{23}\epsilon_1-\lambda^\nu{}_{23}\epsilon_2)+\frac{\mathrm{i}N^3}{4}\delta_{\epsilon_1,\epsilon_2}\epsilon_1\ (\lambda^\mu{}_{23}-\lambda^\nu{}_{23})\sum_{n\neq 0}e^{-\mathrm{i}n(\sigma-\sigma')}\\
&\quad-\ \frac{\mathrm{i}}{4}\delta_{\epsilon_1,-\epsilon_2}\Big(\sum_{n\neq 0}e^{-\mathrm{i}n(\sigma-\sigma')}\Big)\Big(\partial_{\sigma_{\epsilon_1}}Y^3_0|_\Sigma(\tau,\sigma)-\partial_{\sigma'_{-\epsilon_1}}Y^3_0|_\Sigma(\tau,\sigma')\Big)\\
&\quad+\frac{\mathrm{i}\pi\epsilon_1}{2}\ \delta_{\epsilon_1,-\epsilon_2}\ \Big(Y^3_0|_\Sigma(\tau,\sigma')-Y^3_0|_\Sigma(\tau,\sigma)\Big)\partial_\sigma\delta(\sigma-\sigma')\\
&\quad+\frac{\mathrm{i}\pi\epsilon_1}{2}\ \delta_{\epsilon_1,\epsilon_2}\ \Big(2y^3+Y^3_{0\epsilon_1}|_\Sigma(\tau,\sigma)+Y^3_{0\epsilon_1}|_\Sigma(\tau,\sigma')\Big)\partial_\sigma\delta(\sigma-\sigma')\Bigg)\ .
\end{aligned}
$$

It is now possible to match both evaluations (2.93) and (2.95). This will be done in two steps where τ-independent and τ-dependent terms are matched separately. The former results in two equations,

$$[p_H^1, p_{\epsilon_2}^2] = [p_H^2, p_{\epsilon_1}^1] \quad \forall\ \epsilon_1,\ \epsilon_2\ , \tag{2.96}$$

$$\begin{aligned}\frac{\epsilon_1}{2}\ \delta_{\epsilon_1,\epsilon_2}\ y^3 \sum_{m\neq 0} m\ e^{-\mathrm{i}m(\sigma-\sigma')} &= \frac{\mathrm{i}N^3\epsilon_1}{4}\ \delta_{\epsilon_1,\epsilon_2}\ (\lambda^\mu{}_{23} - \lambda^\nu{}_{23}) \sum_{m\neq 0} e^{-\mathrm{i}m(\sigma-\sigma')} \\ &+ \frac{1}{2}\sum_{m\neq 0} e^{-\mathrm{i}m(\epsilon_1\sigma-\epsilon_2\sigma')}\Big([\gamma^\mu_{m\epsilon_1}, \alpha^\nu_{-m\epsilon_2}] - [\gamma^\nu_{-m\epsilon_2}, \alpha^\mu_{m\epsilon_1}]\Big)\ .\end{aligned} \tag{2.97}$$

The last equality can be viewed as a Fourier series in $(\sigma - \sigma')$ and is satisfied by

$$[\gamma^\mu_{m\epsilon_1}, \alpha^\nu_{-m\epsilon_2}] - [\gamma^\nu_{-m\epsilon_2}, \alpha^\mu_{m\epsilon_1}] = \delta_{\epsilon_1,\epsilon_2}\left(y^3 m - \frac{\mathrm{i}N^3\epsilon_1}{2}(\lambda^\mu{}_{23} - \lambda^\nu{}_{23})\right)\ , \tag{2.98}$$

for all $m \neq 0$. Recalling that the whole analysis is restricted to $(\mu,\nu) = (1,2)$ or $(2,1)$, it can be deduced that

$$[\gamma^1_{m\epsilon_1}, \alpha^2_{-m\epsilon_2}] - [\gamma^2_{-m\epsilon_2}, \alpha^1_{m\epsilon_1}] = \delta_{\epsilon_1,\epsilon_2}\left(y^3 m - \frac{\mathrm{i}N^3\epsilon_1}{2}\right)\ . \tag{2.99}$$

Turning to the τ-dependent terms, one first has to notice the rewriting

$$\begin{aligned}&-\ \frac{\mathrm{i}}{4}\delta_{\epsilon_1,-\epsilon_2}\Big(\sum_{n\neq 0} e^{-\mathrm{i}n(\sigma-\sigma')}\Big)\Big(\partial_{\sigma_{\epsilon_1}} Y_0^3|_\Sigma(\tau,\sigma) - \partial_{\sigma'_{-\epsilon_1}} Y_0^3|_\Sigma(\tau,\sigma')\Big) \qquad (2.100)\\ &+ \frac{\mathrm{i}\pi\epsilon_1}{2}\left(\delta_{\epsilon_1,-\epsilon_2}\ \Big(Y_0^3|_\Sigma(\tau,\sigma') - Y_0^3|_\Sigma(\tau,\sigma)\Big) + \delta_{\epsilon_1,\epsilon_2}\ \Big(Y^3_{0\epsilon_1}|_\Sigma(\tau,\sigma) + Y^3_{0\epsilon_1}|_\Sigma(\tau,\sigma')\Big)\right)\ \partial_\sigma\delta(\sigma-\sigma') \\ &= \frac{\mathrm{i}}{8}\sum_{\substack{n,k\neq 0\\ k\neq -n}} e^{-\mathrm{i}(k+n)\tau} e^{-\mathrm{i}\epsilon_1 k\sigma} e^{-\mathrm{i}\epsilon_1 n\sigma'} \alpha^3_{(k+n)\epsilon_1}\ \frac{k-n}{k+n}\ \delta_{\epsilon_1,\epsilon_2} + \frac{\mathrm{i}}{8}\sum_{m\neq 0}\Big(e^{-\mathrm{i}m\sigma_{\epsilon_1}}\alpha^3_{m\epsilon_1} - e^{-\mathrm{i}m\sigma'_{\epsilon_2}}\alpha^3_{m\epsilon_2}\Big)\ .\end{aligned}$$

Matching this with the τ-dependent terms in (2.93) determines the following commutators,

$$\forall\ \epsilon_1,\epsilon_2,\ \forall\ m\neq 0,\ \forall\ n,k\neq 0, k+n\neq 0\ ,$$

$$[\gamma^1_{m\epsilon_1}, p^2_{\epsilon_2}] - \frac{1}{2}[p^2_H, \alpha^1_{m\epsilon_1}] = [\gamma^2_{m\epsilon_1}, p^1_{\epsilon_2}] - \frac{1}{2}[p^1_H, \alpha^2_{m\epsilon_1}] = \frac{\mathrm{i}}{8}\alpha^3_{m\epsilon_1}\ , \tag{2.101}$$

$$[\gamma^1_{k\epsilon_1}, \alpha^2_{n\epsilon_2}] - [\gamma^2_{n\epsilon_2}, \alpha^1_{k\epsilon_1}] = \frac{\mathrm{i}}{4}\frac{k-n}{k+n}\ \delta_{\epsilon_1,\epsilon_2}\ \alpha^3_{(k+n)\epsilon_1}\ . \tag{2.102}$$

As ϵ_1 and ϵ_2 can be chosen arbitrarily, (2.96) and (2.101) lead to

$$[p^1_H, N^2] = [p^2_H, N^1] = [p^1_H, p^2] - [p^2_H, p^1] = 0 \tag{2.103}$$

$$[\gamma^1_{m\epsilon_1}, N^2] = [\gamma^2_{m\epsilon_1}, N^1] = 0 \tag{2.104}$$

$$[\gamma^1_{m\epsilon_1}, p^2] - [p^2_H, \alpha^1_{m\epsilon_1}] = [\gamma^2_{m\epsilon_1}, p^1] - [p^1_H, \alpha^2_{m\epsilon_1}] = \frac{\mathrm{i}}{4}\alpha^3_{m\epsilon_1}\ . \tag{2.105}$$

for all ϵ_1 and $m \neq 0$.

Not all commutators of the expansion coefficients have been determined so far. In particular, there is no information of how y^μ_H commute with other operators. The reason is that strictly speaking it is still unclear whether the full algebra of canonical commutators can be consistently fulfilled, as until here only the combination (2.92) has been investigated. In the following, the canonical commutators and their τ-derivative forms will be fed with the above results to check consistency and to find missing coefficient commutators, in particular from the zero-modes.

The first canonical commutator to be looked at is $[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$ in the form (2.80), where for $(\mu, \nu) = (1, 2)$ it reads

$$\begin{aligned}
0 &= \frac{1}{H}[\partial_\tau Y^1(\tau,\sigma), \partial_\tau Y^2(\tau,\sigma')]|_H \\
&= \frac{\mathrm{i}}{16} \sum_{n,k\neq 0, k\neq -n} \frac{k-n}{k+n} \left(e^{-\mathrm{i}(k\sigma_+ + n\sigma'_+)} \tilde{\alpha}^3_{k+n} + e^{-\mathrm{i}(k\sigma_- + n\sigma'_-)} \alpha^3_{k+n} \right) \\
&\quad - \frac{\mathrm{i}\pi}{4} \left(\tilde{Y}^3_0|_\Sigma(\tau,\sigma') + \tilde{Y}^3_0|_\Sigma(\tau,\sigma) \right) \partial_\sigma \delta(\sigma-\sigma') \\
&\quad + \frac{\mathrm{i}}{4} \left(\partial_\tau Y^3_0(\tau,\sigma) - \partial_\tau Y^3_0(\tau,\sigma') \right) \left(\pi\ \delta(\sigma-\sigma') + \frac{1}{2} \right) .
\end{aligned} \tag{2.106}$$

The right-hand side has been obtained by using a rewriting (2.56) of the H-order solution, several zeroth order commutators and commutators obtained above. The whole equation is satisfied automatically, as can be seen as follows. First, by summing the two equations (2.100) for $\epsilon_1 = \epsilon_2 = \pm 1$ one obtains the relation

$$\begin{aligned}
&\frac{\mathrm{i}}{8} \sum_{n,k\neq 0, k\neq -n} \frac{k-n}{k+n} \left(e^{-\mathrm{i}(k\sigma_+ + n\sigma'_+)} \tilde{\alpha}^3_{k+n} + e^{-\mathrm{i}(k\sigma_- + n\sigma'_-)} \alpha^3_{k+n} \right) \\
&= -\frac{\mathrm{i}}{4} \left(\partial_\tau Y^3_0(\tau,\sigma) - \partial_\tau Y^3_0(\tau,\sigma') \right) + \frac{\mathrm{i}\pi}{2} \left(\tilde{Y}^3_0|_\Sigma(\tau,\sigma') + \tilde{Y}^3_0|_\Sigma(\tau,\sigma) \right) \partial_\sigma \delta(\sigma-\sigma') .
\end{aligned} \tag{2.107}$$

Second, by taking into account a property of the δ-distribution given in (B.6), one can find

$$\left(\partial_\tau Y^3_0(\tau,\sigma) - \partial_\tau Y^3_0(\tau,\sigma') \right) \delta(\sigma-\sigma') = 0 . \tag{2.108}$$

By using these two relations, (2.106) can be confirmed. No new commutators are obtained from it.

The second canonical commutator to be looked at is $[Y^\mu, \mathcal{P}_\nu]$ in the form (2.82), where after plugging in (2.56) and several zeroth order commutators the following relation for $(\mu, \nu) = (1, 2)$ or $(2, 1)$ can be found,

$$\begin{aligned}
\frac{1}{H}[Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')]|_H &= [Y^\mu_0(\tau,\sigma), p^\nu_H + \frac{1}{2}\sum_{n\neq 0} \left(\tilde{g}^\nu_n e^{-\mathrm{i}n\sigma'_+} + \gamma^\nu_n e^{-\mathrm{i}n\sigma'_-} \right)] \\
&+ [y^\mu_H + p^\mu_H\ \tau + \frac{\mathrm{i}}{2} \sum_{m\neq 0} \frac{1}{m} \left(\tilde{g}^\mu_m e^{-\mathrm{i}m\sigma_+} + \gamma^\mu_m e^{-\mathrm{i}m\sigma_-} \right), \partial_\tau Y^\nu_0(\tau,\sigma')] \\
&+ \frac{1}{8} \sum_{n\neq 0} \frac{1}{n} e^{-\mathrm{i}n(\sigma'-\sigma)} \left(\partial_\tau \tilde{Y}^3_0(\tau,\sigma') + N^3 - 4N^3 \lambda^\nu{}_{23} \right) \\
&+ \frac{\mathrm{i}\pi}{4} \delta(\sigma-\sigma') \left(Y^3_0(\tau,\sigma) + Y^3_0(\tau,\sigma') - 2y^3 + 2p^3\tau - N^3(\sigma+\sigma') + 4N^3(\lambda^\mu{}_{23}\sigma + \lambda^\nu{}_{23}\sigma') \right) \\
&- \frac{\mathrm{i}}{8} \left(Y^3_0(\tau,\sigma) - Y^3_0(\tau,\sigma') + N^3(\sigma'-\sigma) - 2\tau\partial_\tau Y^3_0(\tau,\sigma') + 2p^3\tau \right) .
\end{aligned} \tag{2.109}$$

The right-hand side can be further simplified by using first order commutators that have been obtained so far and the δ-distribution property (B.4) as well as

$$\Big(Y_0^3|_\Sigma(\tau,\sigma) - Y_0^3|_\Sigma(\tau,\sigma')\Big)\ \delta(\sigma-\sigma') = 0\ , \tag{2.110}$$

which again follows from (B.6). In total, the following relation has to be fulfilled,

$$\begin{aligned}
\mathrm{i}\pi\ \delta(\sigma-\sigma')\ Y_0^3(\tau,\sigma') &= \frac{1}{H}[Y^\mu(\tau,\sigma),\partial_\tau Y^\nu(\tau,\sigma')]|_H \\
&= [y^\mu, p_H^\nu] + [y_H^\mu, p^\nu] \\
&\quad + \frac{1}{2}\sum_{n\neq 0}\Big(e^{-\mathrm{i}n\sigma'_+}([y^\mu,\tilde{g}_n^\nu] + [y_H^\mu,\tilde{\alpha}_n^\nu]) + e^{-\mathrm{i}n\sigma'_-}\left([y^\mu,\gamma_n^\nu] + [y_H^\mu,\alpha_n^\nu]\right)\Big) \\
&\quad - \frac{\mathrm{i}}{8}\left(4y^3 + Y_0^3|_\Sigma(\tau,\sigma')\right) \\
&\quad + \mathrm{i}\pi\ \delta(\sigma-\sigma')\ Y_0^3(\tau,\sigma')\ .
\end{aligned} \tag{2.111}$$

This can be achieved by setting

$$[y^1, p_H^2] + [y_H^1, p^2] = [y^2, p_H^1] + [y_H^2, p^1] = \frac{\mathrm{i}}{2}y^3\ , \tag{2.112}$$

$$[y^1, \gamma^2_{n\epsilon_1}] + [y_H^1, \alpha^2_{n\epsilon_1}] = [y^2, \gamma^1_{n\epsilon_1}] + [y_H^2, \alpha^1_{n\epsilon_1}] = -\frac{1}{8n}\alpha^3_{n\epsilon_1}\ , \tag{2.113}$$

for $\forall\epsilon_1$, $\forall n\neq 0$.

The last canonical commutator to be looked at is $[Y^\mu, Y^\nu]$ in the form (2.73). Its two parts are given by

$$\begin{aligned}
[Y_0^1(\tau,\sigma), Y_H^2(\tau,\sigma')] &= [Y_0^1(\tau,\sigma), y_H^2 + p_H^2\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} + \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\Big)] \\
&\quad + \frac{\mathrm{i}}{4}\tau(Y_0^3|_\Sigma(\tau,\sigma') + p^3\tau) \\
&\quad + \frac{1}{8}(\tilde{Y}_0^3|_\Sigma(\tau,\sigma') + 2N^3\tau)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)}\ ,
\end{aligned} \tag{2.114}$$

and

$$\begin{aligned}
[Y_H^1(\tau,\sigma), Y_0^2(\tau,\sigma')] &= [y_H^1 + p_H^1\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), Y_0^2(\tau,\sigma')] \\
&\quad - \frac{\mathrm{i}}{4}\tau(Y_0^3|_\Sigma(\tau,\sigma) + p^3\tau) \\
&\quad + \frac{1}{8}(\tilde{Y}_0^3|_\Sigma(\tau,\sigma) - 2N^3\tau)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)}\ .
\end{aligned} \tag{2.115}$$

The sum of these two expressions can be simplified by using some of the first order commutators obtained so far. An intermediate result is given by

$$
\begin{aligned}
0 &= [Y_0^1(\tau,\sigma), Y_H^2(\tau,\sigma')] + [Y_H^1(\tau,\sigma), Y_0^2(\tau,\sigma')] \\
&= [y^1, y_H^2] - [y^2, y_H^1] + \sigma[N^1, y_H^2] - \sigma'[N^2, y_H^1] \\
&\quad + \tau\left([y^1, p_H^2] - [y^2, p_H^1] + [p^1, y_H^2] - [p^2, y_H^1]\right) \\
&\quad + \frac{i}{4}\tau\Bigg(Y_0^3|_\Sigma(\tau,\sigma') - Y_0^3|_\Sigma(\tau,\sigma) + 2\sum_{n\neq 0}\frac{1}{n}\Big([p^1, (\tilde{g}_n^2 e^{-in\sigma'_+} + \gamma_n^2 e^{-in\sigma'_-})] \\
&\qquad - [p^2, (\tilde{g}_n^1 e^{-in\sigma_+} + \gamma_n^1 e^{-in\sigma_-})] + [p_H^1, (\tilde{\alpha}_n^2 e^{-in\sigma'_+} + \alpha_n^2 e^{-in\sigma'_-})] \\
&\qquad - [p_H^2, (\tilde{\alpha}_n^1 e^{-in\sigma_+} + \alpha_n^1 e^{-in\sigma_-})]\Big)\Bigg) \\
&\quad + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big([y^1, (\tilde{g}_n^2 e^{-in\sigma'_+} + \gamma_n^2 e^{-in\sigma'_-})] - [y^2, (\tilde{g}_n^1 e^{-in\sigma_+} + \gamma_n^1 e^{-in\sigma_-})] \\
&\qquad + [y_H^1, (\tilde{\alpha}_n^2 e^{-in\sigma'_+} + \alpha_n^2 e^{-in\sigma'_-})] - [y_H^2, (\tilde{\alpha}_n^1 e^{-in\sigma_+} + \alpha_n^1 e^{-in\sigma_-})]\Big) \\
&\quad + \frac{\mathrm{i}}{16}\sum_{n\neq 0}\frac{1}{n^2}\left(\tilde{\alpha}_n^3(e^{-in\sigma'_+} - e^{-in\sigma_+}) + \alpha_n^3(e^{-in\sigma'_-} - e^{-in\sigma_-})\right) \ .
\end{aligned}
\tag{2.116}
$$

It can be simplified by using (2.101) to cancel the terms in $\frac{\mathrm{i}}{4}\tau$, by using (2.113) to cancel the last three rows, and by using (2.112) to cancel the terms in the third row. The resulting condition is

$$
\begin{aligned}
0 &= \tfrac{1}{H}[Y^1(\tau,\sigma), Y^2(\tau,\sigma')]|_H \\
&= [Y_0^1(\tau,\sigma), Y_H^2(\tau,\sigma')] + [Y_H^1(\tau,\sigma), Y_0^2(\tau,\sigma')] \\
&= [y^1, y_H^2] - [y^2, y_H^1] + \sigma[N^1, y_H^2] - \sigma'[N^2, y_H^1] \ ,
\end{aligned}
\tag{2.117}
$$

that can be fulfilled by setting

$$
\begin{aligned}
&[y^1, y_H^2] - [y^2, y_H^1] = 0 \\
&[N^1, y_H^2] = [N^2, y_H^1] = 0 \ .
\end{aligned}
\tag{2.118}
$$

This ends the derivation of the commutators in the $(\mu,\nu) = (1,2)$ and $(2,1)$ direction, as all canonical commutators have been used. It shall be emphasised once more, that the given relations are only sufficient but not at all necessary. They only provide one possible set of commutators to fulfill the canonical ones, others might as well be possible. Furthermore, there are commutators left undetermined as many relations only fix sums of commutators. In a sense, the whole quantisation procedure is doomed to remain incomplete, because neither an extension to higher orders nor processing other directions than $(1,2)$ can deliver more information on these fixed sums. For example, seeking commutation relations at second order will only increase the number of terms in such sums,

$$
[Y^1, Y^2]|_{H^2} = H^2\Big([Y_0^1, Y_{H^2}^2] + [Y_{H^2}^1, Y_0^2] + [Y_H^1, Y_H^2]\Big) \ . \tag{2.119}
$$

As it will turn out later, the relations obtained here are sufficient to discover some properties of the commutators in the non-geometric frame related by T-duality, though.

2.3 The non-geometric frame

This section shows how to obtain coordinate solutions Z^μ in the Q-flux frame by integrating the worldsheet T-duality rules for going from one frame to the dual one. Furthermore, it is shown to what extent the commutator of two such coordinates can be computed using the quantisation results from the twisted torus frame, but no further assumptions. In order to do so, it is of particular importance that T-duality as it is used in the following allows to express the coordinates Z^μ in terms of the expansion coefficients of the twisted torus coordinates Y^μ. As will be discussed at the end of this section, there will, nevertheless, appear integration constants whose commutation relations are not determinable. Their values can be fixed by employing various physical arguments, which will be done in the next section.

As has been argued in the introduction to this chapter, it is a priori not clear what the canonical commutators in the non-geometric frame should be. The aim of this section thus is to determine the coordinate commutators from canonical commutators in the geometric frame of the twisted torus. Whether T-duality is capable of providing such a relation can be questioned, and in particular it might be considered as a working hypothesis that the integration of the T-duality rules can prepare the correct behaviour of the coordinates Z^μ. One argument in favour of such a proceeding is that the T-duality rules correctly map one set of canonical commutators in the torus with H-flux frame to the set of canonical commutators in the twisted torus frame. At order $\mathcal{O}(H^1)$, one finds

$$\begin{array}{lcl} [\partial_\sigma X^\mu(\tau,\sigma), \partial_{\sigma'} X^\nu(\tau,\sigma')]|_H & & [\partial_\sigma Y^\mu(\tau,\sigma), \partial_{\sigma'} Y^\nu(\tau,\sigma')]|_H \\ [\partial_\sigma X^\mu(\tau,\sigma), \partial_\tau X^\nu(\tau,\sigma')]|_H & \Longleftrightarrow & [\partial_\sigma Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')]|_H \\ [\partial_\tau X^\mu(\tau,\sigma), \partial_\tau X^\nu(\tau,\sigma')]|_H & & [\partial_\tau Y^\mu(\tau,\sigma), \partial_\tau Y^\nu(\tau,\sigma')]|_H \ . \end{array} \tag{2.120}$$

An example of how to proof this will be given in (2.125).

On immediate question is, whether the T-duality rules can directly map the commutators $[Y^1, Y^2]$ and $[Z^1, Z^2]$ into each other. This would make the whole procedure of finding classical solutions superfluous. In fact, it turns out that such a mapping is impossible. T-duality, as has been mentioned already, only relates derivatives of coordinates, which is also the reason for the above mapping to contain only commutators of σ- and τ-derivatives of coordinates.

In summary, it can be stated that T-duality allows to map certain commutators correctly, and therefore it will be taken for granted that it also allows to construct all other commutators from mapped solutions.

2.3.1 Classical solutions

To obtain the classical solution for the string coordinate fields $Z^\mu(\tau,\sigma)$ it is necessary to solve both the equations of motion and the boundary conditions. At first sight, it seems feasible to directly solve the equations of motion, as they formally are identical to the equations of motion in the frame X^μ, up to linear order in H. This can be seen from the respective target space fields, which for the non-geometric frame can be computed by applying the T-duality rules[5] (A.8) in $\mu = 2$ direction. As discussed in the introduction, a non-trivial denominator

[5]Again, it has to be noted that due conventions a sign change in the off-diagonal components occurs.

appears and turns the configuration non-geometric:

$$G = f \begin{pmatrix} \frac{1}{R_1^2} & 0 & 0 \\ 0 & \frac{1}{R_2^2} & 0 \\ 0 & 0 & \frac{R_3^2}{f} \end{pmatrix} , \quad B = f \begin{pmatrix} 0 & -\frac{HZ^3}{R_1^2 R_2^2} & 0 \\ \frac{HZ^3}{R_1^2 R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{2.121}$$

with

$$f = \left(1 + \left(\frac{HZ^3}{R_1 R_2} \right)^2 \right)^{-1} . \tag{2.122}$$

Applying the rescaling (2.15) and expanding only up to order $\mathcal{O}(H^1)$ yields

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(H^2) , \quad B = \begin{pmatrix} 0 & -HZ^3 & 0 \\ HZ^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(H^2) . \tag{2.123}$$

Up to a sign in the B-field, these fields agree to (2.8) after rescaling.

Although it seems that the expansion in H removes the ill-definedness and allows for a direct solution, it is at this stage completely unclear, what the boundary conditions for the coordinates have to be. There are arguments that determine them in a rather sketchy manner, cf. [2], appendix C. But they are not sufficient to clarify the exact expressions in terms of the notation established here, as can be seen by comparing the result of the following analysis (2.147) with the suggested boundary conditions (2.148). In particular, the result (2.147) does not at all match with the boundary conditions for the torus with H-flux frame (2.201), which once more shows that non-geometry can appear in global features despite locally equivalent setups.

To evade the question of how the boundary conditions can be determined properly, the guideline to find classical coordinate solutions $Z^1(\tau, \sigma)$ and $Z^2(\tau, \sigma)$ shall instead be to use the T-duality rules[6] *for the coordinate fields* derived in appendix A, namely (A.10) and (A.11). Here, the coordinates Z^μ shall be constructed by performing a T-duality in the $\mu = 2$ direction on the coordinates Y^μ. Using the rescaling (2.15) and the target space fields for the twisted torus (2.11) the duality rules read, up to order $\mathcal{O}H^2$,

$$\begin{aligned} \partial_\sigma Z^{1,3} &= \partial_\sigma Y^{1,3} \\ \partial_\tau Z^2 &= \partial_\sigma Y^2 - HY^3 \partial_\sigma Y^1 \\ \partial_\sigma Z^2 &= \partial_\tau Y^2 - HY^3 \partial_\tau Y^1 . \end{aligned} \tag{2.124}$$

They, as mentioned above, allow to map commutators in different frames according to table (2.120). The last line, for example can then be proven by

$$\begin{aligned} [\partial_\tau X^1(\tau,\sigma), \partial_\tau X^2(\tau,\sigma')]|_H &= [\partial_\sigma Y^1(\tau,\sigma), \partial_\tau Y^2(\tau,\sigma')]|_H - H\ [Y_0^3 \partial_\sigma Y_0^2(\tau,\sigma), \partial_\tau Y_0^2(\tau,\sigma')] \\ &= \mathrm{i}\pi H\ \left(Y_0^3(\tau,\sigma') - Y_0^3(\tau,\sigma) \right)\ \partial_\sigma \delta(\sigma - \sigma') \\ &= \mathrm{i}\pi H\ \left(X_0^3(\tau,\sigma') - X_0^3(\tau,\sigma) \right)\ \partial_\sigma \delta(\sigma - \sigma') , \end{aligned} \tag{2.125}$$

using (2.81), (2.64) and the setting $Y_0^3 = X_0^3$.

[6] There is a sign change in the last terms of rows two and three of (2.124) due to different conventions in this chapter. See the discussion in appendix A.

To obtain classical coordinate solutions, it is straightforwardly possible to proceed in the same spirit order by order by defining

$$\begin{aligned} Z^1 &= Z_0^1 + HZ_H^1 \\ Z^2 &= Z_0^2 + HZ_H^2 \ . \end{aligned} \tag{2.126}$$

Zeroth order

Separated to different orders, (2.124) at zeroth order simply give

$$\partial_\tau Z_0^1 = \partial_\tau Y_0^1 \ , \ \partial_\sigma Z_0^1 = \partial_\sigma Y_0^1 \ , \quad \partial_\tau Z_0^2 = \partial_\sigma Y_0^2 \ , \ \partial_\sigma Z_0^2 = \partial_\tau Y_0^2 \ , \tag{2.127}$$

that can be integrated to

$$Z_0^1(\tau,\sigma) = z^1 - y^1 + Y_0^1(\tau,\sigma) \tag{2.128}$$

$$\begin{aligned} Z_0^2(\tau,\sigma) &= z^2 - \tilde{y}^2 + \tilde{Y}_0^2(\tau,\sigma) \\ &= z^2 + p^2\sigma + N^2\tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_n^2 e^{-\mathrm{i}n\sigma_+} - \alpha_n^2 e^{-\mathrm{i}n\sigma_-}\right) \ . \end{aligned} \tag{2.129}$$

Integration leaves two undetermined integration constants z^1 and z^2, whereas the former constants y^1 and $\tilde{y}^2$ are removed. These new constants will play an important role later on as there is no possibility to determine the commutator of their operator counterpart with any of the other coefficients. Therefore, the procedure to determine the canonical commutators $[Z^\mu, Z^\nu]$ will always leave some particular freedom of choice.

Strictly speaking, new integration constants can also arise at zeroth or first order in the third coordinate Z^3. It can be obtained by integrating (2.124), namely

$$\partial_\tau Z^3 = \partial_\tau Y^3 \ , \quad \partial_\sigma Z^3 = \partial_\sigma Y^3 \ . \tag{2.130}$$

For both a physical and a technical reason, the following analysis assumes

$$Y^3(\tau,\sigma) = Z^3(\tau,\sigma) \ . \tag{2.131}$$

First, from the physical point of view, the $\mu = 3$ coordinate is the base circle a two-torus is fibered over in the X- and the Y-frame. T-duality transformations in the fiber directions should leave it invariant. Also, the physics given by commutators with the $\mu = 3$ coefficients should be left invariant, and thus it is reasonable to rule out changing integration constants. Second, from a technical perspective, the target space fields G and B in the X- and Y-frame, namely (2.8) and (2.11), are T-dual to each other only if the $\mu = 3$ coordinates are identified, $X^3 = Y^3$. Consistency in this respect thus also supports the above assumption.

First order

To obtain the first order classical solutions Z_H^μ, the following equations have to be integrated,

$$\partial_\tau Z_H^2 = \partial_\sigma Y_H^2 - HY_0^3\partial_\sigma Y_0^1 \ , \quad \partial_\sigma Z_H^2 = \partial_\tau Y_H^2 - HY_0^3\partial_\tau Y_0^1 \ , \tag{2.132}$$

$$\partial_\tau Z_H^{1,3} = \partial_\tau Y_H^{1,3} \ , \quad \partial_\sigma Z_H^{1,3} = \partial_\sigma Y_H^{1,3} \ , \tag{2.133}$$

using the general relations (2.124). As discussed before, it is assumed that $Z^3_H = Y^3_H$, and for Z^1_H one finds

$$Z^1_H(\tau,\sigma) = z^1_H - y^1_H + Y^1_H(\tau,\sigma)\ , \tag{2.134}$$

allowing for an integration constant z^1_H at first order.

The strategy to obtain Z^2_H is as follows. First, the first relation of (2.132) is integrated separately. This leaves an undetermined function of σ. On the other hand, the σ-derivative of the integrated expression has to match the second part of (2.132), which allows to solve for the so far undetermined function.

To begin with, the solution Y^2_H, given by (2.50), shall be straightforwardly rewritten as follows,

$$\begin{aligned} Y^2_H(\tau,\sigma) = {} & y^2_H + p^2_H\ \tau + (p^1p^3 - N^1N^3)\ \frac{\tau^2}{2} \\ & + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}^2_n e^{-\mathrm{i}n\sigma_+} + \gamma^2_n e^{-\mathrm{i}n\sigma_-}\right) \\ & + \frac{\mathrm{i}}{2}\ \tau\ \sum_{n\neq 0}\frac{1}{n}\left((p^1_R\tilde{\alpha}^3_n + p^3_R\tilde{\alpha}^1_n)e^{-\mathrm{i}n\sigma_+} + (p^1_L\alpha^3_n + p^3_L\alpha^1_n)e^{-\mathrm{i}n\sigma_-}\right) \\ & - \frac{1}{8}\sum_{n,m\neq 0}\frac{1}{nm}\left(\tilde{\alpha}^1_m\alpha^3_n + \tilde{\alpha}^3_m\alpha^1_n\right)e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)}\ . \end{aligned} \tag{2.135}$$

To simplify notation the following abbreviations shall be used,

$$\begin{aligned} A^R_n &= p^1_R\tilde{\alpha}^3_n + p^3_R\tilde{\alpha}^1_n\ , \quad & A^L_n &= p^1_L\alpha^3_n + p^3_L\alpha^1_n \\ A_{nm} &= \tilde{\alpha}^1_n\alpha^3_m + \tilde{\alpha}^3_n\alpha^1_m\ , \quad & \tilde{A}_{nm} &= \tilde{\alpha}^1_n\alpha^3_m - \tilde{\alpha}^3_n\alpha^1_m\ . \end{aligned} \tag{2.136}$$

Writing out the first half of (2.132) then gives,

$$\begin{aligned} \partial_\tau Z^2_H = {} & -N^1(y^3 + p^3\tau + N^3\sigma) \\ & + \frac{1}{2}\sum_{n\neq 0}\left(\left(\tilde{g}^2_n + \tau A^R_n - \frac{\mathrm{i}}{n}N^1\tilde{\alpha}^3_n - (y^3 + p^3\tau + N^3\sigma)\tilde{\alpha}^1_n\right)e^{-\mathrm{i}n\sigma_+}\right. \\ & \qquad \left. - \left(\gamma^2_n + \tau A^L_n + \frac{\mathrm{i}}{n}N^1\alpha^3_n - (y^3 + p^3\tau + N^3\sigma)\alpha^1_n\right)e^{-\mathrm{i}n\sigma_-}\right) \\ & - \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\tilde{A}_{nn}e^{-2in\tau} \\ & - \frac{\mathrm{i}}{8}\sum_{n\neq 0, m\neq 0, \pm n}\frac{m+n}{nm}\tilde{A}_{mn}e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)} \\ & - \frac{\mathrm{i}}{4}\sum_{n,m\neq 0}\frac{1}{n}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} - \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right)\ , \end{aligned} \tag{2.137}$$

which, after integration, can be found to be

$$\begin{aligned}
Z_H^2(\tau,\sigma) &= f_H(\sigma) - N^1(y^3+N^3\sigma)\tau - \frac{1}{2}N^1p^3\tau^2 - \frac{\mathrm{i}}{4}\tau\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}^1_{-n}\tilde{\alpha}^3_n - \alpha^1_{-n}\alpha^3_n\right) \qquad (2.138)\\
&+ \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Bigg(\left(\tilde{g}^2_n + (\tau-\frac{\mathrm{i}}{n})A^R_n - \frac{\mathrm{i}}{n}(N^1\tilde{\alpha}^3_n - p^3\tilde{\alpha}^1_n) - (y^3+p^3\tau+N^3\sigma)\tilde{\alpha}^1_n\right)e^{-\mathrm{i}n\sigma_+}\\
&\qquad - \left(\gamma^2_n + (\tau-\frac{\mathrm{i}}{n})A^L_n + \frac{\mathrm{i}}{n}(N^1\alpha^3_n + p^3\alpha^1_n) - (y^3+p^3\tau+N^3\sigma)\alpha^1_n\right)e^{-\mathrm{i}n\sigma_-}\Bigg)\\
&+ \frac{1}{8}\sum_{n\neq 0}\frac{1}{n^2}\tilde{A}_{nn}e^{-2\mathrm{i}n\tau}\\
&+ \frac{1}{8}\sum_{n\neq 0, m\neq 0,\pm n}\frac{1}{nm}\tilde{A}_{mn}e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)}\\
&+ \frac{1}{4}\sum_{n,m\neq 0, m\neq -n}\frac{1}{n(n+m)}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} - \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right)\ .
\end{aligned}$$

The function $f_H(\sigma)$ is a so far undetermined, τ-independent integration constant. The derivative of this whole expression, namely

$$\begin{aligned}
\partial_\sigma Z_H^2(\tau,\sigma) &= f'_H(\sigma) - N^1N^3\tau \qquad (2.139)\\
&+ \frac{1}{2}\sum_{n\neq 0}\Bigg(\left(\tilde{g}^2_n + (\tau-\frac{\mathrm{i}}{n})A^R_n - \frac{\mathrm{i}}{n}(N^1\tilde{\alpha}^3_n - (p^3-N^3)\tilde{\alpha}^1_n) - (y^3+p^3\tau+N^3\sigma)\tilde{\alpha}^1_n\right)e^{-\mathrm{i}n\sigma_+}\\
&\qquad + \left(\gamma^2_n + (\tau-\frac{\mathrm{i}}{n})A^L_n + \frac{\mathrm{i}}{n}(N^1\alpha^3_n + (p^3+N^3)\alpha^1_n) - (y^3+p^3\tau+N^3\sigma)\alpha^1_n\right)e^{-\mathrm{i}n\sigma_-}\Bigg)\\
&- \frac{\mathrm{i}}{8}\sum_{n\neq 0, m\neq 0,\pm n}\frac{m-n}{nm}\tilde{A}_{mn}e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)}\\
&- \frac{\mathrm{i}}{4}\sum_{n,m\neq 0, m\neq -n}\frac{1}{n}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} + \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right)\ ,
\end{aligned}$$

should match the second half of (2.132), namely

$$\begin{aligned}
\partial_\sigma Z_H^2 &= p^2_H - p^1(y^3+N^3\sigma) - N^1N^3\tau \qquad (2.140)\\
&+ \frac{1}{2}\sum_{n\neq 0}\Bigg(\left(\tilde{g}^2_n + (\tau+\frac{\mathrm{i}}{n})A^R_n - \frac{\mathrm{i}}{n}p^1\tilde{\alpha}^3_n - (y^3+p^3\tau+N^3\sigma)\tilde{\alpha}^1_n\right)e^{-\mathrm{i}n\sigma_+}\\
&\qquad + \left(\gamma^2_n + (\tau+\frac{\mathrm{i}}{n})A^L_n - \frac{\mathrm{i}}{n}p^1\alpha^3_n - (y^3+p^3\tau+N^3\sigma)\alpha^1_n\right)e^{-\mathrm{i}n\sigma_-}\Bigg)\ ,\\
&+ \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\tilde{A}_{n-n}e^{-2\mathrm{i}n\sigma}\\
&+ \frac{\mathrm{i}}{8}\sum_{n\neq 0, m\neq 0,\pm n}\frac{n-m}{nm}\tilde{A}_{mn}e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)}\\
&- \frac{\mathrm{i}}{4}\sum_{n,m\neq 0}\frac{1}{n}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} + \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right)\ .
\end{aligned}$$

This can be achieved by setting

$$f'_H(\sigma) = \Big(p^2_H - p^1(y^3 + N^3\sigma) + \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\tilde{A}_{n-n}e^{-2\mathrm{i}n\sigma}\Big) - \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\left[\tilde{\alpha}^1_{-n}\tilde{\alpha}^3_n + \alpha^1_{-n}\alpha^3_n\right] \ . \tag{2.141}$$

Integrating f_H allows to give the full solution for Z^2_H,

$$\begin{aligned}
Z^2_H(\tau,\sigma) = {} & z^2_H + (p^2_H - p^1y^3)\sigma - N^1(y^3+N^3\sigma)\tau - \frac{1}{2}(N^1p^3\tau^2 + p^1N^3\sigma^2) \\
& - \frac{\mathrm{i}}{4}\tau\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}^1_{-n}\tilde{\alpha}^3_n - \alpha^1_{-n}\alpha^3_n\right) - \frac{\mathrm{i}}{4}\sigma\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}^1_{-n}\tilde{\alpha}^3_n + \alpha^1_{-n}\alpha^3_n\right) \\
& + \frac{1}{8}\sum_{n\neq 0}\frac{1}{n^2}(\tilde{A}_{nn}e^{-2\mathrm{i}n\tau} - \tilde{A}_{n-n}e^{-2\mathrm{i}n\sigma}) \\
& + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Bigg(\Big(\tilde{g}^2_n + (\tau - \frac{\mathrm{i}}{n})A^R_n - \frac{\mathrm{i}}{n}(N^1\tilde{\alpha}^3_n - p^3\tilde{\alpha}^1_n) - (y^3 + p^3\tau + N^3\sigma)\tilde{\alpha}^1_n\Big)e^{-\mathrm{i}n\sigma_+} \\
& \qquad - \Big(\gamma^2_n + (\tau - \frac{\mathrm{i}}{n})A^L_n + \frac{\mathrm{i}}{n}(N^1\alpha^3_n + p^3\alpha^1_n) - (y^3 + p^3\tau + N^3\sigma)\alpha^1_n\Big)e^{-\mathrm{i}n\sigma_-}\Bigg) \\
& + \frac{1}{8}\sum_{n\neq 0, m\neq 0, \pm n}\frac{1}{nm}\tilde{A}_{mn}e^{-\mathrm{i}(m\sigma_+ + n\sigma_-)} \\
& + \frac{1}{4}\sum_{n,m\neq 0, m\neq -n}\frac{1}{n(n+m)}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} - \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right) \ .
\end{aligned} \tag{2.142}$$

A new integration constant z^2_H has been inserted. To simplify this expression a little, it shall be noted that the third and the sixth row can be combined to

$$\frac{1}{4}(-Y^3_0|_\Sigma\tilde{Y}^1_0|_\Sigma + Y^1_0|_\Sigma\tilde{Y}^3_0|_\Sigma), \tag{2.143}$$

whereas the rest can be rearranged to give the final result,

$$\begin{aligned}
Z^2_H(\tau,\sigma) = {} & z^2_H + p^2_H\sigma + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}^2_n e^{-\mathrm{i}n\sigma_+} - \gamma^2_n e^{-\mathrm{i}n\sigma_-}\Big) \\
& - \frac{1}{4}\left(Y^3_0|_\Sigma\tilde{Y}^1_0|_\Sigma - Y^1_0|_\Sigma\tilde{Y}^3_0|_\Sigma\right) \\
& + \frac{1}{2}\tau\left(p^1\tilde{Y}^3_0|_\Sigma + p^3\tilde{Y}^1_0|_\Sigma - N^1Y^3_0|_\Sigma - N^3Y^1_0|_\Sigma\right) - (y^3 + p^3\tau + N^3\sigma)\tilde{Y}^1_0|_\Sigma \\
& - p^1y^3\sigma - N^1(y^3+N^3\sigma)\tau - \frac{1}{2}(N^1p^3\tau^2 + p^1N^3\sigma^2) \\
& - \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\left(\sigma_+\tilde{\alpha}^1_{-n}\tilde{\alpha}^3_n - \sigma_-\alpha^1_{-n}\alpha^3_n\right) \\
& + \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2}\left(\left(p^1_L\tilde{\alpha}^3_n - p^3_L\tilde{\alpha}^1_n\right)e^{-\mathrm{i}n\sigma_+} - \left(p^1_R\alpha^3_n - p^3_R\alpha^1_n\right)e^{-\mathrm{i}n\sigma_-}\right) \\
& + \frac{1}{4}\sum_{\substack{n,m\neq 0\\ m\neq -n}}\frac{1}{n(n+m)}\left(\tilde{\alpha}^1_m\tilde{\alpha}^3_n e^{-\mathrm{i}(n+m)\sigma_+} - \alpha^1_m\alpha^3_n e^{-\mathrm{i}(n+m)\sigma_-}\right) \ .
\end{aligned} \tag{2.144}$$

It turns out that for the computations to be followed in the next section it is helpful to rearrange this huge expression once more. Its second line contains oscillator parts of coordinates only, $Y_0^\mu|_\Sigma$, which can be completed to full coordinates Y_0^μ at the cost of adding more terms. Technically, this simplifies some computations in the next section as there the commutator of Z_H^2 with Y_0^1 will be computed, and thus the more full coordinates are made appear the more often canonical commutation relations at zeroth order can be used. Eventually, one finds,

$$
\begin{aligned}
Z_H^2(\tau,\sigma) = {} & z_H^2 + p_H^2\ \sigma + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma_-}\Big) \\
& + \frac{1}{4}\sum_{m,n\neq 0, m\neq -n}\frac{1}{n(n+m)}\left(\tilde{\alpha}_m^1\tilde{\alpha}_n^3 e^{-\mathrm{i}(n+m)\sigma_+} - \alpha_m^1\alpha_n^3 e^{-\mathrm{i}(n+m)\sigma_-}\right) \\
& + \frac{1}{4}\sum_{n\neq 0}\frac{1}{n^2}\Big(\left((p^1+N^1)\tilde{\alpha}_n^3 - (p^3+N^3)\tilde{\alpha}_n^1\right)e^{-\mathrm{i}n\sigma_+} \\
& \qquad + \left((p^3-N^3)\alpha_n^1 - (p^1-N^1)\alpha_n^3\right)e^{-\mathrm{i}n\sigma_-}\Big) \\
& - \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\left(\sigma_+\tilde{\alpha}_{-n}^1\tilde{\alpha}_n^3 - \sigma_-\alpha_{-n}^1\alpha_n^3\right) \\
& + \sigma\tau N^1N^3 + \frac{1}{4}(\tau^2+\sigma^2)(N^1p^3 + p^1N^3) + \frac{1}{4}(y^1\tilde{y}^3 + 3y^3\tilde{y}^1) \\
& + \frac{1}{4}\left(-y^1\tilde{Y}_0^3 + \tilde{y}^1Y_0^3 - 3y^3\tilde{Y}_0^1 - \tilde{y}^3Y_0^1\right) \\
& + \frac{1}{4}\sigma\left(-N^1(\tilde{Y}_0^3-\tilde{y}^3) + p^1(Y_0^3-y^3) - 3N^3(\tilde{Y}_0^1-\tilde{y}^1) - p^3(Y_0^1-y^1)\right) \\
& + \frac{1}{4}\tau\left(p^1(\tilde{Y}_0^3-\tilde{y}^3) - N^1(Y_0^3-y^3) - p^3(\tilde{Y}_0^1-\tilde{y}^1) - 3N^3(Y_0^1-y^1)\right) \\
& + \frac{1}{4}\left(Y_0^1\tilde{Y}_0^3 - Y_0^3\tilde{Y}_0^1\right)\ .
\end{aligned}
\tag{2.145}
$$

Result

To summarise, the following classical coordinate solutions after performing a T-duality in $\mu = 2$ direction have been found:

$$
\begin{aligned}
Z^1(\tau,\sigma) &= z^1 - y^1 + Y_0^1(\tau,\sigma) + H(z_H^1 - y_H^1 + Y_H^1(\tau,\sigma)) \\
Z^2(\tau,\sigma) &= z^2 - \tilde{y}^2 + \tilde{Y}_0^2(\tau,\sigma) + HZ_H^2(\tau,\sigma) \\
Z^3(\tau,\sigma) &= Y_0^3(\tau,\sigma) + HY_H^3(\tau,\sigma)\ ,
\end{aligned}
\tag{2.146}
$$

where the complicated Z_H^2 is given above in (2.144) or (2.145).

As has been discussed before, from pure physical reasoning it is not clear what the boundary conditions for the coordinates Z^μ in a non-geometric frame should be. The adapted strategy here is to use the T-duality rules as a guideline and not assume any boundary conditions in the first place. The given solution for Z_H^2 allows to read off the following, rather complicated, boundary condition,

$$
\begin{aligned}
Z^2(\tau,\sigma+2\pi) = {} & Z^2(\tau,\sigma) + 2\pi p^2 + H\Big(-2\pi N^3(\tilde{Y}_0^1 - \tilde{y}^1) \\
& + 2\pi(p_H^2 - p^1y^3 - p^1N^3\pi) - \frac{\mathrm{i}\pi}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_{-n}^1\tilde{\alpha}_n^3 + \alpha_{-n}^1\alpha_n^3\right)\Big)\ .
\end{aligned}
\tag{2.147}
$$

As a side remark, it shall be noted that this boundary condition - up to a constant shift of the zero mode operators - has the form

$$Z^2(\tau,\sigma+2\pi) = Z^2(\tau,\sigma) - 2\pi H N^3 \tilde{Z}^1(\tau,\sigma) \; . \tag{2.148}$$

This is expected from either a reasoning starting with a doubled worldsheet, cf. section 4.2.4, or a reasoning on monodromies. Details on the latter can be found in [2], appendix C.

2.3.2 Quantisation

As already described, the canonical commutator of two coordinates Z^1 and Z^2 shall now be determined, rather than imposed. This is strictly speaking not a quantisation but rather the reverse procedure to the canonical quantisation in section 2.2.2: the commutation relations of the expansion coefficients are use to construct the full commutator. To start with, the latter shall be divided into different orders in H as

$$\begin{aligned}
[Z^1, Z^2] = {} & [z^1 - y^1, z^2 - \tilde{y}^2] + [z^1 - y^1, \tilde{Y}_0^2] + [Y_0^1, z^2 - \tilde{y}^2] \\
& + H\Big([Z_H^1, z^2 - \tilde{y}^2] + [z^1 - y^1, Z_H^2]\Big) \\
& + H\Big([Z_H^1, \tilde{Y}_0^2] + [Y_0^1, Z_H^2]\Big) \; .
\end{aligned} \tag{2.149}$$

By construction, non-commutativity cannot stem from zeroth-order commutators. Ignoring all expressions that are at least of order $\mathcal{O}(H^1)$ turns the whole setup back to the quantisation of the free string and its T-duals, where non-geometry does not occur. All new effects to be discovered, therefore, must at least be linear in the flux parameter H. In this sense, it is imposed that

$$[Z_0^1, Z_0^2] = 0 \; , \tag{2.150}$$

which, given the form of Z_0^μ, restricts the undetermined commutators at zeroth order to

$$[z^1 - y^1, \text{ any } 0^{\text{th}} \text{ order operator}] = [z^2 - \tilde{y}^2, \text{ any } 0^{\text{th}} \text{ order operator}] = 0 \; . \tag{2.151}$$

This states that z^1 has the same zeroth order commutators as y^1, and z^2 the same as $\tilde{y}^2$.

An immediate consequence is that the first row of (2.149) vanishes, whereas the second row simplifies,

$$\begin{aligned}
& [Z_H^1(\tau,\sigma), z^2 - \tilde{y}^2] + [z^1 - y^1, Z_H^2(\tau,\sigma')] \\
& \quad = [z_H^1 + p_H^1\,\tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), z^2 - \tilde{y}^2] \\
& \quad\quad + [z^1 - y^1, z_H^2 + p_H^2\,\sigma' + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\Big)] \; ,
\end{aligned} \tag{2.152}$$

using (2.134) and (2.144).

The computation of the last row of (2.149) is the most involved and shall presented in two parts. First, only zeroth order commutators are used, whereas in a second step the first order commutators are employed. Fortunately, exactly those sums appear during the calculation that were determinable by the quantisation procedure in the last section. This indicates the correctness of the whole procedure applied in this chapter, despite the complicated expressions and subtle calculations.

To start with, the second term of the last row in (2.149) can be written as

$$
\begin{aligned}
[Y_0^1(\tau,\sigma),\ Z_H^2(\tau,\sigma')] = & \Big[Y_0^1(\tau,\sigma), z_H^2 + p_H^2\ \sigma' + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\Big)\Big] \\
& - \frac{\mathrm{i}}{4}N^3\sum_{n\neq 0}\frac{1}{n^2}e^{-\mathrm{i}n(\sigma'-\sigma)} \\
& + \frac{\mathrm{i}}{8}\sum_{n\neq 0, m\neq -n}\frac{1}{n(n+m)}\Big(\tilde{\alpha}_n^3 e^{-\mathrm{i}n\sigma'_+}e^{-\mathrm{i}m(\sigma'-\sigma)} - \alpha_n^3 e^{-\mathrm{i}n\sigma'_-}e^{\mathrm{i}m(\sigma'-\sigma)}\Big) \\
& + \frac{1}{8}\left(3y^3 + 3N^3\sigma' + p^3\tau + Y_0^3(\tau,\sigma')\right)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)} \\
& + \frac{\mathrm{i}}{4}\sigma'\left(-y^3 + N^3(\sigma-\sigma') - Y_0^3(\tau,\sigma)\right) + \frac{\mathrm{i}}{4}\tau\Big(p^3\sigma + \tilde{Y}_0^3(\tau,\sigma') - \tilde{Y}_0^3(\tau,\sigma)\Big)\ ,
\end{aligned}
\tag{2.153}
$$

where its third row can be condensed to

$$
\frac{1}{4}Y_0^3|_\Sigma(\tau,\sigma)\sum_{p\neq 0}\frac{1}{p}e^{-\mathrm{i}p(\sigma'-\sigma)}\ , \tag{2.154}
$$

by introducing a new index $p = m+n$. The first term of the last row in (2.149) can be written as

$$
\begin{aligned}
[Z_H^1(\tau,\sigma), \tilde{Y}_0^2(\tau,\sigma')] = & \Big[z_H^1 + p_H^1\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), \tilde{Y}_0^2(\tau,\sigma')\Big] \\
& - \frac{1}{8}\left(-y^3 + 3N^3\sigma + p^3\tau + Y_0^3(\sigma)\right)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)} \\
& + \frac{\mathrm{i}}{4}\sigma\left(2\pi N^3 - p^3\tau\right) + \frac{\mathrm{i}}{4}\tau\Big(\tilde{Y}_0^3(\sigma) - \tilde{y}^3\Big) - \frac{\mathrm{i}}{4}N^3(\tau^2+\sigma^2-2\sigma\sigma')\ .
\end{aligned}
\tag{2.155}
$$

Until here, only zeroth order commutators have been used. Taking now the sum of the respective first lines in the last expressions, (2.153) and (2.155), it will contain exactly matching combinations for using the first order commutators of the last section, namely (2.96), (2.99), and (2.101) to (2.105). That is,

$$
\begin{aligned}
& \Big[Y_0^1(\tau,\sigma), z_H^2 + p_H^2\ \sigma' + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\Big)\Big] \\
& \quad + \Big[z_H^1 + p_H^1\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), \tilde{Y}_0^2(\tau,\sigma')\Big] =
\end{aligned}
\tag{2.156}
$$

$$
\begin{aligned}
&= \Big[Y_0^1(\tau,\sigma), z_H^2\Big] + \Big[z_H^1, \tilde{Y}_0^2(\tau,\sigma')\Big] \\
&\quad + \Big[y^1, p_H^2\ \sigma' + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Big(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\Big)\Big] \\
&\quad + \Big[p_H^1\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), \tilde{y}^2\Big] \\
&\quad + \frac{\mathrm{i}}{4}\sigma'(Y_0^3(\sigma) - y^3) - \frac{\mathrm{i}}{4}\tau(\tilde{Y}_0^3(\sigma') - \tilde{y}^3) + \frac{\mathrm{i}}{4}N^3(\tau^2 - \sigma\sigma') \\
&\quad - \frac{\mathrm{i}}{4}N^3\sum_{n\neq 0}\frac{1}{n^2}e^{-\mathrm{i}n(\sigma'-\sigma)} \\
&\quad - \frac{1}{8}\left(2y^3 - N^3(\sigma+\sigma') - 2p^3\tau + Y_0^3(\sigma) + Y_0^3(\sigma')\right)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)} \\
&\quad + \frac{\mathrm{i}}{16}\sum_{n\neq 0}\frac{1}{n^2}\Big(\tilde{\alpha}_n^3(e^{-\mathrm{i}n\sigma'_+} - e^{-\mathrm{i}n\sigma_+}) - \alpha_n^3(e^{-\mathrm{i}n\sigma'_-} - e^{-\mathrm{i}n\sigma_-})\Big)\ .
\end{aligned}
$$

Two types of commutators are left undetermined by construction. First, commutators of z_H^μ with others, that cannot be determined due to their origin as integration constants. Second, commutators of y^μ and $\tilde{y}^\mu$ with first order coefficients. These will cancel out with (2.152).

The compilation of all intermediate steps, namely (2.152), (2.153), (2.155) and (2.156), gives the end result of this section, the commutator of the first two coordinates in the non-geometric frame, that is

$$
\begin{aligned}
\frac{1}{H}&[Z^1(\tau,\sigma), Z^2(\tau,\sigma')] \qquad (2.157)\\
&= [z^1, z_H^2] + [z_H^1, z^2] \\
&\quad + \tau\Bigg([p^1, z_H^2] + [z_H^1, N^2] + [p_H^1, z^2]\Bigg) \\
&\quad + \sigma\Bigg([N^1, z_H^2] + \frac{\mathrm{i}\pi}{2}N^3\Bigg) - \sigma'\Bigg([p^2, z_H^1] + [p_H^2, z^1] + \frac{\mathrm{i}}{2}y^3\Bigg) \\
&\quad + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Bigg(e^{-\mathrm{i}n\sigma_+}\left([\tilde{\alpha}_n^1, z_H^2] + [\tilde{g}_n^1, z^2]\right) + e^{-\mathrm{i}n\sigma'_+}\left([z_H^1, \tilde{\alpha}_n^2] + [z^1, \tilde{g}_n^2]\right)\Bigg) \\
&\quad + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\Bigg(e^{-\mathrm{i}n\sigma_-}\left([\alpha_n^1, z_H^2] + [\gamma_n^1, z^2]\right) - e^{-\mathrm{i}n\sigma'_-}\left([z_H^1, \alpha_n^2] + [z^1, \gamma_n^2]\right)\Bigg) \\
&\quad + \frac{\mathrm{i}}{16}\sum_{n\neq 0}\frac{1}{n^2}\Big(\tilde{\alpha}_n^3(e^{-\mathrm{i}n\sigma'_+} - e^{-\mathrm{i}n\sigma_+}) - \alpha_n^3(e^{-\mathrm{i}n\sigma'_-} - e^{-\mathrm{i}n\sigma_-})\Big) \\
&\quad - \frac{\mathrm{i}}{2}N^3\sum_{n\neq 0}\frac{1}{n^2}e^{-\mathrm{i}n(\sigma'-\sigma)} + \frac{1}{2}N^3(\sigma'-\sigma)\sum_{n\neq 0}\frac{1}{n}e^{-\mathrm{i}n(\sigma'-\sigma)} - \frac{\mathrm{i}}{4}N^3(\sigma'-\sigma)^2\ .
\end{aligned}
$$

Some comments are appropriate here:

- The commutator, irrespective of its undetermined pieces, is dependent on the worldsheet coordinates. Although this could be interpreted as a specific feature of non-geometry, here it shall be assumed that a physically sensible coordinate commutator must at least be independent of the worldsheet coordinates after taking the limit $\sigma' \to \sigma$. As can easily be confirmed in the above expression, such a claim puts restrictions on various commutators, in particular

$$0 = [p^1, z_H^2] + [z_H^1, N^2] + [p_H^1, z^2]$$
$$0 = [N^1, z_H^2] + \frac{\mathrm{i}\pi}{2} N^3 - [p^2, z_H^1] - [p_H^2, z^1] - \frac{\mathrm{i}}{2} y^3 \; . \tag{2.158}$$

Additionally, the oscillator terms in rows five to seven get restricted, as will be discussed later on.

- From a more abstract point of view, one could employ the T-duality rules (2.124) and directly deduce

$$[\partial_\sigma Z^1(\tau,\sigma), \partial_{\sigma'} Z^2(\tau,\sigma')] = [\partial_\sigma Y^1(\tau,\sigma), (\partial_\tau Y^2 - H Y^3 \partial_\tau Y^1)(\tau,\sigma')] = 0 \; , \tag{2.159}$$

having employed (2.64) and (2.85). This implies that the above commutator (2.157) can be written, up to possible contributions from distributions, as

$$[Z^1(\tau,\sigma), Z^2(\tau,\sigma')] = f_1(\tau,\sigma) + f_2(\tau,\sigma') \; , \tag{2.160}$$

where f_1, f_2 are arbitrary functions. Thus, the σ- and σ'-dependence has to be separable. All but the last row of (2.157) immediately show this kind of separability. Still, by using (B.1) and (B.3), one can verify that

$$\begin{aligned} &\partial_\sigma \partial_{\sigma'} \Big(-\frac{\mathrm{i}}{2} N^3 \sum_{n\neq 0} \frac{1}{n^2} e^{-\mathrm{i}n(\sigma'-\sigma)} + \frac{1}{2} N^3 (\sigma'-\sigma) \sum_{n\neq 0} \frac{1}{n} e^{-\mathrm{i}n(\sigma'-\sigma)} - \frac{\mathrm{i}}{4} N^3 (\sigma'-\sigma)^2 \Big) \\ &= \mathrm{i}\pi N^3 \big(\delta(\sigma'-\sigma) + (\sigma'-\sigma) \partial_{\sigma'} \delta(\sigma'-\sigma) \big) \\ &= 0 \; , \end{aligned} \tag{2.161}$$

to be understood in the sense of distributions, as all expressions presented in this section. In conclusion, there is no contradiction to (2.159), even with a number of undetermined commutators. Furthermore, the last row of (2.157) seems to be a particular part of this commutator.

This concludes the technical derivation of the coordinate fields commutator in one particular direction, which shall now be discussed with respect to a possible non-commutativity.

2.4 Non-commutativity

As has been discussed in the introduction to this chapter, it shall be investigated whether non-geometry can become manifest in non-commutativity of the string coordinates. The main result obtained so far is the commutator (2.157) of the coordinates in the two fiber directions. Due to the implementation of T-duality, that technically involves an integration, there are undetermined expressions in the commutator involving integration constants $z^{1,2}$ and $z_H^{1,2}$. Their commutation relations with other expansion coefficients are in principle indeterminable, and a to this stage consistent quantisation is possible for any value of these commutation relations.

From physical reasoning there are certain preferred values for the undetermined commutation relations, and in the following it shall be argued for one set of such. It will lead to a nonzero commutator $[Z^1, Z^2]$. Nevertheless, there are other possibilities to fix the commutation relations, even under the restriction that there must not be any worldsheet dependence after taking the limit $\sigma' \to \sigma$. In particular, it is possible to tune (2.157) to zero, and thus the answer to the question whether the string coordinates are commuting or not thoroughly depends on how the commutation relations of z^μ and z_H^μ are fixed.

2.4.1 Commutativity

It shall now shortly be investigated how commuting coordinates Z^1 and Z^2, namely

$$[Z^1(\tau,\sigma),\ Z^2(\tau,\sigma')] = 0 \tag{2.162}$$

can be achieved. As commented around (2.158), a dependence of the commutator (2.157) on the worldsheet coordinates that does not vanish in the limit $\sigma' \to \sigma$ is considered unphysical. A simple setup can solve this issue for the linear τ- and σ-dependence in the third and fourth row of (2.157),

$$\begin{aligned}
[z_H^2, p^1] + [z^2, p_H^1] &= [z_H^1, N^2] \\
[z_H^1, p^2] + [z^1, p_H^2] &= \frac{\mathrm{i}}{2} y^3 \\
[z_H^2, N^1] &= \frac{\mathrm{i}\pi}{2} N^3 \ .
\end{aligned} \tag{2.163}$$

This is a small limitation of generality, as there are other solutions to the second line of (2.158). For rows five to seven of (2.157) there are at least two possible approaches. First, one could choose to satisfy the weakest claim only and take

$$[z_H^1, \alpha_{n\epsilon}^2] + [z^1, \gamma_{n\epsilon}^2] = \epsilon \left([z_H^2, \alpha_{n\epsilon}^1] + [z^2, \gamma_{n\epsilon}^1] \right) \ . \tag{2.164}$$

The three rows then vanish after taking the limit $\sigma' \to \sigma$. Alternatively, one could choose to have these rows cancelled amongst each other even before taking the limit by setting

$$\begin{aligned}
[z_H^2, \alpha_{n\epsilon}^1] + [z^2, \gamma_{n\epsilon}^1] &= -\epsilon \frac{1}{8n} \alpha_{n\epsilon}^3 \\
[z_H^1, \alpha_{n\epsilon}^2] + [z^1, \gamma_{n\epsilon}^2] &= -\frac{1}{8n} \alpha_{n\epsilon}^3 \ , \ \forall \epsilon,\ \forall n \neq 0 \ .
\end{aligned} \tag{2.165}$$

The last row of (2.157) requires a more careful investigation of the limit $\sigma' \to \sigma$. The last two terms vanish, whereas the first term is

$$-\frac{\mathrm{i}}{2} N^3 \sum_{n\neq 0} \frac{1}{n^2} e^{-\mathrm{i}n(\sigma'-\sigma)} \xrightarrow{\sigma'\to\sigma} -\frac{\mathrm{i}}{2} \frac{\pi^2}{3} N^3 \ . \tag{2.166}$$

The only possibility to compensate this term is to fix the following cumbersome relation,

$$[z^1, z_H^2] + [z_H^1, z^2] = \frac{\mathrm{i}\pi^2}{6} N^3 \ . \tag{2.167}$$

Although technically possible, there are two alarming observations with this setting. First, it seems to violate the reasoning that no physical effect should stem from the zero-modes.

This has been explained around (2.151) and could be interpreted in a slightly generalised fashion by also setting $[z^{1,2}, A_H] = 0$ for some operator at order $\mathcal{O}(H^1)$. Second, there is a simple argument that fixes the value of the above commutation relation. The assumption to be made is, that the zero modes y^μ and y^μ_H can be split into left- and right-moving parts, as it was already done for y^μ in (2.43). Furthermore, all commutators of those shall take a form where the right-hand side is proportional to $\delta_{\epsilon_1,\epsilon_2}$, as has been assumed for the zeroth order commutator (2.72). This will in particular set

$$[y^1_\epsilon, y^2_{H\epsilon}] = [y^1_{-\epsilon}, y^2_{H-\epsilon}] \ , \quad [y^2_\epsilon, y^1_{H\epsilon}] = [y^2_{-\epsilon}, y^1_{H-\epsilon}] \ . \tag{2.168}$$

One can now claim the following equality, keeping in mind that the T-duality between the Y- and the Z-frame has been performed in the $\mu = 2$ direction, i.e. using a gentle generalisation of (2.151),

$$[z^1, z^2_H] + [z^1_H, z^2] = [y^1, \tilde{y}^2_H] + [y^1_H, \tilde{y}^2] \ . \tag{2.169}$$

The right-hand side can be determined using the above assumption,

$$[y^1, \tilde{y}^2_H] + [y^1_H, \tilde{y}^2] = [y^1_L, y^2_{HL}] - [y^1_R, y^2_{HR}] + [y^1_{HL}, y^2_L] - [y^1_{HR}, y^2_R] = 0 \ , \tag{2.170}$$

such that one has

$$[z^1, z^2_H] + [z^1_H, z^2] = 0 \ . \tag{2.171}$$

This is in obvious contradiction to the fixing (2.167). It shall be noted that this is not a technical contradiction as it is based on certain analogies that strictly speaking can be relaxed at will.

In conclusion, commutativity of the coordinates Z^1 and Z^2 can be reached by a particular setup of the unknown commutation relations but seems physically implausible.

2.4.2 Non-commutativity

Taking again as the guiding principle that the commutator of Z^1 and Z^2, (2.157), must not depend on the worldsheet coordinates after taking the limit $\sigma' \to \sigma$, one can adopt all but one fixing of the unknown commutation relations from the preceding subsection. Namely, for the following shall hold equations (2.163) and (2.165). For the commutation relations amongst the zeroth and first order position zero modes, as has been argued above by analogy, it shall be taken (2.171), namely

$$[z^1, z^2_H] + [z^1_H, z^2] = 0 \ . \tag{2.172}$$

This leaves exactly one source of a non-vanishing contribution to the full commutator, namely the last line of (2.157), which in the limit reduces to

$$-\frac{\mathrm{i}}{2} N^3 \sum_{n \neq 0} \frac{1}{n^2} e^{-\mathrm{i}n(\sigma' - \sigma)} \overset{\sigma' \to \sigma}{\longrightarrow} -\frac{\mathrm{i}}{2} \frac{\pi^2}{3} N^3 \ . \tag{2.173}$$

Eventually, the coordinates Z^1 and Z^2 do not commute,

$$[Z^1(\tau, \sigma),\ Z^2(\tau, \sigma')] = -\frac{\mathrm{i}\pi^2}{6} N^3 H \ , \quad \text{for } \sigma' \to \sigma \ . \tag{2.174}$$

This result, to emphasise it once more, does not follow by strict logic, but rather rests on the two physically motivated assumptions that the commutator does not depend on the worldsheet coordinates after taking the limit and that, by analogy, it is plausible to fix (2.172).

It is possible to give another independent argument for fixing the commutation relations (2.163) and (2.165). This may add more convincibility to the overall result of non-commuting coordinates Z^1 and Z^2. The argument basically employs the idea behind (2.172), which was explained in the preceding subsection. As a general guideline, one assumes that the zero mode operators $z^{1,2}$ and $z_H^{1,2}$ can be understood as the T-dual counterparts, after performing the duality along the $\mu = 2$ direction, of $y^{1,2}$ and $y_H^{1,2}$, and therefore have the same commutation relations at order $\mathcal{O}(H^1)$. This, to emphasise it again, has already been argued for in (2.151) at zeroth order. Concretely, the following relations are assumed,

$$\begin{aligned} [z^1 - y^1, \dots] &= 0 \ , \quad [z_H^1 - y_H^1, \dots] = 0 \\ [z^2 - \tilde{y}^2, \dots] &= 0 \ , \quad [z_H^2 - \tilde{y}_H^2, \dots] = 0 \ . \end{aligned} \tag{2.175}$$

From this, not only (2.169) but also other relations can be inferred, as for example

$$[z_H^2, \alpha_{n\epsilon}^1] + [z^2, \gamma_{n\epsilon}^1] = [\tilde{y}_H^2, \alpha_{n\epsilon}^1] + [\tilde{y}^2, \gamma_{n\epsilon}^1] \tag{2.176}$$

$$[z_H^1, \alpha_{n\epsilon}^2] + [z^1, \gamma_{n\epsilon}^2] = [y_H^1, \alpha_{n\epsilon}^2] + [y^1, \gamma_{n\epsilon}^2] = -\frac{1}{8n}\alpha_{n\epsilon}^3 \ , \ \forall\epsilon, \ \forall n \neq 0 \ , \tag{2.177}$$

where the second row already determines the commutation relation from the earlier quantisation result (2.113). The first row can be processed in the following way: First one writes

$$y^2 = y_\epsilon^2 + y_{-\epsilon}^2 \ , \quad \tilde{y}^2 = \epsilon(y_\epsilon^2 - y_{-\epsilon}^2) \ , \tag{2.178}$$

and similar for y_H^2, $\tilde{y}_H^2$. Second, as was already assumed in various places, one takes that commutators between left- and right-moving operators vanish,

$$[A_\epsilon, B_{-\epsilon}] = 0 \ , \tag{2.179}$$

such that the above relation simplifies to

$$[\tilde{y}_H^2, \alpha_{n\epsilon}^1] + [\tilde{y}^2, \gamma_{n\epsilon}^1] = \epsilon[y_\epsilon^2, \alpha_{n\epsilon}^1] + \epsilon[y_\epsilon^2, \gamma_{n\epsilon}^1,] \ . \tag{2.180}$$

Using again (2.113), multiplied by ϵ, fixes the commutation relation

$$\epsilon[y_\epsilon^2, \alpha_{n\epsilon}^1] + \epsilon[y_\epsilon^2, \gamma_{n\epsilon}^1] = -\epsilon\frac{1}{8n}\alpha_{n\epsilon}^3 \ , \tag{2.181}$$

using the same decomposition. In conclusion, (2.165) has been confirmed independently.

The relations (2.175) are even more powerful, as for example the second line of (2.163) can be found straight away,

$$[z_H^1, p^2] + [z^1, p_H^2] \equiv [y_H^1, p^2] + [y^1, p_H^2] = \frac{\mathrm{i}}{2}y^3 \ , \tag{2.182}$$

using the quantisation result (2.112). In the same way, the right-hand side of the first row of (2.163) can be determined from (2.118),

$$[z_H^1, N^2] = [y_H^1, N^2] = 0 \ . \tag{2.183}$$

Arguing in favour of the particular values of the two missing relations in (2.163) is a bit more involved but possible due to an analogy. From a physical perspective, the coordinates X^1

and Y^2 are in a sense equivalent: they are both the starting point for the T-duality to be performed; they have similar boundary conditions with a simple winding that become very complicated, but again similar, after performing the respective dualities,

$$\begin{aligned} X^1 \to Y^1 \text{ with } Y^1(\tau,\sigma+2\pi) &= Y^1(\tau,\sigma) + 2\pi N^1 + 2\pi N^3 H Y^2_0(\tau,\sigma) \\ Y^2 \to Z^2 \text{ with } Z^2(\tau,\sigma+2\pi) &= Z^2(\tau,\sigma) + 2\pi p^2 + 2\pi N^3 H(-\tilde{Y}^1_0(\tau,\sigma)) + \ldots . \end{aligned} \tag{2.184}$$

Technically speaking, the comparison can be made precise by looking at the classical solutions at order $\mathcal{O}(H^1)$ for X^1_H in (2.211), to be derived in the following section, and Y^2_H in (2.50). They map into each other under the following exchange rules,

$$\begin{aligned} x^1_H &\leftrightarrow y^2_H & y^1 &\leftrightarrow \tilde{y}^2 \\ \tilde{x}^1_H &\leftrightarrow \tilde{y}^2_H & p^1 &\leftrightarrow -N^2 \\ p^1_{HX} &\leftrightarrow p^2_H & N^1 &\leftrightarrow -p^2 \\ \gamma^1_{Xm\epsilon} &\leftrightarrow \gamma^2_{m\epsilon} & \alpha^1_{m\epsilon} &\leftrightarrow -\epsilon\alpha^2_{m\epsilon} \ , \end{aligned} \tag{2.185}$$

where the right column details $Y^1_0 \leftrightarrow -\tilde{Y}^2_0$, which is also present in the boundary conditions shown above. The modes x^1_H, $\tilde{x}^1_H$, p^1_{HX} and $\gamma^1_{Xm\epsilon}$ are also introduced in section 2.5.

The assumption to be made here is that all commutators that involve these modes preserve their value under the given exchange rules. Roughly speaking, that can be phrased by saying that the physical content of the theory in the fibre directions is equal before the respective T-duality is performed. In other words, the underlying claim is that *the order in which the two T-dualities are performed does not matter.*

Having such rules at hand, it is easy to fix the two remaining commutators. For the last line of (2.163), one first uses (2.185) to prove

$$[\tilde{y}^2, p^1_{HX}] + [\tilde{y}^2_H, N^1] = -[y^1, p^2_H] - [y^1_H, p^2] \ , \tag{2.186}$$

where in addition it was used that $\tilde{x}^1_H$ can be replaced by y^1_H similar to the rules in (2.175). Eventually, one finds

$$\begin{aligned} [z^2_H, N^1] = [\tilde{y}^2_H, N^1] &= -[\tilde{y}^2, p^1_{HX}] - [y^1, p^2_H] - [y^1_H, p^2] \\ &= -\frac{\mathrm{i}}{2} y^3 - [\tilde{y}^2, p^1_{HX}] = \frac{\mathrm{i}\pi}{2} N^3 \ , \end{aligned} \tag{2.187}$$

where the first step originates from (2.175), the second step follows from the equation just derived, the third step from plugging in (2.112), and the last one from (2.232), which relates various modes by investigating their dependence via the T-duality rules, to be discussed in section 2.5.

The last commutation relation, namely the first row of (2.163), can be obtained by

$$\begin{aligned} [z^2_H, p^1] + [z^2, p^1_H] &= [\tilde{y}^2_H, p^1] + [\tilde{y}^2, p^1_H] \\ &= [\tilde{y}^2_H, p^1] = [N^2, y^1_H] = 0 = [z^1_H, N^2] \ , \end{aligned} \tag{2.188}$$

where the first step follows from (2.175), the second step from (2.233), the third step from applying (2.185), the fourth step from the quantisation result (2.118), and the last step by matching the already found (2.182).

In summary, the particular values (2.163) and (2.165) that the unknown commutation relations have been given to, are now argued for in a completely independent manner. This still does not make a proof as the argument involved particular physically motivated choices, but adds more convincibility to the result of non-commuting coordinates Z^1 and Z^2.

Origin of non-commutativity[7]

The discussion has shown that it can be argued for non-commuting coordinate fields Z^μ in the non-geometric Q-flux frame. More importantly, this feature seems to be a string theory specific appearance, as it is notably connected to the winding number N^3. And indeed, it is possible to reveal the subtle adjustments that lead to a non-vanishing commutator, and to show that they are rooted in the string oscillation modes. This will give an even stronger argument, that strings are necessary to probe the special structure of spacetime in the non-geometric setup, as particles cannot have winding or oscillation modes.

As can be seen from (2.157) and (2.173), the only relevant part of the commutator $[Z^1, Z^2]$ is given by

$$A \equiv -\frac{\mathrm{i}}{2} N^3 \sum_{n\neq 0} \frac{1}{n^2} e^{-\mathrm{i}n(\sigma'-\sigma)} \ . \tag{2.189}$$

This subsection gives a guideline to the details of how A arises in the above analysis, and shows that T-duality plays a dominant role within the explanation. In what follows, first, the origin of A in the non-geometric frame shall be spotted by tracing two different contributions. After that, it is shown how T-duality induces subtle changes such that there is no A in the two geometric backgrounds.

Non-geometric background: There are two different contributions to A, each of them adding one half of it.

a) The first contribution can be seen in the second line of $[Y_0^1, Z_H^2]$, (2.153). It comes from the zeroth order commutator,

$$[\alpha^1_{m\epsilon}, \alpha^1_{n\epsilon}] = m\ \delta_{m,-n} \ , \tag{2.190}$$

where one of the α^1_m stems from Y_0^1, and the other can be traced back to a particular piece in Z_H^2, namely the sixth line of (2.144),

$$+\frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2}\Big(\left[p_L^1\tilde{\alpha}_n^3 - p_L^3\tilde{\alpha}_n^1\right]e^{-\mathrm{i}n\sigma_+} - \left[p_R^1\alpha_n^3 - p_R^3\alpha_n^1\right]e^{-\mathrm{i}n\sigma_-}\Big) \ . \tag{2.191}$$

After using (2.190), one is left with two terms that add up to a piece proportional to $-(p_L^3 - p_R^3) = -N^3$, giving $\frac{1}{2}A$.

This contribution appears as a particular feature of T-duality in the following sense: the above term can be characterized by its $1/n^2$ dependence. Such a dependence originates from the third line of the solution in the twisted torus frame, (2.50),

$$+\,\theta^\mu{}_{\nu\rho}\,\frac{1}{2}\,\tau\,\left(p^\rho Y_0^\nu|_\Sigma - N^\rho\tilde{Y}_0^\nu|_\Sigma + p^\nu Y_0^\rho|_\Sigma - N^\nu\tilde{Y}_0^\rho|_\Sigma\right) \ , \tag{2.192}$$

and the particular form of the T-duality rules (2.132). To be precise, the crucial point is the relation of σ-derivatives on Z_H^2 to τ-derivatives on Y_H^2, and vice versa,

$$\partial_\tau Z_H^2 = \partial_\sigma Y_H^2 + \dots \ , \quad \partial_\sigma Z_H^2 = \partial_\tau Y_H^2 + \dots \ , \tag{2.193}$$

that after integration produces from (2.192), amongst others, terms with a $1/n^2$ dependence.

[7] This subsection is taken from [2], p. 28-30, almost literally as it was predominantly written by myself.

b) The second contribution comes from the commutator (2.99) between first order oscillators γ_n^μ and zeroth order oscillators α_n^μ,

$$[\gamma^1_{m\epsilon_1}, \alpha^2_{-m\epsilon_2}] - [\gamma^2_{-m\epsilon_2}, \alpha^1_{m\epsilon_1}] = \delta_{\epsilon_1,\epsilon_2}\left(y^3 m - \frac{\mathrm{i}N^3\epsilon_1}{2}\right) . \tag{2.194}$$

Applying it to one part of the first lines of (2.153) and (2.155),

$$\begin{aligned}&\left[Y_0^1(\tau,\sigma), \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^2 e^{-\mathrm{i}n\sigma'_+} - \gamma_n^2 e^{-\mathrm{i}n\sigma'_-}\right)\right] \\ &+\left[\frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}_n^1 e^{-\mathrm{i}n\sigma_+} + \gamma_n^1 e^{-\mathrm{i}n\sigma_-}\right), \tilde{Y}_0^2(\tau,\sigma')\right] ,\end{aligned} \tag{2.195}$$

produces pieces that combine into $\frac{1}{2}A$. It has to be emphasised that due to the $\delta_{\epsilon_1,\epsilon_2}$ in (2.194) only commutators with either two right-moving or two left-moving oscillators are nonzero. As also several combinations of γ_n^μ, $\tilde{\gamma}_n^\mu$ with α_n^μ, $\tilde{\alpha}_n^\mu$ appear, the result is very sensitive to the signs they come with. Schematically, it is

$$\begin{aligned}&[\tilde{\alpha}^1 + \alpha^1, \tilde{\gamma}^2 - \gamma^2] + [\tilde{\gamma}^1 + \gamma^1, \tilde{\alpha}^2 - \alpha^2] \\ &\quad = \left([\gamma^2,\alpha^1] - [\gamma^1,\alpha^2]\right) - \left([\tilde{\gamma}^2,\tilde{\alpha}^1] - [\tilde{\gamma}^1,\tilde{\alpha}^2]\right) \\ &\qquad + \left([\gamma^1,\tilde{\alpha}^2] - [\tilde{\gamma}^2,\alpha^1]\right) + \left([\gamma^2,\tilde{\alpha}^1] - [\tilde{\gamma}^1,\alpha^2]\right) .\end{aligned} \tag{2.196}$$

These are exactly the combinations that are available in (2.194). Here, the two possible permutations for $\epsilon_1 = \epsilon_2$ add up, while the terms in the last row are simply zero.

Table 2.1 gives an overview to the fate of the two contributions a) and b), given in different lines, in the various T-dual backgrounds. The rightmost column depicts the discussion for the non-geometric background we have given so far, where the expression $1+1$ for the contribution b) pays tribute to the subtle sign combination explained above. The middle column depicts the situation in the twisted torus frame, to be described below. The first column shows how the various contributions cancel out when one uses T-duality to go back to the torus with H-flux frame. This shall be commented on in the last section 2.5 of this chapter.

Contribution	H-flux	Twisted torus	Non-geometric
a) $[\alpha,\alpha]$	$-\frac{1}{2}$	0	$\frac{1}{2}$
b) $[\alpha,\gamma]$	$\frac{1+1}{4}$	$\frac{1-1}{4}$	$\frac{1+1}{4}$
Sum	**0**	**0**	**1**

Table 2.1: Contributions to (2.189) in units of A.

Geometric backgrounds: For the twisted torus, two things change when recapitulating the above explanations. First, there is no type a) contribution - depicted by 0 in table 2.1.

This is most easily seen by noting the absence of any term with N^3/n^2 dependence in the expression for Y_H^μ, (2.50), that could contribute. Second, the contribution b) is zero due to a sign change. As explained above, in the non-geometric situation there are two pieces coming from only left-moving and only right-moving oscillators. Here, they appear with the opposite sign and cancel out - depicted by $1-1$ in the table 2.1. Schematically, this can be seen from

$$\begin{aligned}
&[\tilde{\alpha}^1+\alpha^1,\tilde{\gamma}^2+\gamma^2]+[\tilde{\gamma}^1+\gamma^1,\tilde{\alpha}^2+\alpha^2] \\
&\qquad = -\left([\gamma^2,\alpha^1]-[\gamma^1,\alpha^2]\right)-\left([\tilde{\gamma}^2,\tilde{\alpha}^1]-[\tilde{\gamma}^1,\tilde{\alpha}^2]\right) \\
&\qquad\quad +\left([\gamma^1,\tilde{\alpha}^2]-[\tilde{\gamma}^2,\alpha^1]\right)-\left([\gamma^2,\tilde{\alpha}^1]-[\tilde{\gamma}^1,\alpha^2]\right)\ ,
\end{aligned} \tag{2.197}$$

where again the last row vanishes and now the opposite sign of the first term on the left-hand side causes the above mentioned cancellation. The sign change exactly is the well-known sign change of the right-moving oscillators due to T-duality. In summary, there is no term (2.189) appearing in the twisted torus frame thanks to subtle adjustments from T-duality. It follows that the only source of non-commutativity has been removed for the coordinates Y^μ, and this gives a nontrivial proof of consistency for the whole procedure.

2.5 The torus with H-flux

This section presents supplementary considerations to the above investigation: First, it determines the classical solutions X^μ up to first order in the H-expansion for the torus with H-flux frame, which was defined in (2.8). Second, it will show how the T-duality rules relate the expansion coefficients of these X^μ with those of Y^μ. This underlines how the equations of motion in each frame, i.e. either the torus with H-flux frame or the twisted torus frame, are mapped to each other under T-duality. Third, it is shown that the quantisation of the twisted torus is consistent with commuting coordinates in the torus with H-flux frame even when taking into account the relations between expansion coefficients in both frames. In conclusion, these observations support the procedure that was applied to obtain commutation relations in the non-geometric frame.

2.5.1 Classical solution

The flat torus with constant H-flux is defined by the target space fields G and B in (2.8),

$$G=\begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}\ ,\quad B=\begin{pmatrix} 0 & HX^3 & 0 \\ -HX^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ , \tag{2.198}$$

that enter the string action (2.5), together with the torus identifications (2.6). By rescaling the coordinates and the flux parameter,

$$X^\mu \to \frac{1}{R_\mu}X^\mu\ ,\quad H\to HR_1R_2R_3\ , \tag{2.199}$$

similarly to (2.15), the resulting equations of motion simplify to

$$\partial_\alpha\partial^\alpha X^\mu(\tau,\sigma)=H\eta^{\mu\lambda}\epsilon_{\lambda\nu\rho}\partial_\sigma X^\nu\partial_\tau X^\rho\ . \tag{2.200}$$

Here, η is just the identity stemming from the diagonal metric, $\eta = \mathrm{diag}(1,1,1)$, and ϵ is the totally antisymmetric ϵ-tensor in three dimensions. The torus identifications are taken to be fulfilled by the following boundary conditions,

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) + 2\pi N_X^\mu \ . \tag{2.201}$$

This allows for winding of the string, and to differentiate between the winding of the string in the twisted torus frame, the winding number N_X^μ is supplemented by an index 'X'.

The solution, again, is found by expanding in the parameter H as it is taken to be small. This is the dilute flux expansion that was explained in more detail in section 2.2. In particular, also the present configuration solves the target space equations of motion up to order $\mathcal{O}(H^1)$. The coordinates are written as

$$X^\mu(\tau, \sigma) = X_0^\mu(\tau, \sigma) + H X_H^\mu(\tau, \sigma) + \mathcal{O}(H^2) \ , \tag{2.202}$$

which separates the equations of motion into two differential equations, one for each order,

$$\partial_\alpha \partial^\alpha X_0^\mu = 0 \tag{2.203}$$

$$\partial_\alpha \partial^\alpha X_H^\mu = \epsilon^\mu{}_{\nu\rho} \partial_\sigma X_0^\nu \partial_\tau X_0^\rho \ , \tag{2.204}$$

with $\epsilon^\mu{}_{\nu\rho} = \eta^{\mu\lambda}\epsilon_{\lambda\nu\rho}$. By taking this expansion in orders of H and additionally Fourier expanding the solution to the boundary conditions (2.201), the coordinates can be written as

$$X^\mu(\tau, \sigma) = N_X^\mu \sigma + \sum_{n\in\mathbb{Z}} b_{Xn}^\mu(\tau) e^{-\mathrm{i}n\sigma} + H\Big(\sum_{n\in\mathbb{Z}} c_{Xn}^\mu(\tau) e^{-\mathrm{i}n\sigma}\Big) \ . \tag{2.205}$$

When inserting this expansion into the equations of motion, one finds at order $\mathcal{O}(H^0)$ the free string solution with winding,

$$X_0^\mu = x^\mu + p_X^\mu \tau + N_X^\mu \sigma + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_{Xn}^\mu e^{-\mathrm{i}n\sigma_+} + \alpha_{Xn}^\mu e^{-\mathrm{i}n\sigma_-}\right) \ , \tag{2.206}$$

where again all expansion coefficients are supplemented by an index 'X' to distinguish them from the twisted torus frame.

At order $\mathcal{O}(H^1)$, the insertion into the equations of motion results in an ordinary differential equation for the coefficients c_{Xn}^μ in terms of τ, as was encountered in the twisted torus frame,

$$\begin{aligned} -\sum_{n\in\mathbb{Z}}\left(\partial_\tau^2 c_{Xn}^\mu + n^2 c_{Xn}^\mu\right) e^{-\mathrm{i}n\sigma} = \epsilon^\mu{}_{\nu\rho}\Big(& N_X^\nu p_X^\rho + \sum_{n\neq 0}\left(-p_R^\nu \tilde{\alpha}_{Xn}^\rho e^{-\mathrm{i}n\sigma_+} + p_L^\nu \alpha_{Xn}^\rho e^{-\mathrm{i}n\sigma_-}\right) \\ & + \frac{1}{2}\sum_{n,m\neq 0}\left(\tilde{\alpha}_{Xn}^\nu \alpha_{Xm}^\rho e^{-\mathrm{i}n\sigma_+ - \mathrm{i}m\sigma_-}\right)\Big) \ . \end{aligned} \tag{2.207}$$

After separation into Fourier modes $n = 0$ and $n \neq 0$, the equations to be solved are

$$-\partial_\tau^2 c_{X,0}^\mu = \epsilon^\mu{}_{\nu\rho}\Big(N_X^\nu p_X^\rho + \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{Xn}^\nu \alpha_{Xn}^\rho e^{-2\mathrm{i}n\tau}\Big) \tag{2.208}$$

$$-\partial_\tau^2 c_{Xn}^\mu - n^2 c_{Xn}^\mu = \epsilon^\mu{}_{\nu\rho}\Big(-p_R^\nu \tilde{\alpha}_{Xn}^\rho e^{-in\tau} + p_L^\nu \alpha_{X-n}^\rho e^{in\tau} + \frac{1}{2}\sum_{k\neq 0,n}\tilde{\alpha}_{Xk}^\nu \alpha_{Xk-n}^\rho e^{-\mathrm{i}(2k-n)\tau}\Big) \ .$$

The respective solutions are

$$c^\mu_{X,0}(\tau) = x^\mu_H + p^\mu_{HX}\tau - \frac{1}{8}\epsilon^\mu{}_{\nu\rho}\big(4N^\nu_X p^\rho_X \tau^2 - \sum_{n\neq 0}\frac{1}{n^2}\tilde{\alpha}^\nu_{Xn}\alpha^\rho_{Xn}e^{-2\mathrm{i}n\tau}\big) \ , \tag{2.209}$$

and

$$\begin{aligned} c^\mu_{Xn}(\tau) =& \frac{\mathrm{i}}{2n}\left(\tilde{g}^\mu_{Xn}e^{-\mathrm{i}n\tau} - \gamma^\mu_{X-n}e^{\mathrm{i}n\tau}\right) \\ &- \frac{1}{8}\epsilon^\mu{}_{\nu\rho}\big(-\frac{4\mathrm{i}}{n}\tau(p^\nu_R\tilde{\alpha}^\rho_{Xn}e^{-\mathrm{i}n\tau} + p^\nu_L\alpha^\rho_{X-n}e^{\mathrm{i}n\tau}) + \sum_{k\neq 0,n}\frac{1}{k(n-k)}\tilde{\alpha}^\nu_{Xk}\alpha^\rho_{Xk-n}e^{-\mathrm{i}(2k-n)\tau}\big) \ . \end{aligned} \tag{2.210}$$

Altogether, the coordinate solution X^μ_H at order $\mathcal{O}(H^1)$ can be given by

$$\begin{aligned} X^\mu_H(\tau,\sigma) = & \ x^\mu_H + p^\mu_{HX}\ \tau + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{g}^\mu_{Xn}e^{-\mathrm{i}n\sigma_+} + \gamma^\mu_{Xn}e^{-\mathrm{i}n\sigma_-}\right) \\ & - \epsilon^\mu{}_{\nu\rho}\ p^\rho_X N^\nu_X\ \frac{\tau^2}{2} \\ & - \epsilon^\mu{}_{\nu\rho}\ \frac{1}{2}\ \tau\ \left(N^\nu_X X^\rho_0|_\Sigma - p^\nu_X \tilde{X}^\rho_0|_\Sigma\right) \\ & - \epsilon^\mu{}_{\nu\rho}\ \frac{1}{4}\tilde{X}^\nu_0|_\Sigma X^\rho_0|_\Sigma \ . \end{aligned} \tag{2.211}$$

As a side remark, it shall be noted that this solution seems to be the first formulation of the most general solution to the equations of motion (2.203) and (2.204), and the boundary conditions (2.201) in an H-expansion up to order $\mathcal{O}(H^1)$ in the literature. Nevertheless, there have been other formulations, as for example in [31], equation (3.8). The difference to the solution presented here stems from the difference in the imposed boundary conditions. Additionally, this work does not restrict the coordinate solutions to the zero modes, contrary to what [31] proposes in equation (3.40).

2.5.2 Relations from T-duality

Given the fact that the coordinate solutions Y^μ in the twisted torus frame are T-dual to the coordinate solutions X^μ obtained here, it might be expected that the particular T-duality rules (2.124) applied in the $\mu = 1$ direction give a relation between the respective expansion coefficients. This is indeed the case, as will be shown in the following.

The most trivial relations can be obtained in the unaffected directions $\mu = 2,3$ at order $\mathcal{O}(H^0)$. They are given by

$$\partial_\tau X^{2,3}_0 = \partial_\tau Y^{2,3}_0 \ , \quad \partial_\sigma X^{2,3}_0 = \partial_\sigma Y^{2,3}_0 \ , \tag{2.212}$$

which reads as

$$p^{2,3}_X = p^{2,3} \ , \quad N^{2,3}_X = N^{2,3} \tag{2.213}$$

$$\alpha^{2,3}_{Xn} = \alpha^{2,3}_n \ , \quad \tilde{\alpha}^{2,3}_{Xn} = \tilde{\alpha}^{2,3}_n \ , \tag{2.214}$$

when resolved into relations between the single coefficients. Furthermore, the position zero modes are chosen to be

$$x^{2,3} = y^{2,3} \ , \tag{2.215}$$

although they are not affected by the above relation as they drop out when applying the derivative. For the $\mu = 1$ direction the T-duality rule reads as

$$\partial_\tau X_0^1 = \partial_\sigma Y_0^1 \ , \quad \partial_\sigma X_0^1 = \partial_\tau Y_0^1 \ , \tag{2.216}$$

and can be fulfilled by

$$p_X^1 = N^1 \ , \quad N_X^1 = p^1 \ , \tag{2.217}$$

which states that T-duality interchanges momentum and winding, as expected, and

$$\alpha_{Xn}^1 = -\alpha_n^1 \ , \quad \tilde{\alpha}_{Xn}^1 = \tilde{\alpha}_n^1 \ , \tag{2.218}$$

which shows that the right-moving oscillators get an extra minus sign, which is expected as well.

At first order $\mathcal{O}(H^1)$, the solutions for the unaffected directions $\mu = 2, 3$ can again be matched by

$$\partial_\tau X_H^{2,3} = \partial_\tau Y_H^{2,3} \ , \quad \partial_\sigma X_H^{2,3} = \partial_\sigma Y_H^{2,3} \ , \tag{2.219}$$

or equivalently, by

$$p_{HX}^{2,3} = p_H^{2,3} \ , \tag{2.220}$$
$$\gamma_{Xn}^{2,3} = \gamma_n^{2,3} \ , \quad \tilde{g}_{Xn}^{2,3} = \tilde{g}_n^{2,3} \ . \tag{2.221}$$

Additionally, one can choose

$$x_H^{2,3} = y_H^{2,3} \tag{2.222}$$

for the zero modes. The relation between expansion coefficients of the $\mu = 1$ direction at first order is more involved, but can be obtained from the same reasoning. As a starting point, one writes out the T-duality rules in that case, namely

$$\partial_\tau X_H^1 = \partial_\sigma Y_H^1 - H Y_0^3 \partial_\sigma Y_0^2 \ , \quad \partial_\sigma X_H^1 = \partial_\tau Y_H^1 - H Y_0^3 \partial_\tau Y_0^2 \tag{2.223}$$

which now contain an additional product term. The result is a rather complicated combination,

$$\begin{aligned} &2\left(p_{HX}^1 - (N^3 y^2 - N^2 y^3 - \pi N^2 N^3)\right) + \sum_{n\neq 0} \left((\tilde{g}_{Xn}^1 - \tilde{g}_n^1) e^{-in\sigma_+} + (\gamma_{Xn}^1 + \gamma_n^1) e^{-in\sigma_-}\right) \\ &= p^3 \tilde{Y}_0^2|_\Sigma - p^2 \tilde{Y}_0^3|_\Sigma + N^3 Y_0^2|_\Sigma - N^2 Y_0^3|_\Sigma - \tilde{Y}_0^3|_\Sigma \partial_\tau Y_0^2|_\Sigma - Y_0^3|_\Sigma \partial_\tau \tilde{Y}_0^2|_\Sigma - 2y^3 \partial_\tau \tilde{Y}_0^2|_\Sigma \end{aligned} \tag{2.224}$$

$$\begin{aligned} &2\left(-p_H^1 + y^3 p^2\right) + \sum_{n\neq 0} \left([\tilde{g}_{Xn}^1 - \tilde{g}_n^1] e^{-in\sigma_+} - [\gamma_{Xn}^1 + \gamma_n^1] e^{-in\sigma_-}\right) \\ &= N^3 \tilde{Y}_0^2|_\Sigma - N^2 \tilde{Y}_0^3|_\Sigma + p^3 Y_0^2|_\Sigma - p^2 Y_0^3|_\Sigma - Y_0^3|_\Sigma \partial_\tau Y_0^2|_\Sigma - \tilde{Y}_0^3|_\Sigma \partial_\tau \tilde{Y}_0^2|_\Sigma - 2y^3 \partial_\tau Y_0^2|_\Sigma \ . \end{aligned} \tag{2.225}$$

It can be solved by first taking the sum and the difference of the two above equations,

$$\begin{aligned} &2\left(p_{HX}^1 - (N^3 y^2 - N^2 y^3 - \pi N^2 N^3) - p_H^1 + y^3 p^2\right) + 2\sum_{n\neq 0} (\tilde{g}_{Xn}^1 - \tilde{g}_n^1) e^{-in\sigma_+} \\ &= 2\left(2p_L^3 Y_{0L}^2|_\Sigma - 2p_L^2 Y_{0L}^3|_\Sigma - (2Y_{0L}^3|_\Sigma + y^3)\partial_\tau Y_{0L}^2|_\Sigma\right) \end{aligned} \tag{2.226}$$

$$\begin{aligned} &2\left(p_{HX}^1 - (N^3 y^2 - N^2 y^3 - \pi N^2 N^3) + p_H^1 - y^3 p^2\right) + 2\sum_{n\neq 0} (\gamma_{Xn}^1 + \gamma_n^1) e^{-in\sigma_-} \\ &= 2\left(-2p_R^3 Y_{0R}^2|_\Sigma + 2p_R^2 Y_{0R}^3|_\Sigma + (2Y_{0R}^3|_\Sigma + y^3)\partial_\tau Y_{0R}^2|_\Sigma\right) \ , \end{aligned} \tag{2.227}$$

and, second, reading off the conditions on the oscillators,

$$\tilde{g}^1_{Xn} = +\Big(\tilde{g}^1_n + \frac{\mathrm{i}}{n}(p^3_L\tilde{\alpha}^2_n - p^2_L\tilde{\alpha}^3_n) - \frac{1}{2}y_3\tilde{\alpha}^2_n - \frac{\mathrm{i}}{2}\sum_{m\neq 0,n}\frac{1}{m}\tilde{\alpha}^3_m\tilde{\alpha}^2_{n-m}\Big) \tag{2.228}$$

$$\gamma^1_{Xn} = -\Big(\gamma^1_n + \frac{\mathrm{i}}{n}(p^3_R\alpha^2_n - p^2_R\alpha^3_n) - \frac{1}{2}y_3\alpha^2_n - \frac{\mathrm{i}}{2}\sum_{m\neq 0,n}\frac{1}{m}\alpha^3_m\alpha^2_{n-m}\Big) \ . \tag{2.229}$$

and zero modes,

$$p^1_{HX} = p^1_H - y^3p^2 + (N^3y^2 - N^2y^3 - \pi N^2N^3) - \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^3_n\tilde{\alpha}^2_{-n} \tag{2.230}$$

$$p^1_{HX} = -p^1_H + y^3p^2 + (N^3y^2 - N^2y^3 - \pi N^2N^3) + \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^3_n\alpha^2_{-n} \ , \tag{2.231}$$

respectively. The latter relations can be simplified to

$$p^1_{HX} = (N^3y^2 - N^2y^3 - \pi N^2N^3) - \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}^3_n\tilde{\alpha}^2_{-n} - \alpha^3_n\alpha^2_{-n}\right) \tag{2.232}$$

$$p^1_H = y^3p^2 + \frac{\mathrm{i}}{4}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}^3_n\tilde{\alpha}^2_{-n} + \alpha^3_n\alpha^2_{-n}\right) \ . \tag{2.233}$$

At this stage, two comments shall be made:

- Whereas the mapping of the first order oscillator modes reflects the sign change for the right-moving part once more, the investigation of the first order zero modes resulted in only one relation between p^1_{HX} and twisted torus frame coefficients. In addition, one finds a restriction (2.233) of the zero mode p^1_H. This is related to the fact that there is no order $\mathcal{O}(H^1)$ winding in the boundary conditions for the torus with H-flux. As T-duality maps winding to momentum modes, or σ- to τ-derivatives respectively, such a constraint in the first order winding sector of X^μ is translated to a constraint on the first order momentum sector of Y^μ. One may note, that the quantisation result (2.103) is nevertheless fully compatible with the constraint (2.233).
- It can be expected that there is a trivial map between the solutions $X^{2,3}$ and $Y^{2,3}$ in the unaffected directions, as was found in the above derivation, because also the equations of motion are mapped to each other under the T-duality rules (2.124). More concretely, one finds

$$Y^2 \text{ e.o.m. (2.19)} \Leftrightarrow X^2 \text{ e.o.m. (2.200)} \tag{2.234}$$

$$Y^3 \text{ e.o.m. (2.20)} \Leftrightarrow X^3 \text{ e.o.m. (2.200)} \ , \tag{2.235}$$

valid to all orders in H. Exemplarily, the derivation of the first statement shall be shown in the following. One has, starting from (2.200),

$$\begin{aligned}\partial_\alpha\partial^\alpha X^2 &= H\epsilon^2{}_{\nu\rho}\partial_\sigma X^\nu\partial_\tau X^\rho \\ &= H\left(-(\partial_\tau Y^1 - HY^3\partial_\tau Y^2)\partial_\tau Y^3 + \partial_\sigma Y^3(\partial_\sigma Y^1 - HY^3\partial_\sigma Y^2)\right) \\ &= H\left(\partial_\alpha Y^1\partial^\alpha Y^3 - HY^3\partial_\alpha Y^2\partial^\alpha Y^3\right) \ .\end{aligned} \tag{2.236}$$

Together with $\partial_\alpha \partial^\alpha X^2 = \partial_\alpha \partial^\alpha Y^2$, this exactly reproduces (2.19). The other direction can be worked out in the same way, starting from (2.19),

$$\begin{aligned}\partial_\alpha \partial^\alpha Y^2 &= H\left(\partial_\alpha Y^1 \partial^\alpha Y^3 - HY^3 \partial_\alpha Y^2 \partial^\alpha Y^3\right) \\ &= H\Big((\kappa_\alpha^\beta \partial_\beta X^1 + HX^3 \partial_\alpha X^2)\partial^\alpha X^3 - HY^3 \partial_\alpha X^2 \partial^\alpha X^3\Big) \\ &= H\epsilon^2{}_{\nu\rho} \partial_\sigma X^\nu \partial_\tau X^\rho \ ,\end{aligned} \tag{2.237}$$

with $\kappa = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The equations of motion for X and Y in the $\mu = 1$ direction, which is the direction of the duality, are not related to each other. Rather, under the T-duality rules (2.124) they are mapped to trivial conditions in the following way,

$$0 = \partial_\sigma(\partial_\tau X^1) - \partial_\tau(\partial_\sigma X^1) \Rightarrow Y^1 \text{ e.o.m. (2.18)} \ , \tag{2.238}$$

$$0 = \partial_\sigma(\partial_\tau Y^1) - \partial_\tau(\partial_\sigma Y^1) \Rightarrow X^1 \text{ e.o.m. (2.200)} \ , \tag{2.239}$$

which is reminiscent of the well-known statement that a duality rotation interchanges field equations and Bianchi identities [69].

The latter remark shows that it is indeed possible to use the coordinate field T-duality rules to obtain the solutions Z^μ in the non-geometric frame, as it can be expected that they also map the equations of motion in the right way.

Consistency of the quantisation

Table 2.1 shows where the contributions to the non-commutativity in the commutator $[Z^1, Z^2]$ stem from. In addition, it shows that these contributions cancel in the case of the twisted torus, as was explained in the preceding section. For the torus with H-flux, both contributions appear as well, cf. the matching (2.228) and (2.229),

$$\tilde{g}^1_{Xn} = \tilde{g}^1_n + \frac{\mathrm{i}}{n}(p^3_L \tilde{\alpha}^2_n - p^2_L \tilde{\alpha}^3_n) + \dots \ , \quad \gamma^1_{Xn} = -\gamma^1_n - \frac{\mathrm{i}}{n}(p^3_R \alpha^2_n - p^2_R \alpha^3_n) + \dots \ , \tag{2.240}$$

and one might expect to find a contradiction to the coordinates X^μ being commutative. Nevertheless, a closer investigation shows that T-duality induces a subtle sign change such that there appears the right cancellation. In particular, that is: The type a) pieces, according to the table, can be identified from some of the $1/n$ dependent terms in (2.240). Commuting these with $X^2_0|_\Sigma$ produces a term proportional to $(p^3_L - p^3_R) = N^3$, similarly to the non-geometric situation, giving here $-\frac{1}{2}A$. For the type b) pieces, a similar combination of signs as in (2.196) leads to two parts adding up,

$$\begin{aligned}[&\tilde{\alpha}^1 - \alpha^1, \tilde{\gamma}^2 + \gamma^2] + [\tilde{\gamma}^1 - \gamma^1, \tilde{\alpha}^2 + \alpha^2] \\ &= \big([\gamma^2, \alpha^1] - [\gamma^1, \alpha^2]\big) - \big([\tilde{\gamma}^2, \tilde{\alpha}^1] - [\tilde{\gamma}^1, \tilde{\alpha}^2]\big) \\ &\quad - \big([\gamma^1, \tilde{\alpha}^2] - [\tilde{\gamma}^2, \alpha^1]\big) - \big([\gamma^2, \tilde{\alpha}^1] - [\tilde{\gamma}^1, \alpha^2]\big) \ .\end{aligned} \tag{2.241}$$

Nevertheless, in total the two different contributions a) and b) appear with opposite signs and cancel out, as depicted in the first column of the table. Again, there is no term (2.189) remaining thanks to a rearrangement of signs, as expected.

2.6 Summary and discussion

This chapter was devoted to shed light on the relation between non-geometry and non-commutativity of the string coordinates. T-duality was used as a generating tool for non-geometric backgrounds, and starting from a constant H-flux background, both T-dual frames of the chain $H \to f \to Q$ have been investigated. The main results can be summarised as:

- There are strong arguments that non-geometry in its simplest appearance causes non-commutativity of the target space coordinates.
- The coordinates of the two fibre directions have a commutator,

$$[Z^1(\tau,\sigma),\ Z^2(\tau,\sigma')] = -\frac{\mathrm{i}\pi^2}{6}N^3 H\ , \quad \text{for } \sigma' \to \sigma\ . \tag{2.242}$$

- The non-commutativity has a purely stringy origin, as it, first, is proportional to the winding number and thus cannot be detected by a point particle, and, second, can be traced back to oscillator modes of the string itself.

The chapter shall be concluded by the discussion of various aspects of these results and how they are connected to ideas in the literature.

No proof: As was discussed at length in section 2.4.2 the derivation arriving at (2.242) can strictly speaking not be regarded as a proof. It involves the fixation of certain commutation relations that are indeterminable by the proposed procedure. Although there have been given strong arguments that partly involve conclusions by analogy, it is possible to decide on another set of commutation relations such that the non-commutativity vanishes completely, see equation (2.167). There does not seem to be a simple way to overcome this problem as the basic obstruction enters in a manifest way: T-duality can always only relate derivatives of coordinate fields, regardless whether it is viewed as a field redefinition on the worldsheet or as a canonical transformation, cf. [15] equation (2.2.45) for the latter.

Direct quantisation: One might try to avoid the use of T-duality transformations and quantise the string in the Q-flux background directly. Keeping the thesis that canonical quantisation does not work in a non-geometric setup, one could employ the method of Dirac [70] for the quantisation of a system with constraints. In such an approach, the boundary conditions for the string can be taken as constraints, similar to the suggestion in [71, 56], where a Dirac quantisation for open strings was performed.
Contrary to the method applied in this chapter, the boundary conditions for the Q-flux frame then have to be determined beforehand. Ignoring constant shifts, they have the form given in equation (2.148), and as was discussed there, it is possible to determine such a behaviour from an investigation of T-duality transformed monodromies [2] [28], that is not shown in this work.

Non-geometric fluxes as source for non-commutativity: It might be suspected that the 'amount' of non-commutativity is related to the corresponding flux Q in the non-geometric frame. This is analogous to the findings in the literature, where for particular setups the string coordinates turned out to have a non-commuativity proportional to the B-field or H-flux, respectively, see for example [52] for the open string case or [28] for the closed string case. More concretely it shall be proposed that in the limit $\sigma' \to \sigma$, one finds

$$[Z^\mu(\tau,\sigma), Z^\nu(\tau,\sigma')] \sim \mathrm{i} \oint_{C^\rho} Q_\rho{}^{\mu\nu}(Z)\mathrm{d}Z^\rho \ , \tag{2.243}$$

where C^ρ is a cycle around which the string is wrapped N^ρ times. This proposal has been spelled out already in the work [4], which will be discussed in the next chapter.

Indeed, it is necessary to set up a framework that is capable of implementing non-geometric fluxes, before one can discuss the validity of (2.243). Nevertheless, there are various arguments that the Q-flux in the frame Z^μ simply has the same numerical value as the H-flux in the frame X^μ for the toroidal example[8], as they are related to each other by T-duality transformations.

One way or the other, cf. (3.247), the Q-flux can be found[9] to be

$$Q_3{}^{12} = -Q_3{}^{21} = -H \ , \tag{2.244}$$

such that the above suggestion indeed comes with a right-hand side

$$\mathrm{i} \oint_{C^3} Q_3{}^{12}\ \mathrm{d}Z^3 = -2\pi\ \mathrm{i}\ H\ N^3 \ , \tag{2.245}$$

in agreement with the result of this chapter up to a numerical factor that reduces to a difference in conventions. In conclusion, it can be stated that:

> There is reasonable evidence that the non-geometric flux Q is the source for non-commutativity in the Q-flux frame.

String geometry: It has become clear that strings probe geometry differently than point particles do. This has to be understood in at least two different ways. First, by noting the dependence of (2.242) on the winding number, or the integration around a cycle in (2.243), the type of non-geometry investigated here can be characterised as a global feature. Local entities that can be embedded in one single coordinate patch of the space-time do not encounter any difference to a geometric situation. Second, one can formulate an uncertainty principle,

$$(\Delta Z^1)^2(\Delta Z^2)^2 \geqslant H^2 < N^3 >^2 \ , \tag{2.246}$$

that would make the fibre directions "fuzzy". In principle, this would make it impossible to determine or measure (in the sense of quantum mechanics) the string position. Still, the string theory is well-defined, in contrast to the difficulties of defining a point particle quantum field theory on fuzzy spaces. It shall be stressed once more that the non-commutativity derived here only appears in the internal part of the full spacetime manifold, i.e. the compactification space. In this sense, the "fuzziness" will never be observable in the four-dimensional effective space directly.

T-duality as canonical transformation: T-duality can be viewed as a canonical transformation [72, 73] and thus it might be expected that the commutators of coordinates are preserved when changing the frame from Y^μ to Z^μ. This at first sight is in conflict with the

[8] See for example [39], where the value is N, cf. their equation (4.11). For Q this is shown around equation (4.16).

[9] The parameter H has to be added here, as it was neglected in (3.247), or the setup (3.235), respectively.

results presented here, but can be resolved by noting two remarks. First, one would strictly speaking only expect $[\partial_\sigma Z^\mu, \partial'_\sigma Z^\nu]$ to be preserved, which is in absolute agreement with the presented findings. Second, [72] for example does not take into account a B-field and therefore is not applicable here.

Boundary conditions: It was shown that the boundary conditions for the Q-flux frame mix coordinates Z^μ and their dual coordinates $\tilde{Z}^\mu$, see (2.148). In the doubled geometry approach, that will be discussed in chapter 4, this behaviour will be implemented manifestly. Especially, it is easily possible to reproduce the results of an investigation of the monodromies as mentioned above, cf. page 140 ff.

(Non)-associativity: A possible check of the given calculation is to find a vanishing associator, defined by $[Z^3,[Z^1,Z^2]]$ plus permutations. As the non-geometric frame with only Q-flux is suspected to belong to the class of locally well-defined setups, the coordinate fields should better be associative. In contrast, for the case of a background with R-flux, which is the last part of the chain $H \to f \to Q \to R$, this property does not hold necessarily, as has been argued for in [30, 28, 31]. In particular, it was conjectured that the non-associativity is proportional to the R-flux itself, in the same way as here the non-commutativity is proportional to the Q-flux.

Up to linear order, the results presented above give a vanishing associator,

$$
\begin{aligned}
[Z^3,[Z^1,Z^2]] &= [Z_0^3,[Z^1,Z^2]|_H] + [HZ_H^3,[Z_0^1,Z_0^2]] \\
&= -\frac{\mathrm{i}}{2}\frac{\pi^2}{3}H[Z_0^3,N^3] \\
&= 0 \ ,
\end{aligned}
\tag{2.247}
$$

which seems to agree with the assumption that there is no R-flux in the Q-flux frame. Still, the other permutations are to be computed from scratch by extending the canonical quantisation in the twisted torus frame to the directions $(\mu,\nu)=(1,3)$ and $(2,3)$, which will not be done in this work.

The generic case of an R-flux obtained by T-dualising the Q-flux frame seems to be unaccessible by the procedure followed in this chapter, for the simple reason that the absence of a further isometry does not allow to apply the T-duality rules once more. Therefore, it is unclear how to connect the mode expansion of Z^μ with a hypothetical solution in the R-flux background W^μ. Nevertheless, it shall be noted that the doubled geometry approach presented in chapter 4 and the target space investigations presented in chapter 3 are both capable of describing R-flux backgrounds and, although this is not further pursued in this work, in principle could make statements about the associator in question. As a small side remark it shall be mentioned that there are attempts to define T-duality rules in non-isometry directions, as for example in [68].

Three-brackets: In [30], the authors find that the commutator of two string coordinates depends on the worldsheet coordinates, similar to the result (2.242) presented here. But in contrast to the reasoning adopted here, where certain commutation relations were fixed by the claim of worldsheet independent commutators, they conclude that the fundamental object should be a three-bracket and do not investigate the commutator any further.

Chapter 3

Effective field theories

This chapter investigates the appearance of non-geometry in the context of effective field theories, namely of double field theory and supergravity, presenting the results of [7, 6, 4]. It pursues the question of how non-geometric fluxes can be made visible although they do not appear in the standard formulation of such theories. In other words, it shall be asked for an uplift of the fluxes that appear in the four-dimensional T-duality invariant superpotential. Furthermore, the connection between non-geometric fluxes and non-geometry itself will be explored in more detail, with particular emphasis on the question of whether non-geometric configurations can be described by an effective field theory, or geometrised, in other words.

The structure of this chapter is as follows:

3.1 introduces the relevant ideas and connections to the existing literature.

3.2 shows how to reveal non-geometric fluxes in double field theory by employing a field redefinition. It provides a geometrisation of non-geometric fluxes in the sense that they appear as parts of geometric quantities such as the connection.

3.3 uses the analogous field redefinition in ten-dimensional supergravity, where again one type of non-geometric flux can be recovered. The connection to double field theory is made by applying a solution of the strong constraint. A simple dimensional reduction offers the relation to the known four-dimensional superpotential. Eventually, the ill-definedness of the effective field theory action for non-geometric configurations is shown to be remedied in certain cases.

3.4 concludes by giving a summary of the results and remarking observations that may lead to future research directions.

3.1 Introduction

The starting point for this investigation is that there had been a lot of observations on non-geometry and non-geometric fluxes in the literature that still lacked a consistent framework. In particular, several proposals for the definition of non-geometric fluxes in higher dimensions had been made, and often they were not consistent with each other. Non-geometry was characterised as a global feature of string theory, that could appear even in simple cases of double T-dualised three-tori. Some authors used the tools of generalised geometry [74, 75] to enlighten these findings, some concentrated on defining a generalised bracket structure that captures the connection between non-geometry and non-geometric fluxes. The following results are particularly important:

- In [23] it was observed that non-geometric backgrounds can be obtained as T-duals of geometric configurations. This was investigated in the framework of ten-dimensional supergravity with the aim of finding new solutions that are not of Calabi-Yau type. The three-torus with H-flux is mentioned as an introductory example, where its first T-dual, the twisted torus, is described in detail. For the second T-dual, which will here make the Q-flux background, it is said that "the supergravity approximation is not adequate to describe this background." The authors propose as a rule of thumb not to dualise twice along the isometry directions left after a particular choice for the b-field. In summary, the paper is one of the first to mention that it seems possible to leave the range of supergravity approximations by simply following the T-duality transformation rules. This in particular motivates a search of better suited supergravity-like string theory approximations.

- The appearance of non-geometric setups can be linked to a nonzero bi-vector β, as was first observed in [76, 77] for the three-torus with constant H-flux. There, the bi-vector appears as parametrisation of so-called β-transformations in $SU(3) \times SU(3)$ structures of generalised geometry and prevents the existence of a global generalised complex form, i.e. a globally valid model. This strengthens the idea that non-geometry is a global feature, i.e. non-detectable locally, and that it can be captured by the object β. Further evidence in that direction has been given in [78] and [79].

- A first attempt to define non-geometric fluxes in higher dimensions was made in [79]. Using generalised geometry, a bracket for generalised vectors is suggested. Fluxes are seen as components of the generalised structure coefficients, and for particular subcases the non-geometric fluxes Q and R are defined as functions of the bivector β. Again, it is supposed that nonzero non-geometric fluxes can only appear globally, whereas they can be gauged away locally.

- Another proposal for the ten-dimensional definition of non-geometric fluxes was made in [53, 54], where a worldsheet model that incorporates a bivector was equipped with a particular bracket, and, again, the structure coefficients are supposed to contain all types of fluxes.

- It was also suggested to construct a generalised covariant derivative that contains non-geometric fluxes, where its nilpotency implies all Bianchi indentities [40, 79]. Such a derivative can be used to obtain the expected four-dimensional potential [78].

To condensate these observations one could state that the appearance of non-geometry is connected to the appearance of a nonzero bivector β, and this bivector can be used to define non-geometric fluxes Q and R. Therefore, the following procedure shall be pursued: In the context of an effective field theory of string theory, namely in ten-dimensional supergravity or in double field theory, a field redefinition has to be found that makes the bivector β appear. The application of this field redefinition is supposed to bring the respective action into a form that contains non-geometric fluxes and that probably can cope with the ill-definedness of non-geometric setups.

The basic idea of a field redefinition stems from [79], where in equation (4.11) it was proposed to replace g and b by new variables $\tilde{g}$ and β that introduce non-geometric fluxes in the structure coefficients of the generalised bracket. The particular form of the field redefinition was derived from the possibility of different equivalent parametrisations of the generalised metric $\mathcal{H}$, that appears as the metric of the generalised tanget space:

$$\mathcal{H} = \mathcal{E}^T \mathcal{E} = \mathcal{E}'^T \mathcal{E}' \ , \quad \mathcal{E}' = K\mathcal{E} \text{ with } K \in O(2D) \ . \tag{3.1}$$

These parametrisations are equivalent to the internal $O(D)$ invariance of an ordinary metric that is expressed in terms of vielbeins.

Eventually, a field redefinition does not introduce new degrees of freedom. In the particular case here, with $(g, b) \to (\tilde{g}, \beta)$, all degrees of freedom in b are expressed in terms of the new degrees of freedom in β. This is supposed to be in line with the fact that the known examples of non-geometry, namely the T-duals of the three-torus with constant H-flux, are created by the application of dualities and thus, as well, do not carry additional degrees of freedom. On the other hand, it might well be that there are more involved non-geometric setups that exceed the range of known effective field theories, such that a mere field redefinition does not suffice. This will also be discussed in the following.

The present investigation will start with the discussion of a field redefinition in double field theory before applying it in the context of supergravity. It therefore deliberately does not follow the original development in [7, 6, 4], which went the opposite way. The reason is that in double field theory, non-geometric fluxes can be introduced more systematically and their definition is better justified from various aspects, whereas the discussion of non-geometry can be held more conveniently for a supergravity theory.

Furthermore, it will be helpful to introduce the field redefinition in this order for another reason: it is possible to perform a simple dimensional reduction that connects the supergravity construction to the known four-dimensional potential. In addition, it turns out that by integrating out the dual coordinates of double field theory, one can connect the respective framework with the supergravity one. In this sense, it will be shown that the provided implementation of non-geometric fluxes in higher dimensions indeed makes an uplift of the proposed [33] fluxes in four dimensions.

Finally, the whole discussion will be restricted to the NSNS sector only. The most obvious reason is that the general idea of a field redefinition rests on the parametrisation invariance of the generalised metric. In double field theory, extra objects have to be introduced [80, 81, 82] to capture fermionic degrees of freedom and it is not clear how to implement a field redefinition there. Furthermore, it will turn out that the field redefinition is tightly connected to T-duality, and there are only very few suggestions in the direction of fermionic T-duality [83, 84]. Accordingly, the following regulation shall be made: whenever "supergravity" is referred to, actually the NSNS part of supergravity is meant.

3.2 Double field theory

This sections shows how to systematically reveal non-geometric fluxes in double field theory. The derivation follows three basic steps:

First, the above mentioned field redefinition is formulated and applied to the double field theory action. The result has to be rewritten in a way that makes fluxes appear, which will be done following covariance under diffeomorphisms as an additional guiding principle. Thus, the second step is to find derivatives that are covariant with respect to one particular part of the double field theory gauge symmetry.

As the field redefined action also contains winding derivatives, this task cannot be solved by merely introducing ordinary Christoffel symbols. Rather, a very specific connection will be used.

In the third step, various observations that can be made in the course of covariantisation will be put together in order to find the correct definition of the non-geometric fluxes. It turns out that R appears as the covariant field strength of β, whereas Q will make the antisymmetric part of the new connection. Additional support for these settings is given by the final rewriting of the action, where the respective objects appear as expected: R as a square, Q in the scalar curvature terms.

3.2.1 Field redefinition

In the following, some basic aspects of double field theory shall be presented to set up the framework, and then the details of the field redefinition are developed. Eventually, these can be summarised by a recipe that consists of simple replacement rules to be applied straightforwardly.

Setup

A spacetime of dimension $D = n + d$ is considered. It shall have the form of a product[1],

$$\mathbb{R}^{n-1,1} \times T^d \ . \tag{3.2}$$

Double field theory is a field theory that is formulated using a doubled set of coordinates. All fields depend on both the coordinates x^i associated to the string momentum modes, and the dual coordinates $\tilde{x}_i$ associated to the winding modes. The index i lies in the range $1, \ldots, D$. Consequently, there are two types of derivatives,

$$\partial_i = \frac{\partial}{\partial x^i} \ , \quad \tilde{\partial}^i = \frac{\partial}{\partial \tilde{x}_i} \ . \tag{3.3}$$

Coordinates and derivatives are grouped into $2D$-objects,

$$X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \ , \quad \partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix} \ , \tag{3.4}$$

with the index M lying in the range $1, \ldots, 2D$. It shall be noted that the $2D$-vector X^M has the dual coordinates in the upper half, followed by the coordinates, which collides for example with the conventions to be used in chapter 4.

[1] Cf. footnote 1 of [18].

Indices M are raised and lowered with the constant matrix

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0 \end{pmatrix} , \tag{3.5}$$

that is also used to define the $O(D,D)$ group,

$$h_{MN} \in O(D,D) \quad \Leftrightarrow \quad h^T \eta h = \eta \ . \tag{3.6}$$

It will often be taken advantage of the fact that η is its own inverse,

$$(\eta^{-1})^{MN} = \eta_{MN} \quad \forall M,N \ . \tag{3.7}$$

There are several formulations of the double field theory action [16, 17, 18, 81]. One that makes many symmetries manifest can be expressed in terms of the generalised metric $\mathcal{H}$, which contains the conventional string theory metric[2] g_{ij} and the antisymmetric tensor b_{ij} in a particular way,

$$\mathcal{H}^{MN} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix} , \tag{3.8}$$

where the inverse metric is denoted with upper indices, $(g^{-1})^{ij} = g^{ij}$.

The generalised metric appears in various theories with different functions. Here it is taken as a mere device to group the physical fields g and b. It indeed also carries properties of a metric in the doubled space, but one has to be careful as the geometry is not the usual one [85, 86]. In the context of generalised geometry, $\mathcal{H}$ is one of the admissible metrics for the generalised tangent space. In the doubled worldsheet model to be presented in chapter 4 it makes one of the possible forms the kinetic term may take. In all formalisms, a conjugation of $\mathcal{H}$ by certain $O(D,D)$ matrices allows to model T-duality transformations, see (A.24).

Using that the generalised metric is symmetric, $\mathcal{H}^T = \mathcal{H}$, one can easily show that it is an $O(D,D)$ matrix as defined above,

$$\mathcal{H}^T\eta\mathcal{H} = \begin{pmatrix} bg^{-1} & g - bg^{-1}b \\ g^{-1} & -g^{-1}b \end{pmatrix} \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} = \eta \ . \tag{3.9}$$

This, in particular, establishes the compatibility between raising and lowering the indices M with η and denoting the inverse of $\mathcal{H}^{MN}$ by $\mathcal{H}_{MN}$,

$$\mathcal{H}_{MN}\mathcal{H}^{NP} = \eta_{MK}\mathcal{H}^{KL}\eta_{LN}\mathcal{H}^{NP} = \eta_{MK}\eta^{KP} = \mathbb{1}_M^P \ . \tag{3.10}$$

Because all fields are taken to depend on the doubled set of coordinates, i.e.

$$g_{ij} = g_{ij}(x,\tilde{x}) \ , \quad b_{ij} = b_{ij}(x,\tilde{x}) \ , \tag{3.11}$$

any consistent double field theory action has to be constrained by the so-called "strong constraint",

$$\partial_i\tilde{\partial}^i \cdot = 0 \ , \tag{3.12}$$

where the dot $\cdot$ does not only stand for any field or gauge parameter, but also for any product of such. In particular, one has

$$\partial_i A\tilde{\partial}^i B + \partial_i B\tilde{\partial}^i A = 0 \ . \tag{3.13}$$

[2] This chapter employs the usual convention that the coordinates x, as well as the metric and the b-field are denoted by lower-case letters in effective field theories.

This guarantees the reduction of the degrees of freedom to the standard ones and, in a sense, makes the core of how double field theory is related to supergravity. Also the consistency of double field theory relies on the strong constraint, as for example its invariance under gauge transformations and also the equivalence of its various formulations can only be shown up to this condition. The strong constraint can be motivated from the closed string theory level matching constraint $L_0 - \bar{L}_0 = 0$, cf. [16].

It is possible to rewrite the strong constraint as

$$\partial^M \partial_M \cdot = 0 \ , \tag{3.14}$$

to show that it is $O(D, D)$ invariant. Moreover, for products one finds

$$\partial^M A \partial_M B = 0 \ . \tag{3.15}$$

From a physical point of view, the strong constraint locally renders all fields independent of either the coordinates or the dual coordinates. This, of course, may not simply generalise to a global solution $\tilde{\partial}^i = 0$ when going from one patch to another, although this solution is often assumed to show certain properties of the theory.

One possible $2D$-dimensional effective action of double field theory is given by

$$\begin{aligned} S \ = \ \int \mathrm{d}x \mathrm{d}\tilde{x}\, e^{-2d} \Big(& \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL}\, \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL}\, \partial_L \mathcal{H}_{MK} \\ & - 2\, \partial_M d\, \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN}\, \partial_M d\, \partial_N d \Big) \ . \end{aligned} \tag{3.16}$$

It has been constructed in [18]. Apart from the string theory metric and antisymmetric tensor, that are packed into the generalised metric, it contains the dilaton field $d(x, \tilde{x})$, that will later on be related to the D-dimensional supergravity dilaton.

One remarkable feature of this action is its manifest $O(D, D)$ invariance, which can easily be checked by noting the proper contraction of all indices. This invariance can be interpreted as T-duality invariance and makes one of the main motivations for investigating double field theory, as has been discussed in the introduction.

Instead of using the above action, the following subsection employs another formulation, developed earlier in [17], that is completely equivalent to (3.16),

$$\begin{aligned} S \ = \ \int \mathrm{d}x \mathrm{d}\tilde{x}\ e^{-2d} \Big(& -\frac{1}{4}\, g^{ik} g^{jl}\, g^{pq} \Big(\mathcal{D}_p \mathcal{E}_{kl}\, \mathcal{D}_q \mathcal{E}_{ij} - \mathcal{D}_i \mathcal{E}_{lp} \mathcal{D}_j \mathcal{E}_{kq} - \bar{\mathcal{D}}_i \mathcal{E}_{pl}\, \bar{\mathcal{D}}_j \mathcal{E}_{qk} \Big) \\ & + g^{ik} g^{jl} \big(\mathcal{D}_i d\ \bar{\mathcal{D}}_j \mathcal{E}_{kl} + \bar{\mathcal{D}}_i d\ \mathcal{D}_j \mathcal{E}_{lk} \big) + 4 g^{ij} \mathcal{D}_i d\, \mathcal{D}_j d \Big) \ , \end{aligned} \tag{3.17}$$

with calligraphic derivatives defined as

$$\mathcal{D}_i \ = \ \partial_i - \mathcal{E}_{ik} \tilde{\partial}^k \ , \quad \bar{\mathcal{D}}_i \ = \ \partial_i + \mathcal{E}_{ki} \tilde{\partial}^k \ . \tag{3.18}$$

The key object of (3.17) is the, from the sigma model point of view, natural combination

$$\mathcal{E}_{ij} = g_{ij} + b_{ij} \ , \tag{3.19}$$

that, to underline it, is neither symmetric nor antisymmetric. A formulation in terms of $\mathcal{E}$ is of great convenience when performing the field redefinition later on.

There are several important properties of the action (3.17), four of which shall be discussed in more detail in the following.

Background independence: The first formulation of a double field theory action in [16] was presented in terms of a fluctuation e_{ij} around a constant background $E_{ij} = G_{ij} + B_{ij}$. This background, nevertheless, entered the action directly and it was not obvious that double field theory inherited background independence from string theory.

Technically speaking, background independence is defined, cf. [17], as that a constant part of the fluctuation field e_{ij} can be absorbed into a change of the background field E_{ij}. Denoting this constant part as χ_{ij}, it shall be claimed that

$$S[E_{ij}, e_{ij} + \chi_{ij}] = S[E_{ij} + \chi_{ij}, e'_{ij}] \ , \tag{3.20}$$

where it is allowed that e_{ij} is changed by a field redefinition. Indeed, it was possible to prove background independence of the original double field theory action, by confirming $\delta S = 0$, up to quadratic order in e_{ij}, under

$$\begin{aligned} \delta e_{ij} &= \chi_{ij} + \mathcal{O}(e^1) \\ \delta E_{ij} &= -\chi_{ij} \ . \end{aligned} \tag{3.21}$$

Moreover, the object $\mathcal{E}$, being

$$\mathcal{E}_{ij} = E_{ij} + e_{ij} + \mathcal{O}(e^2) \ , \tag{3.22}$$

turns out to be background independent in the sense discussed here,

$$\delta\mathcal{E}_{ij} = 0 + \mathcal{O}(e^2) \ . \tag{3.23}$$

This implies that also the metric g_{ij} and its inverse are background independent, as well as the calligraphic derivatives (3.18).

In conclusion, the double field theory action (3.17) is written in terms of background independent objects only and thus itself is background independent in the above sense. It shall be noted, that the dilaton d does not change this result.

T-duality invariance: For an arbitrary $O(D,D)$ matrix element

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D,D) \ , \tag{3.24}$$

the action (3.17) can be shown to be invariant under the transformations

$$\begin{aligned} X &\to h\,X \\ \partial &\to h^{-T}\partial \\ \mathcal{E} &\to (a\mathcal{E} + b)(c\mathcal{E} + d)^{-1} \ , \end{aligned} \tag{3.25}$$

where the dilaton transforms as a scalar $d'(X') = d(X)$. In particular, each term of (3.17) is invariant separately, which will become important later on.

For a spacetime (3.2) with constant background fields, string theory possesses an $O(d,d,\mathbb{Z})$ T-duality symmetry that can here easily be seen as a subgroup of the $O(D,D)$ symmetry of double field theory. In this sense, double field theory, especially in the formulation (3.16), but also as (3.17), is a T-duality invariant framework.

Gauge invariance: Double field theory possesses a gauge symmetry that can be parametrised by an $O(D,D)$ vector $\xi = (\xi^i, \tilde{\xi}_i)$, i.e. $\delta S = 0$ for

$$\delta_\xi \mathcal{E}_{ij} = \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i + \mathcal{L}_\xi \mathcal{E}_{ij} + \mathcal{E}_{ik} \left(\tilde{\partial}^q \xi^k - \tilde{\partial}^k \xi^q \right) \mathcal{E}_{qj} + \mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} \tag{3.26}$$

$$\delta_\xi d = -\frac{1}{2} \left(\partial_i \xi^i + \tilde{\partial}^i \tilde{\xi}_i \right) + \xi^i \partial_i d + \tilde{\xi}_i \tilde{\partial}^i d \ , \tag{3.27}$$

up to the strong constraint. In this expression, $\mathcal{L}_\xi$ denotes the standard Lie derivative with respect to ξ^i and $\mathcal{L}_{\tilde{\xi}}$ the dual Lie derivative with respect to $\tilde{\xi}_i$,

$$\mathcal{L}_\xi \mathcal{E}_{ij} = \xi^k \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \mathcal{E}_{kj} + \partial_j \xi^k \mathcal{E}_{ik} \tag{3.28}$$

$$\mathcal{L}_{\tilde{\xi}} \mathcal{E}_{ij} = \tilde{\xi}_k \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \tilde{\xi}_i \mathcal{E}_{kj} - \tilde{\partial}^k \tilde{\xi}_j \mathcal{E}_{ik} \ . \tag{3.29}$$

To prove the gauge invariance in the form of (3.17), a long and rather elaborate calculation is necessary due to the non-linear transformation behaviour of $\mathcal{E}$, cf. [17]. In contrast, the generalised metric can be found to transform linearly,

$$\delta_\xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} - \left(\partial^M \xi_P - \partial_P \xi^M \right) \mathcal{H}^{PN} + \left(\partial^N \xi_P - \partial_P \xi^N \right) \mathcal{H}^{MP} \ , \tag{3.30}$$

and gauge invariance in the form of (3.16) can be proven more elegantly, cf. [18].

Supergravity reduction: It is possible to reduce (3.17) to the standard NSNS supergravity action in ten dimensions. To halve the number of coordinates a global solution to the strong constraint is imposed, $\tilde{\partial}^i = 0$, that renders all fields independent of the dual coordinates. The reduction then discards a volume factor $\int \mathrm{d}\tilde{x}$. Furthermore, the dilaton field $d(x)$ is turned into the conventional dilaton $\phi(x)$ by a field redefinition,

$$\sqrt{|g|}\, e^{-2\phi} = e^{-2d} \ . \tag{3.31}$$

After dropping the integral of a total derivative, one finds that (3.17) reduces to

$$S_* = \int \mathrm{d}x \sqrt{|g|}\, e^{-2\phi} \Big(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \Big) \ , \tag{3.32}$$

with H-flux

$$H_{ijk} = 3\partial_{[i} b_{jk]} \ , \tag{3.33}$$

and the curvature scalar $\mathcal{R}$ constructed from the metric g.

The gauge transformation (3.26) of double field theory for the considered limit $\tilde{\partial}^i = 0$ reduces to

$$\delta_\xi \mathcal{E}_{ij} = \mathcal{L}_\xi \mathcal{E}_{ij} + 2\partial_{[i} \tilde{\xi}_{j]} \ , \tag{3.34}$$

which is exactly the combination of a diffeomorphism parametrised by the vector field ξ^i and a two-form gauge transformation parametrised by the dual one-form $\tilde{\xi}_i$.

Field redefinition

Inspired by the observations discussed in the introduction to this chapter the first step in the analysis to follow is to implement a field redefinition that makes visible the bivector β, i.e.

$$(g_{ij},\, b_{ij},\, \phi) \quad \to \quad (\tilde{g}_{ij},\, \beta^{ij},\, \tilde{\phi}) \ . \tag{3.35}$$

It shall be expressed by changing the parametrisation of the generalised metric,

$$\mathcal{H}^{MN} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{ij} & -\tilde{g}_{ik}\beta^{kj} \\ \beta^{ik}\tilde{g}_{kj} & \tilde{g}^{ij} - \beta^{ik}\tilde{g}_{kl}\beta^{lj} \end{pmatrix} , \tag{3.36}$$

which corresponds to the internal $O(2D)$ symmetry of the generalised vielbeins that determine it, cf. (3.1).

The notation is as follows: redefined fields get an extra tilde, except β that corresponds to b in the original variables. For the derivatives this does not produce confusion, as the derivative without tilde and the dual one with tilde have to be exchanged under the field redefinition, which will be explained below.

A short calculation brings the four equations in (3.36) into a convenient form: First, one reads off the upper row and finds two equations that determine the old field g and b in terms of the new ones,

$$\begin{aligned} g &= \left(\tilde{g}^{-1} - \beta\tilde{g}\beta\right)^{-1} = (\tilde{g}^{-1} \pm \beta)^{-1}\tilde{g}^{-1}(\tilde{g}^{-1} \mp \beta)^{-1} \\ &= \tfrac{1}{2}\left((\tilde{g}^{-1} + \beta)^{-1} + (\tilde{g}^{-1} - \beta)^{-1}\right) \end{aligned} \tag{3.37}$$

$$b = -(\tilde{g}^{-1} \pm \beta)^{-1}\beta(\tilde{g}^{-1} \mp \beta)^{-1} = \tfrac{1}{2}\left((\tilde{g}^{-1} + \beta)^{-1} - (\tilde{g}^{-1} - \beta)^{-1}\right) . \tag{3.38}$$

For the second equation, on has to use

$$bg^{-1} = -\tilde{g}\beta = -\frac{1}{2}\tilde{g}\left((\tilde{g}^{-1} + \beta) - (\tilde{g}^{-1} - \beta)\right) . \tag{3.39}$$

Second, it is immediately clear that g and b can be expressed as the symmetric and antisymmetric part, respectively, of the inverse of $(\tilde{g}^{-1} + \beta)$, or in other words,

$$(\tilde{g}^{-1} + \beta)^{-1} = g + b = \mathcal{E} . \tag{3.40}$$

By defining

$$\tilde{\mathcal{E}}^{ij} = \tilde{g}^{ij} + \beta^{ij} , \tag{3.41}$$

the field redefinition (3.35) can be brought into the form

$$\tilde{\mathcal{E}}^{-1} = \mathcal{E} . \tag{3.42}$$

The transformation of the dilaton is defined such as to leave the double field theory dilaton field or the NSNS measure, respectively, invariant,

$$\sqrt{|g|}e^{-2\phi} = e^{-2d} = \sqrt{|\tilde{g}|}e^{-2\tilde{\phi}} . \tag{3.43}$$

It now turns out that the $O(D,D)$ invariance of the double field theory action (3.17) can easily be exploited to avoid any computation when performing the field redefinition (3.42). First, one can check that a transformation (3.25) with $h = \eta$ brings $\mathcal{E}$ to its inverse,

$$\mathcal{E}(\tilde{x}, x) \to \mathcal{E}'(\tilde{x}', x') = \mathcal{E}^{-1}(x, \tilde{x}) , \tag{3.44}$$

where the transformation $X \to \eta X$ has been written out explicitly and the indices of the right-hand side are lowered by δ^i_j implicitly. Such a transformation could be regarded as a T-duality transformation in all directions if it was not for the non-compact directions.

Second, by comparing the left- and right-hand side of (the index structure is again implicitly guaranteed by insertions of δ^i_j)

$$(\tilde{g}^{-1}+\beta)(\tilde{x},x)=\tilde{\mathcal{E}}(\tilde{x},x)=\mathcal{E}'(x,\tilde{x})=(g'+b')(x,\tilde{x})\ , \tag{3.45}$$

one can define the rules that are equivalent to the field redefinition (3.42). Once taken into account that each term of the action (3.17) is invariant under any $O(D,D)$ transformation separately, these rules read

$$g'\to\tilde{g}^{-1}\ ,\quad b'\to\beta\ ,\quad \partial_i\leftrightarrow\tilde{\partial}^i\ , \tag{3.46}$$

where the last exchange takes care of undoing the transformation of X.

To summarise, starting from (3.17) one performs an $O(D,D)$ transformation with $h=\eta$ and then replaces all

$$\mathcal{E}'_{ij}\to\tilde{\mathcal{E}}^{ij}\ ,\quad \mathcal{D}'_i\ \to\tilde{\mathcal{D}}^i=\tilde{\partial}^i-\tilde{\mathcal{E}}^{ik}\partial_k\ ,\quad \bar{\mathcal{D}}'_i\ \to\bar{\tilde{\mathcal{D}}}^i=\tilde{\partial}^i+\tilde{\mathcal{E}}^{ki}\partial_k\ . \tag{3.47}$$

The result is

$$\begin{aligned}S'=\int\mathrm{d}x\mathrm{d}\tilde{x}\ e^{-2d}\Big(&-\frac{1}{4}\,\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\left(\tilde{\mathcal{D}}^p\tilde{\mathcal{E}}^{kl}\,\tilde{\mathcal{D}}^q\tilde{\mathcal{E}}^{ij}-\tilde{\mathcal{D}}^i\tilde{\mathcal{E}}^{lp}\tilde{\mathcal{D}}^j\tilde{\mathcal{E}}^{kq}-\bar{\tilde{\mathcal{D}}}^i\tilde{\mathcal{E}}^{pl}\,\bar{\tilde{\mathcal{D}}}^j\tilde{\mathcal{E}}^{qk}\right)\\&+\tilde{g}_{ik}\tilde{g}_{jl}\big(\tilde{\mathcal{D}}^id\ \bar{\tilde{\mathcal{D}}}^j\tilde{\mathcal{E}}^{kl}+\bar{\tilde{\mathcal{D}}}^id\ \tilde{\mathcal{D}}^j\tilde{\mathcal{E}}^{lk}\big)+4\tilde{g}_{ij}\tilde{\mathcal{D}}^id\,\tilde{\mathcal{D}}^jd\Big)\ .\end{aligned} \tag{3.48}$$

In this form it is far from obvious, that the field redefinition has revealed any hidden feature of non-geometric fluxes. Contrariwise, it looks like the rewriting has not helped at all. An additional guiding principle is needed, in order to reformulate the above action in a more enlightening way. This principle is covariantisation.

3.2.2 Covariantisation

In the following, the simple claim of using covariant objects in the field redefined action (3.48) will help to reformulate it. The idea is that, basically, one would like to replace the non-covariant derivatives $\mathcal{D}$ by proper covariant derivatives. On the other hand, double field theory possesses a rather involved gauge symmetry (3.26) that goes far beyond ordinary Riemannian geometry in $2D$ dimensions[3]. It is difficult to have derivatives that are covariant with respect to the full gauge symmetry, and the introduction of ordinary Christoffel symbols is certainly not enough.

Interestingly, one half of the double field theory gauge transformations is sufficient to include ordinary diffeomorphisms for the field redefined metric. Taking this as a guideline, the covariantisation shall be restricted to only that half of the gauge parameter space. In particular, it will then be necessary to find winding derivatives that covariantise diffeomorphisms.

As the preceding section has implemented the field redefinition (3.35), the starting point here is the correspondingly redefined gauge transformation. In particular, by applying (3.47) and its extension to the gauge parameter, namely

$$\xi^i\ \leftrightarrow\ \tilde{\xi}_i\ , \tag{3.49}$$

[3]See also the recent proposals of generalised diffeomorphisms, [86, 85].

to (3.26), one can obtain the gauge transformation rule for the new field $\tilde{\mathcal{E}}$,

$$\delta\tilde{\mathcal{E}}^{ij} = \mathcal{L}_{\tilde{\xi}}\tilde{\mathcal{E}}^{ij} + \tilde{\partial}^i\xi^j - \tilde{\partial}^j\xi^i + \mathcal{L}_\xi\tilde{\mathcal{E}}^{ij} - \tilde{\mathcal{E}}^{ik}\big(\partial_k\tilde{\xi}_l - \partial_l\tilde{\xi}_k\big)\tilde{\mathcal{E}}^{lj} \ . \tag{3.50}$$

The respective Lie derivatives for contravariant objects are defined as

$$\mathcal{L}_{\tilde{\xi}}\tilde{\mathcal{E}}^{ij} = \tilde{\xi}_k\tilde{\partial}^k\tilde{\mathcal{E}}^{ij} + \tilde{\partial}^i\tilde{\xi}_k\tilde{\mathcal{E}}^{kj} + \tilde{\partial}^j\tilde{\xi}_k\tilde{\mathcal{E}}^{ik} \tag{3.51}$$
$$\mathcal{L}_\xi\tilde{\mathcal{E}}^{ij} = \xi^k\partial_k\tilde{\mathcal{E}}^{ij} - \partial_k\xi^i\tilde{\mathcal{E}}^{kj} - \partial_k\xi^j\tilde{\mathcal{E}}^{ik} \ . \tag{3.52}$$

One can see, that ξ^i parametrises diffeomorphisms for $\tilde{g}$ and β in the fourth term of (3.50), whereas transformations parametrised by $\tilde{\xi}_i$ act non-linearly. Following the idea discussed above, only the ξ^i gauge transformations shall be considered from here on, i.e. the other half of the gauge parameter ξ^M is set to zero,

$$\tilde{\xi}_i = 0 \ . \tag{3.53}$$

Of course, as the whole process to follow is only a particular rewriting of the action (3.48), invariance under the remaining half of the gauge transformations is retained, but will be hidden.

In this limit, one can deduce the following transformation rules for the fundamental fields $\tilde{g}$ and β that make the symmetric and antisymmetric part of $\tilde{\mathcal{E}}$, respectively,

$$\begin{aligned}\delta_\xi\tilde{g}_{ij} &= \mathcal{L}_\xi\tilde{g}_{ij} \\ \delta_\xi\beta^{ij} &= \mathcal{L}_\xi\beta^{ij} + \tilde{\partial}^i\xi^j - \tilde{\partial}^j\xi^i \ .\end{aligned} \tag{3.54}$$

The construction of covariant derivatives with respect to these transformations will be pursued step by step in the following. It is therefore useful to denote the non-covariant part of a gauge varied object by

$$\Delta_\xi \equiv \delta_\xi - \mathcal{L}_\xi \ . \tag{3.55}$$

In other words, any object is considered transforming properly, i.e. "covariantly", if it undergoes a diffeomorphism parametrised by the gauge parameter ξ^i. This is indeed the case for the redefined metric,

$$\Delta_\xi\tilde{g}_{ij} = 0 \ , \tag{3.56}$$

but, more importantly, not the case for the redefined antisymmetric tensor,

$$\Delta_\xi\beta^{ij} = \tilde{\partial}^i\xi^j - \tilde{\partial}^j\xi^i \ , \tag{3.57}$$

as both can be seen from (3.54).

Scalars

Any scalar, by definition, transforms covariantly in the above sense. Taking the dilaton $\tilde{\phi}$ as an example, the transformation rule therefore has to be

$$\delta_\xi\tilde{\phi} = \xi^i\partial_i\tilde{\phi} \ , \tag{3.58}$$

to fulfill $\Delta_\xi\tilde{\phi} = 0$. Not surprisingly, taking the winding derivative of a scalar quantity will not result in a covariant object,

$$\delta_\xi(\tilde{\partial}^i\tilde{\phi}) = \tilde{\partial}^i(\xi^j\partial_j\tilde{\phi}) = \xi^j\partial_j(\tilde{\partial}^i\tilde{\phi}) + \tilde{\partial}^i\xi^j\partial_j\tilde{\phi} \ . \tag{3.59}$$

A simple observation helps to find the proper derivative operator: one can add

$$-\partial_j\xi^i\tilde{\partial}^j\tilde{\phi} - \tilde{\partial}^j\xi^i\partial_j\tilde{\phi} = 0 \ , \tag{3.60}$$

which is zero up to the strong constraint in its form (3.13), to the above equation and find

$$\begin{aligned}\delta_\xi(\tilde{\partial}^i\tilde{\phi}) &= \xi^j\partial_j(\tilde{\partial}^i\tilde{\phi}) - \partial_j\xi^i\tilde{\partial}^j\tilde{\phi} + (\tilde{\partial}^i\xi^j - \tilde{\partial}^j\xi^i)\partial_j\tilde{\phi} \\ &= \mathcal{L}_\xi(\tilde{\partial}^i\tilde{\phi}) + \Delta_\xi\beta^{ij}\ \partial_j\tilde{\phi} \ .\end{aligned} \tag{3.61}$$

Given this result, the obvious definition

$$\tilde{D}^i \equiv \tilde{\partial}^i - \beta^{ij}\partial_j \ , \tag{3.62}$$

is indeed the correct first step in finding a fully covariant derivative,

$$\Delta_\xi(\tilde{D}^i\tilde{\phi}) = 0 \ . \tag{3.63}$$

This statement can also be checked by simply writing out the Lie derivative,

$$\begin{aligned}\mathcal{L}_\xi(\tilde{D}^i\tilde{\phi}) &= \xi^k\partial_k(\tilde{D}^i\tilde{\phi}) - \tilde{D}^k\tilde{\phi}\partial_k\xi^i \\ &= \xi^k\partial_k\tilde{\partial}^i\tilde{\phi} - \xi^k\partial_k\beta^{ij}\partial_j\tilde{\phi} - \beta^{ij}\xi^k\partial_j\partial_k\tilde{\phi} - \tilde{\partial}^k\tilde{\phi}\partial_k\xi^i + \beta^{km}\partial_m\tilde{\phi}\partial_k\xi^i \\ &= \delta_\xi(\tilde{\partial}^i\tilde{\phi}) - (\delta_\xi\beta^{ik})\partial_k\tilde{\phi} - \beta^{ik}\delta_\xi(\partial_k\tilde{\phi}) \\ &= \delta_\xi(\tilde{D}^i\tilde{\phi}) \ ,\end{aligned} \tag{3.64}$$

where the strong constraint had to be used in the third line.

Although this is only the first step in the proposed procedure, the commutator of two such derivative operators provides important insights into the structure of the non-geometric fluxes by suggesting their respective definitions. It can be computed as

$$\begin{aligned}\big[\tilde{D}^i, \tilde{D}^j\big]\tilde{\phi} &= (\tilde{\partial}^i - \beta^{im}\partial_m)(\tilde{\partial}^j - \beta^{jk}\partial_k)\tilde{\phi} \ - \ (i \leftrightarrow j) \\ &= -2\tilde{\partial}^{[i}\beta^{j]k}\partial_k\tilde{\phi} - 2\beta^{m[i}\partial_m\beta^{j]k}\partial_k\tilde{\phi} \\ &= -3\big(\tilde{\partial}^{[i}\beta^{jk]} + \beta^{m[i}\partial_m\beta^{jk]}\big)\partial_k\tilde{\phi} - \partial_k\beta^{ij}(\tilde{\partial}^k - \beta^{km}\partial_m)\tilde{\phi} \\ &= -3\tilde{D}^{[i}\beta^{jk]}\partial_k\tilde{\phi} - \partial_k\beta^{ij}\tilde{D}^k\tilde{\phi} \ ,\end{aligned} \tag{3.65}$$

and by defining the following objects,

$$Q_k{}^{ij} \equiv \partial_k\beta^{ij} \tag{3.66}$$

$$R^{ijk} \equiv 3\tilde{D}^{[i}\beta^{jk]} \ , \tag{3.67}$$

will be simplified to

$$\big[\tilde{D}^i, \tilde{D}^j\big] = -R^{ijk}\partial_k - Q_k{}^{ij}\tilde{D}^k \ . \tag{3.68}$$

These objects will indeed become the non-geometric fluxes Q and R later on, but here it shall in particular be noted that R^{ijk} is a covariant object, or in other words can be regarded as the field strength of β^{ij}. To prove this property, one can again simply compute the Lie derivative and compare it with the result of a gauge transformation (3.54),

$$\begin{aligned}\mathcal{L}_\xi R^{ijk} &= \xi^m\partial_m R^{[ijk]} - 3R^{m[ij}\partial_m\xi^{k]} \\ &= 3\xi^m\partial_m\tilde{\partial}^{[i}\beta^{jk]} + 3\xi^m\partial_m\beta^{n[i}\partial_n\beta^{jk]} + 3\xi^m\beta^{n[i}\partial_m\partial_n\beta^{jk]} + 3\tilde{\partial}^m\xi^{[i}\partial_m\beta^{jk]} \\ &\quad - 6\partial_m\xi^{[i}\tilde{\partial}^j\beta^{k]m} + 6\beta^{m[i}\partial_n\xi^j\partial_m\beta^{k]n} - 3\beta^{mn}\partial_m\beta^{[ij}\partial_n\xi^{k]} \\ &= \delta_\xi R^{ijk} \ ,\end{aligned} \tag{3.69}$$

where the strong constraint was used at several places. Eventually, this is

$$\Delta_\xi R^{ijk} = 0 \ . \tag{3.70}$$

The object ${Q_k}^{ij}$ fails to be a covariant 2, 1-tensor, as can be suspected from the appearance of a simple partial derivative. Its definition is given based on similar suggestions in the literature, e.g. (4.13) of [79] or (40) of [54], and will be further justified by its correct appearance in the Bianchi identity (3.105) and in the four-dimensional potential (3.220) later on.

Vectors and Tensors

The above constructed derivative operator $\tilde{D}$ can now be used to define covariant derivatives for any vector V^i or co-vector V_i,

$$\begin{aligned} \tilde{\nabla}^i V^j &= \tilde{D}^i V^j - \check{\Gamma}_k{}^{ij} V^k \\ \tilde{\nabla}^i V_j &= \tilde{D}^i V_j + \check{\Gamma}_j{}^{ik} V_k \ , \end{aligned} \tag{3.71}$$

where a new connection $\check{\Gamma}$ has been introduced, that shall be determined in terms of the physical fields $\tilde{g}$ and β in the following. The covariant derivatives defined here are taken to extend in the usual way to any tensor as

$$\tilde{\nabla}^i T^{j_1 j_2 \dots}_{k_1 k_2 \dots} = \tilde{D}^i T^{j_1 j_2 \dots}_{k_1 k_2 \dots} + \check{\Gamma}_{k_1}{}^{im} T^{j_1 j_2 \dots}_{m k_2 \dots} + \dots - \check{\Gamma}_m{}^{i j_1} T^{m j_2 \dots}_{k_1 k_2 \dots} - \dots \ . \tag{3.72}$$

A first condition on the connection $\check{\Gamma}$ comes from the claim that the derivative of a vector is a covariant tensor, namely

$$\Delta_\xi(\tilde{\nabla}^i V^j) \stackrel{!}{=} 0 \ . \tag{3.73}$$

This will be imposed in two steps: First, by a short calculation, one can determine the failure of $\tilde{D}^i V^j$ to be a covariant tensor,

$$\begin{aligned} \Delta_\xi(\tilde{D}^i V^j) &= \delta_\xi\big((\tilde{\partial}^i + \beta^{mi}\partial_m)V^j\big) - \mathcal{L}_\xi\big((\tilde{\partial}^i + \beta^{mi}\partial_m)V^j\big) \\ &= -V^k \tilde{\partial}^i \partial_k \xi^j + \beta^{im} V^k \partial_m \partial_k \xi^j \\ &= -\tilde{D}^i \partial_k \xi^j V^k \ . \end{aligned} \tag{3.74}$$

Second, one notes the following rules for determining the non-covariant part of a sum of two arbitrary objects A and B, or the contraction of any object ${C_k}^{ij}$ with a covariant vector V^i, respectively,

$$\Delta_\xi(A + B) = \Delta_\xi A + \Delta_\xi B \tag{3.75}$$

$$\Delta_\xi({C_p}^{ij} V^p) = (\Delta_\xi {C_p}^{ij}) V^p \ . \tag{3.76}$$

The rather surprising second relation can be proven as follows:

$$\begin{aligned} \Delta_\xi({C_p}^{ij} V^p) &= (\delta_\xi {C_p}^{ij}) V^p + {C_p}^{ij} \delta_\xi V^p - \mathcal{L}_\xi({C_p}^{ij} V^p) \\ &= (\mathcal{L}_\xi {C_p}^{ij}) V^p + (\Delta_\xi {C_p}^{ij}) V^p + {C_p}^{ij} \mathcal{L}_\xi V^p \\ &\quad - \xi^k \partial_k({C_p}^{ij} V^p) + \partial_k \xi^i {C_p}^{kj} V^p + \partial_k \xi^j {C_p}^{ik} V^p \\ &= (\Delta_\xi {C_p}^{ij}) V^p \ , \end{aligned} \tag{3.77}$$

where in the second row $\Delta_\xi V^p = 0$ was used. Eventually, by applying these two rules to the left-hand side of (3.73) and taking into account the result (3.74), one can conclude that the connection has to transform non-covariantly,

$$\Delta_\xi \breve{\Gamma}_k{}^{ij} = -\tilde{D}^i \partial_k \xi^j \ . \tag{3.78}$$

This implies that the antisymmetric part of $\breve{\Gamma}$ cannot be set to zero, as it, as well, does not transform covariantly. In other words, the connection has to have torsion.

A second condition on the connection shall be that it is metric compatible,

$$\tilde{\nabla}^i \tilde{g}^{jk} = \tilde{D}^i \tilde{g}^{jk} - \breve{\Gamma}_p{}^{ij} \tilde{g}^{pk} - \breve{\Gamma}_p{}^{ik} \tilde{g}^{jp} \stackrel{!}{=} 0 \ . \tag{3.79}$$

This allows to separate the symmetric part as

$$\breve{\Gamma}_k{}^{(ij)} = \tilde{\Gamma}_k{}^{ij} - \tilde{g}_{kl}\Big(\tilde{g}^{pi}\breve{\Gamma}_p{}^{[jl]} + \tilde{g}^{pj}\breve{\Gamma}_p{}^{[il]}\Big) \ , \tag{3.80}$$

where an abbreviation for the Christoffel symbols in terms of the new derivative $\tilde{D}^i$ has been used,

$$\tilde{\Gamma}_k{}^{ij} = \frac{1}{2}\tilde{g}_{kl}\left(\tilde{D}^i \tilde{g}^{jl} + \tilde{D}^j \tilde{g}^{il} - \tilde{D}^l \tilde{g}^{ij}\right) \ . \tag{3.81}$$

To determine the antisymmetric part of $\breve{\Gamma}$ completely, a third condition has to be imposed. It is not derived from an abstract principle but can rather be viewed as a particular choice that is consistent with the construction so far. The idea is to investigate the commutator of two covariant derivatives acting on a scalar. Taking into account the definition as $\tilde{\nabla}^i \tilde{\phi} = \tilde{D}^i \tilde{\phi}$ and the condition (3.73), the result has to be a covariant tensor by construction. Using the result (3.68), this translates to

$$\begin{aligned}\big[\tilde{\nabla}^i, \tilde{\nabla}^j\big]\tilde{\phi} &= \big[\tilde{D}^i, \tilde{D}^j\big]\tilde{\phi} - \breve{\Gamma}_k{}^{ij}\tilde{D}^k\tilde{\phi} + \breve{\Gamma}_k{}^{ji}\tilde{D}^k\tilde{\phi} \\ &= -R^{ijk}\partial_k\tilde{\phi} - \big(Q_k{}^{ij} + 2\breve{\Gamma}_k{}^{[ij]}\big)\tilde{D}^k\tilde{\phi} \ .\end{aligned} \tag{3.82}$$

It can be shown, that the second term of the right-hand side is a covariant object itself:

$$\begin{aligned}\Delta_\xi\Big(\big(Q_k{}^{ij} + 2\breve{\Gamma}_k{}^{[ij]}\big)\tilde{D}^k\tilde{\phi}\Big) &= \Delta_\xi\big(Q_k{}^{ij} + 2\breve{\Gamma}_k{}^{[ij]}\big)\tilde{D}^k\tilde{\phi} \\ &= \big(2\tilde{D}^{[i}\partial_k\xi^{j]} - 2\tilde{D}^{[i}\partial_k\xi^{j]}\big)\tilde{D}^k\tilde{\phi} \\ &= 0 \ .\end{aligned} \tag{3.83}$$

In the first row, (3.76) has been used. The second row used (3.78) and

$$\Delta_\xi Q_k{}^{ij} = 2\tilde{D}^{[i}\partial_k\xi^{j]} \ , \tag{3.84}$$

which can be checked straightforwardly. It is therefore a covariant restriction of the constructed derivatives to set

$$\big[\tilde{\nabla}^i, \tilde{\nabla}^j\big]\tilde{\phi} \stackrel{!}{=} -R^{ijk}\partial_k\tilde{\phi} \ , \tag{3.85}$$

which is the same as

$$\breve{\Gamma}_k{}^{[ij]} = -\frac{1}{2}Q_k{}^{ij} \ . \tag{3.86}$$

This solves for the new connection $\check{\Gamma}$ without any left-over ambiguity, and by adding (3.80) and (3.86), it is given as

$$\check{\Gamma}_k{}^{ij} \;=\; \tilde{\Gamma}_k{}^{ij} + \tilde{g}_{kl}\tilde{g}^{p(i}Q_p{}^{j)l} - \frac{1}{2}Q_k{}^{ij} \;. \tag{3.87}$$

It can now be checked explicitly that this connection transforms as required by (3.78). The calculation is tedious, but straightforward for the symmetric part:

$$\begin{aligned}
\Delta_\xi \check{\Gamma}_k{}^{(ij)} &= \delta_\xi \check{\Gamma}_k{}^{(ij)} - \mathcal{L}_\xi \check{\Gamma}_k{}^{(ij)} \\
&= \tilde{g}_{kl}\tilde{\partial}^m \tilde{g}^{l(i}\partial_m \xi^{j)} + \tilde{g}_{kl}\partial_m \tilde{g}^{l(i}\tilde{\partial}^{j)}\xi^m - \partial_k \tilde{\partial}^{(i}\xi^{j)} - \tilde{g}_{kl}\tilde{g}^{m(i}\partial_m \tilde{g}^{j)}\xi^l \\
&\quad - \tfrac{1}{2}\tilde{g}_{kl}\tilde{\partial}^l \xi^m \partial_m \tilde{g}^{ij} + \tilde{g}_{kl}\tilde{\partial}^l \partial_m \xi^{(i}\tilde{g}^{j)m} - \tfrac{1}{2}\tilde{g}_{km}\partial_l \xi^m \tilde{\partial}^l \tilde{g}^{ij} \\
&\quad - \tilde{g}_{kl}\partial_p \tilde{g}^l \xi^{(i}\tilde{g}^{j)p} + \tilde{g}_{kl}\tilde{g}^{p(i}\partial_p \tilde{\partial}^{j)}\xi^l + \tilde{g}_{kl}\partial_p \partial_m \xi^l \beta^{m(i}\tilde{g}^{j)p} + \tilde{g}_{kl}\beta^{lm}\partial_p \partial_m \xi^{(i}\tilde{g}^{j)p} \\
&\quad - \tilde{g}_{kl}\tilde{\partial}^{[n}\xi^{l]}\partial_n \tilde{g}^{ij} + \tilde{g}_{kl}\beta^{nl}\partial_n \partial_m \xi^{(i}\tilde{g}^{j)m} - \tilde{g}_{kl}\beta^{n(i}\partial_n \partial_m \xi^{j)}\tilde{g}^{ml} \\
&\quad + \tilde{g}_{kl}\tilde{\partial}^n \xi^{(i}\partial_n \tilde{g}^{j)l} - \tilde{g}_{kl}\partial_n \tilde{g}^{l(i}\tilde{\partial}^{j)}\xi^n - \tilde{g}_{kl}\beta^{n(i}\tilde{g}^{j)m}\partial_n \partial_m \xi^l \\
&= -\partial_k \tilde{\partial}^{(i}\xi^{j)} - \beta^{m(i}\partial_k \partial_m \xi^{j)} \\
&= -\tilde{D}^{(i}\partial_k \xi^{j)} \;.
\end{aligned} \tag{3.88}$$

The antisymmetric part can be determined from (3.84) immediately, so that in total one finds

$$\begin{aligned}
\Delta_\xi \check{\Gamma}_k{}^{ij} &= \Delta_\xi \check{\Gamma}_k{}^{(ij)} + \Delta_\xi \check{\Gamma}_k{}^{[ij]} \\
&= -\tilde{D}^{(i}\partial_k \xi^{j)} - \tilde{D}^{[i}\partial_k \xi^{j]} \\
&= -\tilde{D}^i \partial_k \xi^j \;.
\end{aligned} \tag{3.89}$$

This concludes the determination of a derivative for arbitrary tensors, that makes the ξ^i diffeomorphisms manifest. In summary, only two conditions had to be imposed, namely (3.79) and (3.85). A third condition, (3.73), is satisfied automatically.

Torsion

The antisymmetric part of the connection $\check{\Gamma}$ does not transform covariantly, as can be seen from (3.78). Thus, it is not possible to define a torsion tensor in the standard way.

Interestingly, the trace of the connection, to be denoted by $\mathcal{T}^i$,

$$\mathcal{T}^i = \check{\Gamma}_k{}^{ki} = \frac{1}{2}\tilde{g}_{pq}\tilde{D}^i \tilde{g}^{pq} - Q_k{}^{ki} \;, \tag{3.90}$$

is transforming covariantly,

$$\Delta_\xi \check{\Gamma}_k{}^{ki} = -\partial_k \tilde{\partial}^k \xi^i + \beta^{kp}\partial_p \partial_k \xi^i = 0 \;, \tag{3.91}$$

using the strong constraint. From now on, this vector $\mathcal{T}^i$ shall be called torsion. It appears at various places and will be necessary to rewrite the double field theory action in the desired manner later on. Here, two examples of its occurrence shall be given: First, for an integration by parts, one finds

$$\int \mathrm{d}x\mathrm{d}\tilde{x}\sqrt{|\tilde{g}|}\, V_i \tilde{\nabla}^i W = -\int \mathrm{d}x\mathrm{d}\tilde{x}\sqrt{|\tilde{g}|}\, W\left(\tilde{\nabla}^i V_i - 2\mathcal{T}^i V_i\right) \;, \tag{3.92}$$

for an object W and a co-vector V_i. To prove this relation, it is useful to note

$$\tilde{D}^i\sqrt{|\tilde{g}|} = -\frac{1}{2}\sqrt{|\tilde{g}|}\;\tilde{g}_{pq}\tilde{D}^i\tilde{g}^{pq} = -\sqrt{|\tilde{g}|}\;\left(\mathcal{T}^i + Q_k{}^{ki}\right)\;. \tag{3.93}$$

Second, the torsion appears as the inhomogenous part of a "covariant strong constraint",

$$\tilde{\nabla}^i\nabla_i\tilde{\phi} = \mathcal{T}^i\nabla_i\tilde{\phi} \neq 0\;, \tag{3.94}$$

with ∇_i being the ordinary covariant derivative of Riemannian geometry with a Levi-Civita connection based on the metric $\tilde{g}$.

Riemann tensor and curvature

A Riemann tensor for the covariant derivative can be defined by applying the commutator of two derivatives to a co-vector. The straightforward calculation shows

$$\begin{aligned}\left[\tilde{\nabla}^i,\tilde{\nabla}^j\right]V_k &= \left[\tilde{D}^i,\tilde{D}^j\right]V_k + 2\tilde{D}^{[i}\breve{\Gamma}_k{}^{j]p}V_p + 2\breve{\Gamma}_k{}^{[i|m|}\breve{\Gamma}_m{}^{j]p}V_p \\ &\quad + Q_m{}^{ij}\tilde{D}^m V_k + Q_m{}^{ij}\breve{\Gamma}_k{}^{mp}V_p \\ &= -R^{ijp}\nabla_p V_k + \breve{\mathcal{R}}^{ij}{}_k{}^l V_l\;,\end{aligned} \tag{3.95}$$

where (3.68) was used and the Riemann tensor $\breve{\mathcal{R}}^{ij}{}_k{}^l$ is defined as

$$\breve{\mathcal{R}}^{ij}{}_k{}^l = \tilde{D}^i\breve{\Gamma}_k{}^{jl} - \tilde{D}^j\breve{\Gamma}_k{}^{il} + \breve{\Gamma}_k{}^{iq}\,\breve{\Gamma}_q{}^{jl} - \breve{\Gamma}_k{}^{jq}\,\breve{\Gamma}_q{}^{il} + Q_q{}^{ij}\,\breve{\Gamma}_k{}^{ql} - R^{ijq}\,\Gamma^l{}_{qk}\;. \tag{3.96}$$

Given that the left-hand side of (3.95) is covariant, as well as the first term of its right-hand side, one can conclude that the Riemann tensor defined here is a covariant object as well,

$$\Delta_\xi\breve{\mathcal{R}}^{ij}{}_k{}^l = 0\;. \tag{3.97}$$

It is antisymmetric in the first two indices i and j by construction, which is also reflected in the definition. But additionally, it can be shown that $\breve{\mathcal{R}}$ is antisymmetric in the last two indices k and l. This is done in two steps: First, the commutator of two covariant derivatives is applied to a vector,

$$\left[\tilde{\nabla}^i,\tilde{\nabla}^j\right]V^k \;=\; -R^{ijp}\nabla_p V^k - \breve{\mathcal{R}}^{ij}{}_l{}^k V^l\;. \tag{3.98}$$

Second, this is compared to the relation for a co-vector (3.95) with a raised index k,

$$\left[\tilde{\nabla}^i,\tilde{\nabla}^j\right]V^k \;=\; -R^{ijp}\nabla_p V^k + \breve{\mathcal{R}}^{ijkl}V_l\;, \tag{3.99}$$

with the desired result,

$$\breve{\mathcal{R}}^{ijkl} \;=\; -\breve{\mathcal{R}}^{ijlk}\;. \tag{3.100}$$

In contrast to the ordinary Riemann tensor, $\breve{\mathcal{R}}$ has no exchange symmetry between the two index pairs. This can be seen by spelling out the Bianchi identity that follows from the Jacobiator,

$$0 \stackrel{!}{=} \Big(\left[\tilde{\nabla}^i,\left[\tilde{\nabla}^j,\tilde{\nabla}^k\right]\right] + \left[\tilde{\nabla}^j,\left[\tilde{\nabla}^k,\tilde{\nabla}^i\right]\right] + \left[\tilde{\nabla}^k,\left[\tilde{\nabla}^i,\tilde{\nabla}^j\right]\right]\Big)\tilde{\phi} \tag{3.101}$$

$$\begin{aligned}&= -\tilde{\nabla}^{[i}R^{jk]l}\,\partial_l\tilde{\phi} - R^{l[ij}\left[\tilde{\nabla}^{k]},\nabla_l\right]\tilde{\phi} + \breve{\mathcal{R}}^{[ij}{}_l{}^{k]}\,\tilde{\nabla}^l\tilde{\phi} \\ &= -4\tilde{\nabla}^{[i}R^{jkl]}\,\nabla_l\tilde{\phi} + 3\breve{\mathcal{R}}^{[ij}{}_l{}^{k]}\,\tilde{\nabla}^l\tilde{\phi} + \nabla_p R^{ijk}\,\tilde{D}^p\tilde{\phi}\;,\end{aligned} \tag{3.102}$$

where in the second row (3.85) and (3.98) were used. The third row was simplified by noting

$$\big[\tilde{\nabla}^i, \nabla_l\big]\tilde{\phi} = \breve{\Gamma}_l{}^{pi}\partial_p\tilde{\phi} - \Gamma^i{}_{lp}\tilde{D}^p\tilde{\phi}\ , \tag{3.103}$$

and a relation that is implied by the strong constraint,

$$\tilde{D}^l R^{ijk}\,\partial_l\tilde{\phi} + \partial_l R^{ijk}\,\tilde{D}^l\tilde{\phi} = 0\ . \tag{3.104}$$

Furthermore, there is a differential Bianchi identity for the R-flux tensor, namely

$$\tilde{\nabla}^{[i}R^{jkl]} = 0\ , \tag{3.105}$$

that can be checked by straightforwardly spelling out its explicit form,

$$4\,\tilde{\partial}^{[i}R^{jkl]} + 4\,\beta^{p[i}\partial_p R^{jkl]} + 6\,Q_p{}^{[ij}R^{kl]p} = 0\ . \tag{3.106}$$

This relation can be compared to (78) of [87] and (11) of [88], which by construction do not contain a dual derivative. Nevertheless, the agreement with these equations makes further evidence that the definitions of the non-geometric fluxes (3.66) and (3.67) are correct.

Applying the above relation turns the Jacobiator into an algebraic Bianchi identity,

$$3\breve{\mathcal{R}}^{[ij}{}_l{}^{k]} + \nabla_l R^{ijk} = 0\ , \tag{3.107}$$

or, after raising the index l,

$$\breve{\mathcal{R}}^{ijkl} + \breve{\mathcal{R}}^{jkil} + \breve{\mathcal{R}}^{kijl} = \nabla^l R^{ijk}\ . \tag{3.108}$$

The failure of the Riemann tensor $\breve{\mathcal{R}}$ to have an index pair exchange symmetry can be deduced from this Bianchi identity by simply writing out particular index permutations,

$$\breve{\mathcal{R}}^{ijkl} - \breve{\mathcal{R}}^{klij} = \nabla^{[i}R^{j]kl} - \nabla^{[k}R^{l]ij}\ . \tag{3.109}$$

Interestingly, the right-hand side is only nonzero for nonzero R-flux, which fits well the picture that this type of flux arises in geometries that do not even have an ordinary local appearance.

It shall be noted that the above (3.107) can be written out explicitly,

$$\partial_l R^{ijk} = 3\big(\tilde{D}^{[i}Q_l{}^{jk]} - Q_q{}^{[ij}Q_l{}^{k]q}\big)\ , \tag{3.110}$$

and then, in the limit $\tilde{\partial}^i = 0$, exactly reduces to equation (75) in [87].

The construction of a scalar curvature is straightforward. First one can define a Ricci tensor by setting

$$\breve{\mathcal{R}}^{ij} \equiv \breve{\mathcal{R}}^{ki}{}_k{}^j\ . \tag{3.111}$$

This quantity is defined uniquely as there is only one independent non-vanishing contraction of the Riemann tensor, due to its symmetry properties. Explicitly, the Ricci tensor reads

$$\begin{aligned}\breve{\mathcal{R}}^{ij} &= \tilde{D}^k\breve{\Gamma}_k{}^{ij} - \tilde{D}^i\breve{\Gamma}_k{}^{kj} + \breve{\Gamma}_k{}^{ij}\,\breve{\Gamma}_q{}^{qk} - \breve{\Gamma}_p{}^{ki}\,\breve{\Gamma}_k{}^{pj} \\ &= \tilde{D}^k\breve{\Gamma}_k{}^{ij} - \breve{\Gamma}_q{}^{ki}\,\breve{\Gamma}_k{}^{qj} - \tilde{\nabla}^i\mathcal{T}^j\ ,\end{aligned} \tag{3.112}$$

where the first row used (3.86), and the second row used the definition of the torsion tensor $\mathcal{T}^i$ in (3.90). Again, contrary to ordinary differential geometry, the symmetry properties of the Ricci tensor are unusual, which can be inferred from the unusual Bianchi identity (3.107). One can find that the antisymmetric part is non-vanishing and given by the R-flux,

$$\breve{\mathcal{R}}^{[ij]} = -\frac{1}{2}\nabla_k R^{kij} \ . \tag{3.113}$$

This can be checked by tracing (3.109) or by direct computation. Nevertheless, the scalar curvature is defined in the usual way,

$$\breve{\mathcal{R}} \equiv \tilde{g}_{ij}\breve{\mathcal{R}}^{ij} \ . \tag{3.114}$$

It is by construction a scalar under diffeomorphisms parametrised by ξ^i. This completes the collection of covariant objects that are necessary to rewrite the field redefined double field theory action.

3.2.3 Rewriting the action

In the following, it shall be shown that it is possible to use the covariantisation discussed above to rewrite the double field theory action (3.48), which was obtained from performing the field redefinition (3.35).

As a surprising result, all terms occurring organise into simple structures when expressed in terms of the covariant winding derivatives. In particular, the R-flux (3.67) appears in a field strength term, i.e. as a square. Moreover, there will appear two Einstein-Hilbert terms, one that is associated to ordinary derivatives ∂_i, and another that stems from the scalar curvature $\breve{\mathcal{R}}$. The latter contains the Q-flux as antisymmetric part of the connection, such that both non-geometric fluxes have obtained a geometric role.

Because the computation is lengthy and technical, the final result shall be given first. It is

$$\boxed{S' = \int \mathrm{d}x\mathrm{d}\tilde{x}\, \sqrt{|\tilde{g}|}\, e^{-2\tilde{\phi}}\Big(\mathcal{R} + \breve{\mathcal{R}} - \frac{1}{12}R^{ijk}R_{ijk} + 4(\partial\tilde{\phi})^2 + 4\big(\tilde{D}^i\tilde{\phi} + \mathcal{T}^i\big)^2\Big) \ .} \tag{3.115}$$

In principle, this can be shown by simply writing it out and comparing it with (3.48). But as it necessitates many integrations by parts, the reader shall now be guided through this process.

The first step is to introduce the new derivative (3.62), which can be done by noting

$$\tilde{\mathcal{D}}^i = -\tilde{g}^{ij}\partial_j + \tilde{D}^i \ , \quad \bar{\tilde{\mathcal{D}}}^i = \tilde{g}^{ij}\partial_j + \tilde{D}^i \ . \tag{3.116}$$

It is furthermore convenient to note a consequence of the strong constraint, namely

$$\tilde{D}^i A\partial_i B + \partial_i A\tilde{D}^i B = 0 \ . \tag{3.117}$$

for any fields A and B. This helps to rewrite the last term of the action S' in (3.48), as it is

$$\partial_j d\tilde{D}^j d = 0 \ , \tag{3.118}$$

such that the result is

$$4\tilde{g}_{ij}\tilde{\mathcal{D}}^i d\,\bar{\tilde{\mathcal{D}}}^j d = 4\big(\tilde{g}^{ij}\partial_i d\partial_j d + \tilde{g}_{ij}\tilde{D}^i d\tilde{D}^j d\big) \ . \tag{3.119}$$

The other two terms in the second row of (3.48), namely the off-diagonal dilaton terms, can be expanded to

$$\tilde{g}_{ik}\tilde{g}_{jl}\tilde{\mathcal{D}}^i d\,\tilde{\overline{\mathcal{D}}}^j\tilde{\mathcal{E}}^{kl} + \tilde{g}_{ik}\tilde{g}_{jl}\,\tilde{\overline{\mathcal{D}}}^i d\,\tilde{\mathcal{D}}^j\tilde{\mathcal{E}}^{lk} = -2\big(\partial_k d\,\partial_l\tilde{g}^{kl} + \tilde{g}_{jl}\,\partial_k d\,\tilde{D}^j\beta^{kl} - \tilde{g}_{ik}\,\tilde{D}^i d\,\partial_l\beta^{kl} - \tilde{g}_{ik}\tilde{g}_{jl}\tilde{D}^i d\,\tilde{D}^j\tilde{g}^{kl}\big)\ . \tag{3.120}$$

The first term that is quadratic in $\tilde{\mathcal{E}}$ and can be expanded to

$$-\frac{1}{4}\,\tilde{g}_{ik}\,\tilde{g}_{jl}\,\tilde{g}_{pq}\,\tilde{\mathcal{D}}^p\tilde{\mathcal{E}}^{kl}\,\tilde{\mathcal{D}}^q\tilde{\mathcal{E}}^{ij} = -\frac{1}{4}\,\tilde{g}_{ik}\,\tilde{g}_{jl}\tilde{g}^{rs}\partial_r\tilde{g}^{kl}\,\partial_s\tilde{g}^{ij} - \frac{1}{4}\,\tilde{g}_{ik}\,\tilde{g}_{jl}\tilde{g}^{rs}\partial_r\beta^{kl}\,\partial_s\beta^{ij} - \frac{1}{4}\,\tilde{g}_{ik}\,\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\tilde{g}^{kl}\,\tilde{D}^q\tilde{g}^{ij} + \tilde{D}^p\beta^{kl}\,\tilde{D}^q\beta^{ij}\big)\ , \tag{3.121}$$

where the strong constraint (3.13) was used to cancel some terms. Eventually, the introduction of the new derivative (3.62) can be completed by noting that the remaining two terms expand to

$$\frac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{\mathcal{D}}^i\tilde{\mathcal{E}}^{lp}\,\tilde{\mathcal{D}}^j\tilde{\mathcal{E}}^{kq} + \tilde{\overline{\mathcal{D}}}^i\tilde{\mathcal{E}}^{pl}\,\tilde{\overline{\mathcal{D}}}^j\tilde{\mathcal{E}}^{qk}\big) = \frac{1}{2}\tilde{g}_{pq}\,\partial_k\tilde{g}^{lp}\,\partial_l\tilde{g}^{kq} + \frac{1}{2}\tilde{g}_{pq}\,\partial_k\beta^{lp}\,\partial_l\beta^{kq} + \frac{1}{2}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^i\tilde{g}^{lp}\tilde{D}^j\tilde{g}^{kq} + \tilde{D}^i\beta^{lp}\tilde{D}^j\beta^{kq}\big) - \tilde{g}_{jl}\tilde{g}_{pq}\big(\partial_k\beta^{lp}\,\tilde{D}^j\tilde{g}^{kq} + \partial_k\tilde{g}^{lp}\tilde{D}^j\beta^{kq}\big)\ . \tag{3.122}$$

The next step is to collate all terms from (3.119), (3.120), (3.121) and (3.122) that are independent of β and contain only standard derivatives ∂_i. Concretely, these are the respective first terms of each piece. Their sum exactly makes the curvature scalar $\mathcal{R}$ with respect to standard derivatives and the standard connection Γ, plus the kinetic term for the dilaton in terms of the standard derivative, up to a total derivative. Namely, by using

$$\partial_i d = \partial_i\tilde{\phi} + \frac{1}{4}\tilde{g}_{kl}\partial_i\tilde{g}^{kl}\ , \tag{3.123}$$

and the scalar curvature

$$\begin{aligned}\mathcal{R} &= \tilde{g}^{ln}\partial_k\Gamma^k{}_{nl} - \tilde{g}^{lp}\partial_p\Gamma^k{}_{kl} + \Gamma^{pn}{}_n\Gamma^k{}_{kp} - \Gamma^{pn}{}_k\Gamma^k{}_{np}\\ &= \tilde{g}^{lm}\tilde{g}^{ku}\partial_k\partial_m\tilde{g}_{lu} - \tilde{g}^{lu}\tilde{g}^{km}\partial_k\partial_m\tilde{g}_{lu} + \tfrac{1}{2}\partial_m\tilde{g}_{ln}\partial_k\tilde{g}_{pu}\big(2\tilde{g}^{kl}\tilde{g}^{mn}\tilde{g}^{pu}\\ &\quad - \tfrac{1}{2}\tilde{g}^{km}\tilde{g}^{ln}\tilde{g}^{pu} + \tfrac{3}{2}\tilde{g}^{km}\tilde{g}^{np}\tilde{g}^{lu} - \tilde{g}^{mp}\tilde{g}^{kn}\tilde{g}^{lu} - 2\tilde{g}^{mn}\tilde{g}^{kp}\tilde{g}^{lu}\big)\ ,\end{aligned} \tag{3.124}$$

one can show that

$$\begin{aligned}&4\tilde{g}^{ij}\partial_i d\partial_j d - 2\partial_k d\,\partial_l\tilde{g}^{kl} - \frac{1}{4}\,\tilde{g}_{ik}\,\tilde{g}_{jl}\tilde{g}^{rs}\partial_r\tilde{g}^{kl}\,\partial_s\tilde{g}^{ij} + \frac{1}{2}\tilde{g}_{pq}\,\partial_k\tilde{g}^{lp}\,\partial_l\tilde{g}^{kq}\\ &= \mathcal{R}(\tilde{g}) + 4(\partial\tilde{\phi})^2 - \partial_k\left(e^{-2d}\left(-\partial_l\tilde{g}^{lk} - \tilde{g}^{ij}\tilde{g}^{lk}\partial_l\tilde{g}_{ij}\right)\right)\ .\end{aligned} \tag{3.125}$$

This can be cross-checked by noting that the first line of the above equation is what remains from the original double field theory action (3.17) after setting $\tilde{\partial}^i = 0$ and $b = 0$ and replacing $g_{ij} \to \tilde{g}_{ij}$. Using that this action reduces to the standard supergravity action (3.32) proves the right-hand side of the above statement up to the total derivative.

Putting together this part and the remaining pieces, the Lagrangian $\mathcal{L}'$ of the action (3.48) under investigation has been brought to the form

$$\begin{aligned} e^{2d}\mathcal{L}' = \mathcal{R} &+ 4(\partial\tilde{\phi})^2 + 4(\tilde{D}d)^2 - \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\beta^{kl}\tilde{D}^q\beta^{ij} - 2\tilde{D}^i\beta^{lp}\tilde{D}^j\beta^{kq}\big) \\ &+ 2\tilde{g}_{ik}\tilde{g}_{jl}\tilde{D}^i d\tilde{D}^j\tilde{g}^{kl} - 2\tilde{g}_{jl}\partial_k d\,\tilde{D}^j\beta^{kl} + 2\tilde{g}_{ik}\tilde{D}^i d\,\partial_l\beta^{kl} \\ &- \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\,\partial_r\beta^{kl}\,\partial_s\beta^{ij} + \tfrac{1}{2}\tilde{g}_{pq}\partial_k\beta^{lp}\partial_l\beta^{kq} \\ &- \tilde{g}_{jl}\tilde{g}_{pq}\big(\partial_k\beta^{lp}\tilde{D}^j\tilde{g}^{kq} + \partial_k\tilde{g}^{lp}\,\tilde{D}^j\beta^{kq}\big) \\ &- \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\tilde{g}^{kl}\tilde{D}^q\tilde{g}^{ij} - 2\tilde{D}^i\tilde{g}^{lp}\tilde{D}^j\tilde{g}^{kq}\big)\ , \end{aligned} \tag{3.126}$$

where from now on the total derivatives are not shown explicitly. As a next step, the terms linear in the dilaton in the second row of the above expression are removed by another integration by parts, resulting in

$$\begin{aligned} e^{2d}\mathcal{L}' = \mathcal{R}(\tilde{g}) &+ 4(\partial\tilde{\phi})^2 + 4(\tilde{D}d)^2 - \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\beta^{kl}\tilde{D}^q\beta^{ij} - 2\tilde{D}^i\beta^{lp}\tilde{D}^j\beta^{kq}\big) \\ &- \tilde{D}^i\tilde{D}^j\tilde{g}_{ij} - \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\tilde{g}^{kl}\tilde{D}^q\tilde{g}^{ij} - 2\tilde{D}^i\tilde{g}^{lp}\tilde{D}^j\tilde{g}^{kq}\big) \\ &- 2\tilde{g}_{ij}\,\tilde{D}^i\partial_k\beta^{kj} - 2\tilde{D}^i\tilde{g}_{ij}\,\partial_k\beta^{kj} - \tilde{g}_{jl}\tilde{g}_{pq}\,\partial_k\beta^{lp}\tilde{D}^j\tilde{g}^{kq} \\ &- \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\,\partial_r\beta^{kl}\,\partial_s\beta^{ij} - \tfrac{1}{2}\tilde{g}_{pq}\partial_k\beta^{lp}\partial_l\beta^{kq} - \tilde{g}_{ij}\,\partial_p\beta^{pi}\,\partial_q\beta^{qj}\ . \end{aligned} \tag{3.127}$$

The last two terms of the first row can be recognised to be the square of the R-flux tensor, as defined in (3.67),

$$\begin{aligned} -\tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\beta^{kl}\tilde{D}^q\beta^{ij} - 2\tilde{D}^i\beta^{lp}\tilde{D}^j\beta^{kq}\big) &= -\tfrac{3}{4}\tilde{g}_{pi}\tilde{g}_{qj}\tilde{g}_{rk}\tilde{D}^{[p}\beta^{qr]}\tilde{D}^{[i}\beta^{jk]} \\ &= -\tfrac{1}{12}R_{ijk}R^{ijk}\ . \end{aligned} \tag{3.128}$$

Now, the definition of the Q-flux (3.66) can be applied after writing out some of the winding derivatives $\tilde{D}^i$, in particular for the term $(\tilde{D}d)^2$. Furthermore, some additional integrations by parts are performed, this time also in the dual coordinates directions. The result is

$$\begin{aligned} e^{2d}\mathcal{L}' = \mathcal{R} &+ 4(\partial\tilde{\phi})^2 + 4(\tilde{D}\tilde{\phi})^2 - \tfrac{1}{12}R^{ijk}R_{ijk} \\ &- \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\,Q_r{}^{kl}\,Q_s{}^{ij} - \tfrac{1}{2}\tilde{g}_{pq}Q_k{}^{lp}Q_l{}^{kq} - \tilde{g}_{ij}\,Q_p{}^{pi}\,Q_q{}^{qj} \\ &- 2Q_l{}^{lk}\tilde{D}^i\tilde{g}_{ik} - \tilde{g}_{jl}\tilde{g}_{pq}\,Q_k{}^{lp}\tilde{D}^j\tilde{g}^{kq} + Q_p{}^{ip}\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^i\tilde{g}_{kl} \\ &- \tilde{D}^i\tilde{D}^j\tilde{g}_{ij} - \tilde{D}^i\big(\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^j\tilde{g}_{kl}\big) - 2\tilde{g}_{jl}\,\tilde{D}^jQ_k{}^{kl} \\ &- \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\tilde{D}^p\tilde{g}^{kl}\tilde{D}^q\tilde{g}^{ij} - 2\tilde{D}^i\tilde{g}^{lp}\tilde{D}^j\tilde{g}^{kq}\big) - \tfrac{1}{4}\tilde{g}_{ij}\tilde{g}_{kl}\tilde{g}_{mn}\tilde{D}^i\tilde{g}^{kl}\tilde{D}^j\tilde{g}^{mn}\ . \end{aligned} \tag{3.129}$$

The discarded total derivatives are

$$+\,e^{2d}\;\tilde{\partial}^i\Big(e^{-2d}\big(\tilde{D}^j\tilde{g}_{ij} - \tilde{g}_{ij}Q_l{}^{jl} + \tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^i\tilde{g}_{kl}\big)\Big)\ , \tag{3.130}$$

in the dual coordinates directions, and

$$\begin{aligned} +e^{2d}\;\partial_k\Big(e^{-2d}\big(&\tilde{g}_{jl}\tilde{D}^j\beta^{kl} - \beta^{ik}(\tilde{D}^j\tilde{g}_{ij} - \tilde{g}_{ij}Q_l{}^{jl}) \\ &+ \partial_l\tilde{g}^{lk} + \tilde{g}^{ij}\tilde{g}^{lk}\partial_l\tilde{g}_{ij} - \beta^{ik}\tilde{g}_{ij}\tilde{g}^{ml}\tilde{D}^i\tilde{g}_{ml}\big)\Big)\ , \end{aligned} \tag{3.131}$$

in the standard coordinates direction.

The last but one step in this computation is to identify the terms of the new scalar curvature $\check{\mathcal{R}}$, and this considerably simplifies the Lagrangian in the form above, (3.129). A straightforward, but tedious calculation shows

$$\begin{aligned}\check{\mathcal{R}} = &-\tfrac{1}{4}\tilde{g}_{ij}\tilde{g}_{mn}\tilde{g}^{kl}Q_k{}^{mi}Q_l{}^{nj} - \tfrac{1}{2}\tilde{g}_{ij}Q_k{}^{lj}Q_l{}^{ki} - \tilde{g}_{ij}Q_k{}^{ki}Q_l{}^{lj} \\ &+ 2Q_l{}^{lk}\tilde{D}^i\tilde{g}_{ik} - \tilde{g}_{jl}\tilde{g}_{pq}Q_k{}^{lp}\tilde{D}^j\tilde{g}^{kq} + Q_p{}^{jp}\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^i\tilde{g}_{kl} \\ &- \tilde{D}^i\tilde{D}^j\tilde{g}_{ij} + \tilde{D}^i\big(\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^j\tilde{g}_{kl}\big) + 2\tilde{g}_{ij}\tilde{D}^iQ_p{}^{pj} \\ &+ \tfrac{1}{4}\tilde{g}_{ij}\big(\tilde{D}^i\tilde{g}_{kl}\tilde{D}^j\tilde{g}^{kl} - 2\tilde{D}^i\tilde{g}_{kl}\tilde{D}^k\tilde{g}^{lj} - \tilde{g}_{kl}\tilde{g}_{mn}\tilde{D}^i\tilde{g}^{kl}\tilde{D}^j\tilde{g}^{mn}\big) \ ,\end{aligned} \tag{3.132}$$

by simply writing out the definitions (3.112) and (3.114). This expression matches many terms in (3.129), which then can be written compactly as

$$\begin{aligned}e^{2d}\mathcal{L}' = &\mathcal{R} + \check{\mathcal{R}} + 4(\partial\tilde{\phi})^2 + 4(\tilde{D}\tilde{\phi})^2 - \tfrac{1}{12}R^{ijk}R_{ijk} \\ &- 4\tilde{D}^i\big(Q_l{}^{lk}\tilde{g}_{ik}\big) - 2\tilde{D}^i\big(\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^j\tilde{g}_{kl}\big) \ ,\end{aligned} \tag{3.133}$$

up to total derivatives.

The last step is to recognise the second row in the above expression as

$$-4\tilde{D}^i\big(Q_l{}^{lk}\tilde{g}_{ik}\big) - 2\tilde{D}^i\big(\tilde{g}_{ij}\tilde{g}^{kl}\tilde{D}^j\tilde{g}_{kl}\big) = 4\big(\tilde{\nabla}^i\mathcal{T}_i - \mathcal{T}^i\mathcal{T}_i\big) \ , \tag{3.134}$$

which can be rewritten by using the rule for integrations by parts, (3.92), in the form

$$4\int \mathrm{d}x\mathrm{d}\tilde{x}\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\tilde{\nabla}^i\mathcal{T}_i = 4\int \mathrm{d}x\mathrm{d}\tilde{x}\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\big(2\tilde{\nabla}^i\tilde{\phi}\,\mathcal{T}_i + 2\mathcal{T}^i\mathcal{T}_i\big) \ . \tag{3.135}$$

This completes the square

$$4(\tilde{D}\tilde{\phi})^2 + 4\big(\tilde{\nabla}^i\mathcal{T}_i - \mathcal{T}^i\mathcal{T}_i\big) = 4\big(\tilde{D}^i\tilde{\phi} + \mathcal{T}^i\big)^2 \ , \tag{3.136}$$

and the final result is

$$e^{2d}\mathcal{L}' = \mathcal{R} + \check{\mathcal{R}} + 4(\partial\tilde{\phi})^2 + 4\big(\tilde{D}^i\tilde{\phi} + \mathcal{T}^i\big)^2 - \tfrac{1}{12}R^{ijk}R_{ijk} \ , \tag{3.137}$$

which exactly matches (3.115) and thus concludes the proof.

Final result

In summary, it has been shown how to apply the field redefinition (3.35) to the standard double field theory action (3.17), and how to reformulate the result using a new diffeomorphism covariant derivative (3.71), that contains both derivatives and dual derivatives. It is conjectured that the non-geometric flux Q is the antisymmetric part of the new connection $\check{\Gamma}$, whereas R is the covariant field strength of β.

The rewritten action (3.115) is made of terms that are separately invariant under D-dimensional diffeomorphisms of the coordinates x^i, parametrised by the gauge parameters ξ^i. The remaining gauge invariance of double field theory, parametrised by $\tilde{\xi}_i$, is hidden but retained.

As a side remark, it should be noted that one could decide to covariantise the other half of the double field theory gauge transformations. The procedure is exactly the same as it was implemented here, but one would have to work with the original field variables g and b. Eventually, the resulting action would be[4] as (3.115) with all indices reversed, the dilaton redefined and R replaced by H. In a sense, this provides an alternative proof for the full gauge invariance of double field theory.

[4]See p.18 of [4] for details.

3.3 Supergravity

So far, it was possible to reveal non-geometric fluxes in double field theory, but non-geometry itself has not been investigated. This section will show that ten-dimensional supergravity makes a better framework to do so. Furthermore, despite some indications and consistency arguments, it is so far not completely clear whether the objects Q and R are indeed connected to the four-dimensional objects that appear in the T-duality invariant superpotential. Eventually, it cannot be guaranteed that the above findings are independent of the particular doubling of the coordinates that is implemented in double field theory.

This section then suggests four further steps in the analysis. First, the field redefinition (3.35) is implemented in the framework of ten-dimensional supergravity, namely its NSNS part. Again, and independently of the previous considerations, the fluxes Q and R appear in a very particular way, and their definitions agree to the ones given above when cutting the dependence on dual coordinates.

Second, it will be shown how the procedure in ten dimensions can be related to the approach in double field theory by employing particular solutions of the strong constraint. This confirms the consistency of the whole construction and shows that it is not intrinsically relying on the doubling of the coordinates.

Third, a very basic dimensional reduction with two moduli will be performed, that eventually reproduces terms with the correct scaling behaviour. It makes an independent proof that the proposed objects Q and R are indeed suitable to describe non-geometric fluxes.

Finally, non-geometry will be investigated for the particular case of a three-torus with constant H-flux. It turns out that the field redefined supergravity action offers a well-defined framework to describe such a setup. This indicates that, indeed, non-geometric fluxes and non-geometry are closely interrelated. A few generalisations of these findings shall conclude the section, they are supposed to clarify the features of non-geometry in the context of effective field theories.

3.3.1 Field redefinition

In the following, the field redefinition of the preceding chapter is translated into the framework of ten-dimensional supergravity. Its application turns out to be very involved on the calculational level, such that first a subcase of the most general procedure shall be considered in detail. Later on, the full result is given without writing out all the details of the respective derivation. Interestingly, the restriction is not too severe as for example the three-torus setup is part of the subcases.

Concretely, the transformation (3.35),

$$(g_{ij},\, b_{ij},\, \phi) \;\rightarrow\; (\tilde{g}_{ij},\, \beta^{ij},\, \tilde{\phi}) \;, \tag{3.138}$$

shall now be implemented at the level of the supergravity action, namely its NSNS part,

$$S = \int \mathrm{d}x \sqrt{|g|}\, e^{-2\phi} \Big(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{2}|H|^2 \Big) \;, \tag{3.139}$$

noted using the abbreviation

$$|H|^2 \equiv \frac{1}{3!} H_{mnp} H^{mnp} \;. \tag{3.140}$$

Again, the transformation rules for the respective fields can be read off from the change of parametrisation in the generalised metric, (3.36),

$$\mathcal{H}^{MN} = \begin{pmatrix} g_{ij} - b_{ik}g^{kl}b_{lj} & b_{ik}g^{kj} \\ -g^{ik}b_{kj} & g^{ij} \end{pmatrix} = \begin{pmatrix} \tilde{g}_{ij} & -\tilde{g}_{ik}\beta^{kj} \\ \beta^{ik}\tilde{g}_{kj} & \tilde{g}^{ij} - \beta^{ik}\tilde{g}_{kl}\beta^{lj} \end{pmatrix} . \tag{3.141}$$

In this context, $\mathcal{H}$ is not part of the theory itself but rather taken to be a mere bookkeeping device that indicates the correct rules. Technically speaking, the metric and the b-field transform as

$$\begin{aligned} g &= (\tilde{g}^{-1} + \epsilon\beta)^{-1}\tilde{g}^{-1}(\tilde{g}^{-1} - \epsilon\beta)^{-1} \\ b &= (\tilde{g}^{-1} + \epsilon\beta)^{-1}\beta(\tilde{g}^{-1} - \epsilon\beta)^{-1} , \end{aligned} \tag{3.142}$$

with $\epsilon = \pm 1$ as has been shown in (3.37) and (3.38). For later convenience, the following abbreviation shall be introduced,

$$G_{\pm}^{mn} \equiv \tilde{g}^{mn} \pm \beta^{mn} . \tag{3.143}$$

The letter G has been chosen to distinguish this object from the double field theory object $\tilde{\mathcal{E}}$ of the previous chapter which has a dependence on doubled coordinates that do not appear in the supergravity framework here. Using the property

$$G_{\pm}^{T} = G_{\mp} , \tag{3.144}$$

the field redefinitions can be written as

$$\begin{aligned} g_{mn} &= (G_{\pm}^{-1})_{mk}\tilde{g}^{kp}(G_{\pm}^{-1})_{np} \\ g^{mn} &= G_{\pm}^{mk}\tilde{g}_{kp}G_{\pm}^{np} \\ b_{mn} &= (G_{\pm}^{-1})_{mk}\beta^{kp}(G_{\mp}^{-1})_{pn} . \end{aligned} \tag{3.145}$$

To leave the measure invariant the dilaton is defined to transform as

$$e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|} = e^{-2\phi}\sqrt{|g|} . \tag{3.146}$$

This can be worked out as

$$e^{-2\tilde{\phi}} = e^{-2\phi}\sqrt{\frac{|\det g|}{|\det \tilde{g}|}} = e^{-2\phi}(\det(g^{-1}\tilde{g}))^{-1/2} = e^{-2\phi}|\mathbb{1} - \beta\tilde{g}\beta\tilde{g}|^{-1/2} , \tag{3.147}$$

where the second equality used that $\det g$ and $\det \tilde{g}$ have the same sign according to (3.145). The transformation of the dilaton itself can thus be written as

$$\phi = \tilde{\phi} - \frac{1}{4}\mathrm{tr}(\ln(\mathbb{1} - \beta\tilde{g}\beta\tilde{g})) , \tag{3.148}$$

where it was used that for any invertible matrix A one can write

$$\ln(\det A) = \mathrm{tr}(\ln A) , \tag{3.149}$$

and that in this case

$$\mathbb{1} - \beta\tilde{g}\beta\tilde{g} = g^{-1}\tilde{g} \tag{3.150}$$

is invertible.

Simplifying assumption

Having the field redefinitions (3.145) and (3.146) at hand, it is basically a matter of plugging them into the supergravity action (3.139) and work out how the non-geometric fluxes Q and R arise. This indeed has been done in full generality in appendix B of [4], and the result will be shown at the end of this section.

The emphasis in the following is rather on illustrating the procedure and showing parts of the calculational details, as well as, more importantly, keeping track of the total derivatives that arise thereby. These total derivatives will play a key role in the discussion of non-geometric features of the field redefined action. In order to keep the presentation clear, only a subcase shall be investigated here. It is defined by the following constraint,

$$\beta^{km}\partial_m = 0 \ , \tag{3.151}$$

that is supposed to hold for any field. Its application can be supported by the observation that it precisely holds for the T-dual of the twisted torus of chapter 2, a background that is characterised by nonzero Q-flux. This important case of a non-geometric situation can thus be fully captured by the restricted investigation of this subsection, see the discussion around (3.245).

More generally, any background with a block diagonal metric along a base $\mathcal{B}$ and a fibre $\mathcal{F}$, such that all fields have isometries along the fibre $\mathcal{F}$, will satisfy (3.151) if the b-field is purely along $\mathcal{F}$ as well. This can be seen as follows: First, the isometries locally imply that the fields can only depend on the coordinates of the base. Second, the inverse of (3.145), namely

$$\begin{aligned} \tilde{g} &= g - bg^{-1}b \\ \beta &= (g+b)^{-1}b(g-b)^{-1} \ , \end{aligned} \tag{3.152}$$

shows that also β and $\tilde{g}$ are along the fibre $\mathcal{F}$, with a mere dependence on base coordinates. The only nonzero derivatives are thus along the base $\mathcal{B}$, but this exactly proves the assumption (3.151). From (3.146) one can conclude in the same way for the dilaton.

In particular, given this general range of application, it turns out that all three T-dual frames of the preceding chapter 2 are satisfying the constraint (3.151). Namely, these are the torus with constant H-flux, the twisted torus and the above mentioned Q-flux frame. Eventually, although the constraint does not allow for any R-flux, see (3.195) and the discussion below, it still allows for interesting backgrounds that feature non-geometric properties.

One of the main advantages to impose the constraint (3.151) on the technical level is that it implies the following relation,

$$\left(G_\pm^{-1}\right)_{mn}\tilde{g}^{nk}\partial_k = \partial_m \ , \tag{3.153}$$

which can be proven by simply multiplying with $G_\pm$. This equation simplifies the following calculations as it helps to eliminate the object $G_\pm^{-1}$. The final result is supposed to be a valid ten-dimensional action for non-geometric fluxes in terms of $\tilde{g}$ and β, and therefore should better not contain inverses of sums of these fields. It should be noted, though, that also without any simplifying assumption all inverses cancel, but in a more subtle and concealed way.

There are two other useful relations that are implied by (3.151) directly or its structural foundations just discussed. First, one has

$$\partial_k \beta^{nm} \partial_m \cdot = 0 \ , \tag{3.154}$$

which can be proven from

$$\partial_k (\beta^{nm} \partial_m \cdot) = 0 \ . \tag{3.155}$$

Second, one has in addition

$$\partial_k \beta^{km} = 0 \ , \tag{3.156}$$

which can be inferred either from the general reasoning of β being only along a fibre $\mathcal{F}$, or an integration by parts of any term with the structure (3.151). Interestingly, a closer investigation shows that the latter reasoning refers to the former anyway: Assuming F_n and f to be some combinations of fields with an index n plus arbitrary other indices, or arbitrary indices at all, respectively, integration by parts can be written out as

$$\begin{aligned} e^{-2\tilde{\phi}} \sqrt{|\tilde{g}|}\ F_n \beta^{nk} \partial_k f = {} & \partial_k \left(e^{-2\tilde{\phi}} \sqrt{|\tilde{g}|}\ F_n \beta^{nk} f \right) - \partial_k \left(e^{-2\tilde{\phi}} \sqrt{|\tilde{g}|}\ F_n \right) \beta^{nk} f \\ & - e^{-2\tilde{\phi}} \sqrt{|\tilde{g}|}\ F_n (\partial_k \beta^{nk}) f \ . \end{aligned} \tag{3.157}$$

The last term of the first row vanishes by assumption, as well as the left-hand side. To show (3.156) one might discard the total derivative. But, as will be discussed later on in more detail, for non-geometric situations this cannot be done anymore. Still, it is possible to argue that the derivative is only nonzero for a coordinate in the base $\mathcal{B}$, as was discussed above, where the component of β is zero. But with such a reasoning, one is back at proving (3.156) directly. Conclusively, one might say that (3.156) is fully compatible with integration by parts.

Curvature scalar

This and following two subsections show how the field redefinition is worked out for each of the three terms in the action (3.139). As the calculations are lengthy, the material presented will be a bit technical. But to emphasise it once more, they make an independent check of whether the field redefinition used in double field theory can help to reveal non-geometric fluxes also in the ten-dimensional framework alone.

The first part is now concerning the scalar curvature term. Using its basic definition, as in (3.124),

$$\begin{aligned} \mathcal{R} &= g^{ln} \partial_k \Gamma^k{}_{nl} - g^{lm} \partial_m \Gamma^k{}_{kl} + \Gamma^{pn}{}_n \Gamma^k{}_{kp} - \Gamma^{pn}{}_k \Gamma^k{}_{np} \\ &= g^{lm} g^{ku} \partial_k \partial_m g_{lu} - g^{lu} g^{km} \partial_k \partial_m g_{lu} + \tfrac{1}{2} \partial_m g_{ln} \partial_k g_{pu} \big(2 g^{kl} g^{mn} g^{pu} \\ &\quad - \tfrac{1}{2} g^{km} g^{ln} g^{pu} + \tfrac{3}{2} g^{km} g^{np} g^{lu} - g^{mp} g^{kn} g^{lu} - 2 g^{mn} g^{kp} g^{lu} \big) \ , \end{aligned} \tag{3.158}$$

one can show by simply substituting (3.145) that the field redefined version $\widetilde{\mathcal{R}}$ is given by

$$\begin{aligned}\widetilde{\mathcal{R}} = \mathcal{R} &+ \partial_m\tilde{g}_{np}\partial_k\tilde{g}_{rs}\beta(2\tilde{g}^{km}\tilde{g}^{nr}\tilde{g}^{ps} + 2\tilde{g}^{rs}\tilde{g}^{mn}\tilde{g}^{pk} + \tfrac{1}{2}\tilde{g}^{ms}\tilde{g}^{nr}\tilde{g}^{pk}) \\ &+ \tilde{g}_{ln}\partial_k\beta^{kl}\partial_m\beta^{mn} + \tfrac{1}{2}\tilde{g}_{ln}\partial_k\beta^{lm}\partial_m\beta^{nk} \\ &- 2\tilde{g}^{km}\tilde{g}^{pq}\partial_k\partial_m\tilde{g}_{pq} - 2\tilde{g}^{km}(G^{-1})_{pq}\partial_k\partial_m G^{qp} \\ &- \partial_m G^{np}\big(-2\tilde{g}^{rs}\tilde{g}^{km}(G^{-1})_{sn}\partial_k\tilde{g}_{pr} + 2\tilde{g}^{qs}\tilde{g}^{mr}(G^{-1})_{sn}\partial_p\tilde{g}_{qr} - 2\tilde{g}^{mr}\tilde{g}^{ks}(G^{-1})_{pn}\partial_k\tilde{g}_{rs} \\ &\quad - \tilde{g}^{rs}\tilde{g}^{km}(G^{-1})_{pn}\partial_k\tilde{g}_{rs} - \tilde{g}^{mr}\tilde{g}^{qs}(G^{-1})_{ps}\partial_n\tilde{g}_{qr} + \tilde{g}^{km}\tilde{g}^{rs}(G^{-1})_{ps}\partial_k\tilde{g}_{nr}\big) \\ &- \partial_m G^{np}\big((G^{-1})_{qn}\partial_p G^{mq} + g_{qn}\partial_p G^{qm} - \tfrac{1}{2}g_{qp}\partial_n G^{mq}\big) \\ &- \partial_m G^{np}\partial_k G^{rs}\big(-(G^{-1})_{pn}(G^{-1})_{sr} - \tfrac{5}{2}(G^{-1})_{pr}(G^{-1})_{sn} - g_{rn}\tilde{g}_{ps} + \tfrac{1}{2}g_{ps}\tilde{g}_{nr}\big)\tilde{g}^{km}\ .\end{aligned} \tag{3.159}$$

At places where, at this stage, no further simplifications can be achieved, g_{mn} was left as it stands, without expressing it in terms of $\tilde{g}$ and β. At many places the constraint (3.151) and its descendants (3.154) and (3.156) have been used. Furthermore, G^{-1} is taken to be G_+^{-1}.

As later on, all the terms containing G^{-1} shall cancel amongst each other when combining the above with the other two parts of the action, it is necessary to bring (3.159) into a slightly different form. This will be done in two steps, where the first is to note the two identities

$$\partial_m G^{np}\tilde{g}^{rs}\tilde{g}^{km}\Big(-(G^{-1})_{sn}\partial_k\tilde{g}_{pr} + (G^{-1})_{ps}\partial_k\tilde{g}_{nr}\Big) + 2\tilde{g}^{km}\tilde{g}_{ps}g_{nr}\partial_m\tilde{g}^{pn}\partial_k\beta^{sr} = 0 \tag{3.160}$$

$$\partial_m G^{np}\tilde{g}^{qs}\tilde{g}^{mr}\Big((G^{-1})_{sn}\partial_p\tilde{g}_{qr} - (G^{-1})_{ps}\partial_n\tilde{g}_{qr}\Big) - 2g_{qn}\partial_m\tilde{g}^{kn}\partial_k\beta^{mq} = 0\ . \tag{3.161}$$

They can be proven by noting a rewriting of the field redefinition (3.145),

$$\begin{aligned}(G^{-1})_{(ps)} &= \tfrac{1}{2}\big((G^{-1})_{ps} + (G^{-1})_{sp}\big) \\ &= \tfrac{1}{2}\big((\tilde{g}^{-1}+\beta)^{-1} + (\tilde{g}^{-1}-\beta)^{-1}\big)_{ps} \\ &= \big((\tilde{g}^{-1}+\beta)^{-1}\tilde{g}^{-1}(\tilde{g}^{-1}-\beta)^{-1}\big)_{ps} = g_{ps}\ ,\end{aligned} \tag{3.162}$$

cf. the formulations (3.37). This rewriting helps to recover the hidden g in the respective last terms of the above identities.

Now, one can note four identities that resort terms in a particular way,

$$\begin{aligned}\tilde{g}^{rs}\tilde{g}^{km}\partial_m G^{np}\big(-2(G^{-1})_{sn}\partial_k\tilde{g}_{pr} + (G^{-1})_{ps}\partial_k\tilde{g}_{nr}\big) &= -\partial_m G^{np}\tilde{g}^{km}\tilde{g}^{rs}(G^{-1})_{ps}\partial_k\tilde{g}_{nr} \\ &\quad +2\partial_m G^{np}\tilde{g}^{rs}\tilde{g}^{km}\big(-(G^{-1})_{sn}\partial_k\tilde{g}_{pr} + (G^{-1})_{ps}\partial_k\tilde{g}_{nr}\big) \\ \tilde{g}^{qs}\tilde{g}^{mr}\partial_m G^{np}\big(2(G^{-1})_{sn}\partial_p\tilde{g}_{qr} - (G^{-1})_{ps}\partial_n\tilde{g}_{qr}\big) &= \partial_m G^{np}\tilde{g}^{mr}\tilde{g}^{qs}(G^{-1})_{ps}\partial_n\tilde{g}_{qr} \\ &\quad +2\partial_m G^{np}\tilde{g}^{qs}\tilde{g}^{mr}\big((G^{-1})_{sn}\partial_p\tilde{g}_{qr} - (G^{-1})_{ps}\partial_n\tilde{g}_{qr}\big)\end{aligned} \tag{3.163}$$

$$\begin{aligned}\tilde{g}^{km}\partial_m G^{np}\partial_k G^{rs}\big(-g_{rn}\tilde{g}_{ps} + \tfrac{1}{2}g_{ps}\tilde{g}_{nr}\big) &= \tilde{g}^{km}g_{ps}\tilde{g}_{nr}\big(-\tfrac{1}{2}\partial_m G^{np}\partial_k G^{rs} + 4\partial_m\tilde{g}^{pn}\partial_k\beta^{sr}\big) \\ \partial_m G^{np}\big(g_{qn}\partial_p G^{qm} - \tfrac{1}{2}g_{qp}\partial_n G^{mq}\big) &= \tfrac{1}{2}g_{pq}\partial_m G^{kp}\partial_k G^{mq} - 4g_{qn}\partial_m\tilde{g}^{kn}\partial_k\beta^{mq}\ .\end{aligned}$$

Then, the above (3.160) and (3.161) will cancel the respective last terms when taking the sum of all four identities. This sum is now used to simplify the expression for $\widetilde{\mathcal{R}}$,

$$\begin{aligned}\widetilde{\mathcal{R}} =&\mathcal{R} + \partial_k\tilde{g}_{su}\partial_m\tilde{g}_{pq}\big(2\tilde{g}^{km}\tilde{g}^{uq}\tilde{g}^{ps} + 2\tilde{g}^{pq}\tilde{g}^{ks}\tilde{g}^{mu} + \tfrac{1}{2}\tilde{g}^{uq}\tilde{g}^{sm}\tilde{g}^{kp}\big)\\ &+ \tilde{g}_{pq}\partial_k\beta^{pk}\partial_m\beta^{qm} + \tfrac{1}{2}\tilde{g}_{pq}\partial_k\beta^{qm}\partial_m\beta^{pk}\\ &- 2\tilde{g}^{km}\tilde{g}^{pq}\partial_k\partial_m\tilde{g}_{pq} - 2\tilde{g}^{km}(G^{-1})_{pq}\partial_k\partial_m G^{qp}\\ &- \partial_m G^{vl}\big(-2\tilde{g}^{mr}\tilde{g}^{ks}(G^{-1})_{lv}\partial_k\tilde{g}_{rs} - \tilde{g}^{rs}\tilde{g}^{km}(G^{-1})_{lv}\partial_k\tilde{g}_{rs}\\ &\quad + \tilde{g}^{ms}\tilde{g}^{ru}(G^{-1})_{lu}\partial_v\tilde{g}_{rs} - \tilde{g}^{km}\tilde{g}^{rs}(G^{-1})_{ls}\partial_k\tilde{g}_{vr}\big)\\ &- \partial_m G^{vl}\big((G^{-1})_{lq}\partial_v G^{qm} + \tfrac{1}{2}g_{lq}\partial_v G^{mq}\big)\\ &+ \partial_m G^{vl}\partial_k G^{ps}\tfrac{1}{2}\tilde{g}^{km}\big(2(G^{-1})_{lv}(G^{-1})_{sp} + 5(G^{-1})_{sv}(G^{-1})_{lp} + g_{sl}\tilde{g}_{pv}\big)\ .\end{aligned} \tag{3.164}$$

One important observation here is, that the third line contains second derivative terms that cannot be canceled by any other term in the action (3.139). They will be removed by integrations by parts that consequently bring in new terms containing the dilaton. To see which dilaton terms will arise from the kinetic terms this shall be the next step of investigation.

Dilaton term and total derivative

The claim of an invariant measure for the supergravity action lead to the field redefinition (3.148). Its derivative can be computed as follows,

$$\begin{aligned}\partial_m\tilde{\phi} - \partial_m\phi &= \tfrac{1}{4}\partial_m\big(\mathrm{tr}(\ln(\mathbb{1} - \beta\tilde{g}\beta\tilde{g}))\big)\\ &= \tfrac{1}{4}\mathrm{tr}\big((\mathbb{1} - \beta\tilde{g}\beta\tilde{g})^{-1}\partial_m(\mathbb{1} - \beta\tilde{g}\beta\tilde{g})\big)\\ &= \tfrac{1}{4}\mathrm{tr}(G^{-1}\partial_m\beta + \tilde{g}^{-1}G^{-1}\beta\partial_m\tilde{g}) - \tfrac{1}{4}\mathrm{tr}(G^{-T}\partial_m\beta + \tilde{g}^{-1}G^{-T}\beta\partial_m\tilde{g})\\ &= \tfrac{1}{2}(G^{-1})_{kl}\partial_m\beta^{lk} + \tfrac{1}{2}(G^{-1})_{kl}\tilde{g}^{ln}\partial_m\tilde{g}_{np}\beta^{pk}\\ &= \tfrac{1}{2}\tilde{g}^{pq}\partial_m\tilde{g}_{pq} + \tfrac{1}{2}(G^{-1})_{lk}\partial_m G^{kl} \equiv \tfrac{1}{2}C_m\ .\end{aligned} \tag{3.165}$$

Here, the second row used that for every invertible matrix A, in particular for $A = \mathbb{1} - \beta\tilde{g}\beta\tilde{g}$, one can express the derivative that appears as

$$\partial_m\mathrm{tr}(\ln A) = \partial_m\ln(\det A) = \mathrm{tr}(A^{-1}\partial_m A)\ , \tag{3.166}$$

cf. equation (3.149). Using the standard definition of the square, namely

$$(\partial\phi)^2 \equiv g^{km}\partial_k\phi\partial_m\phi\ , \quad (\partial\tilde{\phi})^2 \equiv \tilde{g}^{km}\partial_k\tilde{\phi}\partial_m\tilde{\phi}\ , \tag{3.167}$$

one can find the difference of the squares to be

$$(\partial\tilde{\phi})^2 - (\partial\phi)^2 = (\tilde{g}^{km} - g^{km})\partial_k\tilde{\phi}\partial_m\tilde{\phi} - \tfrac{1}{4}g^{km}C_kC_m + g^{km}C_k\partial_m\tilde{\phi}\ . \tag{3.168}$$

The first term on the right-hand side vanishes by the simplifying constraint (3.151), whereas the others can be written out as

$$\begin{aligned}(\partial\tilde{\phi})^2 = (\partial\phi)^2 &- \tfrac{1}{4}\tilde{g}^{km}\tilde{g}^{pq}\tilde{g}^{uv}\partial_m\tilde{g}_{pq}\partial_k\tilde{g}_{uv} - \tfrac{1}{2}\tilde{g}^{km}\tilde{g}^{pq}(G^{-1})_{uv}\partial_m\tilde{g}_{pq}\partial_k G^{vu}\\ &- \tfrac{1}{4}\tilde{g}^{km}(G^{-1})_{pl}(G^{-1})_{uv}\partial_m G^{lp}\partial_k G^{vu} + \tilde{g}^{km}\tilde{g}^{pq}\partial_k\tilde{g}_{pq}\partial_m\tilde{\phi} + \tilde{g}^{km}(G^{-1})_{pq}\partial_k G^{qp}\partial_m\tilde{\phi}\ .\end{aligned} \tag{3.169}$$

This result shall now be compared with what remains after integrating by parts the second derivative terms in (3.164). For general fields f with arbitrary indices and F^{km} with at least two indices up, an integration by parts has the form

$$\begin{aligned}\int \mathrm{d}^D x\; e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\; F^{km}\partial_k\partial_m f &= \int \mathrm{d}^D x\; \partial_k\big(e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\; F^{km}\partial_m f\big) \\ &\quad + \int \mathrm{d}^D x\; e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\; \Big(\Big(2\partial_k\tilde{\phi} - \tfrac{1}{2}\tilde{g}^{pq}\partial_k\tilde{g}_{pq}\Big)F^{km} - \partial_k F^{km}\Big)\partial_m f\;.\end{aligned} \tag{3.170}$$

This can be applied to the two terms in the third row of (3.164),

$$\begin{aligned}2\tilde{g}^{km}\tilde{g}^{pq}\partial_k\partial_m\tilde{g}_{pq} + 2\tilde{g}^{km}(G^{-1})_{pq}\partial_k\partial_m G^{qp} \;&=\; 4\tilde{g}^{km}\partial_k\tilde{\phi}\big(\tilde{g}^{pq}\partial_m\tilde{g}_{pq} + (G^{-1})_{pq}\partial_m G^{qp}\big) \\ &\quad + \partial_k\tilde{g}_{uv}\partial_m\tilde{g}_{pq}\big(2\tilde{g}^{pq}\tilde{g}^{mu}\tilde{g}^{kv} + \tilde{g}^{km}\big(2\tilde{g}^{pu}\tilde{g}^{vq} - \tilde{g}^{uv}\tilde{g}^{pq}\big)\big) \\ &\quad + (G^{-1})_{pq}\partial_k\tilde{g}_{uv}\partial_m G^{qp}\big(2\tilde{g}^{mu}\tilde{g}^{kv} - \tilde{g}^{km}\tilde{g}^{uv}\big) \\ &\quad + 2\tilde{g}^{km}(G^{-1})_{pu}(G^{-1})_{vq}\partial_k G^{uv}\partial_m G^{qp} \;+\; \text{t.d.}\end{aligned} \tag{3.171}$$

where the total derivative is given by

$$\text{t.d.} = \frac{1}{e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}}\partial_k\Big(e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\; 2\tilde{g}^{km}\left(\tilde{g}^{pq}\partial_m\tilde{g}_{pq} + (G^{-1})_{pq}\partial_m G^{qp}\right)\Big)\;. \tag{3.172}$$

This total derivative turns out to be important later on when discussing the appearance of non-geometry in the field redefined action.

A comparison of (3.171) and (3.169) shows that the first term on the right-hand side of the former exactly matches the last two terms of the latter when taking into account the factor in front of the dilaton square. Even more, by using the derived constraints (3.154) and (3.156), one can bring the effect of the field redefinition on the scalar curvature and the dilaton terms into a quite compact form,

$$\begin{aligned}\widetilde{\mathcal{R}} - \mathcal{R} + 4(\partial\tilde{\phi})^2 - 4(\partial\phi)^2 &= -\tfrac{1}{2}\tilde{g}^{ku}\tilde{g}^{mq}\tilde{g}^{vl}\tilde{g}^{sp}(g_{sl} - \tilde{g}_{sl})\partial_m\tilde{g}_{uv}\partial_k\tilde{g}_{pq} \\ &\quad - \tilde{g}^{km}\partial_m G^{pl}\Big(\tilde{g}_{pq}(G^{-1})_{lr}\partial_k\tilde{g}^{rq} - \tfrac{1}{2}\big(g_{lq}\tilde{g}_{pr} + (G^{-1})_{qp}(G^{-1})_{lr}\big)\partial_k G^{rq}\Big) - \text{t.d.}\end{aligned} \tag{3.173}$$

It turns out that the remaining terms, apart from the total derivative, can be matched completely with what the expansion of the H-flux contribution gives. This part, therefore, makes the last issue of the investigation.

H-flux term

Using its definition and the field redefinition (3.145), the H-flux term of the supergravity action (3.139) can be written out as

$$\begin{aligned}\tfrac{1}{3}H_{kmn} &= \partial_{[k}b_{mn]} \\ &= -(G^{-1}_{-\epsilon})_{p[m}\partial_k\beta^{pq}(G^{-1}_{\epsilon})_{n]q} - 2(G^{-1}_{-\epsilon})_{p[m}\partial_k G^{pq}_{\epsilon}b_{n]q}\;.\end{aligned} \tag{3.174}$$

What makes the computation to follow particularly difficult is the antisymmetrisation appearing in this formula. It necessitates the use of many more indices than before, so for this subsection they can get subindices, as for example in $k_1, k_2, \ldots$. The final result will be expressed with an ordinary set of indices. Furthermore, it turns out to be helpful to leave

$\epsilon = \pm 1$ in G_ϵ^{-1} unspecified at this stage. Later on, this freedom of choice will help to cancel terms of the form $G_{\epsilon_1}^{-1} G_{\epsilon_2}$.
The term $|H|^2$ under investigation is constructed by raising the indices with g, it therefore reads

$$\begin{aligned}\tfrac{2}{3}\,|H|^2 &= g^{k_1k_2} g^{m_1m_2} g^{n_1n_2} \big((G_{\epsilon_1}^{-1})_{m_1p_1} \partial_{k_1} \beta^{p_1q_1} (G_{\epsilon_1}^{-1})_{n_1q_1} + 2 (G_{\epsilon_1}^{-1})_{m_1p_1} \partial_{k_1} G_{\epsilon_1}^{p_1q_1} b_{n_1q_1} \big) \\ &\quad \times \big((G_{-\epsilon_2}^{-1})_{p_2[m_2} \partial_{k_2} \beta^{p_2q_2} (G_{\epsilon_2}^{-1})_{n_2]q_2} + 2 (G_{-\epsilon_2}^{-1})_{p_2[m_2} \partial_{k_2} G_{\epsilon_2}^{p_2q_2} b_{n_2]q_2} \big) \\ &= (1) + (2) + (3)\ ,\end{aligned} \tag{3.175}$$

where the antisymmetrisation of the first factor can be neglected because of the second bracket, and the result is given in three types of terms (1), (2) and (3). They shall now be computed separately, where the notation

$$D_\epsilon^p \equiv G_\epsilon^{pq} \partial_q \tag{3.176}$$

will be used.

$$\begin{aligned}3\,(1) &= 3(g^{-1})^3 (G_{\epsilon_1}^{-1})_{m_1p_1} \partial_{k_1} \beta^{p_1q_1} (G_{\epsilon_1}^{-1})_{n_1q_1} (G_{-\epsilon_2}^{-1})_{p_2[m_2} \partial_{k_2} \beta^{p_2q_2} (G_{\epsilon_2}^{-1})_{n_2]q_2} \\ &= (\tilde g_{p_1p_2} \tilde g_{q_1q_2} \tilde g_{s_1s_2} - \tilde g_{p_1s_2} \tilde g_{q_1q_2} \tilde g_{s_1p_2} - \tilde g_{p_1p_2} \tilde g_{q_1s_2} \tilde g_{s_1q_2})\, D_\epsilon^{s_1} \beta^{p_1q_1} D_\epsilon^{s_2} \beta^{p_2q_2} \\ &= \tilde g_{p_1p_2} \tilde g_{q_1q_2} \tilde g^{s_1s_2} \partial_{s_1} \beta^{p_1q_1} \partial_{s_2} \beta^{p_2q_2} - 2 \tilde g_{q_1q_2} \partial_{p_2} \beta^{p_1q_1} \partial_{p_1} \beta^{p_2q_2} \\ &= \tilde g_{p_1p_2} \tilde g_{q_1q_2} \tilde g^{s_1s_2} \partial_{s_1} \beta^{p_1q_1} \partial_{s_2} \beta^{p_2q_2}\ .\end{aligned} \tag{3.177}$$

Here, the first row used the abbreviation $(g^{-1})^3$ for the first three metric contractions in (3.175) with the respective indices $k_1, k_2, m_1, m_2, n_1, n_2$; the third row used the constraint (3.151), and the fourth row used the derived constraint (3.154). For the second term, one finds

$$\begin{aligned}3\,(2) &= 12(g^{-1})^3 (G_{\epsilon_1}^{-1})_{m_1p_1} \partial_{k_1} G_{\epsilon_1}^{p_1q_1} b_{n_1q_1} (G_{-\epsilon_2}^{-1})_{p_2[m_2} \partial_{k_2} \beta^{p_2q_2} (G_{\epsilon_2}^{-1})_{n_2]q_2} \\ &= 4\,(\tilde g_{p_1p_2} \tilde g_{t_1q_2} \tilde g_{s_1s_2} - \tilde g_{p_1s_2} \tilde g_{t_1q_2} \tilde g_{s_1p_2} - \tilde g_{p_1p_2} \tilde g_{t_1s_2} \tilde g_{s_1q_2})\, \beta^{t_1t_2} (G_\epsilon^{-1})_{q_1t_2} D_\epsilon^{s_1} G_\epsilon^{p_1q_1} D_\epsilon^{s_2} \beta^{p_2q_2} \\ &= 4\epsilon \big(\tilde g_{q_1q_2} - (G_\epsilon^{-1})_{q_1q_2} \big) \big(- \tilde g_{p_1p_2} \tilde g^{s_1s_2} \partial_{s_2} G_\epsilon^{p_1q_1} \partial_{s_1} \beta^{p_2q_2} + \partial_{p_2} G_\epsilon^{p_1q_1} \partial_{p_1} \beta^{p_2q_2} \big) \\ &= 4 \tilde g_{p_1p_2} \tilde g^{s_1s_2} \partial_{s_1} \beta^{p_2q_2} \big(- \tilde g_{q_1q_2} \partial_{s_2} \beta^{p_1q_1} + \epsilon (G_\epsilon^{-1})_{q_1q_2} \partial_{s_2} G_\epsilon^{p_1q_1} \big) \\ &= -4 \tilde g_{p_1p_2} \tilde g_{q_1q_2} \tilde g^{s_1s_2} \partial_{s_1} \beta^{p_1q_1} \partial_{s_2} \beta^{p_2q_2} + 4 \tilde g_{p_1p_2} \tilde g^{s_1s_2} \partial_{s_1} G_\epsilon^{p_1q_1} \big(g_{q_1q_2} \partial_{s_2} G_\epsilon^{p_2q_2} - (G_\epsilon^{-1})_{q_1q_2} \partial_{s_2} \tilde g^{p_2q_2} \big)\ .\end{aligned} \tag{3.178}$$

Again, the constraint was applied. The last row is a rewriting to match the third term, which is the most complicated one given the double appearance of b.

$$\begin{aligned}3\,(3) &= 12(g^{-1})^3 (G_{\epsilon_1}^{-1})_{m_1p_1} \partial_{k_1} G_{\epsilon_1}^{p_1q_1} b_{n_1q_1} (G_{-\epsilon_2}^{-1})_{p_2[m_2} \partial_{k_2} G_{\epsilon_2}^{p_2q_2} b_{n_2]q_2} \\ &= 2 \big(\tilde g_{p_1p_2} \tilde g_{t_1t_2} \tilde g_{s_1s_2} - \tilde g_{p_1s_2} \tilde g_{t_1t_2} \tilde g_{s_1p_2} - \tilde g_{p_1p_2} \tilde g_{t_2s_2} \tilde g_{s_1t_1} \\ &\qquad - \tilde g_{p_1t_1} \tilde g_{p_2t_2} \tilde g_{s_1s_2} + \tilde g_{p_1s_2} \tilde g_{t_2p_2} \tilde g_{s_1t_1} + \tilde g_{p_1t_1} \tilde g_{t_2s_2} \tilde g_{s_1p_2} \big) \\ &\quad \times \big(\delta_{q_1}^{t_2} - (G_\epsilon^{-1})_{q_1u_2} \tilde g^{u_2t_2} \big) \big(\delta_{q_2}^{t_1} - (G_\epsilon^{-1})_{q_2u_1} \tilde g^{u_1t_1} \big) D_\epsilon^{s_1} G_\epsilon^{p_1q_1} D_\epsilon^{s_2} G_\epsilon^{p_2q_2} \\ &= 2 (g_{q_1q_2} - \tilde g_{q_1q_2}) \partial_{p_2} \tilde g^{p_1q_1} \partial_{p_1} \tilde g^{p_2q_2} + 2 \tilde g^{s_1s_2} \partial_{s_1} G_\epsilon^{p_1q_1} \partial_{s_2} G_\epsilon^{p_2q_2} \big(\tilde g_{p_1p_2} \tilde g_{q_1q_2} \\ &\qquad - \tilde g_{p_1q_2} \tilde g_{q_1p_2} + 2 \tilde g_{p_1q_2} (G_\epsilon^{-1})_{q_1p_2} - \tilde g_{p_1p_2} g_{q_1q_2} - (G_\epsilon^{-1})_{q_2p_1} (G_\epsilon^{-1})_{q_1p_2} \big)\ .\end{aligned} \tag{3.179}$$

Expanding the last row makes the promised matching,

$$\begin{aligned}3\,(3) &= -3\,(2) + 2 (g_{q_1q_2} - \tilde g_{q_1q_2}) \partial_{p_2} \tilde g^{p_1q_1} \partial_{p_1} \tilde g^{p_2q_2} \\ &\quad + 2 \tilde g^{s_1s_2} \partial_{s_1} G_\epsilon^{p_1q_1} \big(2 \tilde g_{p_1p_2} (G_\epsilon^{-1})_{q_1q_2} \partial_{s_2} \tilde g^{p_2q_2} - \big(\tilde g_{p_1p_2} g_{q_1q_2} + (G_\epsilon^{-1})_{q_2p_1} (G_\epsilon^{-1})_{q_1p_2} \big) \partial_{s_2} G_\epsilon^{p_2q_2} \big)\ .\end{aligned} \tag{3.180}$$

Putting everything together gives, now back in ordinary indices,

$$\begin{aligned}|H|^2 = {}& \tfrac{1}{2}\tilde{g}_{pq}\tilde{g}_{rs}\tilde{g}^{km}\partial_k\beta^{pr}\partial_m\beta^{qs} + (g_{pq}-\tilde{g}_{pq})\partial_r\tilde{g}^{sp}\partial_s\tilde{g}^{rq} \\ & + \tilde{g}^{km}\partial_k G^{pq}\big(2\tilde{g}_{pr}(G^{-1})_{qs}\partial_m\tilde{g}^{rs} - \big(\tilde{g}_{pr}g_{qs} + (G^{-1})_{sp}(G^{-1})_{qr}\big)\,\partial_m G^{rs}\big)\ .\end{aligned} \tag{3.181}$$

A comparison with the intermediate result (3.173) shows that all but the first term in the above expression are cancelled against what the calculation had given so far. In total one finds

$$\widetilde{\mathcal{R}} - \mathcal{R} + 4(\partial\tilde{\phi})^2 - 4(\partial\phi)^2 - \tfrac{1}{2}|Q|^2 + \tfrac{1}{2}|H|^2 = -\text{t.d.}\ , \tag{3.182}$$

where the definitions

$$|Q|^2 = \frac{1}{2!}Q_k{}^{mn}Q_p{}^{qr}\tilde{g}^{kp}\tilde{g}_{mq}\tilde{g}_{nr}\ , \tag{3.183}$$

and

$$Q_k{}^{mn} \equiv \partial_k\beta^{mn}\ , \tag{3.184}$$

have been chosen to reveal a square term. They agree with the double field theory computation, cf. (3.66), and the related suggestions from the literature mentioned there, but, again, have to be understood as suggestions that will be supported by other observations. This concludes the computation of the field redefined terms.

Final result

In total, the above considerations proof the following result:

$$\boxed{\begin{aligned} S &= \int \mathrm{d}x\sqrt{|g|}e^{-2\phi}\Big(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{2}|H|^2\Big) \\ &= \int \mathrm{d}x\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\Big(\widetilde{\mathcal{R}} + 4(\partial\tilde{\phi})^2 - \frac{1}{2}|Q|^2\Big) + \int \mathrm{d}x\sqrt{|\tilde{g}|}e^{-2\tilde{\phi}}\ \text{t.d.}\end{aligned}} \tag{3.185}$$

where the total derivative in the last term is given in (3.172).

This shows that it is indeed possible to reveal non-geometric fluxes in the framework of supergravity by employing the field redefinition (3.145). As the restriction (3.151) has been assumed to simplify the presentation, only the flux Q is appearing, whereas the R-flux cannot be reconstructed. Furthermore, Q appears in a very particular form that is similar to the former H-flux. These findings can and will be generalised below.

Before drawing the connection between the above result and the respective results for double field theory in the next section, a few further points shall be discussed right away.

Total derivative: In field theories it is common practise to neglect total derivatives as it is assumed that they integrate to zero. This is usually justified by claiming that all fields vanish at infinity or that they have trivial monodromies on compact integration manifolds. In the present investigation, the total derivative that appeared was kept explicit in the final result because exactly the latter claim might be violated. This can be checked explicitly by using the three-torus with dilute flux as an example. Its target space fields were defined in (2.8) of chapter 2, in the rescaled guise they read

$$g = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\ ,\quad b = \begin{pmatrix}0&x_3&0\\-x_3&0&0\\0&0&0\end{pmatrix}\ . \tag{3.186}$$

Although this situation is geometric, its description in terms of $(\tilde{g}, \beta, \tilde{\phi})$ offers non-trivial torus monodromies. Using (3.152), one finds

$$\tilde{g} = \begin{pmatrix} 1+(x_3)^2 & 0 & 0 \\ 0 & 1+(x_3)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \beta = \frac{1}{1+(x_3)^2} \begin{pmatrix} 0 & x_3 & 0 \\ -x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{3.187}$$

and a dilaton

$$\tilde{\phi}(x_3) = \phi(x_3) + \frac{1}{2} \ln(1+(x_3)^2) , \tag{3.188}$$

which all do not transform properly under $x_3 \to x_3 + 2\pi$. The total derivative term can be computed using (3.172) and indeed does not integrate to zero,

$$\int_0^{2\pi} \mathrm{d}x_3 \partial_3 \left(e^{-2\phi} \frac{4x_3}{1+x_3^2} \right) = e^{-2\phi(2\pi)} \frac{8\pi}{1+4\pi^2} . \tag{3.189}$$

In general one will find that the original supergravity action and the field redefined one are only equal up to a constant,

$$S\,[\, g, b, \phi \,] \;=\; S\,[\, \tilde{g}, \beta, \tilde{\phi} \,] + \text{const.} \tag{3.190}$$

On the level of the path integral this constant might matter, as it is not necessarily proportional to 2π.

To summarise, *it is not possible to neglect the total derivative term in general* and claim the complete equivalence of the two actions presented in this section. Rather, one could decide to change the theory by dropping the total derivative at all. This will be discussed for non-geometric backgrounds in section 3.3.4.

Non-tensorial character of Q: One has to be careful with the term that contains the Q-flux. The object Q itself does not transform as a tensor under diffeomorphisms in general, and therefore this term is not completely equivalent to the square of the H-flux. Nevertheless, it can be shown that the constraint (3.151) turns Q into a $(1,2)$-tensor and thus makes $|Q|^2$ transform as a scalar,

$$\delta_\xi \beta^{mn} = \mathcal{L}_\xi \beta^{mn} = \xi^k \partial_k \beta^{mn} - \partial_k \xi^m \beta^{kn} - \partial_k \xi^n \beta^{mk} \tag{3.191}$$

$$\begin{aligned} \Delta_\xi Q_p{}^{mn} &= \delta_\xi(\partial_p \beta^{mn}) - \mathcal{L}_\xi(\partial_p \beta^{mn}) \\ &= 2\partial_p \partial_k \xi^{[m} \beta^{n]k} = 0 \quad \text{for} \quad \beta^{mk}\partial_k = 0 . \end{aligned} \tag{3.192}$$

Relaxing the constraint would add more terms to the field redefined action such that the total invariance under diffeomorphisms is retained. It should be emphasised once more that the restricted action fully applies to the important case of the torus with H-flux and its T-duals discussed in chapter 2.

No simplifying assumption: It is possible to apply the field redefinition (3.145) to the supergravity action (3.139) without any additional assumption. The calculation can be performed in absolutely the same manner as presented here, but will be more complicated.

This has been done in [4], appendix B, with the result

$$\begin{aligned} S = \int \mathrm{d}x \sqrt{|\tilde{g}|} e^{-2\tilde{\phi}} \Big(& \widetilde{\mathcal{R}} + 4(\partial\tilde{\phi})^2 - \tfrac{1}{2}|R|^2 \\ & + 4\tilde{g}_{ij}\beta^{ik}\beta^{jl}\partial_k d\, \partial_l d - 2\partial_k d\, \partial_l \big(\tilde{g}_{ij}\beta^{ik}\beta^{jl}\big) \\ & - \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\, Q_r{}^{kl}\, Q_s{}^{ij} + \tfrac{1}{2}\tilde{g}_{pq} Q_k{}^{lp} Q_l{}^{kq} \\ & + \tilde{g}_{jl}\tilde{g}_{pq}\beta^{jm}\big(Q_k{}^{lp}\partial_m \tilde{g}^{kq} + \partial_k \tilde{g}^{lp}\, Q_m{}^{kq}\big) \\ & - \tfrac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\big(\beta^{pr}\beta^{qs}\partial_r\tilde{g}^{kl}\partial_s\tilde{g}^{ij} - 2\beta^{ir}\beta^{js}\partial_r\tilde{g}^{lp}\partial_s\tilde{g}^{kq}\big)\Big) \\ + \int \mathrm{d}x \sqrt{|\tilde{g}|} e^{-2\tilde{\phi}}\ \text{t.d.} \end{aligned} \tag{3.193}$$

The total derivative is given by

$$t.d. = \frac{1}{e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}}\partial_k \left(e^{-2\tilde{\phi}}\sqrt{|\tilde{g}|}\ \left(\tilde{g}^{km}\tilde{g}^{pq}\partial_m \tilde{g}_{pq} - g^{km}g^{pq}\partial_m g_{pq} - \partial_m (g^{km} - \tilde{g}^{km}) \right)\right) \ , \tag{3.194}$$

where a few g are left for convenience, but actually have to be replaced by (3.145).

As Q will not transform as a tensor without the constraint (3.151), the first term in the third row of the above action has not been written as a square. It is rather the whole set of rows two to four which transforms properly. In contrast to the former result (3.185), the Q-flux is not anymore appearing in a simple form that is similar to the H-flux – which could have been anticipated from the respective result in double field theory.

This is different for the R-flux, which is defined as in the double field theory case (3.67) but without dual derivatives,

$$R^{kmn} \equiv 3\beta^{p[k}\partial_p \beta^{mn]} \ . \tag{3.195}$$

The motivation for this definition is similar to the arguments there. In particular, R transforms as a tensor due to its antisymmetry,

$$R^{kmn} = 3\beta^{p[k}\nabla_p \beta^{mn]} \ , \tag{3.196}$$

where ∇ denotes the standard covariant derivative. In this sense, *the actual equivalent to the term* $|H|^2$ *is* $|R|^2$. Furthermore, it shall be noted that for clarity the abbreviation

$$e^{-2d} = \sqrt{|\tilde{g}|}e^{-2\phi} \tag{3.197}$$

has been written out.

It can easily be checked that the above action (3.193) reduces to the restricted one (3.185) by imposing (3.151), which in particular sets the R-flux to zero. The total derivative term truncates accordingly.

3.3.2 Connection to double field theory

In section 3.2 it was shown that the double field theory action (3.17) can be reduced to the conventional NSNS supergravity action (3.32) by solving the strong constraint in the most straightforward manner, $\tilde{\partial} = 0$. Furthermore, a field redefinition was performed in order to make the non-geometric fluxes Q and R visible, expressed in terms of a new field variable β. This section has shown how the very same field redefinition in the framework of ten-dimensional supergravity is able to make the R-flux visible again, while the Q-flux remains somewhat obscure.

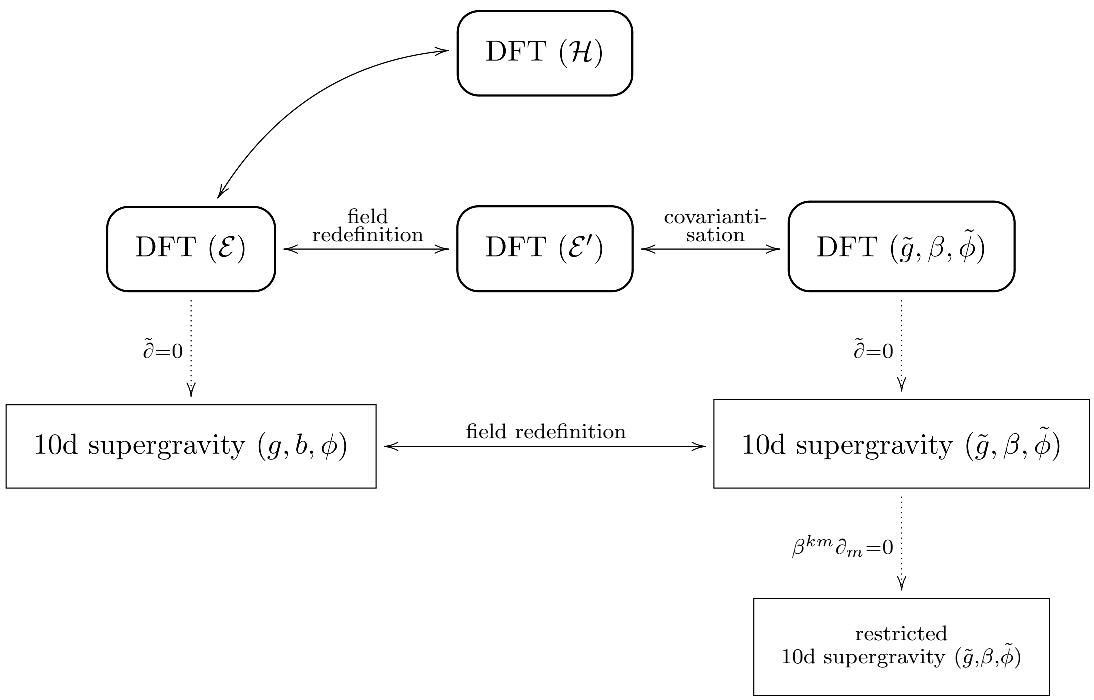
A natural question that might arise, given these results, is whether the field redefined double field theory action can be connected to the field redefined supergravity action. This would make an independent cross-check for all the calculations done, and furthermore provides insight into the geometrical role the Q-flux plays in the ten-dimensional framework. The flux Q in ten dimensions would then be the reduced antisymmetric part of the connection terms.

Indeed, such a link can be drawn. Once the field redefined action (3.115) is reduced by solving the strong constraint, $\tilde{\partial} = 0$, it equals the field redefined supergravity action (3.193). This can be seen by comparing the intermediate result (3.126) with (3.193) directly. Except for an integration by parts to sort out the dilaton terms, these Lagrangians match under the reduction

$$\tilde{\partial} = 0 \quad \Rightarrow \quad \tilde{D}^i = -\beta^{ij}\partial_j \ . \tag{3.198}$$

Again, a volume factor $\int \mathrm{d}\tilde{x}$ has to be discarded[5].

In summary, this chapter has discussed and established the following connections:



The double field theory action in its generalised metric formulation [18] can be restated in terms of the object $\mathcal{E} = g + b$. This formulation simplifies the implementation of the field redefinition that then exchanges $\mathcal{E}$ for $\mathcal{E}' = \mathcal{E}^{-1}$. In particular, each term is invariant separately such that the exchange is roughly speaking only a matter of inverting all indices, cf. (3.47).

The result, a double field theory action in terms of $\mathcal{E}'$, is not straightforwardly helpful regarding the investigation of non-geometric fluxes and non-geometry, and thus has to be rewritten in a D-diffeomorphism covariant manner. This covariantisation offers a formulation in terms of new variables $\tilde{g}$, β and $\tilde{\phi}$ that reveals the R-flux as the covariant field strength of β and the Q-flux as antisymmetric part of the connection.

[5]Some further details can be found in section 2.3 of [4].

These three formulations (3.17), (3.48) and (3.115) are equivalent to each other, which is indicated in the second row of the above diagram. Technically speaking, one has to integrate by parts to switch from one formulation to another, and the respective total derivatives have to be kept in the case of non-geometric backgrounds as will be discussed below. For well-defined setups the equivalence holds, at least classically, without further restriction.

On the level of ten-dimensional supergravity, one can switch directly between a formalism in terms of the usual NSNS variables g, b and ϕ, i.e. the action (3.139), and a new formalism in terms of $\tilde{g}$, β and $\tilde{\phi}$, i.e. the action (3.193), by employing the field redefinition. This is shown in the third line of the above diagram. Again, for non-geometric setups there appear non-trivial total derivatives that have to be treated with care.

In the course of the above investigations, a simplifying assumption was made. It leads to a restricted ten-dimensional supergravity action (3.185), that is shown in the last row of the diagram.

There are two straightforward connections between the double field theory framework and the supergravity framework. Both times, the strong constraint has to be solved by $\tilde{\partial} = 0$ and a volume of $\int \mathrm{d}\tilde{x}$ has to be integrated out. This is depicted by two vertical arrows between the second and the third row in the above diagram.

As a side remark, another observation shall be made here. One other obvious solution to the strong constraint of double field theory obviously is given by

$$\partial_i = 0 \ . \tag{3.199}$$

This will keep the dual coordinates only. The reduced action can be shown to be

$$\tilde{S} = \int \mathrm{d}\tilde{x} \, \sqrt{|\det \tilde{g}^{ij}|} \, e^{-2\phi'} \Big(\mathcal{R}(\tilde{g}^{ij}, \tilde{\partial}) + 4 \, \tilde{g}_{ij} \, \tilde{\partial}^i \phi' \, \tilde{\partial}^j \phi' - \frac{1}{12} R^{ijk} R_{ijk} \Big) \tag{3.200}$$

which is most easily checked by using the intermediate result (3.129). All terms containing Q vanish, as well as the standard curvature scalar $\mathcal{R}$ and the standard dilaton term $(\partial\tilde{\phi})^2$. All derivatives reduce to

$$\tilde{D}^i = \tilde{\partial}^i \ , \tag{3.201}$$

such that the remaining terms organise into a new scalar curvature $\mathcal{R}(\tilde{g}^{ij}, \tilde{\partial})$. Additionally, the R-flux is now given by its reduced form

$$R^{ijk} = 3\tilde{\partial}^{[i} \beta^{jk]} \ , \tag{3.202}$$

and the dilaton ϕ' has to be defined as

$$\sqrt{|\det \tilde{g}^{ij}|} \, e^{-2\phi'} = e^{-2d} \tag{3.203}$$

Another way to check this rewriting is to use (3.115) and

$$e^{-\tilde{\phi}} \sqrt{|\tilde{g}|} \left(\tilde{D}^i \tilde{\phi} + \mathcal{T}^i \right) = \tilde{\partial}^i \left(e^{-\tilde{\phi}} \sqrt{|\tilde{g}|} \right) + \partial_m \left(\beta^{mi} e^{-\tilde{\phi}} \sqrt{|\tilde{g}|} \right) \ . \tag{3.204}$$

The action (3.200) has to be understood as having all upper and lower indices interchanged. In particular, $\tilde{g}^{ij}$ is the metric on a space with coordinates $\tilde{x}_i$; β^{ij} transforms as a two-form under $\tilde{x}$-diffeomorphisms, and R^{ijk} is the same field strength associated to it as H is the field strength associated to b_{ij}. More generally, it has been shown in [17, 80] that (3.200) is precisely equivalent to the standard NSNS supergravity action.

This result could be interpreted in the following way. Keeping only winding derivatives in the field redefined double field theory action (3.115) corresponds to a T-duality in all directions [80, 81]. It is the combination of using the $O(D,D)$ transformed field variable $\tilde{\mathcal{E}}$, c.f. (3.44), and keeping the transformation of the $O(D,D)$ vector X, that was formerly undone in (3.46). As the R-flux after setting $\tilde{\partial}_i = 0$ behaves like the H-flux in the standard supergravity action, it appears justified to conjecture that they are only *dual descriptions of the same physical content*.

This indicates that the non-geometric fluxes that have been introduced in this chapter do not make any new degree of freedom, as one can also expect from applying a field redefinition and as was already supposed in the introduction to this chapter. And indeed, it might be that not all non-geometric setups can be captured by the present framework, as will be further discussed in the last section.

3.3.3 Dimensional reduction

It is possible to obtain a generic four-dimensional scalar potential for the non-geometric fluxes Q and R by taking into account a volume modulus and the four-dimensional dilaton. Although this provides just a very simple dimensional reduction of the higher-dimensional actions presented in this chapter, a comparison to suggested forms for that kind of potentials in the literature can be drawn.

Furthermore, having at hand such a link, one might conjecture that the presented actions indeed provide an uplift of the four-dimensional non-geometric fluxes. This in particular provides further evidence that the suggested definitions of Q and R are the correct ones.

The starting point is the ten-dimensional action (3.193), which now shall be reduced in the setup of an unwarped compactification ansatz

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{M}_6 \ , \tag{3.205}$$

reflected by a metric of the form

$$\tilde{g}_{mn}(x^m) = \begin{pmatrix} \tilde{g}_{\mu\nu}(x^\lambda) & 0 \\ 0 & \tilde{g}_{ij}(x^m) \end{pmatrix} \ . \tag{3.206}$$

Indices μ, ν, λ denote external directions, indices i, j, k denote internal directions, and the index m is supposed to be a ten-dimensional index. All fields are set to their vacuum expectation values denoted by an index (0), where in particular the dilaton is expected to have a constant vacuum expectation value. The fluxes Q and R are restricted to have internal legs only, which can be achieved by setting

$$\beta^{\mu\nu} = \beta^{i\mu} = \beta^{\mu i} = 0 \ , \quad \text{and} \quad \beta^{ij} = \beta^{ij}(x^k) \ . \tag{3.207}$$

Two four-dimensional scalar fields, here denoted as moduli, ρ and φ shall be introduced as fluctuations of the metric, or the dilaton, respectively, around their vacuum expectation values. The former modulus corresponds to volume fluctuations in the internal manifold. In total, the reduction then takes the form

$$\tilde{\phi}(x^m) = \tilde{\phi}^{(0)} + \varphi(x^\mu) \tag{3.208}$$

$$\tilde{g}_{ij}(x^m) = \rho(x^\mu)\, \tilde{g}^{(0)}_{ij}(x^k) \ . \tag{3.209}$$

Two comments shall be made at this stage:

- It is assumed that there exists a background of the prescribed form. This is justified by the fact that the two moduli ρ and φ appear in any compactification, independent of the particular model. Furthermore, having no warping seems less problematic when compared to the analogous situation in type IIB supergravity compactifications, where it simply leads to a large volume limit.
- The vacuum expectation values $\tilde{g}^{(0)}$, $\tilde{\phi}^{(0)}$, $\beta^{(0)}$ are not specified explicitly, i.e. they are simply assumed to be solutions of the equations of motion. For the restricted case, i.e. when (3.151) holds, these equations are written down in (3.225) and (3.226).

Eventually, the rather simplistic approach here is at least sufficient to reveal the scaling behaviour of the non-geometric fluxes in the respective four-dimensional potential, which is what is intended in this section.

Geometric fluxes

To illustrate what kind of information can be obtained from the reduction procedure presented here, first the four-dimensional potential of the standard NSNS supergravity action (3.32) shall be computed. The next subsection then presents the corresponding results for the field redefined action.

Using the same notation and construction as defined above for the original fields ϕ, g and b, the appearing terms show the following dependence on the two moduli ρ and φ,

$$\begin{aligned} \sqrt{|g_{ij}|} &= \rho^3 \sqrt{|g^{(0)}_{ij}|} \\ \mathcal{R}_6 &= \rho^{-1}\,\mathcal{R}^{(0)}_6 \\ H_{ijk}H^{ijk} &= \rho^{-3}\,H^{(0)}_{ijk}H^{(0)\,ijk}\ , \end{aligned} \tag{3.210}$$

where the second equality refers to the internal curvature scalar with $\mathcal{R} = \mathcal{R}_4 + \mathcal{R}_6$. The full action (3.32) can be brought to the form

$$S_E = M_4^2 \int \mathrm{d}^4x\,\sqrt{|g^E|}\left(\mathcal{R}^E_4 + \mathrm{kin} - \frac{1}{M_4^2}V(\rho,\sigma)\right)\ , \tag{3.211}$$

with a potential V given by

$$V(\rho,\sigma) = \sigma^{-2}\left(\rho^{-3}\,V^0_H + \rho^{-1}\,V^0_f\right)\ . \tag{3.212}$$

This reveals the known scaling behaviour of the two contributions V_H and V_f, coming from the H-flux and geometric flux, respectively, in the most accessible way. The particular definitions are

$$\begin{aligned} V^0_H &= \frac{M_4^2}{v_0}\int \mathrm{d}^6x\,\sqrt{|g^{(0)}_{ij}|}\,\frac{1}{12}H^{(0)}_{ijk}H^{(0)ijk} \\ V^0_f &= -\frac{M_4^2}{v_0}\int \mathrm{d}^6x\,\sqrt{|g^{(0)}_{ij}|}\,\mathcal{R}^{(0)}_6\ . \end{aligned} \tag{3.213}$$

A few more conventions entered here:

- The supergravity action has been supplemented with a prefactor

$$\frac{1}{2\kappa^2} = \frac{1}{(2\pi)^7 \alpha'^4} \ . \tag{3.214}$$

- The string coupling was taken to be given in terms of the vacuum expectation value of the (constant) dilaton,

$$g_s = e^{\phi^{(0)}} \ . \tag{3.215}$$

- A four-dimensional dilaton has been defined for convenience,

$$\sigma = \rho^{3/2} \, e^{-\phi} \ . \tag{3.216}$$

- Some constants were repackaged into a four-dimensional Planck mass

$$(M_4)^2 = \frac{v_0}{2\kappa^2 g_s^2} \ , \tag{3.217}$$

  with v_0 being the volume of the internal manifold in its background configuration,

$$v_0 = \int \mathrm{d}^6 x \sqrt{|g_{ij}^{(0)}|} \ . \tag{3.218}$$

  This of course presupposes that the internal manifold is compact.

- The transformation to Einstein frame, denoted by an additional index E on the respective quantities, was performed by rescaling the external metric as

$$g_{\mu\nu} = \sigma^{-2} g_{\mu\nu}^E \ . \tag{3.219}$$

- The kinetic terms of the moduli ρ and φ, i.e. all terms that contain derivatives on these fields, are not of interest in this investigation and thus have been collected under the label “kin”.

More importantly, it has been assumed that the fields ϕ, g and b are well-defined in the sense that they can be integrated over the compact internal manifold. As this might be automatic in the context of ordinary effective field theories, such a claim could be violated in non-geometric configurations where the fields can, for instance, acquire non-trivial monodromies. A solution to this threat is provided by using the field redefined action, as will be discussed in the next section. But before doing so, the reduction procedure shall be applied to precisely that action without differentiating between well- and ill-defined fields.

Non-geometric fluxes

By translating all the above conventions and definition to a field basis $\tilde{\phi}$, $\tilde{g}$ and β, the unrestricted action (3.193) can be brought into the same form (3.211), but with a potential

$$\boxed{V(\rho,\sigma) = \sigma^{-2} \left(\rho^{-1} \, V_f^0 + \rho \, V_Q^0 + \rho^3 \, V_R^0 \right) \ .} \tag{3.220}$$

To get this result, one should take into account that by definition of the reduction procedure all terms containing $\beta^{km}\partial_m\tilde{\phi}$ vanish. In particular, it is

$$-2\beta^{km}\partial_m d = \beta^{km}\partial_m \ln\sqrt{|\tilde{g}|} = \frac{1}{2}\beta^{km}\tilde{g}^{pq}\partial_m\tilde{g}_{pq} \ , \tag{3.221}$$

which helps to verify that the only dilaton kinetic term in (3.193) is the standard one after reduction. Furthermore, the respective terms in the potential can be identified as

$$\begin{aligned}
V_f^0 &= -\frac{M_4^2}{v_0}\int \mathrm{d}^6x\,\sqrt{|\tilde{g}_{ij}^{(0)}|}\;\widetilde{\mathcal{R}}_6^{(0)} \ , \\
V_R^0 &= \frac{M_4^2}{v_0}\int \mathrm{d}^6x\,\sqrt{|\tilde{g}_{ij}^{(0)}|}\;\frac{1}{12}R^{(0)ijk}R^{(0)}_{ijk} \ , \\
V_Q^0 &= -\frac{M_4^2}{v_0}\int \mathrm{d}^6x\,\sqrt{|\tilde{g}_{ij}^{(0)}|}\;\Bigg(-\frac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}^{rs}\,Q_r{}^{kl}\,Q_s{}^{ij} + \frac{1}{2}\tilde{g}_{pq}Q_k{}^{lp}Q_l{}^{kq} \\
&\quad + \tilde{g}_{jl}\tilde{g}_{pq}\beta^{jm}\left(Q_k{}^{lp}\partial_m\tilde{g}^{kq} + \partial_k\tilde{g}^{lp}\,Q_m{}^{kq}\right) \\
&\quad - \frac{1}{4}\tilde{g}_{ik}\tilde{g}_{jl}\tilde{g}_{pq}\left(\beta^{pr}\beta^{qs}\partial_r\tilde{g}^{kl}\partial_s\tilde{g}^{ij} - 2\beta^{ir}\beta^{js}\partial_r\tilde{g}^{lp}\partial_s\tilde{g}^{kq}\right) \\
&\quad + \frac{1}{2\sqrt{|\tilde{g}|}}\;\tilde{g}^{pq}\partial_k\tilde{g}_{pq}\;\partial_m\left(\sqrt{|\tilde{g}|}\tilde{g}_{ij}\beta^{ik}\beta^{jm}\right)\Bigg) \ ,
\end{aligned} \tag{3.222}$$

where in the last equality all fields on the right-hand side are understood as carrying the index $^{(0)}$, i.e. as being vacuum expectation values.

The potential (3.220) contains two new types of scaling behaviour when compared to the standard result (3.212). These correspond to the terms induced by non-geometric fluxes. Interestingly, this makes another independent argument in favour of the definitions of $Q_i{}^{jk}$ and R^{ijk} in (3.184) and (3.195). Even when ignoring the structures that arise by using these definitions, one could simply sort the result of the field redefinition according to the scaling behaviour in the potential (3.220) and find the same result.

Furthermore, the resulting potential can be successfully compared to the literature. In [89] it was argued that the most general potential that can arise from the NSNS sector is given by

$$V(\rho,\sigma) = \sigma^{-2}\left(\rho^{-3}\ V_H^0 + \rho^{-1}\ V_f^0 + \rho\ V_Q^0 + \rho^3\ V_R^0\right) \ , \tag{3.223}$$

with V_Q^0 and V_R^0 being constants depending on the four-dimensional fluxes Q and R. One can straightforwardly recognise that this result is confirmed by the potential derived here. As b was replaced by β in the field redefinition, there is no V_H^0 in (3.220) though.

In other words, one can draw the conclusion that *the field redefined action* (3.193) *provides a ten-dimensional lift of the four-dimensional fluxes Q and R*, since it reproduces the corresponding terms in the potential.

Restricted case

For the case where the field redefinition is simplified by the assumption (3.151), it is possible to draw another link to the literature. Given the setup for a dimensional reduction discussed above, the equations of motion imply constraints on the fluxes when the external manifold is required to be Minkowski or de Sitter. These constraints match the findings that have been suggested for example in [49].

As a starting point, the field redefined action (3.185) shall be slightly generalised in the sense that it is completed with an H-flux term,

$$S = \int \mathrm{d}x e^{-2\phi}\sqrt{|g|}\left(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{2}|Q|^2 - \frac{1}{2}|H|^2\right) \; . \tag{3.224}$$

Such an additional term can never be the result of the field redefinition (3.145) if the respective indices are taken to range over all dimensions. On the other hand, one could assume a product structure of the internal manifold such that there is an H-flux turned on with legs in only one factor, while the field redefinition is applied to the other factor, which then contains Q-flux only. A mixed action like the above would then exactly capture the right degrees of freedom.

Another way to motivate (3.224) is to call it a heuristic approach that helps to take into account the H-flux along the dimensional reduction. In this sense, it can be viewed as being halfway between the original fields (ϕ, g, b) and the completely redefined fields $(\tilde{\phi}, \tilde{g}, \beta)$ obtained from an intermediate choice of parametrisation in (3.36). The a priori appearing additional degrees of freedom are assumed to cancel due to Bianchi identities [87].

The ten-dimensional equations of motion for ϕ and g can be determined as

$$0 = \partial_k\left(8e^{-2\phi}\sqrt{|g|}g^{km}\partial_m\phi\right) + e^{-2\phi}\sqrt{|g|}\left(2\mathcal{R} + 8(\partial\phi)^2 - |Q|^2 - |H|^2\right) \tag{3.225}$$

$$\begin{aligned} 0 = {} & \mathcal{R}_{mn} - \frac{1}{2}g_{mn}\mathcal{R} + 2g_{mn}\left((\partial\phi)^2 - \nabla^2\phi\right) + 2\nabla_m\nabla_n\phi + \frac{1}{4}g_{mn}\left(|H|^2 + |Q|^2\right) \\ & - \frac{1}{4}H_{mpq}H_n{}^{pq} - \frac{1}{4}Q_{mpq}Q_n{}^{pq} + \frac{1}{2}Q^p{}_{mq}Q_{pn}{}^q \; , \end{aligned} \tag{3.226}$$

whereas the equations of motion for b and β are not of interest in the following. They are determined in [7] for an extended definition of Q that here shall not be elaborated on.

Two assumptions of the reduction setup have to be recalled here: first, the spacetime is assumed to have product structure without warping; second, the fluxes H and Q are assumed to have internal legs only. In particular, one has $\mathcal{R} = \mathcal{R}_4 + \mathcal{R}_6$ with

$$\mathcal{R} = g^{mn}\mathcal{R}_{mn} = -\frac{9}{2}\nabla^2\phi + 5(\partial\phi)^2 + \frac{1}{4}|H|^2 + \frac{3}{4}|Q|^2 \tag{3.227}$$

$$\mathcal{R}_4 = g^{\mu\nu}\mathcal{R}_{\mu\nu} = 2(\partial\phi)^2 - \nabla^2\phi - 2\nabla_\mu\nabla^\mu\phi - \frac{1}{2}|H|^2 + \frac{1}{2}|Q|^2 \; , \tag{3.228}$$

obtained by tracing the Einstein equation (3.226).

Having a Minkowski or de Sitter spacetime translates to a non-negative four-dimensional curvature scalar, which for the assumed case of a constant dilaton gives the constraint

$$0 \leqslant \mathcal{R}_4 = -\frac{1}{2}|H|^2 + \frac{1}{2}|Q|^2 \; , \tag{3.229}$$

read off from (3.228) directly. On the other hand, using (3.227), one can solve for the H-flux and replace it by the internal curvature,

$$0 \leqslant -\mathcal{R}_6 + |Q|^2 \; . \tag{3.230}$$

Physically, this can be interpreted by extracting two statements:

- The non-geometric flux Q is capable of compensating negative four-dimensional curvature contributions from other fluxes.

- Due to a balancing effect of the non-geometric flux Q, it is possible to have positively curved internal manifolds.

At this stage, a comparison with the literature is straightforwardly possible. First, one can recognise that under the simplifying assumption (3.151) the terms in the four-dimensional potential, namely (3.222), reduce to

$$V_f \sim \mathcal{R}_6 \ , \quad V_Q \sim \frac{1}{2}|Q|^2 \ , \quad V_R = 0 \ . \tag{3.231}$$

Additionally, one can infer that the contribution of an extra H-flux term as in (3.224) will be

$$V_H \sim \frac{1}{2}|H|^2 \ . \tag{3.232}$$

With these settings, the conditions (3.229) and (3.230) can be found in equation (5.3) of [49]. There, further ingredients like RR-fluxes, sources and warping have been considered, which should be set to zero for comparison.

Although these results promise progress in solving the typical problems of flux compactifications, as for example finding at least Minkowski, if not de Sitter, vacua, a drawback arises immediately when taking into account the dilaton equation of motion (3.225). For a constant dilaton it dictates to solve the above conditions by setting

$$\mathcal{R}_4 = 0 \ , \quad \mathcal{R}_6 = |H|^2 = |Q|^2 \ , \tag{3.233}$$

which does not allow for de Sitter solutions anymore. At this point, it becomes clear that non-geometric fluxes alone do not guarantee the possibility of such spacetimes, the above mentioned other ingredients are still necessary. Of course it is possible that the relaxation of the constraint (3.151) allows for more elaborate setups that do not suffer of such drawbacks, but this rests even more on whether there are genuine solutions of all equations of motion in ten dimensions, that make real non-geometric backgrounds.

3.3.4 Non-geometry

So far, it has been shown how to reveal non-geometric fluxes in a framework of effective field theories, namely in double field theory and in NSNS supergravity. Until here, one could simply notice that as a particular method to rewrite these well-known actions in a more or less useful way. In the following, it will be shown how the new descriptions can be used in actually non-geometric configurations and how it is possible to repair the ill-definedness appearing there.

A case study

It is instructive to first investigate the special case of a three-torus with constant H-flux. This setup allows to reveal some features of non-geometry and how they can be accommodated by the field redefined action. The basic construction has been described in detail in chapter 2 from the perspective of the worldsheet theory, a short comment on how it translates into the target space considerations of this chapter has been given on page 98. The necessary details shall be repeated here.

On the geometric side, the basic construction is a torus fibration $T^2 \times S^1$ that makes three of the internal directions. A full background can be obtained by completing this with additional dimensions, an appropriate external spacetime factor and some other ingredients as shortly discussed in the previous chapter. In the following, the three dimensions shall only serve as a toy model and it will be ignored that they alone do not suffice as a valid string theory setup.

To make up a three-torus, the coordinates $x^{1,2,3}$ are periodically identified,

$$x^i \sim x^i + 2\pi \ , \tag{3.234}$$

where in contrast to chapter 2, all radii are taken to be of unit length for convenience. The metric is thus given by the unit matrix, and a constant H-flux is added by the following b-field[6],

$$g_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad b_A = \begin{pmatrix} 0 & x_3 & 0 \\ -x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{3.235}$$

This defines the first T-duality frame, so the fields are subscribed with an index 'A', and are identical to the ones discussed in (3.186). The only coordinate dependence is on the base fibre x_3, such that there are two isometry directions. These allow for two T-dualities, the corresponding frames are denoted by indices 'B' and 'C', where the former is the twisted torus frame and the latter the non-geometric frame.

The importance of this example is on the one hand underlined by its frequent appearance in the literature (cf. the introduction to this chapter), but on the other hand especially emphasised by the fact, that in all three frames, the simplifying assumption (3.151) is fulfilled. This will be checked explicitly in the following, but can also be deduced from the fibre structure and the coordinate dependence of b and g. Roughly speaking, β defined by (3.152) will never have a component in the x_3 direction that makes the only coordinate dependence of any field in any T-duality frame, i.e.

$$\beta^{ij}\partial_j = 0 \ , \tag{3.236}$$

which exactly is (3.151). This also holds for the dilaton, as in order to have two isometries it depends on x_3 only,

$$\phi_A = \phi_A(x_3) \ . \tag{3.237}$$

A monodromy $x_3 \to x_3 + 2\pi$ induces a simple gauge transformation of b, and the action

$$S = \int \mathrm{d}x \sqrt{|g_A|} e^{-2\phi_A} \Big(\mathcal{R}_A + 4(\partial\phi_A)^2 - \frac{1}{2}|H_A|^2 \Big) \tag{3.238}$$

is well-defined. As has been shown in (3.187) and (3.188), this is not anymore the case for a description with the redefined fields $(\tilde{\phi}, \tilde{g}, \beta)$. A monodromy in the third coordinate induces changes of all three target space fields, that cannot be undone by any symmetry of the field redefined action, in particular not by a diffeomorphism. In conclusion, the standard NSNS supergravity action is capable of describing background A, whereas the Q-flux action (3.185) is not.

The twisted torus frame B can be reached by applying a T-duality in the x_1 direction. Its target space fields read, cf. (2.17),

$$g_B = \begin{pmatrix} 1 & -x_3 & 0 \\ -x_3 & 1 + x_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad b_B = 0 \ , \tag{3.239}$$

[6] The constant H has been dropped compared to chapter 2, and there will be no dilute flux approximation here.

with the dilaton not being changed,

$$\phi_B = \phi_A - \frac{1}{2}\ln\big((g_A)_{11}\big) = \phi_A \ , \tag{3.240}$$

according to its transformation rule (A.12). The standard NSNS supergravity action and its field redefined counterpart conincide in this case, as the field redefinition (3.152) shows,

$$\tilde{g}_B = g_B \ , \quad \beta_B = b_B = 0 \ , \quad \tilde{\phi}_B = \phi_B \ . \tag{3.241}$$

Furthermore, all fields are well-defined, where in particular the monodromy $x_3 \to x_3 + 2\pi$ for g_B can be compensated by a diffeomorphism[7]. In conclusion, background B is geometric and thus not of further interest in the given context - although this property was crucial in chapter 2.

A second T-duality transformation, this time along the x_2 direction, constitutes background C. The target space fields read

$$g_C = \frac{1}{1+x_3^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+x_3^2 \end{pmatrix} \ , \quad b_C = \frac{1}{1+x_3^2}\begin{pmatrix} 0 & -x_3 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{3.242}$$

and

$$\phi_C = \phi_B - \frac{1}{2}\ln\big((g_B)_{22}\big) = \phi_A - \frac{1}{2}\ln(1+x_3^2) \ . \tag{3.243}$$

All three fields are ill-defined, as they have a non-trivial monodromy for $x_3 \to x_3 + 2\pi$. In particular, the b-field cannot be patched over the full base circle by using gauge transformations or diffeomorphisms only. Furthermore, not even the torus volume

$$\mathrm{vol}_3 = \det\ g_C = \frac{1}{(1+x_3^2)^2} \tag{3.244}$$

has a trivial monodromy. But as the fields were defined as T-duals of well-defined ones, a patching with T-duality transformations could make ϕ_C, g_C and b_C well-defined as well. This will be discussed later on in more detail. Accordingly, the background is truly non-geometric and therefore suspected to be describable by the field redefined action (3.185). And indeed, the new field variables are surprisingly simple,

$$\tilde{g}_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad \beta_C = \begin{pmatrix} 0 & -x_3 & 0 \\ x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{3.245}$$

and

$$\tilde{\phi}_C = \phi_C + \frac{1}{2}\ln(1+x_3^2) = \phi_A \ , \tag{3.246}$$

according to (3.146). Additionally, the simplifying assumption (3.151) is fulfilled, as in $\beta^{ij}\partial_j$ only $j = 1, 2$ is generically nonzero but the fields depend on x_3 only. The metric $\tilde{g}_C$ and the dilaton $\tilde{\phi}_C$ are well-defined under the monodromy $x_3 \to x_3 + 2\pi$, whereas β_C seems to be problematic due to the lack of a proper gauge symmetry in this formalism - in contrast

[7] Cf. (2.12), which can also be formulated as a lattice action, e.g. (2.24) in [90]. I thank D. Andriot for pointing that out.

to the double field theory approach of the preceding section (cf. (3.50) with dual derivatives appearing). Nevertheless, the field redefined action is not touched by this subtlety as only the constant non-geometric flux Q enters with

$$(Q_C)_3{}^{12} = -(Q_C)_3{}^{21} = -1 \; . \tag{3.247}$$

In other words,

$$S = \int \mathrm{d}x \sqrt{|\tilde{g}_C|} e^{-2\tilde{\phi}_C} \left(\widetilde{\mathcal{R}}_C + 4(\partial\tilde{\phi}_C)^2 - \frac{1}{2}|Q_C|^2 \right) \tag{3.248}$$

is capable of describing the non-geometric background C, whereas the original NSNS supergravity action is not. The two descriptions differ by a total derivative (3.172) that does not integrate to zero,

$$\int_0^{2\pi} \mathrm{d}x \partial_3 \left(e^{-2\phi_A} \frac{4x_3}{1+x_3^2} \right) = e^{-2\phi_A(2\pi)} \frac{8\pi}{1+4\pi^2} \; , \tag{3.249}$$

and it is strictly speaking a change of theory when switching between the two actions. This point shall be discussed in more detail later on.

To conclude the present case study, one can record at least three important observations:

- The fluxes arrange into the proposed T-duality chain along the different frames. A constant H-flux in the first frame A is transformed into constant geometric flux in frame B (visible as $\mathcal{R}_B = -1/2$) with no other fluxes, that finally is transformed into constant Q-flux in frame C visible after the field redefinition. In summary, it is

$$H_A \quad \overset{T_1}{\longleftrightarrow} \quad f_B \quad \overset{T_2}{\longleftrightarrow} \quad Q_C \; . \tag{3.250}$$

  This is also good evidence for that the suggested definition of the Q-flux is correct.

- Non-geometry appears through non-trivial monodromies of the fields and hinders the definition of a regular patching along the manifold.

- There is a preferred field basis for each frame. The geometric frames A and B have well-defined actions in terms of the original NSNS supergravity fields ϕ, g and b, whereas the non-geometric frame has a well-defined action in terms of the redefined fields $\tilde{\phi}$, $\tilde{g}$ and β.

These observations can be put into a broader context, which will be done in the following.

General considerations

The above case study motivates the following line of thought, that due to its broad implications shall be formulated as carefully as possible in a scheme of suppositions and theses.

- Supposition 1: Non-geometric configurations can be obtained by T-duality transformations on geometric configurations with a well-defined NSNS supergravity field content, i.e. a proper target space interpretation.

- Supposition 2: Such non-geometric configurations, obtained by a T-duality transformation, are ill-defined in the context of NSNS supergravity and cannot be described by it, although they are well-defined in string theory.

- Main thesis: There is a preferred field basis for any field configuration, whose respective action is well-defined and at most differs by a total derivative from the NSNS supergravity action.

These statements shall be defended one by one and critically evaluated against other evidence in the remainder of this section.

Supposition 1: The first supposition basically serves to set the range of applicability for the following considerations. Due to its logical structure it is strictly speaking already proven by the observations in the above case study: There is at least one non-geometric configuration, namely the three-torus with Q-flux, that has been generated by the application of a T-duality transformation to a well-defined geometric configuration, namely the three-torus with constant H-flux. The more important impact of supposition 1 comes from the implicit exclusion of non-geometric configurations that arise in a different way. For example, there are genuinely stringy constructions that are supposed to be non-geometric, stemming from conformal field theory considerations. Asymmetric orbifolds make such examples [91]. At first sight, they are out of range for effective field theories like the NSNS supergravity used here, and therefore shall be ignored in the following.

Another question is, how far reaching such an exclusion of other constructions will be. In other words: Is the set of all non-geometric configurations in string theory in any way exhausted by the ones obtained from T-duality? The most obvious answer to that question is of course negative, given at least the asymmetric orbifold construction mentioned here. But this statement might be relativised when taking the perspective of four-dimensional effective field theories. A first hint comes from the fact that the non-geometric fluxes constructed here enter the four-dimensional potential in the right way. There, they were originally added 'by hand' [33] to make it T-duality invariant, as was already discussed in the first chapter . In this sense, a theory that is capable of uplifting the Q- and R-flux terms, here obtained in (3.222), exhausts all possibilities.

The case study above has shown a slight drawback, though. A background with non-geometric H-flux (3.242) was translated into a well-defined description with nonzero Q-flux via the field redefinition. The field redefined action (3.248) does not contain any H-flux anymore. What is in principle impossible in that framework, is to have nonzero H- and Q- or R-flux at the same time. As the b-field is always traded off against β, it cannot be elsewise. On the other hand, there are indications [92] that such configurations make valid string backgrounds and it is clear that they are not describable by the approach presented in this chapter.

A more detailed discussion of how the field redefinition provides an uplift of different configurations in the four-dimensional setup helps to clarify the situation. This can be conveniently done by classifying gauged supergravities using the embedding tensor formalism[8]. The embedding tensor Θ lies in the representation of the global duality group of the ungauged supergravity and encodes which subgroup of that is promoted to a local symmetry. Accordingly, it encodes the masses and couplings of the respective gauged supergravity. Any two embedding tensors related to each other by a duality transformation lead to physically equivalent theories. More precisely, a theory with fields Φ and embedding tensor Θ is equivalent to a theory with redefined fields $h(\Phi)$ and the duality transformed embedding tensor $\tilde{\Theta} = h(\Theta)$,

[8]See [37] and references therein.

where h is the duality group element. In terms of actions, this is

$$S_{\text{gauged sugra}}\left[\Phi,\Theta\right] = S_{\text{gauged sugra}}\left[h(\Phi),h(\Theta)\right] \ . \tag{3.251}$$

In other words, physically *in*equivalent theories are exactly in one-to-one correspondence to different orbits of the duality group.

For the present case, the duality group should be the T-duality group $O(6,6)$ and, indeed, there are flux compactifications of the ten-dimensional supergravity (3.32) to four dimensions whose duality group contains exactly that one. Nevertheless, such compactifications do not make complete orbits under $O(6,6)$ and it shall here be concluded, that the method presented in this chapter provides the necessary completion. This is summarised by the following diagram, which can be thought of as the completion of the diagram on p. 101, leaving out its second row:

$$\begin{tikzcd}
& \boxed{\text{Double field theory } (\mathcal{H})} \arrow[dl, dotted, bend right] \arrow[dr, dotted, bend left] & \\
\boxed{\text{10d supergravity } (g,b,\phi)} \arrow[rr, leftrightarrow, "\text{field redefinition}"] \arrow[d, dotted, "\text{flux compactification}"] & & \boxed{\text{10d supergravity } (\tilde{g},\beta,\tilde{\phi})} \arrow[d, dotted, "\text{flux compactification}"'] \\
\boxed{\text{4d gauged supergravity } \Theta} \arrow[rr, leftrightarrow, "h\in O(6{,}6)\text{ duality}"'] & & \boxed{\text{4d gauged supergravity } \tilde{\Theta}=h(\Theta)}
\end{tikzcd}$$

The standard set of field variables (g,b,ϕ) provides one part of the duality group orbit in four dimensions, indicated by the embedding tensor Θ. A field redefinition in the higher-dimensional description to some new variables $(\tilde{g},\beta,\tilde{\phi})$ corresponds to a change of the embedding tensor $\Theta \to \tilde{\Theta} = h(\Theta)$ by an element from the missing part of the duality group orbit. As has been shown in the previous section, this change is induced by a reparametrisation of the generalised metric $\mathcal{H}$ in the respective double field theory. Such a reparametrisation arises naturally, as $\mathcal{H}$ is in general an $O(d,d)$ element, with $d = 6$ in this case.

Eventually, it is possible to answer the question raised above in slightly more detail. A theory using the field redefined action is part of a duality group orbit that has at least some subset within the region of geometric configurations in the space of all possible configurations, or embeddings, respectively. Put differently, supposition 1 restricts the range of applicability of the presented procedure to configurations that are generated by T-duality transformations, and this is the same as to consider only such orbits that have overlap with geometric configurations. This is shown pictorially in figure 3.3.4.

To conclude, it shall be noted that there might be solutions to the equations of motion for the redefined fields $(\tilde{g},\beta,\tilde{\phi})$ that are not related to any geometric configuration. These could then be regarded as elements of the respective duality group orbits in four dimensions, that have no overlap with the geometric configuration space. It thus might be possible to relax supposition 1.

Supposition 2: The second supposition clarifies the status of non-geometric configurations. As they are assumed to be generated by T-duality transformations, which leave the

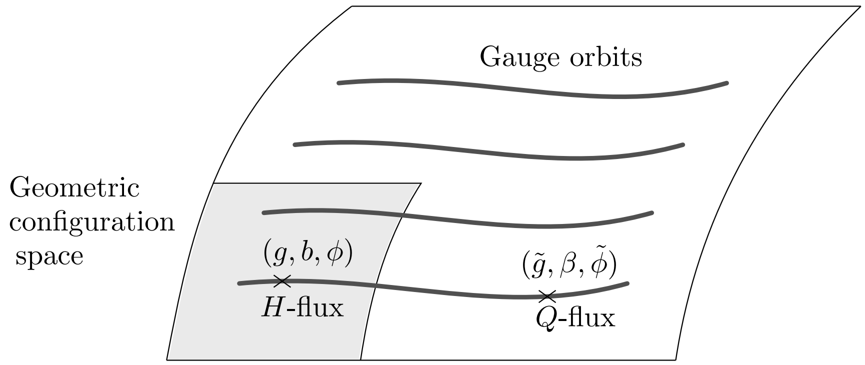


Figure 3.1: Gauge orbits and the field redefinition

path integral of the respective string theory, i.e. worldsheet model, invariant, they are valid configurations from that perspective. On the other hand, the example of a three-torus with H-flux has shown that from the perspective of the target space geometry this is not clear anymore. Rather, the fields acquire non-trivial monodromies under the torus periodicities, e.g.

$$g_C(x_3) \to g'_C(x_3) = g_C(x_3 + 2\pi) \ , \tag{3.252}$$

where g'_C is not diffeomorphism equivalent to g_C. But, as the structure group is restricted to be $GL(d)$ at most, it must be when considering the overlap of two patches in the base circle that contain the identified points $0 \sim 2\pi$. Strictly speaking, the metric g_C fails to be a proper tensor.

It follows straightforwardly that if the structure group was enlarged to $O(d,d)$ the monodromies would become unproblematic. The transition function needed in the overlap of two patches then simply consists of a stack of three operations: the first is the reverse T-duality transformation that was used to create the non-geometric configuration, it returns g_C to the geometric g_A. The second operation is a diffeomorphism, or element of the geometric structure group, that is needed to patch in the geometric version of the overlap. Eventually, the last operation is the inverse of the first and returns the patched field g'_A to its non-geometric counterpart g'_C.

The target space theories used in this chapter do not allow for such an enlarged structure group and thus are incapable of describing non-geometric configurations in the above sense. As a side remark it shall be noted that this makes part of the motivation to find doubled sigma models with $O(d,d)$ covariance, which will be explained in more detail in chapter 4.

Supposition 2 generalises these observations, as it states that non-geometry, restricted in the sense of supposition 1, will *always* manifest itself in the form of problematic monodromies. This has two consequences:

- An integration of fields over the whole manifold becomes problematic. This comes as follows [93]. Given an open covering $\{U_i\}$ for M, and a function $f : M \to \mathbb{R}$ divided by a partition of unity subordinate to the covering, the integration is defined as

$$\int_M f\sqrt{|g|}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{10} \equiv \sum_i \int_{U_i} f_i \sqrt{|g|}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{10} \ . \tag{3.253}$$

  A different covering will not change the result because the volume element is invariant under a change of coordinates. On the other hand, for a non-geometric configuration this is not sufficient as was shown above. The volume element is not invariant under T-duality transformations that are necessary to define the patching. Different coverings will result in different values for the integral, which will then become ill-defined. Strictly speaking, this will turn the action ill-defined and finally cast doubt on the theory itself.

- From a similar reasoning, it becomes clear that total derivatives on compact directions will in general not integrate to zero anymore. This was exemplified in (3.249). Furthermore, the change from the standard NSNS supergravity action to the field redefined action with Q- and R-flux becomes non-trivial. The correct identification is (3.185), or (3.193) in the general case, which explicitly keeps the integrated total derivative term. Accordingly, it is, without further assumptions, not correct to state that the two actions are equivalent.

Although these two consequences have already been observed in the case study, the purpose of restating them here is to generalise these observations to all cases of non-geometry in the sense of supposition 1.

Main thesis: After the context has been prepared carefully, the main statement is formulated under the status of a thesis. It basically says that the ill-definedness of a non-geometric configuration can be cured by using the field redefined action (3.193). This generalises the observation that there is a preferred, i.e. well-defined, field basis in each frame of the three-torus example. To be precise, the claim here shall be: There is always a field redefinition such that the obtained action is well-defined and at most differs by a total derivative from the original one.

The total derivative might not integrate to zero, so it could be called ill-defined. Roughly speaking, one could say that the ill-definedness of the theory is sourced out into the total derivative term by using the field redefinition. In other words, using an appropriate set of field variables will shift the problem of non-geometry to the process of changing to these variables. One subtlety in this reasoning has already appeared in the case study: It might be that all terms in the action are well-defined, but some of the basic fields are not. Namely, β in (3.245) still has a problematic monodromy, where on the other hand it only appears in the form of a constant Q, which is unproblematic.

In order to strengthen the main thesis, it is very helpful to consider a generalisation of the procedure in this chapter. Actually, the two field bases (g, b, ϕ) and $(\tilde{g}, \beta, \tilde{\phi})$ can in some sense be considered as particular examples for a whole variety of possible choices. The generalised metric used to define the field redefinition can be parametrised by generalised vielbeins,

$$\mathcal{H} = \mathcal{E}^T \mathbb{1}_{2d} \mathcal{E} \ . \tag{3.254}$$

Accordingly, the two field bases were obtained from particular choices,

$$\mathcal{E} = \begin{pmatrix} e & 0 \\ -e^{-T} b & e^{-T} \end{pmatrix} \ , \quad \tilde{\mathcal{E}} = \begin{pmatrix} \tilde{e} & -\tilde{e}\beta \\ 0 & \tilde{e}^{-T} \end{pmatrix} \ , \tag{3.255}$$

with e and $\tilde{e}$ being ordinary vielbeins for g and $\tilde{g}$, respectively. As there is a whole $O(2d)$ symmetry in the freedom to parametrise $\mathcal{H}$, there is also other field bases that might be used to find the *preferred* field basis for a given non-geometric configuration[9].

[9] A similar reasoning can be found in [94].

Given that it is possible to find the right field redefinition such that there only remains an ill-defined total derivative term, it shall here be proposed that this term is dropped. Of course, that is, strictly speaking, a change of theory. But there are some arguments that help to justify such a procedure:

- As already mentioned, the field redefined action - without any total derivative term - seems to be reducible to the expected four-dimensional potential.
- In the double field theory framework, the field redefined action allows for a geometric interpretation of the non-geometric fluxes, once the total derivative is dropped. This is in good agreement with the fact that non-geometric backgrounds are valid string backgrounds and, in a sense, should not be special.
- It is not surprising that the NSNS supergravity as a theory of point particles has to be changed when considering non-geometric setups, because there the one-dimensionality of the string becomes crucial.

This concludes the discussion of the proposed theses about how non-geometry could be dealt with by the field redefinition applied to effective field theories as double field theory and supergravity. It shall not be kept secret that some of the claims made here are rather extensive, if not bold. To find more examples that supply the case study is therefore highly important and subject of current research work [92, 91].

3.4 Summary and discussion

This chapter has presented an investigation of non-geometry and non-geometric fluxes in the context of effective field theories of string theory. The two major results shall be phrased as:

- A field redefinition can reveal non-geometric fluxes in double field theory and supergravity.
- Non-geometry can be dealt with, at least in some cases, by using the field redefined theories.

It has been shown that there is a close connection between the results in double field theory and supergravity, as they can be related by solving the strong constraint and integrating out the dual coordinates. Furthermore, the ten-dimensional supergravity framework allows for a generic dimensional reduction that reveals the correct scaling behaviour of the non-geometric fluxes. This supplies evidence for the proposed definitions and completes the interrelations between higher-dimensional frameworks and the four-dimensional stage. In particular, it shows that there is a geometric interpretation of non-geometric degrees of freedom when working with the $2D$ dimensions of double field theory.

Non-geometry itself has been exemplified by the three-torus with H-flux and its T-duals. The suspected duality chain from geometric H-flux to geometric f-flux to non-geometric Q-flux has been confirmed, and, indeed, the occurrence of nonzero Q-flux implied non-geometric behaviour of the physical fields. In the supergravity framework, this typical ill-definedness can be cured by using the redefined variables, which motivates the notion of a preferred field basis. For setups that exceed the special case of three-tori it is assumed that such preferred field bases also exist and suffice to remedy any non-geometric bulkiness.

To complete the discussion, a few more points shall be presented.

Degrees of freedom: Introducing additional field variables in the effective field theory action rises the question about degrees of freedom. Here, one could simply reply that a field redefinition does not change the number of degrees of freedom. On the one hand, such a restriction is desirable, as there must not be new fields given the well-defined set of string modes that are taken over to the supergravity framework.

On the other hand, some non-geometric setups are created by T-duality transformations of certain backgrounds, and T-duality mixes scales such that is is not clear anymore what the set of string modes to be described should be. In principle, it could be that a low-energy effective action of non-geometric string setups has to take into account more modes than supergravity or double field theory. The present framework does not provide such an extension, as for example the field redefinition always trades H-flux off against non-geometric flux, but at least all non-geometric setups that are within the T-duality orbit of a geometric background can be dealt with.

Backgrounds with nonzero R-flux have to be considered with care. To generate them by T-dualities one has to perform transformations in non-isometry directions, which, strictly speaking, is not possible as long as one argues along the lines of Buscher. This is in accordance with the usual statement that R-flux backgrounds even lack a local description [33]. Nevertheless, there are suggestions to use Buscher T-duality rules in non-isometry directions "formally" [92]. In particular, backgrounds with several types of fluxes turned on at the same time where obtained by employing the double field theory framework of this chapter and the notion of "non-geometric" branes. Following this idea implies to add an H-flux term in the field redefined supergravity action (3.193), and thus to add more degrees of freedom.

In [95], such a procedure was indeed suggested as an extension of the framework presented here. It was compared to the democratic formalism of supergravity [96], and so some additional constraints have been introduced to keep the number of degrees of freedom. In [94], the authors propose another point of view, where the standard supergravity and the field redefined supergravity in terms of β are only limiting cases of a general field redefinition. They have shown that there exists a subtle mathematical structure that captures all cases, in particular ones with H- and non-geometric flux at the same time, and that the number of degrees of freedom is reduced by Bianchi identities.

Remedy non-geometry: The presented field redefinition allows to shift ill-defined terms that appear in the action into a total derivative. One option then is to drop such a total derivative, and, strictly speaking to change the theory, in order to have a well-defined framework. This was proposed here, but at least three other options have to be noted:

- It might not be necessary to keep the notion of Riemannian geometry with transition functions in $GL(D)$. One could allow for more involved constructions that embed the effective field theory in a well-defined manner also for non-geometric setups. An example of such is the Lie algebroid construction of [94].

- Double field theory offers involved structures, like the gauge symmetry (3.30) or $O(D, D)$ invariance, that here have not been considered in order to deal with ill-defined fields. So far, it is not exactly clear how to employ these peculiarities of double field theory to deal with non-geometry in $2D$ dimensions, but ideas can be found in chapter 4 of [86].

General considerations of how to define a geometry framework for double field theory beyond the standard one are given in [85].

- Eventually, one could take the existence of well-defined actions for non-geometric setups seriously and suppose that these are effective theories that belong to a particular formulation of a string worldsheet theory. In particular, one might suspect that such a sigma model has to have doubled coordinate fields and can be connected to double field theory as its respective effective field theory. Then, roughly speaking, the extra coordinates could help to make non-geometric configurations well-defined, for example by providing a geometrisation of T-duality.

In particular, the ideas of the last point shall be pursued further in chapter 4.

Non-geometric Q- and geometric f-flux: It was shown that in the supergravity framework (3.193), the non-geometric flux R appeared as the equivalent of the former geometric flux H. The remaining terms containing β could not be sorted into a square of the non-geometric flux Q, which was compared to the situation in the double field theory framework, where Q made a part of the connection and was hidden in the curvature scalar.

The idea that the same identification can be done in the supergravity framework was pursued in [95], where it literally turned out that Q is the actual analogue of the former geometric flux f. This can most easily be seen from[10]

$$\mathcal{R} = -\tfrac{1}{4}(\eta^{ad}\eta_{be}\eta_{cg}Q_a{}^{bc}Q_d{}^{eg} + 2\eta_{cd}Q_a{}^{bc}Q_b{}^{ad} + 2R^{acd}f^b{}_{cd}\eta_{ab}) \ , \tag{3.256}$$

where Q appears as part of a second curvature scalar that stems from the analogue of the Levi-Civita spin connection.

Gauge transformations of β: The double field theory framework allowed for gauge transformations of the new field β, see (3.54), which use the dual coordinates. It seems that this is not possible for the supergravity framework: the Kalb-Ramond field b was replaced by β, and whereas the former has the usual gauge transformations, the latter has none due to the lack of dual coordinates. In other words, there is no analogue of the invariance

$$\mathrm{d}b \to \mathrm{d}b \quad \text{for} \quad b \to b + \mathrm{d}\Lambda \tag{3.257}$$

for bivectors, as there is no analogous derivative. It seems that the action (3.193) has lost the former gauge symmetry and therefore carries too many degrees of freedom.

This issue has been investigated more closely in [95]. It has been shown that the b-field gauge transformation is hidden by the field redefinition. The former

$$b \to b + s \ , \quad \text{with} \quad s_{mn} = \partial_{[m}\Lambda_{n]} \tag{3.258}$$

becomes

$$\begin{aligned}
&\tilde{g} \to \big(\mathbb{1} + (\tilde{g}^{-1} + \beta)\, s\big)^T \tilde{g}\big(\mathbb{1} + (\tilde{g}^{-1} + \beta)\, s\big) \\
&\beta \to \big(\mathbb{1} + (\tilde{g}^{-1} + \beta)\, s\big)^{-1}\big(\beta - (\tilde{g}^{-1} + \beta)\, s\, (\tilde{g}^{-1} + \beta)^T\big)\big(\mathbb{1} + (\tilde{g}^{-1} + \beta)\, s\big)^{-T} \ ,
\end{aligned} \tag{3.259}$$

using (3.145). The field redefined action is then invariant under this transformation. It is alarming, but expected, that the metric transforms under a former gauge transformation.

[10] Equation (4.15) on p. 31 of [95]

However, the authors have shown that one can introduce a differential constraint such that the above reduces to an invariant metric and a shifted β in analogy to the standard behaviour of g and b.

Inverses in the field redefinition: In general, it seems problematic to have inverses of matrices appearing in the particular equations that define the field redefinition, as it is not guaranteed that they exist. A closer look shows that for only two objects existence has to be assumed,

$$(\tilde{g}^{-1} \pm \beta)^{-1} , \quad (\mathbb{1} - \beta\tilde{g}\beta\tilde{g})^{-1} . \tag{3.260}$$

There are at least two arguments in favour of such an assumption:

- For the case of a three-torus with H-flux and its T-duals, the respective objects do exist.
- The respective final results in the double field theory and the supergravity framework do not contain any questionable inverse. It seems plausible, that one can find adapted field redefinitions for cases with no inverse that produce the same result.

Scherk-Schwarz reductions: There have been recent attempts to find a direct connection between double field theory and four-dimensional gauged supergravities. The most successful approaches [97, 98] use Scherk-Schwarz reductions, that compactify the $2D$-dimensional double field theory with $D = n + d$ on doubled twisted tori $T^{d,d}$. To avoid the doubling of external spacetime directions, one takes vanishing dual radii, i.e. the effective space is $n + 2d$ dimensional. The fields are chosen to have a very particular dependence on the according coordinates, as for example the generalised metric is restricted to the following ansatz,

$$\mathcal{H}_{mn}(x, \mathbb{Y}) = U^p{}_m(\mathbb{Y})\mathcal{H}_{pq}(x)U^q{}_n(\mathbb{Y}) . \tag{3.261}$$

Coordinates x refer to the external coordinates, i.e. the four-dimensional target space, and the coordinates $\mathbb{Y}$ refer to the internal doubled space. The "twists" U make generalised internal vielbeins that encode the geometry of the compactification manifold.

In general, it was found that double field theory then reduces to the electric sector of $\mathcal{N} = 4$ gauged supergravity. The RR part has been included as suggested in [80, 81]. In addition, all types of (non-)geometric fluxes H, f, Q and R have been identified with particular gaugings of the effective theory, such that, as suspected here, double field theory provides an uplift of these fluxes in higher dimensions.

The detailed procedure is different for the two approaches: Whereas [97] takes the generalised metric formulation (3.16) but enhances the global symmetry group $O(D, D)$ to $O(D, D + N)$ by adding N vector fields, [98] first rewrites the double field theory action in terms of structure functions F_{abc} that then are chosen to form particular flux backgrounds. Both formulations indicate that it is possible to relax the strong constraint without violating the necessary consistency conditions in the four-dimensional theory. This has also been pursued further in [99, 100, 101]

Eventually, it remains an open question whether the formulation of double field theory presented in this chapter can be reduced in the same way and whether the fluxes Q and R can then directly be identified with the corresponding four-dimensional quantities. At least, it has become clear that all types of fluxes are available from double field theory, also from the formulation of this chapter.

Chapter 4

Doubled geometry on the worldsheet

This chapter introduces a novel worldsheet theory with doubled coordinate fields, providing a basic setup to clarify features of non-geometry on the level of a sigma model. Its main properties are T-duality covariance, an automatic reduction of the degrees of freedom, and compatibility with the standard sigma model. The theory will be analysed both classically and at one-loop level in the quantised version. Although being motivated by the literature, it provides a new approach, that has been first suggested in [3] and further developed in [1].

The structure of this chapter is as follows:

4.1 introduces the relevant ideas and connections to the existing literature.

4.2 reviews the basic construction on the classical level. Emphasis is put on the new symmetries appearing, how $O(D, D)$ and diffeomorphisms can be embedded, and, finally, how non-geometry and non-geometric fluxes can be treated.

4.3 derives the doubled target space equations of motion from claiming Weyl invariance at one-loop level and explains how the reduction to the usual number of degrees of freedom follows automatically.

4.4 gives a summary of the results obtained, and remarks observations that may lead to future research directions.

4.1 Introduction

The preceding chapters have shown that the notion of non-geometry can be nicely exemplified by the three-torus with constant H-flux. The canonical quantisation of this setup has revealed non-commutativity of the target space, depending on the winding of the closed string probing it. On the level of effective field theories, it was shown that a field redefinition with the form of a T-duality transformation helped to remedy the ill-definedness of the action. Both observations involve the existence of dual coordinates, either directly by using winding, or indirectly by invoking T-duality.

It turned out that the use of double field theory simplified the implementation of the field redefinition and helped to extend its applicability. Doubling the coordinates turned T-duality into a manifest global symmetry of the effective field theory. The additional coordinates allowed to "geometrise" non-geometry by parametrising it with non-geometric fluxes Q and R, which could then be interpreted as geometric quantities.

One might be interested in how non-geometry can be detected on the level of a sigma model, and it seems to be an obvious supposition that such a sigma model should better implement a manifest version of T-duality. This, consequently, necessitates a doubling of the coordinate fields, as has been shown for various approaches in the literature. Eventually, any of such doubled models shall be compatible or, in a sense, identical to the standard sigma model of string theory, and the doubled degrees of freedom have to be reduced. There are two main variants of implementing this:

- Based on earlier works [102, 69], Tseytlin proposed a duality symmetric doubled worldsheet model [103, 104] where coordinates and dual coordinates are interpreted as the respective conjugate momenta. This automatically reduces the degrees of freedom correctly, but on the other hand comes with the drawback of loosing manifest Lorentz invariance on the worldsheet. One can impose additional constraints to recover it, cf. [105, 106], but this complicates the derivation of the target space equations of motion [107, 108].

- The approach of Hull [26, 109, 110] implements doubled coordinate fields where the degrees of freedom are reduced by an additional constraint (a "polarisation"), which is imposed by hand. Dual coordinates are conjugate to the winding number of closed strings. The doubled geometry allows for coordinate patching with T-duality transformations and has consequently be named "T-fold".

Indeed, there have been various attempts to investigate the features of non-geometry using worldsheet models, including their doubled versions:

- In [111], the model of Tseytlin is equipped with an additional constraint to preserve Lorentz invariance, and so-called twisted doubled tori are presented as solutions of it. These are group manifolds that allow the embedding of geometric and non-geometric fluxes. The doubled three-torus with H-flux and a particular chiral Wess-Zumino-Witten model are used to exemplify this. Non-geometry reveals itself in the form of non-local coordinate monodromies.

- [106] computes the one-loop effective action of the Tseytlin model and restricts the analysis to the Lorentz invariant class of [111]. The connection between twisted doubled tori and gauged supergravities is discussed, and non-geometry again appears by turning on particular fluxes.

- Halmagyi proposes a first order worldsheet model [53, 54] that is obtained from a Legendre transformation of the standard sigma model. The coordinates are not doubled, but, still, T-duality is realised covariantly for the Hamiltonian of the theory. A lift to the corresponding membrane theory offers the possibility to embed all four types of fluxes, such that non-geometry can be investigated in the sense of non-vanishing Q- or R-flux.

- Hull offers a framework where non-geometry is built in manifestly [26, 110]. T-folds allow for transition functions that include T-duality transformations and are thus an extension of the ordinary notion of a manifold. Non-geometric backgrounds can be embedded naturally, at least for the toroidal case.

The idea that shall be pursued in this chapter is to develop a worldsheet model that implements some of these features, especially the T-duality covariance from doubled coordinate fields, but also overcomes the two main disadvantages that have been faced so far, namely that a constraint has to be put in by hand, and that Lorentz invariance is lost. This will in particular clarify how the theory can be quantised, so that the procedure of obtaining the target space equations of motion becomes unambiguous.

To get rid of the extra degrees of freedom that are introduced by the doubling of the coordinate fields, a gauge symmetry is assumed, and a gauge fixing will be implemented by a Lagrange multiplier term in the action. Such a procedure can be motivated from a step-by-step generalisation of the gauging procedure of Buscher [112, 113], according to ideas given in [114]:

By introducing a covariant derivative and a gauge field, one can promote the standard sigma model to a gauged form, that is the origin for obtaining the T-dual model. Departing from the usual gauge fixing, a non-Lorentz invariant gauge choice leads to the doubled sigma model of Tseytlin. At this stage, it is possible to find another gauge choice that is Lorentz invariant but leaves one gauge field component unfixed. This component appears as a Lagrange multiplier, and, consequently, shall be interpreted as the gauge fixing term for yet another gauge symmetry. The generalisation of this gauge symmetry makes the ansatz that shall be taken as the proposed doubled worldsheet theory.

Eventually, the procedure promises to have a fully Lorentz invariant doubled worldsheet model that is in the same spirit but not identical to the existing proposals in the literature. Apart from enlightening these differences, one might hope to find a possibility of systematically including non-geometric setups and of embedding fluxes, in particular to recover the T-duality chain with four types of fluxes. Furthermore, after the determination of the doubled target space equations of motion, one could hope to find similarities to double field theory, which makes the most important doubled target space theory in the literature. It could even be conceivable that the proposed doubled worldsheet model provides the origin of double field theory in the same way as the standard sigma model of string theory provides the origin of certain supergravity theories.

4.2 Basic construction

This section introduces a novel worldsheet theory with a doubled target space geometry. Such an "invention" must not be unmotivated, and indeed, there is an underlying line of thought that starts at the well-known worldsheet theory of the bosonic string. It employs a particular generalisation of the Buscher gauging procedure, and finally provides the necessary equipment to implement a doubled target space geometry. The new model comes with new symmetries and a possibility to embed a global $O(D, D)$ in- or covariance. Furthermore, it is possible to reveal many known features of non-geometry in a unified way. All this will be developed in consecutive subsections from the perspective of a classical or semi-classical theory (some aspects of the path integral and the BRST symmetry are considered as well) before the next section will develop the target space equations of motion from a one-loop quantisation.

4.2.1 Motivation

The following line of thought shows how one might justify the particular form of the worldsheet action (4.39) for doubled coordinates, to be presented in the next subsection. Although the procedure will not be proven to be unique or complete, it is regarded as the most general ansatz of that kind. Of course, there are steps in the derivation that are, strictly speaking, not compelling, but each of them shall be thoroughly motivated. On the other hand, it is indeed possible to skip the arguments and simply take (4.39) as a starting point by definition.

In the first step, it shall now be shown how to go from the standard sigma model of string theory to its gauged version derived by Buscher [112, 113]. For convenience, the notation is slightly different from the preceding chapters, and some definitions are simply given to make that clear. The presented procedure generalises the discussion in appendix A.

The D coordinate fields X^μ of the bosonic string are described by the standard sigma model in the following action[1],

$$S = \int \mathrm{d}^2\sigma \; \partial_L X^T E \partial_R X \; . \tag{4.1}$$

Here and in the following, a matrix notation shall be employed for indices $\mu, \nu, \ldots$, i.e.

$$\partial_L X^T E \partial_R X = \partial_L X^\mu E_{\mu\nu} \partial_R X^\nu \; . \tag{4.2}$$

The worldsheet derivatives are defined according to the standard worldsheet metric as

$$\partial_{L/R} = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1) \; . \tag{4.3}$$

This corresponds to the light-cone coordinates $\sigma_{L/R} = (\sigma_0 \pm \sigma_1)/\sqrt{2}$. An index $a, b, \ldots$ refers to these coordinates by $a = L/R$. The target space fields are packaged in E as[2]

$$E_{\mu\nu}(X) = g_{\mu\nu}(X) + b_{\mu\nu}(X) \; , \tag{4.4}$$

and represent the metric on a D-dimensional manifold and an antisymmetric Kalb-Ramond field with field strength

$$H_{\mu\nu\kappa} = 3\partial_{[\mu} b_{\nu\kappa]} \; . \tag{4.5}$$

The path integral for this theory can be written as

$$Z = \int \mathcal{D}[X] \sqrt{\det E(X)} \; e^{\mathrm{i}S} \; , \tag{4.6}$$

[1] A factor of $-1/(2\pi\alpha')$ compared to (2.5) is dropped here and in the following.

[2] Compared to chapter 2, the target space fields are here denoted by lower-case letters in order to leave the upper-case ones for later use.

where some motivation for the non-standard measure will be given later on. One can then find that field redefinitions of the coordinate fields induce diffeomorphisms on the target space fields,

$$X^\mu \to f^\mu(X) \ , \quad g \to (\partial f)^T g(\partial f) \ , \quad b \to (\partial f)^T b(\partial f) \ , \tag{4.7}$$

for functions $f : M \to M$ and with the abbreviation

$$(\partial f)^\mu{}_\nu = \partial_\nu f^\mu \ . \tag{4.8}$$

One should note, that the partial derivative here is a target space derivative.

It is now possible to show that there are dual models that describe the same dynamics but different target space geometries, as was discussed in chapter 1. In this case, these are T-dual models. A necessary condition for T-duality is the existence of isometries for the target space fields. In general, it is possible to define dual models for any number of such symmetries, but in the following it shall be assumed that there are exactly D isometries. This is the maximum number of isometries and forces the target space fields to be constant. The restriction will be relaxed during the process of generalisation later on.

The D isometries are promoted to gauge symmetries by introducing a covariant derivative

$$D_a X = \partial_a X + V_a \ , \tag{4.9}$$

with gauge connection V_a^μ. Under the infinitesimal transformation

$$X \to X - \xi \ , \quad V_a \to V_a + \partial_a \xi \ , \tag{4.10}$$

for a transformation parameter $\xi(\sigma)$, the covariant derivative D_a remains invariant. Given that the target space fields are locally not dependent on the coordinate fields, the full kinetic term remains invariant,

$$D_L X^T E D_R X \to D_L X^T E D_R X \ . \tag{4.11}$$

In order to keep the same number of degrees of freedom on the worldsheet, the gauge field V is required to be pure gauge. That is implemented by adding a Lagrange multiplier term to the action (4.1),

$$S = \int \mathrm{d}^2\sigma \ \partial_L X^T E \partial_R X + \tilde{X}^T F \ . \tag{4.12}$$

The Langrange multiplier itself is denoted by $\tilde{X}_\mu$ and can be identified with the dual coordinates later on. It is multiplied by the field strength

$$F = \partial_R V_L - \partial_L V_R \ , \tag{4.13}$$

which itself is invariant under the transformation (4.10), such that the action remains invariant as well.

In order to write down the path integral for the gauged theory, one has to choose a gauge fixing which is implemented by adding another Lagrange multiplier term,

$$S_{gf.} = \int \mathrm{d}^2\sigma \ \tilde{B}^T G \ , \tag{4.14}$$

with a multiplier $\tilde{B}$ and a general gauge fixing condition G. The latter will play a crucial role in the line of thought to follow as it allows to obtain different theories from different gauge fixings. Such a presentation was partly discovered in the appendix of [114], and in particular differs from the way Buscher originally has demonstrated the existence of T-dualities.

Eventually, the path integral is given by

$$Z = \int \mathcal{D}[X, \tilde{X}, V, \tilde{B}, b, c] \sqrt{\det E}\; e^{\mathrm{i}S} \tag{4.15}$$

and

$$S = \int \mathrm{d}^2\sigma\; D_L X^T E D_R X + \tilde{X}^T F + \tilde{B}^T G + b^T \delta_c G\ , \tag{4.16}$$

where the b and c ghosts due to gauge fixing have been added. Their action is determined by replacing the gauge parameter ξ by c in the variation of the gauge fixing condition, i.e. in $\delta_\xi G$.

Dual model

One admissible choice for the gauge fixing condition is $G = X$. Accordingly, the integrals over $\tilde{B}$ and X implement $X = 0$, and the ghosts b and c can be integrated trivially. The integration of the V_a finally leaves a path integral

$$Z = \int \mathcal{D}[\tilde{X}] \sqrt{\det \tilde{E}}\; \exp\left(\mathrm{i}\int \mathrm{d}^2\sigma\; \partial_L \tilde{X}^T \tilde{E} \partial_R \tilde{X}\right)\ , \tag{4.17}$$

that exactly describes the T-dual configuration with

$$\tilde{g} + \tilde{b} = \tilde{E} = E^{-1} = (g+b)^{-1}\ . \tag{4.18}$$

Here, $\tilde{g}$ and $\tilde{b}$ denote the symmetric and antisymmetric part of the matrix $\tilde{E}$ and shall be taken as metric and Kalb-Ramond field of a new target space M' with coordinate fields $\tilde{X}$. Compared to the introduction of the T-duality rules in appendix A, this would correspond to a T-duality transformation in all directions. Of course, that fits well with the fact that the original background was assumed to have D isometries. One should also note the similarity of (4.18) to the field redefinition (3.42) used in chapter 3.

It is of course also possible to dualise only in $d < D$ directions by splitting the action (4.1) into two parts and applying the gauging procedure to only one of them. This would necessitate a block diagonal structure of the metric according to which coordinates are isometry directions. Indeed, [112, 113] considered only one isometry direction.

One should note, that the measure of the dual path integral (4.17) has transformed covariantly. From the Gaussian integrals appearing, one finds an additional factor of $1/\det E$ such that the former $\sqrt{\det E}$ has been replaced by $\sqrt{\det \tilde{E}} = 1/\sqrt{\det E}$, which motivates the departure from the usual $\sqrt{\det g}$ already in (4.6).

Tseytlin model

Although the theories (4.6) and (4.17) are dual to each other, also on the one-loop quantum level [113], the duality is not made manifest but rather appears by a particular gauge fixing. Tseytlin [103, 104] proposed a model that keeps coordinates and dual coordinates such that one part of the kinetic terms is invariant under T-duality transformations and the other part is covariant.

The gauged formulation (4.15) allows to recover Tseytlin's model by fixing the so-called axial gauge, $G = V_1 = (V_L - V_R)/\sqrt{2}$. Integrating over $\tilde{B}$ and V_1 enforces the gauge fixing, whereas V_0 can be integrated to get the path integral of the following action,

$$S = \int \mathrm{d}^2\sigma \ -\frac{1}{2}\partial_1 Y^T \mathcal{H} \partial_1 Y - \frac{1}{2}\partial_1 Y^T \eta \partial_0 Y + b^T \partial_1 c \ . \tag{4.19}$$

Coordinates and dual coordinates (i.e. the former Lagrange multiplier $\tilde{X}$) have been arranged into a $2D$ vector,

$$Y^m = \begin{pmatrix} X^\mu \\ \tilde{X}_\mu \end{pmatrix} \ . \tag{4.20}$$

Notably, the first part of the action contains the generalised metric $\mathcal{H}$ that also played an important role in the preceding chapter, whereas the second part contains the $O(D,D)$ invariant metric η. These matrices are given by

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \quad \mathcal{H} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \ . \tag{4.21}$$

It can be easily seen that this formalism implements T-duality in the form of global $O(D,D)$ transformations: Given, that the coordinate is changed as

$$Y \to Y' = MY \ , \quad M \in O(D,D) \ , \tag{4.22}$$

where the latter statement is equivalent to

$$M^T \eta M = \eta \ , \tag{4.23}$$

the action (4.19) remains invariant if $\mathcal{H}$ transforms covariantly,

$$\mathcal{H} \to \mathcal{H}' = M^T \mathcal{H} M \ . \tag{4.24}$$

Expressing this in terms of how the target space fields g and b transform, the T-duality rules (2.124) are reproduced[3].

However, the action (4.19) does not have manifest Lorentz invariance anymore. It is only recovered on-shell [104], and one has to check for the one-loop level explicitly [105], and may have to claim extra conditions [106]. An obvious reason for that is the gauge fixing condition not being Lorentz invariant, as the appearance of the non-light-cone index shows.

Lorentz invariant gauge fixing

Precisely for 2-dimensional field theories, there is a simple way to restore Lorentz invariance in the gauge fixing condition introduced above, and it leads to a very fruitful formulation. As suggested in [3], one can choose $G = V_L$. This is a Lorentz invariant choice as the corresponding gauge condition $V_L = 0$ does not transform under Lorentz transformations,

$$V_L \to e^\lambda V_L \ , \quad \text{with} \quad \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \to \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \ . \tag{4.25}$$

[3] cf. for example p. 21 of [115]

Accordingly, the integration of $\tilde{B}$ and V_L implements this gauge $V_L = 0$ and the resulting action reads

$$S = \int \mathrm{d}^2\sigma \; \partial_L X^T E \partial_R X + W_L^T V_R - \partial_L c^T b \; . \tag{4.26}$$

Integrating V_R would classically enforce

$$W_L^T = \partial_L X^T E + \partial_L \tilde{X}^T = 0 \; , \tag{4.27}$$

hence it acts as a Lagrange multiplier itself.

One can read (4.26) as a gauge fixed action where (4.27) defines the gauge condition, and the interesting point is to infer the actual gauge symmetry that is fixed by it. When taking the ghost term in (4.26) as coming from (4.27), one can at least conclude

$$\delta_\xi W_L^T = \partial_L \xi^T \; , \tag{4.28}$$

which will be realised, for example, by

$$\delta_\xi X = 0 \; , \quad \delta_\xi \tilde{X} = \xi \; . \tag{4.29}$$

In this case, the gauge parameter has to be a contravariant object, i.e. $\xi = \xi_\mu$. In fact, this gauge transformation could have been anticipated from the gauged theory (4.15), as the pure gauge requirement classically is equally well implemented for a shifted $\tilde{X}$ from (4.29).

One remark has to be made at this stage: It is possible to perform a change of variables on (4.26),

$$\tilde{X} \to \tilde{X} - E^T X \; , \tag{4.30}$$

which leads to the following factor in the corresponding path integral,

$$Z_{\text{ch.bos.}} = \int \mathcal{D}[\tilde{X}, V_R] \exp \mathrm{i} \int V_R^T \partial_L \tilde{X} \; . \tag{4.31}$$

Such an integral can be seen as a chiral boson, that at least in some cases turns out to be highly problematic[4]. However, the full path integral corresponding to (4.26) does not encounter any problems as the above contribution is exactly cancelled by the ghost term.

Generalisation

So far, three different choices for the gauge fixing $G = 0$ of the gauged standard sigma model (4.15) have been worked out. In a sense, they all describe the same physical theory[5] and might be considered as specialisations for particular purposes, such as T-duality covariance.

However, one can put forward a gentle generalisation in the following sense: The gauge symmetry (4.29) is kept as it is, whereas the gauge fixing (4.27) is allowed to have arbitrary coordinate dependence in the form of general $D \times D$ matrix functions K and L,

$$W_L^T = \partial_L X^T K(Y) + \partial_L \tilde{X}^T L(Y) \; . \tag{4.32}$$

[4] Thanks to O. Hohm and a referee of the Physical Review Letters for pointing this out. See also the note [116] on [117].

[5] Of course, this has to be checked more carefully at the quantum level, in particular for higher-loop level, as discussed for example in [112], p.4.

The coordinates Y are defined as in (4.20). To retain the construction, the ghost action has to be changed to

$$S_{\mathrm{gh.}} = -\int \mathrm{d}^2\sigma \, \left(\partial_L c^T L + \partial_L X^T K_{,}{}^{\mu} c_\mu + \partial_L \tilde{X}^T L_{,}{}^{\mu} c_\mu\right) b \ , \tag{4.33}$$

where target space derivatives are denoted by a comma, $K_{,}{}^{\mu} = \partial_\mu K$.

Such a generalisation was investigated in [3], but here only one further aspect shall be considered before moving on to the full generalisation that has been advocated in [1]. It is possible to redefine the field V_R that here plays the role of a Lagrange multiplier,

$$V_R \to V_R + \kappa \partial_R X + \lambda \partial_R \tilde{X} \ , \tag{4.34}$$

where $\kappa = \kappa(Y)$ and $\lambda = \lambda(Y)$ are arbitrary matrix functions. Implementing this change into the generalisation of (4.26), one obtains the following action,

$$S = \int \mathrm{d}^2\sigma \, \frac{1}{2}\partial_L Y^T (\mathcal{G} + \mathcal{C})\partial_R Y + \partial_L Y^T \begin{pmatrix} K \\ L \end{pmatrix} V_R \ , \tag{4.35}$$

which comes together with the ghost part (4.33). The kinetic term is given by

$$\frac{1}{2}(\mathcal{G} + \mathcal{C}) = \begin{pmatrix} E + K\kappa & K\lambda \\ L\kappa & L\lambda \end{pmatrix} \ . \tag{4.36}$$

It can be shown, that this matrix can be brought to either an $O(D, D)$ invariant or covariant form, involving η or the generalised metric $\mathcal{H}$, respectively. These forms will be obtained also from the more general model of the next subsection.

As a main result, the line of thought in this section has shown that with very gentle generalisations one can go from the standard sigma model to a model that contains a doubled set of coordinates Y^m. The redundancy of this doubling is removed by fixing a particular gauge symmetry (4.29), and the model is capable of revealing $O(D, D)$ covariant or invariant behaviour.

4.2.2 Action and symmetries

For any generalisation of the above reasoning, it is clear: A sigma model with the doubled number of coordinate fields, that still describes the same degrees of freedom as the standard sigma model for the bosonic string, has to contain a constraint that makes half of the fields redundant.

The previous section has motivated a gauge symmetry (4.29) for the coordinate fields in order to establish such a redundancy. However, this gauge symmetry involved the doubled coordinates in a very particular way, namely only the dual coordinates were allowed to transform. Here, this shall be extended to the most general form of a gauge symmetry of this kind. It can be written as

$$\delta_\xi Y = \mathcal{K}(Y)\, \xi \ , \tag{4.37}$$

where the gauge parameters are D local fields, labeled by an index $\alpha = 0, \dots, D-1$. This exactly means that there are D redundant degrees of freedom, as will be discussed in more detail later on. Accordingly, the index structure of the accompanying $2D \times D$ matrix function is given by $\mathcal{K}^{m\alpha}$. In order to have a closed algebra of transformations, these matrix functions have to fulfill a structure equation,

$$\mathcal{K}^{m\alpha}{}_{,p}\mathcal{K}^{p\beta} - \mathcal{K}^{m\beta}{}_{,p}\mathcal{K}^{p\alpha} = f^{\alpha\beta}{}_{\gamma}(Y)\,\mathcal{K}^{m\gamma} \ . \tag{4.38}$$

The structure coefficients are allowed to have coordinate dependence at this stage, a fact that will be founded later on, see equation (4.65). Eventually, the $\mathcal{K}^{\alpha}$ bear a striking resemblance to ordinary Killing vectors and from now on shall be called Killing vectors, although the Killing equation itself will be derived only later on, see (4.64).

Similar to the proposed formulation (4.35), the following action,

$$\boxed{S = \int \mathrm{d}^2\sigma \ \frac{1}{2}\partial_L Y^T \mathcal{E} \partial_R Y + \mathcal{W}_L V_R \ .} \tag{4.39}$$

shall be taken as the most general form of a doubled worldsheet theory with gauge fixing term. It has to be supplemented by a ghost term

$$S_{\text{gh.}} = \int \mathrm{d}^2\sigma \ \delta_c \mathcal{W}_L b_R \ , \tag{4.40}$$

where the symbol δ_c stands for the variation under (4.37) with ξ being replaced by the ghost c.

The matrix $\mathcal{E}$ can be split into its symmetric and antisymmetric part, $\mathcal{E}_{mn} = \mathcal{G}_{mn} + \mathcal{C}_{mn}$, where $\mathcal{G}$ can be considered as a metric on a $2D$-dimensional target space. On the other hand, one has to keep in mind that the physical target space remains D-dimensional. The fields V_R^μ with $\mu = 0, \ldots, D-1$ are Lagrange multiplier fields, as has already been supposed in the preceding section. In generalisation of (4.27), they are multiplied by

$$\mathcal{W}_L = \partial_L Y^T \ \mathcal{Z}(Y) \ , \tag{4.41}$$

such that they classically enforce a gauge fixing $\mathcal{W}_L = 0$. This will fix all gauge invariances iff the $D \times D$ matrix $\mathcal{K}^T\mathcal{Z}$ is invertible. In fact, even if at first sight there might be more involved gauge fixing conditions, $\mathcal{W}_L = 0$ is indeed the most general one that is compatible with the conformal symmetry of the action,

$$\sigma_L \to \sigma_L' = h_L(\sigma_L) \ , \quad \sigma_R \to \sigma_R' = h_R(\sigma_R) \ , \tag{4.42}$$

where h_R and h_L are independent holomorphic and antiholomorphic functions.

As a remark, it shall be noted that the choice $G = V_L$ in order to get (4.26) could also have been $G = V_R$, which then would lead to a similar model with all indices L exchanged by R and vice versa. The same applies to the general model presented here. Luckily, the content of the theory is not touched by such an exchange, which renders the arbitrariness harmless.

To conclude the introduction of the new sigma model (4.39), it shall be emphasised once more that it is not in direct connection to the standard sigma model (4.1). Rather it was constructed from a series of generalisations and the following investigation will show what exactly the relation to known theories is. First of all, it shall be examined what kind of symmetries the model (4.39) contains, both classically and also on the quantum level. This will allow to draw a connection between these, as it turns out: non-standard, symmetries and the ordinary diffeomorphism and gauge invariance.

Symmetries

There is a plurality of field redefinitions on the level of the path integral that can be performed to reveal symmetries in the proposed model (4.39). The four most important ones, that are allowed by the conformal symmetry mentioned above, shall be listed and commented on in the following.

- **Doubled diffeomorphisms**: In the same way as D-dimensional diffeomorphisms can be obtained for the standard sigma model, see (4.7), a field redefinition of the doubled coordinate fields will induce their doubled counterparts,

$$Y^m \to \mathcal{F}^m(Y) \ , \quad \mathcal{G} \to (\partial\mathcal{F})^{-T}\,\mathcal{G}\,(\partial\mathcal{F})^{-1} \ , \quad \mathcal{C} \to (\partial\mathcal{F})^{-T}\,\mathcal{C}\,(\partial\mathcal{F})^{-1} \ , \tag{4.43}$$

  where $(\partial\mathcal{F})^m{}_n = \partial_n\mathcal{F}^m$. The Killing vectors and the gauge fixing parameters have to transform as well,

$$\mathcal{K} \to (\partial\mathcal{F})\mathcal{K} \ , \quad \mathcal{Z} \to (\partial\mathcal{F})^{-T}\,\mathcal{Z} \ . \tag{4.44}$$

  The first rule keeps track of the change in the gauge transformation (4.37), the second is to keep the gauge fixing term in (4.39) invariant. How conventional D-dimensional diffeomorphisms can be embedded will be discussed later on, see p. 138 ff.
  It is the doubled diffeomorphisms that put an obstruction on connecting the model with double field theory straightforwardly, as will be discussed around (4.82) and around (4.173).

- **Redefinition of the Lagrange multipliers**: It is possible to redefine the Lagrange multipliers V_R by multiplying them by an arbitrary $D \times D$ matrix function $\rho^\mu{}_\nu(Y)$,

$$V_R \to \rho V_R \ . \tag{4.45}$$

  To keep the action inert, the gauge fixing parameters have to change accordingly,

$$\mathcal{Z} \to \mathcal{Z}\rho^{-1} \ . \tag{4.46}$$

  Together with the doubled diffeomorphisms, this ρ-transformation will be implemented covariantly in the construction of the Feynman rules later on. In particular, it is possible to define derivatives that contain $\mathcal{Z}$ and are still covariant under ρ-transformations, see (4.131).
  Furthermore, it turns out that ρ plays an important role in the discussion of non-geometry, see (4.104).

- **Shift of the Lagrange multipliers**: Instead of a multiplicative transformation, it is also possible to shift the Lagrange multipliers V_R analogous to (4.34),

$$V_R \to V_R + \mathcal{U}(Y)\,\partial_R Y \ , \tag{4.47}$$

  where $\mathcal{U}^n{}_m(Y)$ is a $D \times 2D$ matrix function. To keep the theory inert, one has to shift the kinetic term as well,

$$\mathcal{E} \to \mathcal{E} - 2\mathcal{Z}\mathcal{U} \ . \tag{4.48}$$

  This offers the possibility to construct particular forms of the action, and will turn out to be of topmost importance in the discussion of the relation to other theories. Notably,

the doubled metric $\mathcal{G}$ can be brought to the form of the generalised metric $\mathcal{H}$, offering a starting point to explore possible connections to double field theory.
Furthermore, it is this shift symmetry of the Lagrange multipliers that will be recast in the discussion of quantum symmetries, namely the BRST algebra. There, it connects to the transformation of the b-ghost, see (4.59).

- **Redefinition of the Killing vectors**: In view of (4.37), it is finally possible to redefine the gauge parameters,
$$\xi \to \omega(Y)\xi \ , \tag{4.49}$$
with ${\omega_\alpha}^\beta(Y)$ being a $D \times D$ matrix function. Of course, the Killing vectors have to transform contrariwise,
$$\mathcal{K} \to \mathcal{K}\omega^{-1} \ . \tag{4.50}$$
This turns out to be important when identifying the redundant coordinates. In short, it allows to have the physical coordinate fields in the first D entries of Y^m, see the discussion around (4.69).
As a side remark, one should note that the structure coefficients ${f^{\alpha\beta}}_\gamma$ have to change as well in order to preserve the structure equation (4.38),
$${f^{\alpha\beta}}_\gamma \ \to \ {\omega_\gamma}^\nu \, {f^{\kappa\lambda}}_\nu \, {(\omega^{-1})_\kappa}^\alpha \, {(\omega^{-1})_\lambda}^\beta + {\omega_\gamma}^\nu {(\omega^{-1})_\nu}^{[\alpha}{}_{,p} \, {(\omega^{-1})_\delta}^{\beta]} \, \mathcal{K}^{p\delta} \ . \tag{4.51}$$
Consequently, it is not possible to keep the structure coefficients constant without explicitely excluding the above transformation. This is one reason for allowing coordinate dependent structure coefficients, but another argument can be found in the discussion of the BRST algebra, see p. 134.

Again, one should note that the above list is not exhausting all possible symmetries, but only pointing out the most prominent and novel ones. For example, the antisymmetric tensor field has a gauge symmetry $\mathcal{C} \to \mathcal{C} + \mathrm{d}\Xi$, in the same way as the standard sigma model offers $b \to b + \mathrm{d}\lambda$. This symmetry will be discussed further when showing a possibility to embed fluxes in the doubled sigma model, see p. 144.

Eventually, it shall be noted that the gauge transformation (4.37) itself has not been added to the above list because it is obviously broken by the gauge fixing term $\mathcal{W}_L V_R$ in the action (4.39). Furthermore, depending on the particular form of $\mathcal{E}$, it may happen that the kinetic term itself is invariant under (4.37) only upon imposing the gauge fixing constraint $\mathcal{W}_L = 0$.

BRST symmetry

In the path integral formalism, the gauge symmetry (4.37) has to undergo a Faddeev-Popov gauge fixing for an off-shell quantum description. It will therefore reappear in the form of an BRST symmetry algebra, which shall now be constructed by following only two general instructions:

- The BRST transformations are nilpotent.
- The full quantum action is left invariant.

As a further guiding principle, all fields of the theory shall be classified according to their conformal weight (Q) and their "ghost charge" (R). These are listed in table 4.1. In particular, there appear b_R^α ghost fields[6] which are associated to the gauge fixing conditions $\mathcal{W}_{L\alpha} = 0$, and each gauge parameter ξ_α of the classical gauge symmetry is replaced by ϵc_α which consists of ghosts c_α and a fermionic parameter ϵ.

The following derivation makes use of these charges, as they have to be preserved under any BRST transformation. This allows to determine the most general expressions, which are then further restricted by the two above rules.

Field	Y	V_R	c	b_R	$\partial_R Y$	$\partial_R c$	ε
Q	0	0	1	-1	0	1	-1
R	0	1	0	1	1	1	0

Table 4.1: Ghost charge and conformal weight of the fields

The coordinate fields Y^m have $Q = R = 0$, so the transformation can only contain Y itself, c and ε. As it should be first order in ϵ, there can only be one c as well to compensate the Q charge. Eventually, the most general expression is

$$\delta_\epsilon = \epsilon \mathcal{K}^{m\alpha}(Y)\, c_\alpha \ . \tag{4.52}$$

It coincides precisely with the classical gauge symmetry (4.37) after the exchange of ξ by ϵc, as expected. Taking into account the nilpotency condition leads to

$$\begin{aligned} 0 = \delta_{\epsilon'}\delta_\epsilon Y^m &= \delta_{\epsilon'}\left(\epsilon \mathcal{K}^{m\alpha}(Y)\, c_\alpha\right) \\ &= -\frac{1}{2}\epsilon\,\epsilon'\left(\mathcal{K}^{m\alpha}{}_{,p}\mathcal{K}^{p\beta} - \mathcal{K}^{m\beta}{}_{,p}\mathcal{K}^{p\beta}\right) c_\alpha c_\beta + \epsilon \mathcal{K}^{m\alpha}\delta_{\epsilon'} c_\alpha \ . \end{aligned} \tag{4.53}$$

By using the structure equation (4.38), this determines the transformation of the c_α,

$$\delta_\epsilon c_\alpha = \frac{1}{2} f^{\alpha\beta}{}_\gamma(Y)\, c_\alpha c_\beta \ , \tag{4.54}$$

whose nilpotency in turn leads to

$$f^{\kappa[\alpha}{}_\lambda\, f^{\beta\gamma]}{}_\kappa + \mathcal{K}^{p[\alpha}\, f^{\beta\gamma]}{}_{\lambda,p} = 0 \ . \tag{4.55}$$

This equation is the Jacobi identity for constant structure coefficients f, and can be considered as a generalised Jacobi identity for non-constant structure coefficients, as such are allowed in the present construction.

The b_R ghost carries charges $R = -Q = 1$ which leaves three different terms for the general transformation rule,

$$\delta_\epsilon b_R^\alpha = \epsilon A^\alpha{}_\beta(Y) V_R^\beta + \epsilon B^{\alpha\beta}{}_\gamma(Y) c_\beta b_R^\gamma + \epsilon \mathcal{Q}^\alpha{}_m(Y)\partial_R Y^m \ . \tag{4.56}$$

Without restriction of generality, the matrix function A can be absorbed in the definition of b_R. Furthermore, the second term shall be dropped, $B = 0$, because the investigation of the BRST invariance of the action (4.39) focuses on the kinetic and gauge fixing terms, but the B-term will only involve ghost fields. Thus, the doubled target space properties that will be revealed do not change under this simplification. The transformation rule then reads

$$\delta_\epsilon b_R^\alpha = \epsilon\, V_R^\alpha + \epsilon\, \mathcal{Q}^\alpha{}_m\, \partial_R Y^m \ . \tag{4.57}$$

[6]In the following, indices $\alpha, \beta, \ldots$ and $\mu, \nu, \ldots$ are treated as being of the same type.

Again, the nilpotency condition $\delta_{\epsilon'}\delta_\epsilon b_R^\alpha = 0$ leads to the transformation rule for another field, this time for the Lagrange multiplier V_R^α,

$$\delta_\epsilon V_R^\alpha = -\epsilon \left(\mathcal{Q}^\alpha{}_{m,p}\mathcal{K}^{p\beta} + \mathcal{Q}^\alpha{}_p \mathcal{K}^{p\beta}{}_{,m} \right) c_\beta \partial_R Y^m - \epsilon \mathcal{Q}^\alpha{}_p \mathcal{K}^{p\beta} \partial_R c_\beta \ . \tag{4.58}$$

It is somewhat contrary to the expectation from standard quantum field theory, where the gauge fixing Lagrange multiplier transforms trivially, i.e. $\delta_\epsilon V_R = 0$. Interestingly, such a behaviour can be achieved by applying a $\mathcal{U}$-transformation (4.47) with $\mathcal{U} = \mathcal{Q}$,

$$\delta_\epsilon (V_R^\alpha + \mathcal{Q}^\alpha{}_m \partial_R Y^m) = 0 \ . \tag{4.59}$$

This already indicates that the BRST parameter $\mathcal{Q}$ makes the quantum counterpart of $\mathcal{U}$ in the classical symmetries.

With a set of general BRST transformation rules for all fields, it is now possible to investigate how the action (4.39) transforms. The requirement to keep it invariant will then lead to additional conditions on the target space fields. Using (4.52), (4.54), (4.57) and (4.58) on the full quantum action, i.e. on (4.39) plus its ghost part (4.40), gives the variation

$$\begin{aligned} \delta_\epsilon S = \epsilon \int \mathrm{d}^2\sigma \ & \partial_L Y^m \partial_R c_\beta \, \mathcal{K}^{p\beta} \ \left(\tfrac{1}{2}\mathcal{E}_{mp} - \mathcal{Z}_{m\mu}\mathcal{Q}^\mu{}_p\right) + \partial_L c_\beta \partial_R Y^m \, \mathcal{K}^{p\beta} \ \left(\tfrac{1}{2}\mathcal{E}_{pm} - \mathcal{Z}_{p\mu}\mathcal{Q}^\mu{}_m\right) \\ & + \partial_L Y^m \partial_R Y^n c_\beta \ \Big(\tfrac{1}{2}\mathcal{K}^{p\beta}{}_{,m}\mathcal{E}_{pn} + \tfrac{1}{2}\mathcal{K}^{p\beta}{}_{,n}\mathcal{E}_{mp} + \tfrac{1}{2}\mathcal{K}^{p\beta}\mathcal{E}_{mn,p} \\ & - \mathcal{Z}_{m\mu}\mathcal{Q}^\mu{}_{n,p}\mathcal{K}^{p\beta} - \mathcal{Z}_{m\mu,p}\mathcal{Q}^\mu{}_n \mathcal{R}^{p\beta} - \mathcal{Z}_{m\mu}\mathcal{Q}^\mu{}_p \mathcal{K}^{p\beta}{}_{,n} - \mathcal{Z}_{p\mu}\mathcal{Q}^\mu{}_n \mathcal{K}^{p\beta}{}_{,m} \Big) \ , \end{aligned} \tag{4.60}$$

By defining

$$\mathcal{E} = \tilde{\mathcal{E}} + 2\mathcal{Z}\mathcal{Q} \ , \tag{4.61}$$

the condition $\delta_\epsilon S = 0$ can be recast in the form of two equations,

$$\mathcal{K}^{p\alpha}{}_{,m}\,\tilde{\mathcal{E}}_{pn} + \mathcal{K}^{p\alpha}{}_{,n}\,\tilde{\mathcal{E}}_{mp} + \mathcal{K}^{p\alpha}\,\tilde{\mathcal{E}}_{mn,p} = 0 \tag{4.62}$$

$$\tilde{\mathcal{E}}\,\mathcal{K} = \mathcal{K}^T \tilde{\mathcal{E}} = 0 \ . \tag{4.63}$$

These equations offer a set of important conclusions.

- The first equation (4.62) has the form of the standard Killing equation for each index α,

  $$\mathcal{L}_{\mathcal{K}^\alpha}\tilde{\mathcal{E}} = 0 \ . \tag{4.64}$$

  This conclusively justifies to denote $\mathcal{K}$ as Killing vectors, although one has to keep in mind that it refers to the metric $\tilde{\mathcal{E}}$ and not to $\mathcal{E}$ itself.

- At first sight, it seems problematic to allow for non-constant structure coefficients $f^{\alpha\beta}{}_\gamma(Y)$ in the defining algebra of Killing vectors (4.38), because these will lead to extra terms in (4.62). Even a simple multiplication by a scalar function, $f\mathcal{K}^\alpha$, adds terms of the form

  $$\partial_m f \ \mathcal{K}^{p\alpha}\tilde{\mathcal{E}}_{pn} + \partial_n f \ \mathcal{K}^{p\alpha}\tilde{\mathcal{E}}_{mp} \ . \tag{4.65}$$

  Here, the second condition (4.63) sets exactly these terms to zero, such that, eventually, non-constant structure coefficients are alright. In particular, this is consistent with the transformations (4.50).

- The definition (4.61) shows once more that $\mathcal{Q}$ is the quantum counterpart of $\mathcal{U}$ in the classical symmetries, as the relation between $\mathcal{E}$ and $\tilde{\mathcal{E}}$ is precisely a $\mathcal{U}$-transformation (4.48) with $\mathcal{U} = \mathcal{Q}$.

- The last two equations (4.63) are projection equations that reduce the target space degrees of freedom in $\tilde{\mathcal{E}}$ from $2D \times 2D$ to D^2, to be discussed in more detail in section 4.3.2.

This concludes the investigation of the BRST symmetry algebra for the doubled worldsheet theory (4.39).

4.2.3 Embedding of $O(D, D)$ and D-dimensional diffeomorphisms

Due to the presence of a Lagrange multiplier term, the doubled worldsheet model constructed in this chapter allows for different representations of the same theory. More precisely, it is possible to bring the kinetic term into other forms by applying $\mathcal{U}$-transformations (4.47), without changing the physical content. This is founded in the fact that the gauge symmetry (4.37) renders half of the $2D$ coordinates redundant, which is then reflected on the side of the $2D$-dimensional target space.

One particularly interesting feature of this freedom is the possibility to obtain an $O(D, D)$ covariant, or invariant, respectively, rewriting of the theory. The former reveals the generalised metric $\mathcal{H}$, an object that played an important role in the preceding chapters. An $O(D, D)$ transformation of the generalised metric can be interpreted as a T-duality transformation on the D-dimensional target space fields g and b. In this sense, the formalism becomes T-duality covariant. The invariant rewriting is interesting as the kinetic term has a constant target space metric η, and the physical content of the theory is repackaged solely into the Lagrange multiplier term $\mathcal{W}_L V_R$. T-duality transformations on the D-dimensional target space fields can be recovered as fractional linear transformations on $E = g + b$.

The following discussion is restricted to single coordinate patches. That has various reasons, but can be quickly seen by applying a $2D$-dimensional diffeomorphism to the kinetic term,

$$\partial_L Y^T \,\eta\, \partial_R Y \quad \rightarrow \quad \partial_L Y'^T \,(\partial F)^{-T} \eta (\partial F)^{-1}\, \partial_R Y \ . \tag{4.66}$$

A former manifest $O(D, D)$ invariance is disguised by the additional transformation matrices. Furthermore, the various rewritings to be discussed rest on a particular representation of the Killing vectors. Although such can be achieved locally in any case, a global statement would severely restrict the $2D$ target space manifold. Basically, it would imply that the Killing vectors are integrable and thus only manifolds with D linearly independent integrable vector fields are admissible anymore[7].

The first observation is that the set of Killing vectors $\{\mathcal{K}^\alpha\}$ spans an involutive distribution[8], as their algebra closes in the sense of (4.38). By Frobenius' theorem, there exists a coordinate chart U for every point in the $2D$ target space manifold, such that in U the Killing vectors take the form

$$\mathcal{K} = \begin{pmatrix} 0 \\ K \end{pmatrix} \ . \tag{4.67}$$

[7] Thanks to J. Gray for an enlightening discussion on this topic.

[8] The mathematical background has been taken from [118].

The $D \times D$ matrix K is invertible as the D Killing vectors are supposed to be linearly independent. To be precise, the theorem ensures that there exists a coordinate system in U such that for coordinate functions $x_1, \ldots, x_{2D}$ the slices

$$x_m = \text{const.} \quad \forall\, m \in \{D+1, \ldots, 2D\} \ , \tag{4.68}$$

are integral manifolds of the above mentioned involutive distribution. On the other hand, in this coordinate system, the tangent space of the submanifold that is parameterised by the remaining second D coordinates x_m coincides with the span $\langle \mathcal{K}^\alpha \rangle$.

Using the invertibility of K, a redefinition (4.50) with $\omega = K$ brings the Killing vectors to the simple form

$$\mathcal{K} = \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} \ . \tag{4.69}$$

This has an important implication: Locally, the general theory presented here is equivalent to its preliminary version discussed in the previous section. The general gauge symmetry (4.37), locally, can be brought to the form of (4.29), and the identification of coordinates and dual coordinates is as in (4.20). From the point of view of the target space, the D-dimensional physical part is nested within the first half of Y^m.

The BRST conditions (4.63) enforce the object $\tilde{\mathcal{E}} = \mathcal{E} - 2\mathcal{Z}\mathcal{Q}$ to have only one $D \times D$ block,

$$\tilde{\mathcal{E}} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{4.70}$$

This shows that, locally, the theory has a preferred form of the kinetic term $\tilde{\mathcal{E}}$, as then the BRST transformation of the Lagrange multipliers V_R becomes trivial, and the BRST condition (4.62) becomes a standard Killing equation. In other words, locally, there is a particular $\mathcal{U}$-transformation that brings the theory to the standard sigma model form when choosing $* = 2E = 2(g+b)$ in the above.

As a starting point for the investigation of the possible rewritings, exactly this form shall be chosen, i.e.

$$\mathcal{E} = \begin{pmatrix} 2(g+b) & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{4.71}$$

for (4.39). Following the similarity to the preliminary model (4.26) a bit further, the gauge fixing condition shall be chosen to be

$$\mathcal{Z} = \begin{pmatrix} E \\ \mathbb{1} \end{pmatrix} \ , \tag{4.72}$$

which is the same as (4.27). Potentially, there are other choices, but this particular one is necessary for the arguments to follow. Roughly speaking, the upper half of $\mathcal{Z}$ corresponds to the initial upper left corner of $\mathcal{E}$.

By performing particular $\mathcal{U}$-transformations, it is possible to obtain other representations of the kinetic term. Two of such shall now be discussed.

Invariant representation: With $\mathcal{U} = \begin{pmatrix} \mathbb{1} & 0 \end{pmatrix}$, the $O(D,D)$ invariant representation

$$\mathcal{E} = \mathcal{G} + \mathcal{C} = -\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{4.73}$$

can be obtained. Its antisymmetric part $\mathcal{C}$ is a mere total derivative and can be neglected. All the physical components of the theory, namely the D-dimensional target space fields g and b, are now encoded in the gauge fixing term, i.e. in $\mathcal{Z}$. This shows in particular how subtle the relationship between the obvious $2D$ target space field $\mathcal{E}$ and its physical counterpart E in half the dimensions really is. A full discussion of the connection between these two has to be postponed until the doubled worldsheet theory of this chapter has been carried through the quantisation process.
As for the Tseytlin model (4.19), a global $O(D,D)$ transformation on the coordinates, $Y \to MY$, leaves the kinetic term invariant, but now changes the gauge fixing condition to

$$\mathcal{Z} \to M\mathcal{Z} = \begin{pmatrix} \alpha E + \beta \\ \gamma E + \delta \end{pmatrix} , \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(D,D) . \tag{4.74}$$

It is possible to retain its original form (4.72) by an additional ρ-transformation (4.45) of the Lagrange multipliers V_R,

$$V_R \to (\gamma E + \delta) V_R , \quad \mathcal{Z} \to \mathcal{Z}' = \begin{pmatrix} (\alpha E + \beta)(\gamma E + \delta)^{-1} \\ \mathbb{1} \end{pmatrix} . \tag{4.75}$$

Taking into account that $\mathcal{Z}$ alone carries the physical components of the theory, this precisely reproduces the $O(D,D)$ transformation behaviour (3.25) of double field theory in its formulation [17], namely

$$E \to (\alpha E + \beta)(\gamma E + \delta)^{-1} . \tag{4.76}$$

As a side remark, it should be noted that the ρ-transformation used here, with $\rho = \gamma E + \delta$ seems to resemble the anchor map discussed in [88] and [94], see (3.20) in the latter. Though, the connection to these frameworks is not yet clear.

Covariant representation: A transformation with

$$\mathcal{U} = \begin{pmatrix} \mathbb{1} + g^{-1} b & -g^{-1} \end{pmatrix} \tag{4.77}$$

brings the kinetic term into the following form,

$$\mathcal{E} = \mathcal{H} + \mathcal{C} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{4.78}$$

Again, $\mathcal{C}$ is nothing more than a total derivative and can be ignored. This time, the D-dimensional target space fields appear explicitly in the form of the generalised metric $\mathcal{H}$, that also plays a prominent role in double field theory in the formulation [18].
Global $O(D,D)$ transformations of the coordinates Y induce T-duality transformations on the target space fields g and b from

$$\mathcal{H} \to \mathcal{H}' = M^T \mathcal{H} M . \tag{4.79}$$

This feature was already present in the Tseytlin formalism (4.19), and justifies the denotation 'covariant representation'.

The doubled sigma model presented here offers possibilities to reveal objects that also appear in double field theory, like the generalised metric $\mathcal{H}$. That naturally rises the question of how close the connection between those theories is, or even whether the doubled sigma model can be regarded as the worldsheet description corresponding to the double field theoretic target space model.

A first step to address this issue is to check whether the so-called "gauge transformation" of double field theory, cf. (3.30), can be found on the worldsheet model presented here. One could expect that it stems from an infinitesimal diffeomorphism transformation of the coordinates Y,

$$Y^m \to Y^m + \xi^m(Y) \ , \tag{4.80}$$

but for the kinetic term this only leads to the infinitesimal form of how any tensor should transform, here shown in the covariant representation,

$$\delta_\xi \mathcal{H}_{mn} = \xi^p \mathcal{H}_{mn,p} + \mathcal{H}_{pn}\xi^p{}_{,m} + \mathcal{H}_{mp}\xi^p{}_{,n} \ . \tag{4.81}$$

In other words, and that is no surprise, only the ordinary Lie derivative is reproduced. In contrast, double field theory has a gauge symmetry,

$$\delta_{\mathrm{DFT}}\, \mathcal{H}_{mn} = \xi^p \mathcal{H}_{mn,p} + \mathcal{H}_{mp}(\xi^p{}_{,n} - \xi_{n,}{}^p) + (\xi^p{}_{,m} - \xi_{m,}{}^p)\mathcal{H}_{pn} \ , \tag{4.82}$$

that is sometimes regarded as a "generalised Lie derivative"[9]. The additional terms can never be reproduced from any worldsheet theory with the structure discussed here, contrary to what is at few places claimed in the literature[10]. And as this finding only relies on the transformation behaviour of the coordinates, it for example also applies to the Tseytlin type models.

As an implication of the findings above, it seems unclear where the D-dimensional diffeomorphisms and the b-field gauge transformations remain. In double field theory, the mentioned generalised gauge transformation can also be made visible in a formalism with $E = g + b$ as a fundamental variable, cf. (3.26), and the D-dimensional transformations appear when applying the strong constraint to those, cf. (3.34). Here, at least the b-field transformations are missing due to the missing terms in the standard $2D$ Lie derivative.

One option to recover the correct D-dimensional transformation behaviour could be to embed D-dimensional diffeomorphisms into the $2D$-dimensional ones. This shall be discussed now. Again, only the special case of the covariant representation with the generalised metric $\mathcal{H}$ shall be considered, as it offers the most direct access to the target space fields g and b. Of course, the standard sigma model form could serve for these purposes as well, but then the doubling of the coordinates is purely formal and the following discussion becomes trivial.

A natural guess for such an embedding would be

$$\mathcal{F}(Y) = \begin{pmatrix} f(X) \\ \tilde{X} \end{pmatrix} \ , \tag{4.83}$$

where $f(X)$ denotes the diffeomorphism of the coordinates X^μ. According to (4.43), the kinetic term then transforms as

$$\mathcal{H} \to \begin{pmatrix} (\partial f)^{-T} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \mathcal{H} \begin{pmatrix} (\partial f)^{-1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \ . \tag{4.84}$$

But given that each of the four blocks in $\mathcal{H}$ has a particular transformation behaviour under D-dimensional diffeomorphisms, for instance the upper left one as a (2,0)-tensor, one would rather expect the following,

$$\mathcal{H} \to \begin{pmatrix} (\partial f)^{-T} & 0 \\ 0 & (\partial f)^T \end{pmatrix} \mathcal{H} \begin{pmatrix} (\partial f)^{-1} & 0 \\ 0 & \partial f \end{pmatrix} \ . \tag{4.85}$$

[9]A detailed account can for example be found in [85].

[10]See for example [119], where a private communication with the author confirmed a calculational error.

This is incompatible with the embedding above.

As a next move, one might try to change (4.83). The most general ansatz reads

$$\mathcal{F}(Y) = \begin{pmatrix} f(X) \\ \tilde{F}(Y) \end{pmatrix} . \tag{4.86}$$

It leads to the following condition

$$\partial \mathcal{F} = \begin{pmatrix} \partial_X f & 0 \\ \partial_X \tilde{F} & \partial_{\tilde{X}} \tilde{F} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \partial f & 0 \\ 0 & (\partial f)^{-1} \end{pmatrix} , \tag{4.87}$$

when enforcing the wanted transformation of $\mathcal{H}$. The off-diagonal terms lead to $\tilde{F}(Y) = \tilde{F}(\tilde{X})$. But then the lower right component claims

$$\partial_{\tilde{X}} \tilde{F}(\tilde{X}) = (\partial f)^{-1}(X) , \tag{4.88}$$

which can only be fulfilled for constant f, that do not exhaust all the D-dimensional diffeomorphisms.

The same can independently be found by looking at the gauge fixing term of (4.39). Under the general embedding ansatz (4.86), it transforms as

$$\partial_L Y^T \mathcal{Z} V_R \quad \to \quad \partial_L Y^T \begin{pmatrix} \partial_X f & 0 \\ \partial_X \tilde{F} & \partial_{\tilde{X}} \tilde{F} \end{pmatrix} \begin{pmatrix} E \\ \mathbb{1} \end{pmatrix} V_R , \tag{4.89}$$

for the particular choice of $\mathcal{Z}$ that was argued for in (4.72). In order to have E transforming as a (2,0)-tensor in D dimensions, the $2D$ diffeomorphism has to be accompanied by a ρ-transformation (4.45),

$$V_R \to \partial f \; V_R . \tag{4.90}$$

But then the last line of (4.89) on the one hand gives $\partial_X \tilde{F} = 0$, and on the other hand gives

$$\partial_{\tilde{X}} \tilde{F}(\tilde{X}) = (\partial f)^{-1}(X) , \tag{4.91}$$

to keep the lower $\mathbb{1}$ of $\mathcal{Z}$ inert. This is the same contradiction to the assumption of a general diffeomorphism f that was found already above.

As a side remark, it shall now be shown how it is still possible to give a manifest realisation of D-dimensional diffeomorphisms in the doubled worldsheet theory of this chapter. The construction rests on the introduction of particular diffeomorphism covariant derivatives, but will not provide any embedding into $2D$-dimensional diffeomorphisms.

Additionally to the transformation $X \to f(X)$, the dual coordinates $\tilde{X}$ are now required to behave as contravariant vectors,

$$\tilde{X}_\mu \to \tilde{X}_\nu (\partial f^{-1})^\nu{}_\mu , \quad (\partial f)^\nu{}_\mu = f^\nu{}_{,\mu} . \tag{4.92}$$

This prevents any connection to transformations of the form $Y \to \mathcal{F}(Y)$, as was shown above. Still, it leads to the desired invariance if one introduces worldsheet derivatives

$$D_a Y^m = \begin{pmatrix} \delta^\mu_\rho & 0 \\ -\gamma^\rho_{\nu\kappa} \tilde{X}_\rho & \delta^\lambda_\nu \end{pmatrix} \begin{pmatrix} \partial_a X^\kappa \\ \partial_a \tilde{X}_\lambda \end{pmatrix} , \tag{4.93}$$

with $\gamma^\rho_{\nu\kappa}$ being the D-dimensional Christoffel symbols with respect to the metric $g_{\mu\nu}(X)$ in $\mathcal{Z}$. The doubled worldsheet theory has to be rewritten as

$$S = \int \mathrm{d}^2\sigma \; \frac{1}{2} D_L Y^T \mathcal{E}_D D_R Y + D_L Y^T \mathcal{Z}_D V_R \;, \tag{4.94}$$

with redefined target space fields

$$\mathcal{E} = \mathcal{A}^T \mathcal{E}_D \mathcal{A} \;, \quad \mathcal{Z} = \mathcal{A}^T \mathcal{Z}_D \;. \tag{4.95}$$

The matrix $\mathcal{A}$ is given by the first matrix factor on the right-hand side of (4.93). By construction, this action is invariant under D-dimensional diffeomorphisms and, in addition, leads to the transformation behaviour (4.85) for $\mathcal{E}_D$.

On the other, the use of covariant derivatives (4.93) hides other symmetries of the doubled worldsheet theory, in particular the $2D$-dimensional diffeomorphisms. Insofar, this approach will not be followed any further.

4.2.4 Non-geometry and fluxes

At this stage, the theory of a doubled worldsheet has been developed far enough to consider how non-geometry and non-geometric fluxes can be modeled. Again, the torus with H-flux serves as a guiding example both for its simplicity and its connections to the techniques of the preceding chapters.

Non-geometry

For simplicity, the $O(D, D)$ invariant representation (4.73) will be taken as a starting point. Furthermore, the antisymmetric constant tensor $\mathcal{C}$ will be set to zero simply being a total derivative. For the gauge fixing term, (4.72), $\mathcal{Z}^T = (E^T, \mathbb{1})$, will be chosen, where the D-dimensional target space fields are as for example in (3.235), i.e.

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;, \quad b = z \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = z\omega \;, \tag{4.96}$$

with ω defined as an abbreviation for the constant matrix appearing in b. The coordinates $Y^m = (x, y, z, \tilde{x}, \tilde{y}, \tilde{z})$ are the doubled version of X^μ. Eventually, the full action, as it should be condisered here, then reads

$$S = \int \mathrm{d}^2\sigma \; -\frac{1}{2}\partial_L Y^T \eta \partial_R Y + \partial_L Y^T \begin{pmatrix} g + b \\ \mathbb{1} \end{pmatrix} V_R \;. \tag{4.97}$$

As the D-dimensional target space in this case shall be a T^3, three coordinate periodicities are assumed (cf. (3.234) and (2.6), where for simplicity the factors 2π and R_i are dropped here):

$$T_i : \; Y \to Y + \begin{pmatrix} e_i \\ 0 \end{pmatrix} \;, \quad i = 1, 2, 3 \;. \tag{4.98}$$

They only involve the coordinates, whereas it will remain undefined at this point, whether the dual coordinates are supplied with boundary conditions as well. The kinetic term in (4.97) is obviously invariant under any of the periodicities, because it only involves derivatives of the coordinate fields. In contrast, as the b-field encounters a gauge transformation under the third periodicity,

$$T_z : \; b(z) \to b(z) + \omega \; , \tag{4.99}$$

the gauge fixing term will change,

$$T_z : \; \partial_L Y^T \mathcal{Z} V_R \to \partial_L Y^T \left(\mathcal{Z} + \begin{pmatrix} \omega \\ 0 \end{pmatrix} \right) V_R \; . \tag{4.100}$$

In the standard sigma model, this would not cause a problem, because the kinetic term is invariant under gauge transformations of the b-field. Here, it seems more involved to retain the particular form of the action.

Interestingly, it is possible to find an additional $O(D,D)$ transformation Ω such that a modified periodicity[11]

$$T'_z : \; Y \to \Omega Y + \begin{pmatrix} e_z \\ 0 \end{pmatrix} \; , \tag{4.101}$$

with

$$\Omega = \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \in O(D,D) \; , \tag{4.102}$$

will leave the gauge fixing term invariant, $T'_z : \partial_L Y^T \mathcal{Z} V_R \to \partial_L Y^T \mathcal{Z} V_R$. In particular, T'_z is identical to T_z for the coordinates, whereas the dual coordinates undergo an additional transformation. In conclusion, it is exactly this additional transformation that renders the three-torus well-defined. The full $2D$-dimensional target space, in contrast, has no simple geometric interpretation as T'_z mixes dual coordinates with coordinates.

This shows two things: First, the relation between the $2D$-dimensional target space and its D-dimensional counterpart is subtle and not straightforwardly determinable, although the discussion of the quantisation to one-loop will bring a little more insight into this. Second, only from the D-dimensional perspective, as in the preceding chapters, one can say that the torus with H-flux is a geometric frame that will be turned into a non-geometric one after performing two T-dualities. From the perspective of the doubled sigma model, non-geometry appears right away but for certain representations hidden in the periodicities of the dual coordinates. This is especially accordant with the claim that the doubled sigma model has a T-duality invariant form.

These statements can be worked out a bit further by investigating the situation for the non-geometric dual of the above setup. The corresponding T-duality transformation along the x and y direction is represented by a global $O(D,D)$ transformation,

$$Y \to MY \; , \quad M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in O(D,D) \; . \tag{4.103}$$

It leaves invariant the kinetic term of (4.97), but changes the gauge fixing term $\mathcal{Z} \to M^T \mathcal{Z}$. There are two possibilities to proceed:

[11] Such periodicities have been found from other approaches for example in [110], equation (3.29), or in [120], equation (3.12). See also the appendix C, equation (C.15) in [2].

- One could enforce the standard form of $\mathcal{Z}$ by employing a ρ-transformation (4.45), as a special case of (4.75),

$$V_R \to (g+b)^{-1} V_R \ , \quad M^T \mathcal{Z} = \begin{pmatrix} \mathbb{1} \\ g+b \end{pmatrix} \to \begin{pmatrix} (g+b)^{-1} \\ \mathbb{1} \end{pmatrix} \ . \tag{4.104}$$

  The action of T_z on $(g+b)^{-1}$ is highly non-trivial and interpreting the latter as the new D-dimensional target space fields brings this into the same lines of discussion as in section 3.3.4 of the preceding chapter. A patching seems not to be possible anymore, such that the definition of non-geometry from the D-dimensional perspective applies.

- One could modify the periodicity T_z again and find a well-defined patching while keeping the new $\mathcal{Z}' = M^T \mathcal{Z}$. An obvious example of this is given by

$$T_z'' : \ Y \to M^{-1} \Omega M Y + \begin{pmatrix} e_z \\ 0 \end{pmatrix} \ , \tag{4.105}$$

  with $M^{-1}\Omega M \in O(D,D)$. For the z-coordinate, this still reads as $z \to z+1$. The gauge fixing term now transforms invariantly,

$$\partial_L Y^T \mathcal{Z}' V_R \to \partial_L Y^T M^T \Omega^T M^{-T} M^T \left(\mathcal{Z} + \begin{pmatrix} \omega \\ 0 \end{pmatrix} \right) V_R = \partial_L Y^T \mathcal{Z}' V_R \ . \tag{4.106}$$

  This makes a particular example of how a patching[12] with T-dualities can render non-geometric configurations well-defined. The doubled sigma model allows to reveal such a feature of non-geometry in a particularly clear way[13].
  As a side remark, it should be noted that also the original model (4.97) is compatible with (4.105). Again, the kinetic term does not change as it is $O(D,D)$ invariant, whereas for the gauge fixing term one finds

$$T_z'' : \ \partial_L Y^T \mathcal{Z} V_R \to \partial_L Y^T M^T \Omega^T M^{-T} \left(\mathcal{Z} + \begin{pmatrix} \omega \\ 0 \end{pmatrix} \right) V_R = \partial_L Y^T \mathcal{Z} V_R \ . \tag{4.107}$$

  This does not surprise at all, as by definition of the various objects

$$M^{-1} \Omega M = \Omega \ , \tag{4.108}$$

  it is $T_z'' = T_z'$. Eventually, instead of the original T_z one could have imposed T_z'' from the beginning.

To conclude the investigation of the three-torus with H-flux example, it can be shown that the formalism reveals the same features of non-geometry in different representations. In all of them, the gauge fixing term remains the same as the particular choice of $\mathcal{Z}$ is necessary to switch representation. Therefore, only the kinetic term shall be considered further.

[12] Strictly speaking, it only shows how a specific choice of the periodicities renders the monodromies of the target space fields compatible with the symmetries of the theory. The precise choice of coordinate patches and transition functions shall not be developed here. But it can be noted that the Killing vectors are trivially of the form $\mathcal{K}^T = (0, \mathbb{1})$ and the above statements about locality of such a choice can be relaxed.

[13] Another approach is the T-fold construction of [26] or also [110], cf. in particular equation (3.36) of the latter.

In the standard sigma model formulation (4.71), i.e.

$$\mathcal{E} = \begin{pmatrix} 2(g+b) & 0 \\ 0 & 0 \end{pmatrix} , \tag{4.109}$$

the modified periodicity (4.101) leaves a negligible total derivative,

$$T'_z :\ \partial_L Y^T \mathcal{E} \partial_R Y \to \partial_L Y^T \mathcal{E} \partial_R Y + \partial_L x \partial_R y - \partial_L y \partial_R x , \tag{4.110}$$

using $\Omega^T \mathcal{E} \Omega = \mathcal{E}$. As expected, the torus with H-flux is well-defined at this level. The T-dual formulation is also well-defined, trivially checked by noting

$$\Omega^T \tilde{\mathcal{E}} \Omega = \tilde{\mathcal{E}} , \quad \tilde{\mathcal{E}} = M^T \mathcal{E} M . \tag{4.111}$$

As expected, a T-duality transformation M might spoil the identification of the upper left corner of $\mathcal{E}$ with the D-dimensional target space field $g+b$, or explicitly

$$\tilde{\mathcal{E}} = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.112}$$

In the covariant representation (4.78), the kinetic term is, surprisingly, again invariant under the periodicity $T'_z = T''_z$,

$$\begin{aligned} \partial_L Y^T \mathcal{H} \partial_R Y \ \to\ & \partial_L Y^T \begin{pmatrix} \mathbb{1} & -\omega \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} g - (b+\omega) g^{-1} (b+\omega) & (b+\omega) g^{-1} \\ -g^{-1}(b+\omega) & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \partial_R Y \\ = & \partial_L Y^T \begin{pmatrix} g - b g^{-1} b & b g^{-1} \\ -g^{-1} b & g^{-1} \end{pmatrix} \partial_R Y = \partial_L Y^T \mathcal{H} \partial_R Y . \end{aligned} \tag{4.113}$$

This also shows that the T-dual representation is compatible,

$$T'_z :\ \partial_L Y^T \mathcal{H}' \partial_R Y \to \partial_L Y^T \mathcal{H}' \partial_R Y , \quad \mathcal{H}' = M^T \mathcal{H} M . \tag{4.114}$$

To finish the investigation of non-geometry in the doubled sigma model, the above observations shall be arranged into four statements:

- The example of the three-torus with H-flux can be implemented in the doubled sigma model using various representations (standard sigma model, covariant, invariant).
- The torus periodicities can be formulated such that they are well-defined in every representation.
- A shift of the base circle coordinate $z \to z+1$ is then accompanied by a particular $O(D,D)$ transformation Ω that involves the dual coordinates only.
- Non-geometry in the sense of a patching with T-duality transformations only appears if one insists on a particular rewriting of the periodicity.

In general, one would expect the torus periodicities neither to be $O(D, D)$ invariant nor to be the same in different representations related by $\mathcal{U}$-transformations. In the above case, two additional properties helped to establish such features though: First, the particular matrix Ω and the T-duality transformation matrix M commute, as recognised in (4.108). Second, the periodicity for the z-coordinate always remained to be $z \to z + 1$, such that all fields transformed in a particularly simple way. For more involved situations one can suspect that non-geometry might not be detectable as easily as in the toroidal example.

A rather far-reaching conclusion can be drawn from the last of the four above statements. It can be understood in two ways. Either one takes the position of a very strict definition of non-geometry and claims that not only particular T-duals of the torus with H-flux are non-geometric, but rather all of them are. Or one relaxes the definition of non-geometry and claims that there is no non-geometry in the framework of a doubled sigma model, even for the frame that was labeled the Q-flux frame. This can be read as the statement that *the doubled sigma model is capable of resolving the peculiarities of non-geometry.*

Fluxes

Generally, the gauge fixing term cannot be kept inert under b-field gauge transformations. In the above discussion, it was possible to find global $O(D, D)$ transformations of the coordinates to retain the form $\mathcal{Z}^T = (E^T, \mathbb{1})$ under constant shifts of b. For an arbitrary gauge transformation $b \to b + \mathrm{d}\lambda$ this might be unfeasible as the theory is not in- or covariant under *local* $O(D, D)$ transformations of the coordinates. Also ρ-transformations (4.45) do not help as ρ possibly cannot be chosen such that it compensates the gauge transformation *and* keeps the lower entry of $\mathcal{Z}$ at its value $\mathbb{1}$.

Luckily, the $2D$ antisymmetric tensor field $\mathcal{C}$ offers a gauge symmetry, as has already been mentioned above,

$$\delta\mathcal{C}(Y) = \mathrm{d}\Xi(Y) \ , \tag{4.115}$$

for an arbitrary one-form Ξ. This is a generalisation of the standard sigma model b-field gauge transformations and indicates that fluxes could be encoded in the generalised field strength[14]

$$\mathcal{H} = \mathrm{d}\mathcal{C} \ . \tag{4.116}$$

To define the four known types of fluxes, a new notation shall be introduced for the following discussion. The matrix $\mathcal{C}$ will be written as

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{\mu\nu} & \mathcal{C}_\mu{}^\nu \\ \mathcal{C}^\mu{}_\nu & \mathcal{C}^{\mu\nu} \end{pmatrix} \ , \tag{4.117}$$

according to

$$Y^m = \begin{pmatrix} X^\mu \\ \tilde{X}_\mu \end{pmatrix} \ . \tag{4.118}$$

Taking the index structure of the fluxes as a guideline, they shall be defined as

$$\begin{aligned} H_{\mu\nu\rho} &= \mathcal{H}_{\mu\nu\rho} \ , \quad & f^\mu{}_{\nu\rho} &= \mathcal{H}^\mu{}_{\nu\rho} \ , \\ Q_\rho{}^{\mu\nu} &= \mathcal{H}^{\mu\nu}{}_\rho \ , \quad & R^{\mu\nu\rho} &= \mathcal{H}^{\mu\nu\rho} \ , \end{aligned} \tag{4.119}$$

[14]Not to be confused with the generalised metric $\mathcal{H}_{mn}$, that only has two indices.

where the derivatives are arranged into

$$\partial_\mu = \partial_{m=\mu} \ , \quad \partial^\mu = \partial_{m=D+\mu} \ , \tag{4.120}$$

according to the new index notation. For the Q-flux, the antisymmetry $\mathcal{H}^{\mu\nu}{}_\rho = \mathcal{H}_\rho{}^{\mu\nu}$ was used.

To illustrate this embedding, again, the three-torus with H-flux shall be considered. It has to be implemented in the standard sigma model representation, i.e. with

$$\mathcal{E} = \mathcal{G} + \mathcal{C} = 2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} z\omega & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{4.121}$$

where the above conventions and abbreviations have been used. In particular, it is $\mathcal{C}_{\mu\nu} = b_{\mu\nu}$ and, according to the definition (4.119),

$$H_{123} = \partial_{[1}\mathcal{C}_{23]} = 2 \cdot 1 \ , \quad f^\mu{}_{\nu\rho} = Q_\rho{}^{\mu\nu} = R^{\mu\nu\rho} = 0 \ . \tag{4.122}$$

The factor of 2 is due to the normalisation of the kinetic term. Other T-duality frames can be obtained by sequently applying the $O(D,D)$ matrices,

$$M_i = \begin{pmatrix} \mathbb{1} - e_i e_i^T & e_i e_i^T \\ e_i e_i^T & \mathbb{1} - e_i e_i^T \end{pmatrix} \ , \quad i = x, y, z \ , \tag{4.123}$$

corresponding to T-duality transformations along the x, y, z directions. In particular, the transformation used in (4.103) is given by $M = M_x M_y$. In the second frame one finds

$$\mathcal{C} \to M_x^T \mathcal{C} M_x \ , \quad f^1{}_{23} = 2 \cdot 1 \ , \quad H_{\mu\nu\rho} = Q_\rho{}^{\mu\nu} = R^{\mu\nu\rho} = 0 \ . \tag{4.124}$$

The non-geometric frame eventually has fluxes

$$\mathcal{C} \to M_y^T M_x^T \mathcal{C} M_x M_y \ , \quad Q_3{}^{12} = 2 \cdot 1 \ , \quad H_{\mu\nu\rho} = f^\mu{}_{\nu\rho} = R^{\mu\nu\rho} = 0 \ . \tag{4.125}$$

As a side remark, it shall be noted that on this level of the discussion, also the third T-duality transformation is possible and indeed leads to nonzero R-flux,

$$\mathcal{C} \to M_z^T M_y^T M_x^T \mathcal{C} M_x M_y M_z \ , \quad R^{123} = 2 \cdot 1 \ , \quad H_{\mu\nu\rho} = f^\mu{}_{\nu\rho} = Q_\rho{}^{\mu\nu} = 0 \ , \tag{4.126}$$

using that the coordinate transformation induces $z \to \tilde{z}$. The compatibility of this with the torus periodicities shall not be investigated any further here.

As an important observation, one can note: In each frame, exactly one type of flux is nonzero but constant, and indeed the discussed chain of fluxes appears,

$$H_{123} \xrightarrow{M_x} f^1{}_{23} \xrightarrow{M_y} Q_3{}^{12} \xrightarrow{M_z} R^{123} \ . \tag{4.127}$$

At least for the toroidal example this exactly meets the expectations. In particular, it appears that the four types of fluxes are only different views of the same underlying object, namely the tensor field $\mathcal{C}$.

In general, it seems possible to turn on several types of fluxes at once, which, as discussed in the introduction, is desirable for phenomenological reasons. One could simply start with a model that has a more involved matrix $\mathcal{C}$. On the other hand, it is not clear whether such more general configurations are within the solution space of string theory, or in other words, how exactly they reduce upon enforcing a D-dimensional target space perspective.

Unfortunately, there are more challenges to a naive embedding of the fluxes (4.119). First, one has to stick to one particular representation of the doubled sigma model. Any $\mathcal{U}$-transformation (4.48) will change the flux content, and once more it depends on the exact relation between $2D$-dimensional fields and their D-dimensional counterparts how to interpret this fact. It will partly be discussed in more detail in section 4.3.2. Second, it has not been clarified yet what the relation is between geometric flux f, encoded in the above sense, and its counterpart that can be defined from the metric using vielbeins. For these reasons the ideas given here should be taken as preliminary suggestions only.

4.3 Doubled target space equations of motion

The aim of this section is to derive the equations of motion for the doubled target space fields. They will be obtained from an investigation of the one-loop renormalisation of the doubled worldsheet theory. The leading question will be how it organises the reduction of the degrees of freedom from doubled to standard, and it will turn out that there automatically appear certain projection operators.

In the following, only an overview of the derivation shall be given, whereas the emphasis lies on the interpretation of the resulting equations. A detailed account on the calculational background[15] is given in [1].

4.3.1 Derivation

To derive the target space equations of motion, the analysis follows four steps: First, a set of derivatives is constructed, which are both $2D$ diffeomorphism and ρ-transformation covariant. Second, these derivatives are used to implement a covariant background / quantum splitting in order to expand the action (4.39). Third, all possible vertices and propagators are constructed in order to, fourth, obtain all possible one-loop diagrams. These are then used to determine the conditions for conformal invariance, which, in turn, are interpreted as target space equations of motion.

Covariant derivatives

The doubled worldsheet theory (4.39) possesses several symmetries, and it is desirable to implement them covariantly in the background / quantum splitting to follow. However, the $\mathcal{U}$-transformations (4.47) shall deliberately not be considered for the following reason: At various stages of the computation, it is necessary to have an invertible doubled metric $\mathcal{G}$. On the other hand, some representations explicitly break invertibility, as for example the standard sigma model representation (4.71). It therefore shall be assumed that any non-invertible metric $\mathcal{G}$ is turned into an invertible one by a suitably chosen $\mathcal{U}$-transformation. Employing a $\mathcal{U}$-transformation co- or invariant formalism would obstruct such an assumption and shall, therefore, not be considered. A more detailed discussion on this can be found in the last section 4.4.

[15]The renormalisation procedure has been carried out in [121] as part of a master's thesis, and shall thus only be reviewed in the following.

At least, it is possible to set up a formalism that is covariant with respect to $2D$ diffeomorphisms and ρ-transformations (4.45) that change V_R and $\mathcal{Z}$. The first step in this is to define corresponding derivatives that show such a covariant behaviour. They differ from the standard covariant derivatives and thus one has to differentiate between two types:

- $\mathcal{D}_a$ and $\mathcal{D}_m$ are covariant with respect to $2D$-dimensional diffeomorphisms.
- ∇_a and ∇_m are covariant with respect to both diffeomorphisms and ρ-transformations.

For objects that do not transform under ρ-transformations, there is no difference between the two types. This, in particular, holds for the coordinate fields,

$$\nabla_a Y^m = \mathcal{D}_a Y^m = \partial_a Y^m \ . \tag{4.128}$$

Furthermore, the double derivative is supposed to act as expected,

$$\nabla_a \nabla_b Y^m = \mathcal{D}_b \mathcal{D}_a Y^m = (\delta^m{}_n \partial_b + \Gamma^m{}_{kn} \partial_b Y^k) \partial_a Y^n \ , \tag{4.129}$$

with Christoffel symbols $\Gamma^m{}_{kn}$ defined with respect to the metric $\mathcal{G}$ on the doubled target space. This makes the first occasion where an invertible $\mathcal{G}$ is necessary.

For doubled target space tensors, the covariant derivative in spacetime directions then reads

$$\nabla_p T^m = \mathcal{D}_p T^m = \partial_p T^m + \Gamma^m{}_{pk} T^k \ , \tag{4.130}$$

where its extension to more indices is taken to be the standard one.

The first object that does transform under ρ-transformations is V_R. And indeed, it is possible to introduce a covariant derivative,

$$\nabla_k V_R = \partial_k V_R + \mathcal{Z}_\parallel^T \mathcal{D}_k \mathcal{Z} V_R \ , \tag{4.131}$$

with a new object

$$\mathcal{Z}_\parallel = \mathcal{G}^{-1} \mathcal{Z} (\mathcal{Z}^T \mathcal{G}^{-1} \mathcal{Z})^{-1} \ . \tag{4.132}$$

Again, the existence of the respective inverses is assumed, as discussed above. To check the covariance under (4.45), one can first note that $(\mathcal{Z}^T\mathcal{G}^{-1}\mathcal{Z})^{-1} \to \rho(\mathcal{Z}^T\mathcal{G}^{-1}\mathcal{Z})^{-1}\rho^T$, and then straightforwardly plug in,

$$\begin{aligned} \nabla_k V_R \ \to \ & \rho \partial_k V_R + (\partial_k \rho) V_R + \rho (\mathcal{Z}^T \mathcal{G}^{-1} \mathcal{Z})^{-1} \mathcal{Z}^T \mathcal{G}^{-1} \big((\mathcal{D}_k \mathcal{Z}) \rho^{-1} + \mathcal{Z} \mathcal{D}_k \rho^{-1} \big) \rho V_R \\ & = \rho \nabla_k V_R \ , \end{aligned} \tag{4.133}$$

where in the second line it was used that $\partial_k \rho^{-1} = -\rho^{-1}(\partial_k \rho)\rho^{-1}$.

It is possible to find the respective ρ-transformation covariant derivative for $\mathcal{Z}$ by extending the above one, as would be usually done for a contravariant tensor,

$$\nabla_k \mathcal{Z}_{m\mu} = \partial_k \mathcal{Z}_{m\mu} - \Gamma^l{}_{km} \mathcal{Z}_{l\mu} - (\mathcal{Z}_\parallel^T)^{\nu n} \mathcal{D}_k \mathcal{Z}_{n\mu} \mathcal{Z}_{m\nu} \ . \tag{4.134}$$

This derivative can be compactly written as

$$\nabla_k \mathcal{Z} = \mathcal{P}_\perp \mathcal{D}_k \mathcal{Z} \ , \tag{4.135}$$

by noting the interesting fact that it contains a projection operator

$$\mathcal{P}_\perp = \mathbb{1} - \mathcal{Z} \mathcal{Z}_\parallel^T = \mathbb{1} - \mathcal{Z} (\mathcal{Z}^T \mathcal{G}^{-1} \mathcal{Z})^{-1} \mathcal{Z}^T \mathcal{G}^{-1} \ . \tag{4.136}$$

Indeed, the projection properties can be checked easily. First, one defines the opposite projector $\mathcal{P}_{\|} = \mathbb{1} - \mathcal{P}_{\perp}$. Then, it follows

$$\mathcal{P}_A^2 = \mathcal{P}_A \ , \quad \mathrm{tr}(\mathcal{P}_A) = D \ , \tag{4.137}$$

for both $A = \perp, \|$. Furthermore, the subspaces are complementary, as can be seen from

$$\mathcal{P}_{\perp} + \mathcal{P}_{\|} = \mathbb{1} \ , \quad \mathcal{P}_{\|}\mathcal{P}_{\perp} = 0 \ . \tag{4.138}$$

The notation is further justified by stating

$$\mathcal{P}_{\|}\mathcal{Z} = \mathcal{Z} \ , \quad \mathcal{P}_{\perp}\mathcal{Z} = 0 \ . \tag{4.139}$$

In other words, the projectors define D-dimensional subspaces that lie parallel, or perpendicular, respectively, to the gauged fixed directions defined by $\mathcal{Z}$. It is worth noting that the projector $\mathcal{P}_{\perp}$ arose automatically and was not imposed from the consideration of the degrees of freedom.

For later convenience, also the projected metric and the projection of the inverse metric shall be defined,

$$\mathcal{G}_{\perp} = \mathcal{P}_{\perp}\mathcal{G} \ , \quad \mathcal{G}_{\perp}^{-1} = \mathcal{G}^{-1}\mathcal{P}_{\perp} \ . \tag{4.140}$$

They will appear in the propagators of the quantum fields.

Background / quantum splitting

The general approach of quantising a non-linear sigma model to be followed here is the background field method [122, 123, 124]. It rests on the split of any field into a background piece and a quantum fluctuation. The refined procedure, called "normal coordinate expansion", [125, 126] avoids the breaking of diffeomorphism invariance by using covariant derivatives.

According to the procedure described in [126], the coordinate fields Y and the Lagrange multipliers V_R are extended to one-parameter families with the properties of geodesic curves, i.e. $Y(\sigma; s)$ and $V_R(\sigma; s)$ with $s \in [0,1]$. Here, a slight generalisation is employed, as the covariant derivatives that have been defined above are not only diffeomorphism, but also ρ-transformation covariant.

Technically speaking, the fields are now subject to the following differential conditions,

$$\begin{aligned} 0 &= \nabla_s^2 Y^m(\sigma; s) = \left(\delta^m{}_l \partial_s + \Gamma^m{}_{kl}\dot{Y}^k(\sigma; s)\right)\dot{Y}^l(\sigma; s) \\ 0 &= \nabla_s^2 V_R(\sigma; s) = \nabla_s \left(\partial_s V_R(\sigma; s) + \mathcal{Z}_{\|}^T \mathcal{D}_m \mathcal{Z}\dot{Y}^m(\sigma; s) V_R(\sigma; s)\right) \ , \end{aligned} \tag{4.141}$$

and boundary conditions,

$$Y(\sigma; 0) = Y(\sigma) \qquad \nabla_s Y(\sigma; 0) = y(\sigma) \tag{4.142}$$

$$V_R(\sigma; 0) = V_R(\sigma) \qquad \nabla_s V_R(\sigma; 0) = v_R(\sigma) \ . \tag{4.143}$$

The covariant quantum fields are denoted by $y(\sigma)$ and $v_R(\sigma)$, respectively, whereas $Y(\sigma)$ and $V_R(\sigma)$ refer to the background fields. A dot signalises the derivative with respect to the affine parameter s,

$$\dot{Y}(\sigma; s) = \partial_s Y(\sigma; s) = \nabla_s Y(\sigma; s) \ . \tag{4.144}$$

In order to keep the equations readable, all further dependencies on the coordinate fields in Γ, $\mathcal{Z}_{\|}$, $\mathcal{Z}$ and $\mathcal{D}$ have been omitted. By construction, the full quantum fields are given by

$$\begin{aligned} Y(\sigma;1) &= Y(\sigma) + y(\sigma) + \sum_{n>1} \frac{1}{n!}\partial_s^n Y(\sigma;0) \\ V_R(\sigma;1) &= V_R(\sigma) + v_R(\sigma) + \sum_{n>1} \frac{1}{n!}\partial_s^n V_R(\sigma;0) \ . \end{aligned} \tag{4.145}$$

To obtain a covariant expansion of the doubled sigma model action (4.39), the procedure of [127] is applied. First, the same dependence on the affine parameter s is assumed for the action itself, $S = S(s)$. Second, by taking into account (4.145), the full quantum action is given by $S = S(1)$. Therefore, it is possible to employ the Taylor expansion

$$S = S(1) = \sum_{n\geqslant 0} \frac{1}{n!}\partial_s^n S(0) \ . \tag{4.146}$$

Because the action itself is a scalar quantity, all partial derivatives ∂_s can be replaced by the covariant derivatives ∇_s that then will act on the fields $Y(\sigma;s)$ and $V_R(\sigma;s)$ as described above. This will result in an expansion in terms of covariant objects only, given the differential conditions (4.141).

By construction, the zeroth order of the expansion (4.146) gives the original action in terms of the background fields Y and V_R. The first order can only contain terms with one quantum field y or v_R, it would therefore not contribute to any one particle irreducible (1PI) process. Accordingly, second order is the first relevant part, and, as the whole analysis here is restricted to one loop processes, it is also sufficient. Concretely, the second order term is given by

$$\begin{aligned} S_2 = \int \mathrm{d}^2\sigma \ &\frac{1}{2}\mathcal{G}_{kl}\nabla_L y^k \nabla_R y^l + \frac{1}{2}\mathcal{Z}_{m\mu}\Big(\nabla_L y^m v_R^\mu - y^m \nabla_L v_R^\mu\Big) \\ &+ \frac{1}{2}\Big(\Big(\mathcal{R}_{mkln} + \frac{1}{2}\nabla_{(k}\mathcal{H}_{l)mn}\Big)\partial_L Y^m \partial_R Y^n + \Big(\nabla_{(k}\nabla_{l)}\mathcal{Z}_{m\mu} + \mathcal{R}^p{}_{klm}\mathcal{Z}_{p\mu}\Big)\partial_L Y^m V_R^\mu\Big) y^k y^l \\ &+ \frac{1}{4}\mathcal{H}_{klm}\Big(\partial_R Y^m y^k \nabla_L y^l - \partial_L Y^m y^k \nabla_R y^l\Big) + \nabla_k \mathcal{Z}_{l\mu}\, V_R^\mu\, y^k \nabla_L y^l \\ &+ \Big(\nabla_k \mathcal{Z}_{m\mu} - \frac{1}{2}\nabla_m \mathcal{Z}_{k\mu}\Big)\partial_L Y^m y^k v_R^\mu \ . \end{aligned} \tag{4.147}$$

It was used that the connection for the covariant derivative $\mathcal{D}_p$ is metric compatible,

$$\mathcal{D}_p \mathcal{G}_{mn} = 0 \ , \tag{4.148}$$

and that it can be used to define a Riemann tensor on the doubled target space,

$$[\mathcal{D}_p, \mathcal{D}_q]T^p = \mathcal{R}^p{}_{rpq}T^r \ . \tag{4.149}$$

As above, $\mathcal{H}_{mnp}$ denotes the field strength of the antisymmetric tensor field $\mathcal{C}$ appearing in the kinetic term.

Propagators and vertices

The first line of (4.147) determines the kinetic terms of the quantum fields y and v_R. After the application of a dimensional regularisation scheme with an IR regulator m^2 in $d = 2 - 2\epsilon$ dimensions, and $\eta \to \hat{\eta} = \mathrm{diag}(1, \dots, -1)$, it reads

$$S_{\mathrm{kin}} = \mu^{d-2} \int \mathrm{d}^d\sigma \; \frac{1}{4}\hat{\eta}^{ab}\nabla_a y^T \mathcal{G} \nabla_b y + \frac{1}{4} m^2 y^T \mathcal{G} y + \frac{1}{2} v_R^T \mathcal{Z}^T \nabla_L y - \frac{1}{2} y^T \mathcal{Z} \nabla_L v_R \; , \tag{4.150}$$

where μ makes a regularisation scale to retain the mass dimension. This expression has to be translated into Fourier space by defining appropriate Fourier transforms of the covariant worldsheet derivatives, and by assuming that the loop momenta p_a are much larger than any scale corresponding to the dependence of $\mathcal{Z}$ and $\mathcal{G}$ on the worldsheet coordinates. From the result,

$$S_{\mathrm{kin}} = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d \mu^{d-2}} \; \begin{pmatrix} y^T(-p) & v_R^T(-p) \end{pmatrix} \; \Delta^{-1} \begin{pmatrix} y(p) \\ v_R(p) \end{pmatrix} \; , \tag{4.151}$$

one can identify the inverse propagator for y and v_R,

$$\Delta^{-1} = \begin{pmatrix} \frac{1}{2}\mathcal{G}(p^2 + m^2) & -\mathrm{i}\mathcal{Z} p_L \\ \mathrm{i}\mathcal{Z}^T p_L & 0 \end{pmatrix} \; . \tag{4.152}$$

Under the above assumption, $\mathcal{Z}$ and $\mathcal{G}$ are constant, and one can use the standard algebraic relation for block matrices to invert Δ^{-1}, with the result

$$\Delta = \begin{pmatrix} \mathcal{G}_\perp^{-1} \frac{2}{p^2+m^2} & \mathcal{Z}_\parallel \frac{1}{\mathrm{i}p_L} \\ -\mathcal{Z}_\parallel^T \frac{1}{\mathrm{i}p_L} & -(\mathcal{Z}^T \mathcal{G}^{-1} \mathcal{Z})^{-1} \frac{p^2+m^2}{2p_L^2} \end{pmatrix} \; . \tag{4.153}$$

Again, it was assumed that $\mathcal{G}$ is invertible, as already discussed above. The propagator Δ offers two interesting observations:

- Independently of the construction (4.131), the projector $\mathcal{P}_\perp$ and the projected inverse metric $\mathcal{G}_\perp^{-1}$ appear, due to the algebraic inversion of Δ^{-1}. This has an important physical impact and will be discussed in section 4.3.2.

- There appears a nontrivial $\langle v_R^\mu v_R^\nu \rangle$ propagator, in contrast to the vanishing lower right corner of Δ^{-1}. Fortunately, it will not contribute to any divergent diagram considered in the following.

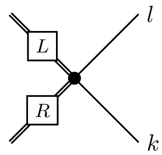
Eventually, one can identify the following three different propagators,

$$\langle y^m y^n \rangle = \; m \; \rule[0.5ex]{4cm}{0.4pt} \; n \; = -\mathrm{i}(\mathcal{G}_\perp^{-1})^{mn} \frac{2}{p^2 + m^2} \tag{4.154}$$

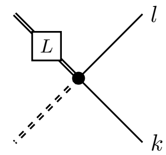
$$\langle y^m v_R^\nu \rangle = \; m \; \rule[0.5ex]{2cm}{0.4pt}\text{- - - - -} \; \nu \; = -(Z_\parallel)^{m\nu} \frac{1}{p_L} \tag{4.155}$$

$$\langle v_R^\mu v_R^\nu \rangle = \; \mu \; \text{- - - - - - - - - - - -} \; \nu \; = \mathrm{i}\Big((\mathcal{Z}^T \mathcal{G}^{-1} \mathcal{Z})^{-1}\Big)^{\mu\nu} \frac{p^2 + m^2}{2p_L^2} \; . \tag{4.156}$$

The corresponding two-point quantum vertices can be read off from the last three rows of (4.147). There are five different ones, two of them to be shown here exemplarily,



$\frac{\mathrm{i}}{2}(\mathcal{R}_{mkln} + \frac{1}{2}\nabla_{(k}\mathcal{H}_{l)mn})\partial_L Y^m \partial_R Y^n y^k y^l$

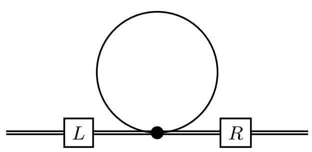


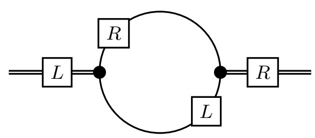
$\frac{\mathrm{i}}{2}(\nabla_{(k}\nabla_{l)}\mathcal{Z}_{m\mu} + \mathcal{R}^p{}_{klm}\mathcal{Z}_{p\mu})\partial_L Y^m V_R^\mu y^k y^l$.

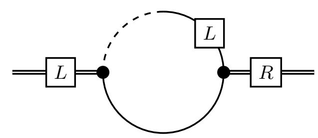
Double lines correspond to background fields, single lines to quantum fields. Solid or dashed lines are chosen according to the propagators shown above. A boxed letter L or R designates the respective worldsheet derivative.

One-loop diagrams

One can use the basic diagrammatic constituents described so far to construct one-loop diagrams. A power counting shows that only diagrams with at most two vertices can have divergent contributions. The detailed analysis shows that there are three relevant diagrams for the renormalisation of the kinetic term, i.e. diagrams proportional to $\partial_L Y^m \partial_R Y^n$,







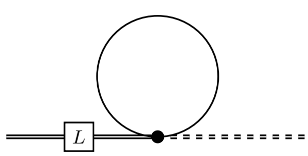
that lead to the expression

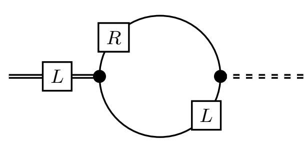
$$\Gamma_{\text{kin}} = I_1 \int \mathrm{d}^2\sigma \, \big((\mathcal{R}_{ijkl} + \tfrac{1}{2}\nabla_i\mathcal{H}_{ljk})(\mathcal{G}_\perp^{-1})^{il} - \tfrac{1}{4}\mathcal{H}_{pmj}\mathcal{H}_{nqk}(\mathcal{G}_\perp^{-1})^{mn}(\mathcal{G}_\perp^{-1})^{pq} + \mathcal{H}_{pmk}(\nabla_q\mathcal{Z}_{j\nu} - \tfrac{1}{2}\nabla_j\mathcal{Z}_{q\nu})(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{Z}_\parallel)^{m\nu}\big)\partial_L Y^j \partial_R Y^k \ . \tag{4.157}$$

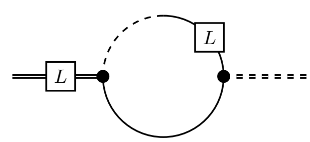
The divergent integral I_1 is given by

$$I_1 = \frac{1}{4\pi}\Gamma(\epsilon)\left(4\pi\frac{\mu^2}{m^2}\right)^\epsilon \ . \tag{4.158}$$

All contributions from the gauge fixing term, i.e. all terms proportional to $\partial_L Y^m V_R^\nu$, stem from three different diagrams,







They add up to the divergent expression

$$\Gamma_{\text{gf}} = I_1 \int \mathrm{d}^2\sigma \, \big((\nabla_{(i}\nabla_{j)}\mathcal{Z}_{k\nu} + \mathcal{R}^m{}_{ijk}\mathcal{Z}_{m\nu})(\mathcal{G}_\perp^{-1})^{ij} + \mathcal{H}_{pmk}\nabla_n\mathcal{Z}_{q\nu}(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{G}_\perp^{-1})^{mn} - 4\nabla_{[p}\mathcal{Z}_{m]\nu}(\nabla_q\mathcal{Z}_{k\mu} - \tfrac{1}{2}\nabla_k\mathcal{Z}_{q\mu})(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{Z}_\parallel)^{m\mu}\big)\partial_L Y^k V_R^\nu \ . \tag{4.159}$$

Eventually, it can be argued that the ghost sector does not contribute any divergent expression.

Target space equations of motion

Instead of explicitly determining the necessary counterterms for $\mathcal{G}$, $\mathcal{C}$ and $\mathcal{Z}$, the target space equations of motion are here taken to be the constraints keeping Weyl invariance anomaly free, cf. [128].

One can easily show that the doubled worldsheet action (4.39) is invariant under Weyl transformations,

$$\gamma(\sigma) \to e^{2\phi(\sigma)}\gamma(\sigma) \ , \tag{4.160}$$

for any conformal factor $\phi(\sigma)$. To this end, one has to replace the constant Minkowski metric η by a conformal gauge worldsheet metric γ,

$$\gamma(\sigma) = e^{2\phi(\sigma)}\eta \ . \tag{4.161}$$

This necessitates the inclusion of an Einstein-Hilbert term on the worldsheet,

$$S_{\mathrm{EH}} = \int \mathrm{d}^2\sigma \, \sqrt{\gamma} R(\gamma)\Phi(Y) \ , \tag{4.162}$$

involving the dilaton $\Phi(Y)$ that was neglected so far. Taking into account the transformation behavior of the curvature scalar R under (4.160), and restoring a factor of $1/\alpha'$, one recovers the doubled worldsheet action in conformal gauge as

$$S_\phi = \frac{1}{\alpha'}\int \mathrm{d}^2\sigma \ \frac{1}{2}\partial_L Y^T \mathcal{E}\partial_R Y + \mathcal{W}_L V_R - 4\alpha'\phi\partial_L\partial_R\Phi \ . \tag{4.163}$$

For infinitesimal ϕ one can determine the renormalised effective action in conformal gauge as

$$\begin{aligned}\Gamma_\phi &= \frac{1}{\alpha'}\int \mathrm{d}^d\sigma \ e^{-2\epsilon\phi}\left(\mathcal{L} - 4\alpha'\phi\partial_L\partial_R\Phi + \frac{1}{4\pi\epsilon}\mathcal{L}_{\mathrm{ct}}\right) + \frac{\alpha'}{4\pi\epsilon}\mathcal{L}_{\mathrm{div}} \\ &= \frac{1}{2\pi\alpha'}\int \mathrm{d}^d\sigma \ 2\pi\mathcal{L} - \phi(\mathcal{L}_{\mathrm{ct}} + 2\alpha'\partial_L\partial_R\Phi) + \frac{1}{2\epsilon}(\mathcal{L}_{\mathrm{ct}} + \alpha'\mathcal{L}_{\mathrm{div}}) \ .\end{aligned} \tag{4.164}$$

The original doubled worldsheet Lagrangian is here denoted by $\mathcal{L}$, the (not explicitly determined) counterterm Lagrangian by $\mathcal{L}_{\mathrm{ct}}$ and the divergent contributions from one-loop diagrams by $\mathcal{L}_{\mathrm{div}}$. It has to be noted that the introduction of the conformal gauge does not change the prefactors of the divergent contributions.

Finiteness of the theory can now be turned into the condition $\mathcal{L}_{\mathrm{ct}} + \alpha'\mathcal{L}_{\mathrm{div}} = 0$, whereas conformal invariance is equivalent to $\mathcal{L}_{\mathrm{ct}} + 2\alpha'\partial_L\partial_R\Phi = 0$, both simply read off from the action (4.164). Combining these conditions gives

$$\mathcal{L}_{\mathrm{div}} - 2\partial_L\partial_R\Phi = 0 \ , \tag{4.165}$$

which can be split into three equations of motion for the doubled target space. Before that, one has to insert the classical field equation for the coordinate fields Y,

$$\begin{aligned}\nabla_L\nabla_R Y^n = &-\frac{1}{2}\mathcal{G}^{nm}\mathcal{H}_{mpq}\partial_L Y^p\partial_R Y^q \\ &+ \mathcal{G}^{nm}\big(\mathcal{D}_{[m}\mathcal{Z}_{p]\nu}\partial_L Y^p V_R^\nu + \mathcal{Z}_{m\nu}\partial_L V_R^\nu\big) \ ,\end{aligned} \tag{4.166}$$

into the second term of

$$\partial_L\partial_R\Phi = \nabla_m\nabla_n\Phi\partial_L Y^m\partial_R Y^n + \nabla_m\Phi\nabla_L\nabla_R Y^m \ . \tag{4.167}$$

The three equations in (4.165), finally, arise from sorting terms according to the three different background field structures they contain. Corresponding to the kinetic term, one finds

$$\boxed{\begin{aligned}&\Big(\mathcal{R}_{ijkl}+\frac{1}{2}\nabla_i\mathcal{H}_{ljk}\Big)(\mathcal{G}_\perp^{-1})^{il}-\frac{1}{4}\mathcal{H}_{pmj}\mathcal{H}_{nqk}(\mathcal{G}_\perp^{-1})^{mn}(\mathcal{G}_\perp^{-1})^{pq}\\&\qquad+\mathcal{H}_{pmk}\Big(\nabla_q\mathcal{Z}_{j\nu}-\frac{1}{2}\nabla_j\mathcal{Z}_{q\nu}\Big)(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{Z}_\parallel)^{m\nu}-2\nabla_j\nabla_k\Phi+\nabla_m\Phi\mathcal{G}^{mn}\,\mathcal{H}_{njk}=0\ .\end{aligned}}\tag{4.168}$$

Corresponding to the gauge fixing term, one finds

$$\boxed{\begin{aligned}&\Big(\nabla_{(i}\nabla_{j)}\mathcal{Z}_{k\nu}+\mathcal{R}^m{}_{ijk}\mathcal{Z}_{m\nu}\Big)(\mathcal{G}_\perp^{-1})^{ij}+\mathcal{H}_{pmk}\,\nabla_n\mathcal{Z}_{q\nu}(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{G}_\perp^{-1})^{mn}\\&\qquad-4\nabla_{[p}\mathcal{Z}_{m]\nu}\Big(\nabla_q\mathcal{Z}_{k\mu}-\frac{1}{2}\nabla_k\mathcal{Z}_{q\mu}\Big)(\mathcal{G}_\perp^{-1})^{pq}(\mathcal{Z}_\parallel)^{m\mu}-4\partial_m\Phi\mathcal{G}^{mn}\partial_{[n}\mathcal{Z}_{k]\nu}=0\ .\end{aligned}}\tag{4.169}$$

Because of its special structure, the last remaining term in (4.165) has to vanish by itself,

$$\boxed{\partial_m\Phi(\mathcal{G}^{-1})^{mn}\mathcal{Z}_{n\nu}=0\ .}\tag{4.170}$$

This ends the derivation of the doubled target space equations of motion. It shall be noted, that compared to the standard case one obtains three instead of one equation. Furthermore, there are many additional terms that come from the gauge fixing term $\mathcal{W}_L V_R$ in the action. Eventually, there is no dilaton equation of motion, as this would necessitate to compute the renormalisation of the worldsheet Einstein-Hilbert term (4.162).

Double field theory

To conclude this section, it shall be shortly investigated how the above equations of motion are related to the double field theory equations of motion.

The initial hope was that the doubled worldsheet model presented here is able to provide a sigma model origin of double field theory. But as discussed around (4.82), this is not the case: The double field theory gauge symmetry cannot be reproduced as a symmetry of the action for any doubled worldsheet model that has $2D$ diffeomorphism covariance. And as the equations of motion are expected to inherit the symmetries of the action, it seems that there cannot be an agreement with double field theory.

Indeed, the target space equations of motion found above confirm this finding. To compare them with double field theory, one could decide to change to the generalised metric representation (4.78) with $\mathcal{C}$ being constant. The first equation of motion (4.168) then reads

$$\mathcal{R}_{ijkl}(\mathcal{G}_\perp^{-1})^{il}-2\nabla_j\nabla_k\Phi=0\ .\tag{4.171}$$

Given the canonical choice of $\mathcal{Z}$ in (4.72), the projector $\mathcal{P}_\perp$ takes the form

$$\mathcal{P}_\perp=\frac{1}{2}\big(\mathbb{1}-\mathcal{H}\eta\big)\ ,\tag{4.172}$$

such that with $\mathcal{G}_\perp^{-1} = \mathcal{H}^{-1}\mathcal{P}_\perp$ there appears one term with the form of the Ricci tensor in (4.171),

$$\mathcal{R}_{jk} + \mathcal{R}_{ijkl}\eta^{il} + 4\nabla_j\nabla_k\Phi = 0 \ . \tag{4.173}$$

Accordingly, from the side of double field theory the most promising candidate for a comparison is the equation of motion (4.56) of [18], which is written in terms of the generalised Ricci tensor $\mathcal{R}^{\mathrm{DFT}}$,

$$\mathcal{R}^{\mathrm{DFT}}_{jk}(\mathcal{H}, \Phi) = 0 \ . \tag{4.174}$$

A detailed analysis shows that, as suspected, it is not possible to match this equation with (4.173). Further comments on this will be made in the last section of this chapter.

4.3.2 Physical interpretation

The doubled worldsheet model described in this chapter is designed to accommodate string theory degrees of freedom in a doubled target space, in order to render T-duality a manifest feature. Eventually, the number of degrees of freedom compared to the standard sigma model of string theory must not change. This was taken into account by imposing invariance under a gauge transformation of the $2D$ coordinate fields (4.37). A set of D Killing vectors rendered half of the coordinates unphysical. Like in other doubled theories, e.g. in [26], the choice of unphysical directions, often called "section condition", is not unique. The present theory indeed incorporates such a freedom by allowing for different sets of Killing vectors.

A more involved question is how the target space degrees of freedom reduce in the correct way. As usual, all functions of the coordinate fields that appear in the action (4.39) are interpreted as target space fields. These are the doubled metric $\mathcal{G}$ and the antisymmetric tensor field $\mathcal{C}$. As the remaining field $\mathcal{Z}$ only parametrises the gauge fixing for the coordinate gauge transformation, and thus is not part of the physical content of the theory, it shall not be interpreted as a dynamical target space field.

In principle, $\mathcal{G}$ and $\mathcal{C}$ make $4D^2$ target space degrees of freedom, which is far too many. On the other hand, the doubled worldsheet model comes with new symmetries. One of those is the $\mathcal{U}$-transformation (4.47), that can be seen as a local, i.e. Y-dependent, transformation (4.48) of the target space fields. In particular, it can be used to bring the the kinetic term into a form involving the $O(D, D)$ covariant generalised metric $\mathcal{H}$, that indeed carries the right number of degrees of freedom. In conclusion, one can expect that the new symmetries truncate the physical degrees of freedom in the right way.

Using the path integral formalism, a gauge symmetry of the coordinate fields necessitates BRST transformations involving ghost fields. The classical $\mathcal{U}$-transformations reappear in the form of a parameter $\mathcal{Q}$ that quantifies the indeterminacy of the actual BRST transformation rules (4.57). One could say that the ambiguities in the BRST symmetry of the double worldsheet theory lead to target space $\mathcal{U}$-gauge transformations.

But not only that: It was shown, that claiming BRST invariance of the action is equivalent to two conditions (4.62) and (4.63) on the Killing vectors $\mathcal{K}$. The second condition allows to rewrite the kinetic term in the standard sigma model form (4.71), such that it carries only D^2 instead of $4D^2$ physical degrees of freedom. Furthermore, the first condition reduces locally, i.e. for trivial Killing vectors (4.69), to $\partial_{\tilde{X}}\tilde{\mathcal{E}} = 0$. This means that the physical target space fields encoded in the remaining $D \times D$ matrix do not depend on the dual coordinates. In that sense, the doubled worldsheet model encodes exactly as many degrees of freedom as the standard sigma model of string theory.

The whole counting becomes more obvious by inspecting the relation between the arbitrary $2D \times 2D$ kinetic term $\mathcal{E}$ and the physical object $\tilde{\mathcal{E}}$ that is subject to the two above mentioned conditions. It is given by (4.61),

$$\mathcal{E} = \tilde{\mathcal{E}} + 2\mathcal{Z}\mathcal{Q} \,. \tag{4.175}$$

As discussed, $\tilde{\mathcal{E}}$ contains only D^2 physical degrees of freedom. Indeed, $\mathcal{Z}$ and $\mathcal{Q}$ remove $3D^2$ degrees of freedom from the original $4D^2$ ones in $\mathcal{E}$, as they each contribute $2D^2$ entries, from which D^2 components can be removed by a ρ-transformation (4.45) of $\mathcal{Z}$.

The discussion so far is summarised in figure 4.1: In the upper row, $2D$ coordinates are reduced to D by the supposed gauge invariance, whereas in the lower row, $4D^2$ target space degrees of freedom are reduced to D^2 by the conditions from BRST invariance.

$$\begin{tikzcd}
\boxed{Y^m(\sigma)} \arrow[rr, "\text{Gauging (4.37)}"] & & \boxed{X^\mu(\sigma)} \\
\boxed{\mathcal{E}_{mn}(Y)} \arrow[r, "\text{Projection (4.63)}"] & \boxed{\tilde{\mathcal{E}}_{mn}(Y)} \arrow[r, "\text{Killing equation (4.62)}"] & \boxed{\tilde{\mathcal{E}}_{mn}(X) \cong E_{\mu\nu}(X)}
\end{tikzcd}$$

Figure 4.1: Reduction of the unphysical degrees of freedom

A reduction of the degrees of freedom is visible also in the target space equations of motion obtained by the procedure described in this section. First, already in the determination of ρ-covariant derivatives a projector $\mathcal{P}_\perp$ appeared. It exactly projects into the D-dimensional subspace that lies perpendicular to the directions that have been fixed by $\mathcal{Z}$ in the gauge fixing term of the doubled worldsheet action. Independently, precisely the same projector appeared in the propagator of the coordinate quantum fluctuations y^m. Accordingly, the target space equations of motion (4.168) and (4.169) contain projected metrics.

In particular, the first term of (4.168) attracts attention: It strongly resembles the standard sigma model case where the Riemann tensor is contracted by an inverse metric to give the Ricci tensor. Here, instead, the contraction is tied by a projected inverse metric, and the same goes for the contraction in the flux term. It seems that the number of degrees of freedom is reduced in the right way.

Another interesting appearance of an automatic reduction of degrees of freedom is given by the third equation, (4.170). It demands D combinations of derivatives acting on the dilaton to vanish, leaving a dependence on the other D coordinate combinations only. This equation is analogous to the BRST conditions but this time uses the gauge fixing parameter $\mathcal{Z}$ instead of the Killing vectors $\mathcal{K}$ to determine the unphysical directions.

To conclude this discussion, table 4.2 lists all objects appearing in the doubled worldsheet theory and contrasts the roles they play in the worldsheet interpretation with the roles they play in the target space interpretation. Additionally, the number of components, or degrees of freedom, is specified and one can easily check that they add up in the right way.

On the side of the coordinates, the $2D$ coordinate fields Y are reduced by gauging to D coordinates X that make the actual target space manifold. The vectors $\mathcal{K}$ parametrise the gauge transformations on the worldsheet, but can be seen as Killing vectors for the target space. Furthermore, they enter the projection equations (4.63) that restrict the target space

Object	**#(comp.)**	**Worldsheet interpretation**	**Target space interpretation**
Y	$2D$	Doubled coordinate fields	Doubled geometry coordinates
X	D	Worldsheet coordinate fields	Target space manifold coordinates
$\mathcal{K}$	D	Gauge transformations	Doubled geometry Killing vectors and projectors
$\mathcal{E}$	$4D^2$	Kinetic and WZ terms of the doubled worldsheet theory	Doubled geometry metric $\mathcal{G}$ and antisymmetric tensor field $\mathcal{C}$
$\tilde{\mathcal{E}} \cong E$	D^2	Projected version of the doubled kinetic and WZ terms	Target space metric g and antisymmetric tensor field b
$\mathcal{Z}$	D^2	Gauge fixing parameters (modulo ρ–transformations)	Non–physical parts of $\mathcal{E}$
$\mathcal{Q}$	$2D^2$	BRST ghost transformation parameters	

Table 4.2: Objects appearing in the doubled worldsheet theory

fields. Exactly D components are contained in $\mathcal{K}$, such that together with X they add up to the $2D$ components of Y.

On the side of the target space fields, in principle $\mathcal{E}$ carries $4D^2$ independent components for the kinetic and WZ terms, which make the doubled target space metric $\mathcal{G}$ and antisymmetric tensor field $\mathcal{C}$. As discussed, there are non-physical degrees of freedom in $\mathcal{E}$, that are removed by the gauge fixing parameters $\mathcal{Z}$ and the BRST ghost transformation parameters $\mathcal{Q}$. Eventually, it remains the projected object $\tilde{\mathcal{E}}$ that indeed can be identified with the D-dimensional target space metric and b-field, i.e. with $E = g + b$. The counting shows, that its D^2 components, together with $2D^2$ components in $\mathcal{Q}$ and D^2 components in $\mathcal{Z}$, remaining after taking into account ρ-transformations, exactly add up to the $4D^2$ components of the original $\mathcal{E}$.

4.4 Summary and discussion

In this chapter, a worldsheet theory with doubled coordinate fields has been developed. It reveals features of $O(D, D)$ covariance and possesses, apart from the usual diffeomorphism covariance, a set of novel symmetries. One of those can be used to rewrite the kinetic term such that also an $O(D, D)$ invariant form is available.

Half of the worldsheet degrees of freedom are rendered redundant by the assumption of a gauge symmetry in the coordinate fields, (4.37). It is fixed by a Lagrange multiplier term that affects the theory in many places. Such a gauge fixing separates the described ansatz from many worldsheet approaches in the literature where often additional constraints are introduced by hand.

An investigation of the one-loop renormalisation and the requirement of Weyl invariance have lead to the target space equations of motion for the doubled target space. Partly, they appear as generalisations of the usual standard sigma model beta-functions, partly, there appear new terms that are related to the Lagrange multiplier term.

Interestingly, there automatically appear certain projection operators at various places, in particular in the propagators of the quantum fields and in the target space equations of motion, which reduce the actual degrees of freedom in the right way.

The main results of this chapter can be summarised as:

The present model of a doubled worldsheet

- is capable of describing non-geometric configurations properly,
- can embed non-geometric fluxes in a agreement with the T-duality chain of fluxes,
- does not reproduce the double field theory equations of motion.

A few drawbacks have been identified: First of all, the description of non-geometric setups has only been exemplified for the well-known toroidal case, that was also used in the other chapters. How non-geometry may be detectable in full generality and, more importantly, describable, is not obvious at the present stage. Second, to model the T-duality chain of fluxes, the doubled antisymmetric tensor $\mathcal{C}$ has been used, although the other results were obtained for $\mathcal{E} = \mathcal{H}$, i.e. without any antisymmetric part. Furthermore, it seems in principle possible to have solutions with several types of fluxes turned on. This would add extra degrees of freedom, whose proper reduction is hidden in the formalism. A systematic study of the solution space has not been done so far. Finally, there are discrepancies in the target space interpretation compared to double field theory, such that it seems questionable whether the doubled worldsheet model makes a good basis for this theory.

To end the chapter, a few further aspects shall be discussed, also with respect to future directions of research.

Invertibiliy of $\mathcal{G}$: In the construction of the covariant derivatives and also when determining the propagators, it was assumed that the doubled metric $\mathcal{G}$ is invertible. Non-invertible metrics shall be turned into invertible ones by a suitable $\mathcal{U}$-transformation. It might seem questionable that this is a well-defined procedure, but it can be argued that this is indeed the case: One can directly determine the propagators for the non-invertible standard sigma model metric (4.71). There are two independent ones, either for the coordinate quantum fields x^μ, or mixed ones between the dual coordinates $\tilde{x}^\mu$ and v_R. These findings can be compared to the general case by parametrising an infinitesimal $\mathcal{U}$-transformation, that brings the metric to an invertible form, and taking the limit of vanishing $\mathcal{U}$ in the results. Eventually, the propagators agree in this limit.

$\mathcal{U}$-invariance: It is clear that $\mathcal{U}$-transformations can change the appearance of the equations of motion quite heavily. Though, the physical content of these equations must not change. It arises the question, whether there is a proof of invariance under $\mathcal{U}$-transformations. A first positive sign is the appearance of the projected inverse metric $\mathcal{G}_\perp^{-1}$, as it can be shown to be invariant under $\mathcal{U}$-transformations. Unfortunately, other objects are changing radically, like for example the Riemann tensor $\mathcal{R}_{ijkl}$. So far, it was not possible to find an representation of the equations of motion with $\mathcal{U}$-transformation invariant objects only. One might conclude that the whole quantisation procedure should be done in an $\mathcal{U}$-transformation covariant way.

Given the remarks about invertibility above, that implies that one has to work with other objects than the doubled metric $\mathcal{E}$.

Double field theory: On the level of the $2D$ target space equations of motion, the presented doubled worldsheet theory and double field theory disagree. Roughly speaking, the reason could be that the comparison was made before reduction constraints like the Killing equation (4.62) or the strong constraint of double field theory have been employed. It still cannot be excluded that the different theories finally agree on the physical content.

Strong constraint: In double field theory, the equations of motion have to be restricted by the strong constraint, cf. (3.14). Although it is not possible to derive this restriction from the doubled worldsheet model of this chapter, there are some similarities to the conditions found here. A very simple solution to the strong constraint is given by $\partial_{\tilde{X}} = 0$, applying to all fields and products of such. Indeed, BRST invariance implies a very similar condition for the worldsheet model: From (4.62) it follows for the kinetic term that $\partial_{\tilde{X}}\tilde{\mathcal{E}} = 0$, as was discussed in the previous section. It would have been an encouraging agreement if one found the same restriction for the other remaining field, namely the dilaton. Unfortunately, its equation of motion reduces to

$$\partial_X \Phi + E \partial_{\tilde{X}} \Phi = 0 \ , \tag{4.176}$$

for the canonical choice of $\mathcal{Z}$ and for $\mathcal{E} = \mathcal{H}$. Setting $\partial_{\tilde{X}}\Phi = 0$ would then imply a constant dilaton, which is a too severe restriction of the solution space. Ultimately, it seems that the doubled worldsheet model implies restrictions of the target space fields, but these are not equivalent to the strong constraint of double field theory.

Off-shell doubling: One could ask whether the idea of doubling the coordinate fields is going into the right direction. Similar to the approaches of Tseytlin [103, 104] and Hull [109, 26], the doubled worldsheet model postulates an off-shell doubling. It could be that this is not necessary, as for example the standard sigma model offers an on-shell variant. Left- and right-moving coordinate fields are treated independently, at which target space coordinates and dual coordinates can be constructed from the respective zero modes. It seems that double field theory is indeed more closely related to such an on-shell construction than to the off-shell doubled coordinates Y. This might offer an explanation for the discrepancy in the target space equations of motion.

Fluxes: There seems to be an ambiguity in how to describe target space fluxes. One option is to use the generalised antisymmetric tensor field $\mathcal{C}$, see the discussion on p. 144. The other option is to keep only b and use for example the covariant representation with $\mathcal{G} = \mathcal{H}$. As discussed, this second option leaves open the question of how the gauge transformations of b are realised, especially inside $\mathcal{Z}$. It seems that further investigations of the interplay between standard sigma model symmetries and the $\mathcal{U}$-transformations of the doubled worldsheet model are necessary. Additionally, it could be an option to use the Killing structure coefficients $f(Y)$ to solve this problem by embedding flux degrees of freedom there.

Conclusion

String theory offers a new microscope to examine the basic entities in nature. As always with new instruments in physics, it was expected to find something novel. Geometry itself can be probed by it, and indeed, new spectacular properties of spacetime have been discovered. It is no longer mandatory that physics takes place on the background of a Riemannian manifold. Rather, string theory symmetries relating different geometries are allowed to connect coordinate patches to each other. Such constructions have been collected under the name of non-geometry, and this thesis has shed a little more light on the mysteries of those.

Since quite some time it was suspected that non-geometries do not arise without dramatic effects on the four-dimensional effective theories. New objects were assumed to appear, denoted as non-geometric fluxes, and they seem to be helpful, for example, when constructing de Sitter vacua. Though, it was not known how to systematically describe non-geometric fluxes in dimensions higher than four, especially in the ten-dimensions of effective supergravity theories. In addition, it was unclear whether the connection between non-geometry and non-geometric fluxes really is one-to-one. Here, such a connection was worked out and a framework was provided that describes non-geometric fluxes in higher dimensions.

The present research work has revealed a couple of independent results that complement the known facts. They were discussed in detail in the respective chapters, but the most important statements shall here be phrased as:

- Non-geometry can imply non-commutativity of the coordinates, that then is sourced by non-geometric fluxes.

- Non-geometric fluxes appear in non-geometric setups, and both can be described in effective theories.

- It is possible to find a worldsheet model of string theory that manifestly incorporates non-geometry.

Quite some of the questions posed in the introduction are now answered by these statements. But there also appeared new questions, and a few of them shall be presented shortly:

Non-associativity: Non-geometry is suspected to have more dramatic effects on the target space structure in certain cases. R-flux backgrounds make such examples. The investigation of chapter 2 will not be able to deal with that kind of non-geometric flux, as it cannot be obtained from regular T-duality transformations. A direct quantisation of the coordinates using the closed string monodromies as Dirac constraints, cf. [71, 56], might offer access to non-associative coordinates.

Beyond the three-torus example: Many observations in the framework of double field theory or supergravity have been made for the particular example of the three-torus with H-flux or its T-duals. Although in some cases there are arguments for valid generalisations, it would be desirable to test the techniques developed in chapter 3 against novel non-geometric backgrounds. Possibly, the introduction of brane sources for non-geometric fluxes [92] helps to construct such backgrounds with several types of fluxes turned on at once. It then might be particularly interesting whether these can be geometrised in the sense of the present double field theory construction.

Doubled models: The proposal of chapter 4 defines a worldsheet model with doubled coordinate fields. Nevertheless, it does not reproduce double field theory as effective theory, at least not directly. As discussed, the main reason seems to be that double field theory rather arises from an on-shell doubling, whereas the proposed theory here offers an off-shell doubling that includes not only zero-modes but all string excitations and therefore is much more general. How big the agreement between both approaches actually is and which framework is suited better to study non-geometry has to be shown in further research work.

Manifolds: Non-geometry exceeds the notion of a manifold and one might ask whether there is a mathematical framework that allows to capture this fact, and whether there are quantum field theories for such a framework. A promising approach is that of Lie algebroids, studied in [129], others are: supergravity as generalised geometry [130], doubled geometry of double field theory [86, 85], or matrix models [63]. In any case, it seems that the new microscope initiated a fundamental change of how physical theories have to be formulated.

There is much to be discovered.

Peter Patalong
November 27, 2013

Appendix A

Technical review of T-duality

This section presents the sigma-model approach to T-duality for one isometry direction in the target space fields and a summary of the relation between T-duality and the group $O(D, D)$. It follows the historical approach [112, 113]. More details about various perspectives on T-duality can be found for example in [115], in [15], or in [13].

Transformation of the target space fields

Having an isometry direction in the target space fields implies that the respective action is invariant under a shift of the coordinate fields $X^\mu \to X^\mu + k^\mu$, where k^μ is a single Killing vector field. More precisely, the metric is constant under the Lie derivative in the k-direction, and the antisymmetric tensor field B at most undergoes a gauge transformation:

$$\mathcal{L}_k G = 0 \ , \quad \mathcal{L}_k B = \mathrm{d}\omega \ . \tag{A.1}$$

One can then choose adapted coordinates such that all fields are simply independent of one coordinate X^ι. This uses the gauge invariance of the action under $B \to B + \mathrm{d}\omega$.

The following calculations can be simplified by introducing light-cone coordinates, i.e. $\sigma_{L/R} = (\sigma_0 \pm \sigma_1)/\sqrt{2}$ and $\partial_{L/R} = (\partial_0 \pm \partial_1)/\sqrt{2}$. For simplicity, the prefactor of $-1/(4\pi\alpha')$ compared to 2.5 is dropped. This agrees with the conventions in chapter 4, and the standard string worldsheet action then reads

$$S = \int \mathrm{d}^2\sigma \ E_{\mu\nu}(X)\partial_L X^\mu \partial_R X^\nu \ , \tag{A.2}$$

with the target space fields packaged into $E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$.

To make visible the dual model, one first has to rewrite the action in a first order form. This can be done by gauging the above assumed isometry, i.e. by introducing a gauge connection V_a and covariant derivative in the isometry direction $\mu = \iota$,

$$D_a X^\iota = \partial_a X^\iota + V_a \ . \tag{A.3}$$

This implies that the action remains invariant under

$$X^\iota \to X^\iota - \xi \ , \quad V_a \to V_a + \partial_a \xi \ , \tag{A.4}$$

when replacing the $\mu = \iota$ partial derivative by the covariant one.

In order to keep the correct number of degrees of freedom, one has to enforce a pure gauge condition by adding a Lagrange multiplier term,

$$S_V = \int \mathrm{d}^2\sigma \; \tilde{X}(\partial_L V_R - \partial_R V_L) \; , \tag{A.5}$$

with Lagrange multiplier $\tilde{X}$. The classical equation of motion for $\tilde{X}$ then is the pure gauge condition. Therefore, the original action can be recovered by plugging this equation of motion into the gauged action and shifting X^ι appropriately, which is allowed by the isometry.

Furthermore, it is possible to obtain the T-dual action by integrating out V_a, i.e. by plugging in its equations of motion. They read

$$\begin{aligned} V_L &= +\frac{1}{G_{\iota\iota}}\partial_L\tilde{X} - \partial_L X^\iota - \frac{E_{\kappa\iota}}{G_{\iota\iota}}\partial_L X^\kappa \\ V_R &= -\frac{1}{G_{\iota\iota}}\partial_R\tilde{X} - \partial_R X^\iota - \frac{E_{\iota\kappa}}{G_{\iota\iota}}\partial_R X^\kappa \; , \end{aligned} \tag{A.6}$$

where the index convention is $\kappa, \lambda \in \{1, \ldots, D\} - \{\iota\}$. The result is

$$S' = \int \mathrm{d}^2\sigma \; \left(E_{\kappa\lambda} - \frac{E_{\iota\kappa}E_{\lambda\iota}}{G_{\iota\iota}}\right)\partial_L X^\lambda \partial_R X^\kappa + \frac{1}{G_{\iota\iota}}\partial_L\tilde{X}\partial_R\tilde{X} - \frac{E_{\kappa\iota}}{G_{\iota\iota}}\partial_L X^\kappa \partial_R\tilde{X} + \frac{E_{\iota\kappa}}{G_{\iota\iota}}\partial_L\tilde{X}\partial_R X^\kappa \; . \tag{A.7}$$

One can now interpret $\tilde{X}$ as the new coordinate field in the dualised direction $\mu = \iota$, replacing the old X^ι. Under this assumption, the action S' can be repackaged in the standard form of the original one, (A.2), by replacing the target space fields in the following way:

$$\boxed{E_{\kappa\lambda} \to E_{\kappa\lambda} - \frac{E_{\iota\kappa}E_{\lambda\iota}}{G_{\iota\iota}} \; , \quad G_{\iota\iota} \to \frac{1}{G_{\iota\iota}} \; , \quad E_{\kappa\iota} \to -\frac{E_{\kappa\iota}}{G_{\iota\iota}} \; , \quad E_{\iota\kappa} \to \frac{E_{\iota\kappa}}{G_{\iota\iota}} \; .} \tag{A.8}$$

These are the T-duality rules defining the transformation of the target space fields, which are sometimes denoted as “Buscher rules”.

At this stage it is important to note that this result is very sensitive to the conventions used. Here, the setup of chapter 4 was implemented to make clear the connection to the procedure of generalisation there. This comes with the costs of having a collision with the conventions in chapter 2. To implement those, one should have chosen $S_V \to -S_V$ in (A.5) and reversed the sign of ϵ_{ab} on the worldsheet. As a result, the last two rules in (A.8) change sign. Despite this difference, the original conventions will be used throughout the rest of this section, as they agree with the literature.

Transformation of the coordinate fields

It is possible to bring the T-duality rules into another form, that directly accesses the coordinate fields X^μ. One can use the symmetry (A.4) to gauge fix V_a to zero. The equations of motion (A.6) then read

$$\begin{aligned} \partial_L\tilde{X} &= +G_{\iota\iota}\partial_L X^\iota + E_{\kappa\iota}\partial_L X^\kappa \\ \partial_R\tilde{X} &= -G_{\iota\iota}\partial_R X^\iota - E_{\iota\kappa}\partial_R X^\kappa \; . \end{aligned} \tag{A.9}$$

Summing these, one can rewrite them for $\tau = \sigma_0$ and $\sigma = \sigma_1$

$$\boxed{\begin{aligned} \partial_\tau \tilde{X} &= G_{\iota\iota}\partial_\sigma X^\iota + G_{\iota\kappa}\partial_\sigma X^\kappa - B_{\iota\kappa}\partial_\tau X^\kappa \\ \partial_\sigma \tilde{X} &= G_{\iota\iota}\partial_\tau X^\iota + G_{\iota\kappa}\partial_\tau X^\kappa - B_{\iota\kappa}\partial_\sigma X^\kappa \ . \end{aligned}} \tag{A.10}$$

This gives a prescription of how the dual sigma model with a new coordinate field $\tilde{X}$ is obtained from the old model with a coordinate field X^ι. Again, one has to note that the respective last terms would have opposite signs for the other conventions mentioned above.

Taking seriously the reinterpretation of the dualised action, one, strictly speaking, cannot identify the remaining coordinates X^κ in the original model with the ones in the dual model. The argument relies on the structural similarity of the actions, but, for example, in the case of constant target space fields, one only has

$$\partial_\tau X^\kappa_{\text{new}} = \partial_\tau X^\kappa_{\text{old}} \ , \quad \partial_\sigma X^\kappa_{\text{new}} = \partial_\sigma X^\kappa_{\text{old}} \ , \tag{A.11}$$

as the bare coordinates X^κ do not appear. This is therefore assumed to hold in the general case, too.

Transformation of the dilaton

It should be mentioned that the full path integral treatment of the above procedure comes with some subtleties, see for example [131]. In particular, the change of variables in a Polyakov path integral formulation induces a Jacobian that after regularisation leads to a shift of the dilaton[1] (which was neglected for simplicity above),

$$\tilde{\Phi} = \Phi - \frac{1}{2}\log|G_{\iota\iota}| \ . \tag{A.12}$$

In combination with

$$|\tilde{G}| = \frac{1}{G_{\iota\iota}^2}|G| \ , \tag{A.13}$$

this leads to the invariant combination

$$e^{-2\tilde{\Phi}}\sqrt{|\tilde{G}|} = e^{-2\Phi}\sqrt{|G|} \ , \tag{A.14}$$

that will play a prominent role in double field theory.

The T-duality group $O(D, D)$

For several isometries, all possible T-duality transformations in the above sense form a group. This subsection shows[2] how to determine that group.

The idea is to consider a compactification of closed string theory[3] on a D-dimensional torus T^D such that the target space fields $G_{\mu\nu}$ and $B_{\mu\nu}$ have D isometries. For the case of maximal D, they are constant. In any case, the setup allows for a whole variety of different

[1] See [13], equation (14.16).
[2] Mainly following [13], chapter 10.
[3] The following assumes $\alpha' = 1$ for simplicity.

compactifications and it turns out that the set of all such is smaller than naively expected - it is reduced by all possible T-duality transformations.

A D-dimensional torus can be defined by claiming the following identification of points,

$$X^\mu \sim X^\mu + 2\pi n^i e_i^\mu \ , \tag{A.15}$$

which defines a lattice Λ_D generated by D linearly independent vectors $\{e_i^\mu\}$. One can then perform a mode expansion of the string coordinates and determine the center of mass position and momentum of the string. The canonical commutation relations imply, that π_μ has to lie on the dual lattice Λ_D^*,

$$\pi_\mu = G_{\mu\nu} p^\nu + B_{\mu\nu} n^i e_i^\nu = m_i e_\mu^{*i} \tag{A.16}$$

with p^ν being the zero mode momentum. In total, there are two quantised quantities: the string momentum, associated to m_i, and the string winding around the compact toroidal directions, associated to n^i.

The mode expansion allows to determine the mass formula from the associated Hamiltonian. By lifting G and B to the lattice,

$$g_{ij} = G_{\mu\nu} e_i^\mu e_j^\nu \ , \quad b_{ij} = B_{\mu\nu} e_i^\mu e_j^\nu \ , \tag{A.17}$$

and by defining

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} \ , \tag{A.18}$$

it reads

$$m^2 = K^T \mathcal{H} K + 2(N_L + N_R - 2) = p_L^2 + p_R^2 + 2(N_L + N_R - 2) \ , \tag{A.19}$$

with $K^T = (n^T, m^T)$ being the collection of the integers that determine string momentum and winding. The left- and right-moving momentum vectors are defined as

$$(p_\mu)_{L/R} = \frac{1}{\sqrt{2}} e_\mu^{*i} \left(m_i \pm g_{ij} n^j - b_{ij} n^j \right) \ . \tag{A.20}$$

The most important observation is that the mass formula can be formulated in a way such that the generalised metric appears. This is the key point for identifying the duality group that reduces the set of inequivalent compactifications. Equally important is the appearance of the level-matching condition in a very particular guise, namely

$$N_R - N_L = m_i n^i = \frac{1}{2} K^T \eta K \ , \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{A.21}$$

It can be shown that the momentum vectors $p_{L/R}$ span an even self-dual Lorentzian lattice for any choice of the background parameters g_{ij} and b_{ij}. These are D^2 parameters and each choice leads to a different lattice. Any such lattice can be obtained by an $O(D,D)$ rotation of a reference lattice, but one has to mod out the $O(D)$ invariance of each momentum vector p_L and p_R. The set of different configurations is then given by

$$\frac{O(D,D)}{O(D) \times O(D)} \tag{A.22}$$

with the expected dimension D^2. This set has to be reduced by modding out all T-duality transformations that by construction identify physically equivalent configurations. A closer inspection of the mass formula reveals discrete symmetry transformations with $K \to MK$ that leave it invariant, and the level-matching condition shows that these have to be elements of $O(D, D; \mathbb{Z})$,

$$M^T \eta M = \eta \ . \tag{A.23}$$

The particular formulation of the mass formula allows to define the covariant transformation of the background parameters g and b in the simple form

$$\mathcal{H} \to M^T \mathcal{H} M \ . \tag{A.24}$$

Eventually, some of these transformations of G and B can be reduced to the T-duality rules derived above. Proving this is non-trivial in full generality, but an example has been worked out in appendix B of [2]. The general proof can be found, for instance, in chapter 4.2 of [115]. All T-duality transformation matrices have the form[4]

$$M = \begin{pmatrix} \mathbb{1} - e_i & e_i \\ e_i & 1 - e_i \end{pmatrix} \ , \quad (e_i)_{mn} = \delta_{im}\delta_{in} \ . \tag{A.25}$$

Finally, the moduli space of the D-dimensional torus compactification for closed strings then is

$$\frac{O(D, D)}{O(D) \times O(D)} \ / \ O(D, D; \mathbb{Z}) \ . \tag{A.26}$$

To conclude, it can be stated that, in general, all T-duality transformations lie in the group $O(D, D)$, which is the continuous analogue of $O(D, D; \mathbb{Z})$, which was recovered in the toroidal case here.

[4] The notation is borrowed from [129], equation (4.41).

Appendix B

Technicalities

B.1 Notation

Object	Description	Defined in
τ, σ	Worldsheet coordinates	(2.5)
$\mathrm{d}^2\sigma = \mathrm{d}\sigma\mathrm{d}\tau$	Worldsheet measure	(2.5)
$\eta^{\alpha\beta}$	Worldsheet metric as Minkowski metric	(2.5)
$\varepsilon^{\alpha\beta}$	Worldsheet totally antisymmetric tensor	(2.5)
X^μ, X^i, $\mathcal{X}^\nu$	Coordinate fields in the standard sigma model	(2.5)
Y^μ, Z^μ	Coordinate fields for the duals of the torus with H-flux	page 24
N^μ	Winding number for the closed string mode expansion	(2.21)
$G_{\mu\nu}(X)$	Target space metric in the standard sigma model	(2.5)
$B_{\mu\nu}(X)$	Kalb-Ramond field	(2.5)
∂_α	Worldsheet derivatives	(2.5)
H_3	3-form field strength for B	
R_μ	Radii of the three-torus with H-flux	(2.8)
$\mathcal{R}$, $\mathcal{R}_{\mu\nu}$	Scalar curvature, Ricci tensor	
g_{mn}	Target space metric in effective field theories	
b_{mn}	Kalb-Ramond field in effective field theories	
$\mathcal{H}_{MN}$	Generalised metric	(3.8)
d	Double field theory dilaton	(3.43)
x, $\tilde{x}$	Ordinary and dual target space coordinates	(3.4)
∂_m, $\tilde{\partial}^m$	Target space derivative, dual derivative	(3.3)

B.2 Representation of the δ-distribution

1. The δ-distribution appearing in the canonical commutation relations of section 2.2.2 is represented as

$$\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n \in \mathbb{N}^*} \cos(nx) = \frac{1}{2\pi} \sum_{n\mathbb{Z}} e^{-inx} \ . \tag{B.1}$$

2. Its derivative can be defined as

$$f(x)\partial_x \delta(x) \equiv -\delta(x)\partial_x f(x) \tag{B.2}$$

for any function $f(x)$ with compact support. Accordingly, since $\sigma \in [0, 2\pi]$, one has

$$(\sigma' - \sigma)\ \partial_\sigma \delta(\sigma - \sigma') = \delta(\sigma - \sigma')\ . \tag{B.3}$$

3. Additionally, it holds that

$$(\sigma - \sigma')\ \delta(\sigma - \sigma') = 0\ . \tag{B.4}$$

4. For any function $u(\tau, \sigma) = \sum_{n \in \mathbb{Z}} u_n\ e^{-\mathrm{i} n \sigma_+}$ with constant coefficients u_n, one can show that

$$\sum_{k \neq 0} e^{-\mathrm{i} k(\sigma - \sigma')}(u(\tau, \sigma) - u(\tau, \sigma')) = -(u(\tau, \sigma) - u(\tau, \sigma'))\ . \tag{B.5}$$

   Using (B.1), this proves

$$\delta(\sigma - \sigma')(u(\tau, \sigma) - u(\tau, \sigma')) = 0\ , \tag{B.6}$$

   which is also valid if σ_+ in $u(\tau, \sigma)$ is exchanged for σ_-.

All relations have to be understood in the sense of distributions, i.e. they are only valid when appropriately integrated against a test function.

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