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ABSTRACT. In this work we investigate the problem of preventing the incompressible 3D Navier-Stokes from developing singularities with the control of one component of the velocity field only in L^∞ norm in times with values in a scaling invariant space. We introduce a space "almost" invariant under the action of the scaling such that if one component measured in this space remains small enough, then there is no blow up.

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1. INTRODUCTION

The purpose of this paper is the investigation of the possible behaviour of a solution of the incompressible Navier-Stokes equation in $\mathbb{R}^3$ near the (possible) blow up time. Let us recall the form of the incompressible Navier-Stokes equation

$$(NS) \qquad \begin{cases} \partial_t v + \operatorname{div}(v \otimes v) - \Delta v + \nabla p = 0\,, \\ \operatorname{div} v = 0 \quad \text{and} \quad v|_{t=0} = v_0\,, \end{cases}$$

where $v = (v^1, v^2, v^3)$ stands for the velocity of the fluid and p for the pressure.

It is well known that the system has two main properties related to its physical origin:

- the scaling invariance which the fact that if (t, x) is a solution on $[0, T]$ then for any positive real number λ, the vector field $u_\lambda(t, x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x)$ is a solution on $[0, \lambda^{-2}T]$;
- the dissipation of energy which writes

$$(1) \qquad \frac{1}{2}\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(t')\|^2_{L^2} dt' \leq \frac{1}{2}\|v_0\|^2_{L^2}.$$

The first type of results which describe the behaviour of a (regular) solution just before the blow up are those which are a consequence of existence theorem for initial data in spaces more regular than the scaling. The seminal text [?] of J. Leray already pointed out in 1934 that the life span $T^\star(u_0)$ of the regular solution associated with an initial data in the Sobolev space $H^1(\mathbb{R}^3)$ is greater than $c\|\nabla u_0\|^{-4}_{L^2}$; then applying this result with $u(t)$ as an initial data gives immediatly that, if $T^\star(u_0)$ is finite, then

$$(2) \qquad \|\nabla u(t)\|^4_{L^2} \geq \frac{c}{T^\star(u_0) - t} \quad \text{which implies that} \quad \int_0^{T^\star(u_0)} \|\nabla u(t)\|^4_{L^2} dt = \infty.$$

More generally, it is very classical result that for any γ in $]0, 1[$ we have

$$(3) \qquad T^\star(u_0) \geq c_\gamma \|u_0\|^{-\frac{1}{\gamma}}_{\dot{H}^{\frac{1}{2}+2\gamma}} \quad \text{which leads to} \quad \|u(t)\|_{\dot{H}^{\frac{1}{2}+2\gamma}} \geq \frac{c_\gamma}{(T^\star(u_0) - t)^\gamma}\,.$$

Let us notice that the formula is scaling invariant abd comes from the resolution of (NS) with a fixed point argument follwing the Kato method. Morever, E. Poulon proved in [] that a regular initila data exists which blows up at finite time then an initial data $\underline{u}_0$ exists in the unit sphere of $\dot{H}^{\frac{1}{2}+\gamma}$ such that

$$T^\star(\underline{u}_0) = \inf\bigl\{T^\star(u_0)\,,\ u_0 \in \dot{H}^{\frac{1}{2}+\gamma}\,,\ \|u_0\|_{\dot{H}^{\frac{1}{2}+\gamma}} = 1\bigr\}\,.$$

Assertion (3) can be generalized to the norm associated with the greatest space which is translation invariant, continuously included the space of tempered distribution $\mathcal{S}'(\mathbb{R}^3)$ and the norm has the same scaling as $\dot{H}^{\frac{1}{2}+2\gamma}$. As pointed out by Y. Meyer in Lemma 9 of [?], this space is the Besov space $\dot{B}^{-1+2\gamma}_{\infty,\infty}$ which can be defined as the space of distributions such that

$$\|u\|_{\dot{B}^{-1+2\gamma}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}-2\gamma}\|e^{t\Delta}u\|_{L^\infty}. \tag{4}$$

is finite. The generalization of the bound given by (3) to this norms can be done (see for instance Theorem1.3 of [?]) we recall here

Theorem 1.1. *For any γ in the interval $]0,1/2[$, a constant c_γ exists such that for any regular initial data u_0, its life span $T^\star(u_0)$ satisfies*

$$T^\star(u_0) \geq c_\gamma \|u_0\|^{-\frac{1}{\gamma}}_{\dot{B}^{-1+2\gamma}_{\infty,\infty}} \quad \textit{which leads to} \quad \|u(t)\|_{\dot{B}^{-1+2\gamma}_{\infty,\infty}} \geq \frac{c_\gamma}{(T^\star(u_0)-t)^\gamma}\,\cdot \tag{5}$$

This result as an analog for a global regularity uder the smallness condition which is the Koch and Tataru theorem (see [?]) which claims that an initial data which have a small norm in the space $BMO^{-1}(\mathbb{R}^3)$ generates a global unique solution (which turns out to be as regular as the initial data). The space $BMO^{-1}(\mathbb{R}^3)$ is a very slightly bigger space than $\dot{B}^{-1}_{\infty,2}(\mathbb{R}^3)$ defined by

$$\|u\|^2_{\dot{B}^{-1}_{\infty,2}} \stackrel{\text{def}}{=} \int_0^\infty \|e^{t\Delta}u\|_{L^\infty} dt < \infty \tag{6}$$

and very slightly smaller than the space $\dot{B}^{-1}_{\infty,\infty}$. Let us notice that classical space $\dot{H}^{\frac{1}{2}}$ of $L^3(\mathbb{R}^3)$ are continuous embedded in $BMO^{-1}(\mathbb{R}^3)$.

Let us point out that the proof of all these results do not use the special structure of (NS) and in particular all the above results are true for any systems of the type

$$(GNS) \quad \partial_t u - \Delta u + \sum_{i,j} A_{i,j}(D)(u^i u^j)$$

where $A_{i,j}(D)$ are smooth homogenenous Fourier multipliers of order 1. The problem investigated here is to improve the description of the behavior of the solution near a possible blow up using the special structure of the non linear term of the Navier-Stokes equation.

One major achievement in this field is the work [?] by L. Escauriaza, G. Serëgin and V. Sverak which proves that

$$T^\star(u_0) < \infty \Longrightarrow \limsup_{t\to T^\star(u_0)} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = \infty. \tag{7}$$

A different context consists in formulating a condition which involves only one component of the velocity field. The first result in that direction is obtained in a pioneer work by J. Neustupa and P. Penel (see [?]) but the norm involved was not scaling invariant. A lot of works (see [?, 4, ?, ?, ?, ?, ?, ?, ?]) established conditions of the type

$$\int_0^{T^\star} \|v^3(t,\cdot)\|^p_{L^q} dt = \infty \quad \text{or} \quad \int_0^{T^\star} \|\partial_j v^3(t,\cdot)\|^p_{L^q} dt = \infty$$

with relations on p and q which do not make these quantities scaling invariant.

The first result in that direction using scaling invariant condition has been proved by the first and the third author in [6]. It claims that for any regular intial data with derivative in $L^{\frac{3}{2}}(\mathbb{R}^3)$, then for any unit vector σ of $\mathbb{R}^3$, we have

$$
(8) \qquad T^\star < \infty \Longrightarrow \int_0^{T^\star} \|v(t)\cdot\sigma\|^p_{\dot H^{\frac12+\frac2p}}\, dt = \infty\,.
$$

for any p in the interval $]4,6[$. It has been generalized by Z. Zhang and the first and the third author for any p greater than 4 in [8]. This is the analog of the integral condition of (2) for only a component.

The motivation of the issue raised in this paper is what happens for the above criteria for $p=\infty$? in other term, it is possible to extend L. Escauriaza, G. Serëgin and V. Sverak criteria (7) only for on component. This question sems to ambitious for the time being. Indeed, following the work [**?**] by G. Koch, F. Planchon and the second author[1] one way to understand L. Escauriaza, G. Serëgin and V. Sverak in the following : assume that a solution exists such that the $\dot H^{\frac12}$ norm remains bounded near the blow up time. The first step consists in proving that the solution tends weakly to 0 when t tends to the blow up time. The second step consists in proving a backforward uniqueness result which implies that the solution is 0 which of course contradicts the fact that its blows up at finite time.

The first step relies in particular on the fact that the Navier-Stokes system (NS) is globally wellposed for small data in $\dot H^{\frac12}$. In our context, the equivalent is that if $\|v_0\cdot\sigma\|_{\dot H^{\frac12}}$ is small enough for some unit vector σ of $\mathbb{R}^3$, then there is a global regular solution. Such a result, assuming it is true, seems out of reach for the time being.

The result we prove in this paper is that if there is a blow up, it is not possible that a component of the velocity field tends to 0 to fast. More precisely, we are going to prove the following theorem.

Theorem 1.2. *A positive constant c_0 exists so that for any initial data v_0 in $H^1(\mathbb{R}^3)$ with associate solution v of (NS) blowing up at a finite time $T^\star$, for any unit vector σ of $\mathbb{R}^3$, there holds*

$$
\forall t < T^\star\,,\qquad \sup_{t'\in[t,T^\star[} \|v(t')\cdot\sigma\|_{\dot H^{\frac12}} \geq c_0 \log^{-\frac12}\Big(e+\frac{\|v(t)\|^4_{L^2}}{T^\star-t}\Big)\cdot
$$

The other result we prove here in that if we reinforce slightly the $\dot H^{\frac12}$ norm remains (almost) scaling invariant.

Definition 1.1. *Let $E>0$ be given, let us define $\dot H^{\frac12}_{\log,E}$ the space of distributions a in the homogeneous space $\dot H^{\frac12}(\mathbb{R}^3)$ such that*

$$
\|a\|^2_{\dot H^{\frac12}_{\log,E}} \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi|\log^2\big(E|\xi|+e\big)|\widehat a(\xi)|^2 d\xi < \infty\,.
$$

Our theorem is the following.

Theorem 1.3. *Let $E>0$ be given. A positive constant c_0 exists such that if v the solution of (NS) associated with an initial data v_0 belongs to $H^1(\mathbb{R}^3)$, and the maximal time of existence $T^\star(v_0)$ is finite, then*

$$
\forall \sigma\in\mathbb{S}^2\,,\ \limsup_{t\to T^\star(v_0)} \|v(t)\cdot\sigma\|_{\dot H^{\frac12}_{\log},E} \geq c_0\,.
$$

`Take care of the constant` E

Let us make some comments about the result and in particular about the $\|\cdot\|\log,E$ norm. It is possible to bound the life span by this norm is the following way.

[1]See also [**?**] for a more elementary approach

Theorem 1.4. *Let $E>0$ be given. Let v_0 be an initial data in $H^{\frac12}_{\log,E}$. Then the maximal time of existence $T^\star$ of the solution v to (NS) in the space $C[0,T^\star[;H^{\frac12})$ satisfies*

$$
T^\star \geq cE^2\exp\bigl(-C\|v_0\|_{\dot H^{\frac12}_{\log,E}}\bigr) \stackrel{\rm def}{=} T(E). \tag{9}
$$

Morever, we have, if $T^\star$ is finite

$$
\|v(t)\|_{\dot H^{\frac12}_{\log,E}} \geq c\log\Bigl(\frac{E^2}{T^\star-t}\Bigr)\cdot
$$

Proof. It is wellknown that a criteria having regular solution of (NS) up a time T is that

$$
I(T) \stackrel{\rm def}{=} \int_0^T\int_{\mathbb{R}^3} e^{-t|\xi|^2}|\xi|^3|\widehat v_0(\xi)|^2 d\xi dt = \varepsilon \quad \text{with} \quad \varepsilon << 1. \tag{10}
$$

For some parameter λ which will be choosen later on, let write that

$$
\begin{aligned}
I(T) &\leq \frac{1}{\log^2\Bigl(\frac{\lambda}{\sqrt T}+e\Bigr)}\int_0^T\int_{\sqrt T|\xi|\geq\lambda} e^{-t|\xi|^2}|\xi|^3\log^2(|\xi|E+e)|\widehat v_0(\xi)|^2 d\xi dt\\
&\qquad\qquad\qquad\qquad + \int_0^T\int_{\sqrt T|\xi|\leq\lambda} e^{-t|\xi|^2}|\xi|^3|\widehat v_0(\xi)|^2 d\xi dt\\
&\leq \frac{\|v_0\|^2_{\dot H^{\frac12}_{\log,E}}}{\log^2\Bigl(\frac{\lambda}{\sqrt T}+e\Bigr)} + \lambda\int_{\mathbb{R}^3}\Bigl(\int_0^\infty e^{-t|\xi|^2}\frac{1}{t^{\frac12}|\xi|}|\xi|^2dt\Bigr)|\xi|\,|\widehat v_0(\xi)|^2 d\xi dt\\
&\leq \frac{\|v_0\|^2_{\dot H^{\frac12}_{\log,E}}}{\log^2\Bigl(\frac{\lambda}{\sqrt T}+e\Bigr)} + \lambda\|v_0\|^2_{\dot H^{\frac12}}\cdot
\end{aligned}
$$

Choosing

$$
\lambda = \frac{\varepsilon}{2\|v_0\|^2_{\dot H^{\frac12}}} \quad \text{and} \quad T=\frac{\varepsilon}{2}E\|v_0\|^{-2}_{\dot H^{\frac12}}\exp\bigl(-C\|v_0\|_{\dot H^{\frac12}_{\log,E}}\bigr).
$$

ensures Condition (13) once observed that as $\|v_0\|^2_{\dot H^{\frac12}_{\log,E}}$ is greater than $\|v_0\|^2_{\dot H^{\frac12}}$ (and thus large) it is not relevant in the above formula defining T. □

`Explain the reason of this shifting`

The structure of the paper is the following:
in the first section, we reduce the proof of Theorem 1.2 and Theorem 1.3 to the proof of two lemmas related of the estimation of expression of the type

$$
\int_{\mathbb{R}^3}\partial_i v^j(x)\partial_3 v^j(x)\partial_i v^j(x)dx
$$

where i is in $\{1,2\}$. These expressions show up when we do L^2 energy estimates for $\nabla_{\rm h}v$. Here we face the difficulty that we cannot control $\partial_3 v$ in any sense using only $\|\nabla v\|_{L^2}$ and $\|\nabla\partial_3 v\|_{L^2}$.

Before going on, let us introduce some notation that will be used in all that follows. By $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. We denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^3)$ inner product of a and b. $L^p_T(L^q_{\rm h}(L^r_{\rm v}))$ stands for the space $L^p([0,T];L^q(\mathbb{R}_{x_{\rm h}};L^r(\mathbb{R}_{x_3})))$ with $x_{\rm h}=(x_1,x_2)$, and $\nabla_{\rm h}=(\partial_{x_1},\partial_{x_2})$,

$\Delta_{\rm h}=\partial_{x_1}^2+\partial_{x_2}^2$. Finally, we always denote $\big(c_{k,\ell}\big)_{k,\ell\in\mathbb{Z}^2}$ (resp. $(c_j)_{j\in\mathbb{Z}}$) to be a generic element of $\ell^2(\mathbb{Z}^2)$ (resp.$\ell^2(\mathbb{Z})$) so that $\displaystyle\sum_{k,\ell\in\mathbb{Z}^2} c_{k,\ell}^2=1$ (resp. $\displaystyle\sum_{j\in\mathbb{Z}} c_j^2=1$).

2. The life span of (NS) computed with log-type norms

The goal of this section is to present the proof of Theorem 1.4.

Proof of Theorem 1.4. We simply prove an *a priori* estimate for a regular solution v of (NS) on $[0,T^\star[$ and skip the classical regularization process. Let us define

$$\overline{T}\stackrel{\rm def}{=}\sup\Big\{T<T^\star\ /\ \forall\ t\leq T\,,\ \|v(t)\|_{\dot H^{\frac12}_{\log,E}}\leq 2\|v_0\|_{\dot H^{\frac12}_{\log,E}}\Big\}\,, \tag{11}$$

and also $v_{\sharp,\Lambda}\stackrel{\rm def}{=}\mathcal{F}^{-1}\big(\mathbf{1}_{\{E|\xi|\geq\Lambda\}}\widehat{v}(\xi)\big)$ and $v_{\flat,\Lambda}\stackrel{\rm def}{=}v-v_{\sharp,\Lambda}$ for some positive constant Λ to be chosen later on. Then we observe from Definition 1.1 that

$$\begin{aligned}\|v_{\sharp,\Lambda}\|_{\dot H^{\frac12}}^2\leq&\int_{\{E|\xi|\geq\Lambda\}}\frac{1}{\log^2\big(E|\xi|+e\big)}|\xi|\log^2\big(E|\xi|+e\big)|\widehat{v}(\xi)|^2d\xi\\ \leq&\frac{1}{\log^2\Lambda}\int_{\{E|\xi|\geq\Lambda\}}|\xi|\log^2\big(E|\xi|+e\big)|\widehat{v}(\xi)|^2d\xi\\ \leq&\frac{1}{\log^2\Lambda}\|v\|_{\dot H^{\frac12}_{\log,E}}^2\,.\end{aligned} \tag{12}$$

Let us proceed to the energy type estimate for (NS) in the $\|\cdot\|_{\dot H^{\frac12}_{\log,E}}$ norm. This gives

$$\frac12\frac{d}{dt}\|v(t)\|_{\dot H^{\frac12}_{\log,E}}^2+\|\nabla v(t)\|_{\dot H^{\frac12}_{\log,E}}^2=-(v\cdot\nabla v|v)_{\dot H^{\frac12}_{\log,E}}\,. \tag{13}$$

We claim that

$$\big|(v\cdot\nabla v|v)_{\dot H^{\frac12}_{\log,E}}\big|\leq C\min\Big\{\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}^2\,,\ \|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\Big\}\,. \tag{14}$$

Assuming (14) to be true, then using Bernstein's inequality (see [1, Lemma 2.1], or Lemma A.1 for an anisotropic version used later in this paper), we infer from (11) and (12) that, for any t less than $\overline{T}$,

$$\begin{aligned}\big|(v\cdot\nabla v|v)_{\dot H^{\frac12}_{\log,E}}\big|\ \leq\ & C\big(\|v_{\sharp,\Lambda}\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}^2+\|v_{\flat,\Lambda}\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\big)\\ \leq\ &\big(C\|v\|_{\dot H^{\frac12}_{\log,E}}(\log\Lambda)^{-1}+\frac14\big)\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}^2+C\|v_{\flat,\Lambda}\|_{L^\infty}^2\|v\|_{\dot H^{\frac12}_{\log,E}}^2\\ \leq\ &\big(C\|v_0\|_{\dot H^{\frac12}_{\log,E}}(\log\Lambda)^{-1}+\frac14\big)\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}^2+\frac{C_0\Lambda^3}{E^3}\|v\|_{L^2}^2\|v\|_{\dot H^{\frac12}_{\log,E}}^2\,.\end{aligned}$$

Now let us choose

$$\Lambda=\exp\big(C_1\|v_0\|_{\dot H^{\frac12}_{\log,E}}\big)$$

for some large enough constant C_1. We deduce from the conservation of energy and from (13) that for any t less than $\overline{T}$,

$$\frac{d}{dt}\|v(t)\|_{\dot H^{\frac12}_{\log,E}}^2+\|\nabla v(t)\|_{\dot H^{\frac12}_{\log,E}}^2\leq\frac{C_0}{E^3}\exp\big(3C_1\|v_0\|_{\dot H^{\frac12}_{\log,E}}\big)\|v_0\|_{L^2}^2\|v(t)\|_{\dot H^{\frac12}_{\log,E}}^2\,.$$

Then Gronwall's Lemma implies that for any t less than $\overline{T}$,

$$\|v(t)\|^2_{\dot H^{\frac12}_{\log,E}} \leq \|v_0\|^2_{\dot H^{\frac12}_{\log,E}} \exp\Big(C_0 \frac{t\|v_0\|^2_{L^2}}{E^3} \exp\big(3C_1\|v_0\|_{\dot H^{\frac12}_{\log,E}}\big)\Big).$$

This ensures (9).

Now let us prove (**??**). In view of (9), we have

$$\frac{d}{dE}T(E) = T(E)\Big(\frac{3}{E} - C\frac{d}{dE}\|v_0\|_{\dot H^{\frac12}_{\log,E}}\Big)\,.$$

And we observe from Definition 1.1 that

$$\begin{aligned}\frac{d}{dE}\|v_0\|_{\dot H^{\frac12}_{\log,E}} &= \|v_0\|^{-1}_{\dot H^{\frac12}_{\log,E}} \int_{\mathbb{R}^3} \frac{|\xi|^2\log(E|\xi|+e)}{E|\xi|+e} |\widehat{v}_0(\xi)|^2\,d\xi\\ &\leq \frac{1}{E}\|v_0\|^{-1}_{\dot H^{\frac12}_{\log,E}} \int_{\mathbb{R}^3} |\xi|\log(E|\xi|+e)|\widehat{v}_0(\xi)|^2\,d\xi\\ &\leq \frac{\|v_0\|_{\dot H^{\frac12}}}{E}\,,\end{aligned}$$

since $\|v(t)\|_{\dot H^{\frac12}} \leq \|v(t)\|_{\dot H^{\frac12}_{\log,E}}$, so this gives rise to

$$\frac{d}{dt}T(E) \geq \big(3 - C\|v_0\|_{\dot H^{\frac12}}\big)\frac{T(E)}{E}\,.$$

Then under the assumption that $\|v_0\|_{\dot H^{\frac12}} \leq \frac{2}{C}$ we obtain

$$\frac{d}{dt}T(E) \geq \frac{T(E)}{E}\,,$$

so that $T(E) \to \infty$ as $E \to \infty$.

To conclude the proof of the theorem we use again the fact that $\|v(t)\|_{\dot H^{\frac12}} \leq \|v(t)\|_{\dot H^{\frac12}_{\log,E}}$, so we deduce from (13) and (14) that

$$\frac{d}{dt}\|v(t)\|^2_{\dot H^{\frac12}_{\log,E}} + \|\nabla v(t)\|^2_{\dot H^{\frac12}_{\log,E}} \leq C\|v(t)\|_{\dot H^{\frac12}_{\log,E}}\|\nabla v(t)\|^2_{\dot H^{\frac12}_{\log,E}}\,,$$

which implies that for $t \leq \overline{T}$

$$\frac{d}{dt}\|v(t)\|^2_{\dot H^{\frac12}_{\log,E}} + \big(1 - 2C\|v_0\|_{\dot H^{\frac12}_{\log,E}}\big)\|\nabla v(t)\|^2_{\dot H^{\frac12}_{\log,E}} \leq 0\,.$$

Thus in particular if $\|v_0\|_{\dot H^{\frac12}_{\log,E}} \leq c_0 \leq \frac{1}{4C}$, we achieve

$$\|v(t)\|_{\dot H^{\frac12}_{\log,E}} \leq \|v_0\|_{\dot H^{\frac12}_{\log,E}} \quad \text{for} \quad t \leq \overline{T}\,.$$

This in turn shows that $\overline{T} = T^\star = \infty$. The theorem is proved, up to the proof of (14).

Proof of (14). We observe from Definition 1.1 that

$$a \in \dot H^{\frac12}_{\log,E} \Longrightarrow \|\Delta_j a\|_{L^2} \lesssim c_j 2^{-\frac{j}{2}}\log^{-1}\big(E2^j + e\big)\|a\|_{\dot H^{\frac12}_{\log,E}}\,. \tag{15}$$

Applying Bony's decomposition (A.2) to $v\cdot\nabla v$ gives

$$v\cdot\nabla v = \sum_{k=1}^3 \Big(T_{v^k}\partial_k v + T_{\partial_k v}v^k + R(v^k,\partial_k v)\Big)\,.$$

Considering the support of the Fourier transform of the terms in $T_{v^k}\partial_k v$ and due to the fact that $\|S_j v\|_{L^\infty} \lesssim c_j 2^j \|v\|_{\dot H^{\frac12}}$, we have

$$\begin{aligned}
\|\Delta_j T_{v^k}\partial_k v\|_{L^2} &\lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}v\|_{L^\infty}\|\Delta_{j'}\nabla v\|_{L^2}\\
&\lesssim \sum_{|j'-j|\le 4} c_{j'}2^{\frac{j'}{2}}\log^{-1}\big(E2^{j'}+e\big)\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\\
&\lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}.
\end{aligned}$$

Similarly noting that $\|S_j v\|_{L^\infty} \lesssim \|v\|_{L^\infty}$, we find

$$\begin{aligned}
\|\Delta_j T_{\partial_k v}v^k\|_{L^2} &\lesssim \sum_{|j'-j|\le 4} c_{j'}2^{\frac{j'}{2}}\log^{-1}\big(E2^{j'}+e\big)\|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\\
&\lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}.
\end{aligned}$$

Along the same lines, we have

$$\|\Delta_j T_{\partial_k v}v^k\|_{L^2} \lesssim \sum_{|j'-j|\le 4}\|S_{j'-1}\nabla v\|_{L^\infty}\|\Delta_{j'}v\|_{L^2}.$$

Then due to $\|S_j \nabla v\|_{L^\infty} \lesssim c_j 2^{2j}\|v\|_{\dot H^{\frac12}}$ and $\|S_j\nabla v\|_{L^\infty} \lesssim c_j 2^j \|v\|_{L^\infty}$, by applying Bernstein's inequality, we obtain

$$\|\Delta_j T_{\partial_k v}v^k\|_{L^2} \lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\min\Big(\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}, \|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\Big).$$

Finally, we get, by applying Bernstein's inequality again, that

$$\begin{aligned}
\|\Delta_j R(v^k,\partial_k v)\|_{L^2} &\lesssim 2^{\frac{3j}{2}}\sum_{j'\ge j-3}\|\Delta_{j'}v\|_{L^2}\|\widetilde\Delta_{j'}\nabla v\|_{L^2}\\
&\lesssim 2^{\frac{3j}{2}}\sum_{j'\ge j-3} c_{j'}2^{-j'}\log^{-1}\big(E2^{j'}+e\big)\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\\
&\lesssim 2^{\frac{3j}{2}}\log^{-1}\big(E2^j+e\big)\sum_{j'\ge j-N_0} c_{j'}2^{-j'}\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\\
&\lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}.
\end{aligned}$$

On the other hand due to $\operatorname{div} v = 0$, one has $\sum_k R(v^k,\partial_k v) = \operatorname{div} R(v,v)$ so

$$\begin{aligned}
\|\Delta_j \sum_k R(v^k,\partial_k v)\|_{L^2} &\lesssim 2^j \sum_{j'\ge j-N_0}\|\Delta_{j'}v\|_{L^\infty}\|\widetilde\Delta_{j'}v\|_{L^2}\\
&\lesssim 2^j\sum_{j'\ge j-N_0} c_{j'}2^{-\frac{j'}{2}}\log^{-1}\big(E2^{j'}+e\big)\|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\\
&\lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}.
\end{aligned}$$

As a result, it turns out that

$$\|\Delta_j(v\cdot\nabla v)\|_{L^2} \lesssim c_j 2^{\frac{j}{2}}\log^{-1}\big(E2^j+e\big)\min\Big(\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}, \|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\Big). \tag{16}$$

Let us now return to the proof of (14). Observing that

$$(v\cdot\nabla v|v)_{\dot H^{\frac12}_{\log,E}} \sim \sum_{j\in\mathbb{Z}} 2^j \log^2\bigl(E2^j+e\bigr)\bigl(\Delta_j(v\cdot\nabla v)|\Delta_j v\bigr)_{L^2}\,,$$

from which, using also (16), we deduce that

$$\begin{aligned}\bigl|(v\cdot\nabla v|v)_{\dot H^{\frac12}_{\log,E}}\bigr| &\leq \sum_{j\in\mathbb{Z}} 2^j\log^2\bigl(E2^j+e\bigr)\|\Delta_j(v\cdot\nabla v)\|_{L^2}\|\Delta_j v\|_{L^2}\\ &\lesssim \sum_{j\in\mathbb{Z}} c_j^2 \min\Bigl(\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}, \|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\Bigr)\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\\ &\lesssim \min\Bigl(\|v\|_{\dot H^{\frac12}}\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}, \|v\|_{L^\infty}\|v\|_{\dot H^{\frac12}_{\log,E}}\Bigr)\|\nabla v\|_{\dot H^{\frac12}_{\log,E}}\,.\end{aligned}$$

This completes the proof of (14), hence of the theorem. □

3. Anisotropic lower bounds at blow up

The goal of this section is to present the proof of Theorems 1.2 and 1.3. We choose $\sigma = (0,0,1)$ in what follows. Let us define

$$I(v^{\rm h},v^3) \stackrel{\rm def}{=} \sum_{i,m=1}^{2}\int_{\mathbb{R}^3}\partial_i v^3\partial_3 v^m(x)\partial_i v^m(x)dx\,. \tag{17}$$

Let us state the two main lemmas leading to the theorem.[2]

Lemma 3.1. *For any positive constant E_0, we have that*

$$I(v^{\rm h},v^3)\lesssim \|v^3\|_{\dot H^{\frac12}}\biggl(\log^{\frac12}\Bigl(\frac{\|\nabla v^3\|_{L^2}^2 E_0}{\|v^3\|^4_{\dot H^{\frac12}}}+e\Bigr)\|\nabla_{\rm h} v\|^2_{\dot H^1}+\frac{\|\partial_3 v^{\rm h}\|^2_{L^2}}{E_0}\|\nabla v^3\|^2_{L^2}\biggr)\,. \tag{18}$$

Lemma 3.2. *Let $E_0\sim\|v_0\|^2_{L^2}$, we have that*

$$I(v^{\rm h},v^3)\lesssim \|v^3\|_{\dot H^{\frac12}_{\log}}\|\nabla_{\rm h}v^{\rm h}\|^2_{\dot H^1}+\|v^3\|_{\dot H^{\frac12}}\frac{\|\partial_3 v^{\rm h}\|^2_{L^2}}{E_0^2}\,. \tag{19}$$

We admit these lemmas for the time being.

Proof of Theorem 1.2. As in [4], we perform L^2 energy estimate in the momentum equation of (NS) with $-\Delta_{\rm h} v$. This can be interpreted as a $\dot H^1$ energy estimate for the horizontal

[2] Here E_0, a scaling parameter which will be chosen later, is homogeneous to a kinetic energy.

variables. Indeed we have

$$
\frac{1}{2}\frac{d}{dt}\|\nabla_{\rm h}v\|_{L^2}^2+\|\nabla_{\rm h}v\|_{\dot H^1}^2=\sum_{j=1}^3\mathcal{E}_j(v)\quad\text{with}
$$

$$
\begin{aligned}
&\mathcal{E}_1(v)\stackrel{\rm def}{=}-\sum_{i=1}^2\bigl(\partial_iv^{\rm h}\cdot\nabla_{\rm h}v^{\rm h}\big|\partial_iv^{\rm h}\bigr)_{L^2}\,,\\
&\mathcal{E}_2(v)\stackrel{\rm def}{=}-\sum_{i=1}^2\bigl(\partial_iv^{\rm h}\cdot\nabla_{\rm h}v^3\big|\partial_iv^3\bigr)_{L^2}\,,\\
&\mathcal{E}_3(v)\stackrel{\rm def}{=}-\sum_{i=1}^2(\partial_iv^3\partial_3v^{\rm h}\big|\partial_iv^{\rm h})_{L^2}\quad\text{and}\\
&\mathcal{E}_4(v)\stackrel{\rm def}{=}-\sum_{i=1}^2(\partial_iv^3\partial_3v^3\big|\partial_iv^3)_{L^2}\,.
\end{aligned}
\tag{20}
$$

Let $\mathrm{div}_{\rm h}\, v^{\rm h}\stackrel{\rm def}{=}\partial_1v^1+\partial_2v^2$. A direct computation shows that

$$
\mathcal{E}_1(v)=-\int_{\mathbb{R}^3}\mathrm{div}_{\rm h}\, v^{\rm h}\Bigl(\sum_{i,j=1}^2(\partial_iv^j)^2+\partial_1v^2\partial_2v^1-\partial_1v^1\partial_2v^2\Bigr)\,dx\,,
$$

which together with $\mathrm{div}\, v=0$ ensure that

$$
\mathcal{E}_1(v)=\int_{\mathbb{R}^3}\partial_3v^3\Bigl(\sum_{i,j=1}^2(\partial_iv^j)^2+\partial_1v^2\partial_2v^1-\partial_1v^1\partial_2v^2\Bigr)\,dx\,.
$$

Then it follows from the laws of product in Besov spaces (see [1]) that

$$
\begin{aligned}
|\mathcal{E}_1(v)|&\lesssim\|\partial_3v^3\|_{\dot B^{-\frac12}_{2,\infty}}\|(\nabla_{\rm h}v^{\rm h})^2\|_{\dot B^{\frac12}_{2,1}}\\
&\lesssim\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2\,.
\end{aligned}
\tag{21}
$$

Similarly, we have

$$
\begin{aligned}
|\mathcal{E}_2(v)|&\lesssim\|\nabla_{\rm h}v^3\|_{\dot B^{-\frac12}_{2,\infty}}\|\nabla_{\rm h}v^{\rm h}\cdot\nabla_hv^3\|_{\dot B^{\frac12}_{2,1}}\\
&\lesssim\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}\|\nabla_{\rm h}v^3\|_{\dot H^1}\,,
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
|\mathcal{E}_4(v)|&\lesssim\|\partial_3v^3\|_{\dot B^{-\frac12}_{2,\infty}}\|(\nabla_{\rm h}v^3)^2\|_{\dot B^{\frac12}_{2,1}}\\
&\lesssim\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^3\|_{\dot H^1}^2\,.
\end{aligned}
\tag{23}
$$

It remains to handle the estimate of $\mathcal{E}_3(v)$. It follows from Lemma 3.1 that

$$
\mathcal{E}_3(v)\lesssim\|v^3\|_{\dot H^{\frac12}}\biggl(\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h}v\|_{L^2}^2E_0}{\|v^3\|_{\dot H^{\frac12}}^4}+e\Bigr)\|\nabla_{\rm h}v\|_{\dot H^1}^2+\frac{\|\partial_3v^{\rm h}\|_{L^2}^2}{E_0}\|\nabla v^3\|_{L^2}^2\biggr)\,.
\tag{24}
$$

Then by inserting (21-24) into (20) gives rise to

$$
\begin{aligned}
\frac12\frac{d}{dt}\|\nabla_{\rm h}v\|_{L^2}^2+\|\nabla_{\rm h}v\|_{\dot H^1}^2\lesssim\|v^3\|_{\dot H^{\frac12}}\biggl(&\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h}v\|_{L^2}^2E_0}{\|v^3\|_{\dot H^{\frac12}}^4}+e\Bigr)\|\nabla_{\rm h}v\|_{\dot H^1}^2\\
&+\frac{\|\partial_3v^{\rm h}\|_{L^2}^2}{E_0}\|\nabla v^3\|_{L^2}^2\biggr)\,.
\end{aligned}
\tag{25}
$$

Now, a positive real number m being given, let us consider $\widetilde{T} < T^\star$ and $T_\star$ such that

$$
(26) \qquad \sup_{t\in[0,\widetilde{T}[} \|v^3(t)\|_{\dot H^{\frac12}} \leq m \quad \text{and} \quad T_\star \stackrel{\rm def}{=} \sup\bigl\{t\in[0,\widetilde{T}[\,/\ \|\nabla_{\rm h} v\|^2_{L^\infty([0,t];L^2)} \leq 2\|\nabla_{\rm h} v_0\|^2_{L^2}\bigr\}\,.
$$

Note that $\operatorname{div} v = 0$, we have

$$
(27) \qquad \begin{aligned} \|\nabla v^3(t)\|^2_{L^2} =& \|\nabla_{\rm h} v^3(t)\|^2_{L^2} + \|\partial_3 v^3(t)\|^2_{L^2} \\ =& \|\nabla_{\rm h} v^3(t)\|^2_{L^2} + \|\operatorname{div}_{\rm h} v^{\rm h}(t)\|^2_{L^2} \leq 2\|\nabla_{\rm h} v_0\|^2_{L^2} \ \ \forall t\leq T_\star\,. \end{aligned}
$$

Then for any t less than $T_\star$, Inequality (25) writes

$$
(28) \qquad \begin{aligned} &\frac{d}{dt}\|\nabla_{\rm h} v\|^2_{L^2} + 2\|\nabla_{\rm h} v\|^2_{\dot H^1} \\ &\leq C_0 m\biggl(\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}E_0}{m^4}+e\Bigr)\|\nabla_{\rm h} v\|^2_{\dot H^1} + \frac{\|\partial_3 v^{\rm h}\|^2_{L^2}}{E_0}\|\nabla_{\rm h} v_0\|^2_{L^2}\biggr)\,. \end{aligned}
$$

So that by time integration of (28) from $[0,t]$ and using the L^2 energy estimate on v, we infer that, for any t less than $T_\star$,

$$
\begin{aligned} \|\nabla_{\rm h} v(t)\|^2_{L^2} \ &+\ 2\int_0^t \|\nabla_{\rm h} v(t')\|^2_{\dot H^1}dt' \\ &\leq\ \|\nabla_{\rm h} v_0\|^2_{L^2} + C_0 m\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}E_0}{m^4}+e\Bigr)\int_0^t \|\nabla_{\rm h} v(t')\|^2_{\dot H^1}dt' \\ &\qquad\qquad + C_0 m\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}}{E_0}\int_0^t \|\partial_3 v^{\rm h}(t')\|^2_{L^2}dt' \\ &\leq\ \|\nabla_{\rm h} v_0\|^2_{L^2} + C_0 m\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}E_0}{m^4}+e\Bigr)\int_0^t \|\nabla_{\rm h} v(t')\|^2_{\dot H^1}dt' \\ &\qquad\qquad + C_0 m\frac{\|v_0\|^2_{L^2}\|\nabla_{\rm h} v_0\|^2_{L^2}}{E_0}\,\cdot \end{aligned}
$$

Choosing $E_0 = \dfrac{\|v_0\|^2_{L^2}}{2C_0 m}$ in the above inequality implies that

$$
\begin{aligned} &\|\nabla_{\rm h} v(t)\|^2_{L^2} + \int_0^t \|\nabla_{\rm h} v(t')\|^2_{\dot H^1}dt' \\ &\qquad \leq \frac32\|\nabla_{\rm h} v_0\|^2_{L^2} + C_1 m\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}\|v_0\|^2_{L^2}}{m^5}+e\Bigr)\int_0^t \|\nabla_{\rm h} v(t')\|^2_{\dot H^1}dt'\,. \end{aligned}
$$

Let us assume that

$$
C_1 m\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}\|v_0\|^2_{L^2}}{m^5}+e\Bigr) \leq \frac12\,\cdot
$$

This implies that m is smaller than $1/2C_1$ and thus that

$$
(29) \qquad m\log^{\frac12}\Bigl(\frac{\|\nabla_{\rm h} v_0\|^2_{L^2}\|v_0\|^2_{L^2}}{c}+e\Bigr) \leq \frac{1}{2C_2}\,\cdot
$$

Then we infer that, for any t less than $T_\star$,

$$
\|\nabla_{\rm h} v(t)\|^2_{L^2} \leq \frac32\|\nabla_{\rm h} v_0\|^2_{L^2}\,.
$$

This implies that $T_\star = \widetilde{T}$ and thus that $\widetilde{T}$ is less than the blow-up time $T^\star$. Moreover, thanks to (27) and Theorem 1.4 of [6], the solution $v(t)$ is smooth for $t\leq\widetilde{T}$. By contraposition and (29), this implies that

$$\sup_{t\in[0,T^\star[} \|v^3(t)\|_{\dot H^{\frac12}} \geq c_1 \log^{-\frac12}\Big(\frac{\|\nabla_{\rm h} v_0\|_{L^2}^2\|v_0\|_{L^2}^2}{c_0}+e\Big)\,. \tag{30}$$

Let us translate this inequality in time. Let us define

$$m(t) \stackrel{\rm def}{=} \sup_{t'\in[t,T^\star[} \|v^3(t')\|_{\dot H^{\frac12}}\,.$$

Inequality (30) writes

$$\|\nabla_{\rm h} v(t)\|_{L^2}^2\|v(t)\|_{L^2}^2 \geq c_2 \exp\Big(\frac{c_3}{m^2(t)}\Big)\,.$$

Because of the energy estimate, we get, by integrating the above inequality over $[t,T^*[$, that

$$\begin{aligned}\frac12\|v(t)\|_{L^2}^4 &\geq \int_t^{T^\star} \|\nabla_{\rm h} v(t')\|_{L^2}^2\,dt'\|v\|_{L^\infty_t(L^2)}^2\\ &\geq \int_t^{T^\star} \|\nabla_{\rm h} v(t')\|_{L^2}^2\|v(t')\|_{L^2}^2\,dt'\\ &\geq c_2\int_t^{T^\star} \exp\Big(\frac{c_3}{m^2(t')}\Big)dt'\,.\end{aligned}$$

The function $t\longmapsto \exp\Big(\dfrac{c_3}{m^2(t')}\Big)$ is a non decreasing function. Thus

$$\frac12\|v(t)\|_{L^2}^4 \geq c_2\exp\Big(\frac{c_3}{m^2(t)}\Big)(T^\star-t).$$

This writes

$$m(t)\geq c\log^{-\frac12}\Big(e+\frac{\|v(t)\|_{L^2}^4}{T^\star-t}\Big)$$

and the theorem is proved provided that we prove Lemma 3.1. □

Proof of Theorem 1.3. The proof follows exactly the same lines as the previous one, replacing the estimate of $\mathcal{E}_3(v)$ by Lemma 3.2 instead of Lemma 3.1. Estimate (25) becomes

$$\frac12\frac{d}{dt}\|\nabla_{\rm h} v(t)\|_{L^2}^2+\|\nabla_{\rm h} v\|_{\dot H^1}^2 \leq C\Big(\|v^3\|_{\dot H^{\frac12}_{\log}}\|\nabla_{\rm h} v\|_{\dot H^1}^2+\|v^3\|_{\dot H^{\frac12}}\frac{\|\partial_3 v^{\rm h}\|_{L^2}^2}{E_0^2}\Big) \tag{31}$$

so by time integration and thanks to the energy estimate we find that as long as

$$t\leq T_*\stackrel{\rm def}{=}\Big\{T\in]0,T^\star[\,/\sup_{t\in[0,T]}\|v^3(t)\|_{\dot H^{\frac12}_{\log}}\leq\frac{1}{2C}\ \Big\}\,,$$

there holds

$$\begin{aligned}\|\nabla_{\rm h} v(t)\|_{L^2}^2+\int_0^t\|\nabla_{\rm h} v(t')\|_{\dot H^1}^2\,dt' &\leq \|\nabla_{\rm h} v_0\|_{L^2}^2+\frac{1}{E_0^2}\int_0^t\|\partial_3 v^{\rm h}(t')\|_{L^2}^2\,dt'\\ &\leq \|\nabla_{\rm h} v_0\|_{L^2}^2+\frac{\|v_0\|_{L^2}^2}{E_0^2}\,,\end{aligned}$$

which together with (27) ensures that

$$\sup_{t\in[0,T_*]}\|\nabla v^3(t)\|_{L^2}^2\leq\|\nabla_{\rm h} v_0\|_{L^2}^2+\frac{\|v_0\|_{L^2}^2}{E_0^2}\,.$$

Then Theorem 1.4 of [6] implies that the solution $v(t)$ is smooth for $t\leq T_*$. Theorem 1.3 follows by contraposition. □

Let us now present the proof of Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1 . We first get, by applying Bony's decomposition (A.2) to $\partial_i v^3 \partial_i v^m$ with i, m belnging to $\{1,2\}$ and then using Leibniz formula, that

$$
\begin{aligned}
\partial_i v^3 \partial_i v^m &= T_{\partial_i v^m} \partial_i v^3 + T_{\partial_i v^3} \partial_i v^m + R(\partial_i v^3, \partial_i v^m) \\
&= \partial_i T_{\partial_i v^m} v^3 + \mathrm{A}(v^3, v^m) \quad \text{with} \\
\mathrm{A}(v^3, v^m) &= -T_{\partial_i^2 v^m} v^3 + T_{\partial_i v^3} \partial_i v^m + R(\partial_i v^3, \partial_i v^m)\,.
\end{aligned} \tag{32}
$$

Applying Lemma A.1 gives

$$
\|S_j(\partial_i v^m)\|_{L^\infty} \lesssim c_j 2^{j\left(\frac{3}{2} - s_2\right)} \|\nabla_{\rm h} v\|_{\dot H^{s_2}} \quad \forall\, s_2 < \frac{3}{2}\,,
$$

and there holds

$$
\begin{aligned}
\|\Delta_j T_{\partial_i v^m} v^3\|_{L^2} &\lesssim \sum_{|j'-j|\leq 4} \|S_{j'-1}(\partial_i v^m)\|_{L^\infty} \|\Delta_{j'} v^3\|_{L^2} \\
&\lesssim \sum_{|j'-j|\leq 4} c_{j'} 2^{j\left(\frac{3}{2} - s_1 - s_2\right)} \|\nabla_{\rm h} v\|_{\dot H^{s_2}} \|v^3\|_{\dot H^{s_1}} \\
&\lesssim c_j 2^{j\left(\frac{3}{2} - s_1 - s_2\right)} \|\nabla_{\rm h} v\|_{\dot H^{s_2}} \|v^3\|_{\dot H^{s_1}}\,,
\end{aligned}
$$

that is

$$
\|T_{\partial_i v^m} v^3\|_{L^2} \lesssim \|\nabla_{\rm h} v\|_{\dot H^{s_2}} \|v^3\|_{\dot H^{s_1}} \quad \text{with} \quad s_1 + s_2 = \frac{3}{2} \quad \text{and} \quad s_2 < \frac{3}{2}\,. \tag{33}
$$

Then by using integrating by parts, we obtain

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^3} \partial_i T_{\partial_i v^m} v^3 \partial_3 v^m \, dx \Big| &= \Big| \int_{\mathbb{R}^3} T_{\partial_i v^m} v^3 \partial_i \partial_3 v^m \, dx \Big| \\
&\leq \|T_{\partial_i v^m} v^3\|_{L^2} \|\partial_i \partial_3 v^m\|_{L^2} \\
&\lesssim \|v^3\|_{\dot H^{\frac{1}{2}}} \|\nabla^{\rm h} v\|_{\dot H^1}^2\,.
\end{aligned} \tag{34}
$$

Next we claim that

$$
\|\mathrm{A}(v^3, v^m)\|_{\dot B^{s_1+s_2-1}_{1,2}} \lesssim \|v^3\|_{\dot H^{s_1}} \|\partial_i v^m\|_{\dot H^{s_2}} \quad \text{with} \quad s_1 + s_2 > 1 \quad \text{and} \quad s_1, s_2 \in]0,1]\,. \tag{35}
$$

Indeed due to $s_1 + s_2 > 1$, it is easy to observe from Lemma A.1 that

$$
\begin{aligned}
\|\Delta_j R(\partial_i v^3, \partial_i v^m)\|_{L^1} &\lesssim \sum_{j' \geq j-3} \|\Delta_{j'} \partial_i v^3\|_{L^2} \|\widetilde{\Delta}_{j'} \partial_i v^m\|_{L^2} \\
&\lesssim \sum_{j' \geq j-3} c_{j'} 2^{-j'(s_1+s_2-1)} \|v^3\|_{\dot H^{s_1}} \|\partial_i v^m\|_{\dot H^{s_2}} \\
&\lesssim c_j 2^{-j(s_1+s_2-1)} \|v^3\|_{\dot H^{s_1}} \|\partial_i v^m\|_{\dot H^{s_2}}\,.
\end{aligned}
$$

Similarly since $s_2 \leq 1$, one has

$$
\begin{aligned}
\|\Delta_j T_{\partial_i^2 v^m} v^3\|_{L^1} &\lesssim \sum_{|j'-j|\leq 4} \|S_{j'-1} \partial_i^2 v^m\|_{L^2} \|\Delta_{j'} v^3\|_{L^2} \\
&\lesssim c_j 2^{-j(s_1+s_2-1)} \|v^3\|_{\dot H^{s_1}} \|\partial_i v^m\|_{\dot H^{s_2}}\,.
\end{aligned}
$$

Finally due to $s_1 \leq 1$, we have $\|S_j\partial_i v^3\|_{L^2} \lesssim 2^{j(1-s_1)}\|v^3\|_{\dot H^{s_1}}$, from which, we infer

$$\begin{aligned}\|\Delta_j T_{\partial_i v^3}\partial_i v^m\|_{L^1} &\lesssim \sum_{|j'-j|\leq 4}\|S_{j'-1}\partial_i v^3\|_{L^2}\|\Delta_{j'}\partial_i v^m\|_{L^2}\\ &\lesssim \sum_{|j'-j|\leq 4} c_{j'}2^{-j'(s_1+s_2-1)}\|v^3\|_{\dot H^{s_1}}\|\partial_i v^m\|_{\dot H^{s_2}}\\ &\lesssim c_j 2^{-j(s_1+s_2-1)}\|v^3\|_{\dot H^{s_1}}\|\partial_i v^m\|_{\dot H^{s_2}}\,.\end{aligned}$$

This results in (35).

Let us now deal with the estimate of $\int_{\mathbb{R}^3}\mathrm{A}(v^3,v^m)\partial_3 v^m\,dx$. The main problem is that when $v^3\in \dot H^{\frac12}, \partial_i v^m \in \dot H^1$, (35) implies that $\mathrm{A}(v^3,v^m)\in \dot B^{\frac12}_{1,2}$, which can imbedded into $\big(\dot B^0_{1,\infty}\big)_{\rm h}\big(\dot B^{\frac12}_{1,2}\big)_{\rm v}$ (see (37) below). Yet it follows from Lemma A.1 that

$$\big(\dot B^0_{1,\infty}\big)_{\rm h}\big(\dot B^{\frac12}_{1,2}\big)_{\rm v}\hookrightarrow \big(\dot B^{-1}_{2,\infty}\big)_{\rm h}\big(\dot B^{\frac12}_{1,2}\big)_{\rm v}\hookrightarrow \big(\dot B^{-1}_{2,\infty}\big)_{\rm h}(L^2_{\rm v})\,.$$

While $\partial_3 v^{\rm h}$ only belongs to $\dot H^{1,0}$ so that the product $\mathrm{A}(v^3,v^m)\partial_3 v^m$ does not make sense in the sense of distributions. The idea consists in decomposing $\partial_3 v^m$ in a term containing only low horizontal frequencies, a term containing only intermediate horizontal frequencies and a term containing only high horizontal frequencies. More precisely, for a couple of positive real numbers (λ,Λ) such that $\lambda\leq\Lambda$, let us define

$$\begin{aligned} a_{\flat,\lambda} \stackrel{\rm def}{=} \mathcal{F}^{-1}(\mathbf{1}_{B_{\rm h}(0,\lambda)}\widehat a)\,,\ a_{\natural,\lambda,\Lambda} \stackrel{\rm def}{=} \mathcal{F}^{-1}\big((\mathbf{1}_{B_{\rm h}(0,\Lambda)}-\mathbf{1}_{B_{\rm h}(0,\lambda)})\widehat a\big)\quad\text{and}\\ a_{\sharp,\Lambda} = \mathcal{F}^{-1}(\mathbf{1}_{B^c_{\rm h}(0,\Lambda)}\widehat a)\,.\end{aligned}\tag{36}$$

Let us study first low horizontal frequencies. Let us write that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\mathrm{A}(v^3,v^m)\partial_3 v^m_{\flat,\lambda}\,dx\Big| &\leq \sum_{k,\ell\in\mathbb{Z}^2}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell \mathrm{A}(v^3,v^m)\|_{L^2}\|\widetilde\Delta^{\rm h}_k\widetilde\Delta^{\rm v}_\ell\partial_3 v^m_{\flat,\lambda}\|_{L^2}\\ &\lesssim \sum_{\substack{2^k\leq\lambda\\ \ell\in\mathbb{Z}}} c_{k,\ell}2^k 2^{\frac{\ell}{2}}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell\mathrm{A}(v^3,v^m)\|_{L^1}\|\partial_3 v^m\|_{L^2}.\end{aligned}$$

Yet notice that for any $a\in\dot B^s_{p,r}(\mathbb{R}^3)$ with $s>0$, we have

$$\begin{aligned}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell a\|_{L^p} &\lesssim \sum_{j\geq \ell-N_0}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell\Delta_j a\|_{L^p}\lesssim \sum_{j\geq\ell-N_0}\|\Delta_j a\|_{L^p}\\ &\lesssim \sum_{j\geq \ell-N_0} c_{j,r}2^{-js}\|a\|_{\dot B^s_{p,r}}\lesssim c_{\ell,r}2^{-\ell s}\|a\|_{\dot B^s_{p,r}},\end{aligned}\tag{37}$$

where $(c_{j,r})_{j\in\mathbb{Z}}$ is a generic element of $\ell^r(\mathbb{Z})$ so that $\sum_{j\in\mathbb{Z}}c^r_{j,r}=1$. (35) along with (37) ensures that that

$$\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell\mathrm{A}(v^3,v^m)\|_{L^1}\lesssim c_\ell 2^{-\frac{\ell}{2}}\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^m\|_{\dot H^1}.\tag{38}$$

As a result, it comes out

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\mathrm{A}(v^3,v^m)\partial_3 v^m_{\flat,\lambda}\,dx\Big| &\lesssim \sum_{\substack{2^k\leq\lambda\\ \ell\in\mathbb{Z}}} c_{k,\ell}c_\ell 2^k\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^m\|_{\dot H^1}\|\partial_3 v^m\|_{L^2}\\ &\lesssim \lambda\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^m\|_{\dot H^1}\|\partial_3 v^m\|_{L^2}\\ &\lesssim \|v^3\|_{\dot H^{\frac12}}\big(\lambda^2\|\partial_3 v^m\|^2_{L^2}+\|\nabla_{\rm h}v^m\|^2_{\dot H^1}\big).\end{aligned}\tag{39}$$

Whereas by applying Lemma A.1, we write

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \mathrm{A}(v^3,v^m)\partial_3 v^{\mathrm{h}}_{\natural,\lambda,\Lambda}\,dx\Big| &\leq \sum_{k,\ell\in\mathbb{Z}^2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\mathrm{A}(v^3,v^m)\|_{L^2}\|\widetilde{\Delta}_k^{\mathrm{h}}\widetilde{\Delta}_\ell^{\mathrm{v}}\partial_3 v^{\mathrm{h}}_{\natural,\lambda,\Lambda}\|_{L^2}\\ &\lesssim \sum_{\substack{\lambda\leq 2^k\leq\Lambda\\ \ell\in\mathbb{Z}}} c_{k,\ell}2^{\frac{\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\mathrm{A}(v^3,v^m)\|_{L^1}\|\nabla_{\mathrm{h}}\partial_3 v^{\mathrm{h}}\|_{L^2},\end{aligned}$$

which together with (38) implies

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \mathrm{A}(v^3,v^m)\partial_3 v^{\mathrm{h}}_{\natural,\lambda,\Lambda}\,dx\Big| &\lesssim \sum_{\substack{\lambda\leq 2^k\leq\Lambda\\ \ell\in\mathbb{Z}}} c_{k,\ell}c_\ell\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\mathrm{h}}v^{\mathrm{h}}\|_{\dot H^1}\|\nabla_{\mathrm{h}}\partial_3 v^m\|_{L^2}\\ &\lesssim \log^{\frac12}\Big(\frac{\Lambda}{\lambda}\Big)\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\mathrm{h}}v^m\|^2_{\dot H^1}\,.\end{aligned}\tag{40}$$

Now let us study the case of high horizontal frequencies. Let us write that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \mathrm{A}(v^3,v^m)\partial_3 v^{m}_{\sharp,\Lambda}\,dx\Big| &\leq \sum_{k,\ell\in\mathbb{Z}^2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\mathrm{A}(v^3,v^m)\|_{L^{\frac43}_{\mathrm{h}}(L^2_{\mathrm{v}})}\|\widetilde{\Delta}_k^{\mathrm{h}}\widetilde{\Delta}_\ell^{\mathrm{v}}\partial_3 v^{m}_{\sharp,\Lambda}\|_{L^4_{\mathrm{h}}(L^2_{\mathrm{v}})}\\ &\lesssim \sum_{\substack{\Lambda\leq 2^k\\ \ell\in\mathbb{Z}}} c^2_{k,\ell}2^{-\frac{k}{2}}\|\mathrm{A}(v^3,v^m)\|_{(\dot B^{\frac12}_{1,2})_{\mathrm{h}}(\dot B^{\frac12}_{1,2})_{\mathrm{v}}}\|\nabla_{\mathrm{h}}\partial_3 v^m\|_{L^2}.\end{aligned}$$

Yet it follows from Lemma A.2 that

$$\dot B^1_{1,2}\hookrightarrow (\dot B^{\frac12}_{1,2})_{\mathrm{h}}(\dot B^{\frac12}_{1,2})_{\mathrm{v}},$$

from which and (35), we deduce that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \mathrm{A}(v^3,v^m)\partial_3 v^{\mathrm{h}}_{\sharp,\Lambda}\,dx\Big| &\lesssim \Lambda^{-\frac12}\|\mathrm{A}(v^3,v^m)\|_{\dot B^1_{1,2}}\|\nabla_{\mathrm{h}}\partial_3 v^m\|_{L^2}\\ &\lesssim \Lambda^{-\frac12}\|\nabla v^3\|_{L^2}\|\nabla_{\mathrm{h}}v^m\|^2_{\dot H^1}\,.\end{aligned}\tag{41}$$

Summing up (34) and (39) to (41), we achieve

$$\begin{aligned}|I(v^{\mathrm{h}},v^3)| \leq C_0\Big(&\|v^3\|_{\dot H^{\frac12}}\big(\lambda^2\|\partial_3 v^{\mathrm{h}}\|^2_{L^2}+\|\nabla_{\mathrm{h}}v^{\mathrm{h}}\|^2_{\dot H^1}\big)\\ &+\log^{\frac12}\Big(\frac{\Lambda}{\lambda}\Big)\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\mathrm{h}}v\|^2_{\dot H^1}+\Lambda^{-\frac12}\|\nabla v^3\|_{L^2}\|\nabla_{\mathrm{h}}v\|^2_{\dot H^1}\Big)\,.\end{aligned}\tag{42}$$

Choosing λ and Λ in the above inequality such that

$$\lambda=\frac{\|\nabla v^3\|_{L^2}}{E_0^{\frac12}}\quad\text{and}\quad \Lambda=\frac{\|\nabla v^3\|^2_{L^2}}{\|v^3\|^2_{\dot H^{\frac12}}}$$

gives rise to (18). □

Proof of Lemma 3.2 . We first get, by using Bony's decomposition (A.2) for both horizontal and vertical variables to $\partial_i v^3\partial_i v^m$, that

$$\partial_i v^3\partial_i v^m=\big(T^{\mathrm{h}}+R^{\mathrm{h}}+\bar T^{\mathrm{h}}\big)\big(T^{\mathrm{v}}+R^{\mathrm{v}}+\bar T^{\mathrm{v}}\big)(\partial_i v^3,\partial_i v^m)\,.$$

Then the proof of Lemma 3.2 will be based on the following claims, which we shall present a general version for the sake of the proof of Lemma 4.1 below. More precisely, we claim that for any $p \in [2, \infty]$, there holds

$$\|T^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{\dot H^{-1,0}} \lesssim \|v^3\|_{\dot H^{\frac12+\frac2p}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{1-\frac2p}}; \tag{43}$$

$$\|\bar T^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{\dot H^{-1,0}} \lesssim \|v^3\|_{\dot H^{\frac12+\frac2p}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{1-\frac2p}}; \tag{44}$$

$$\|R^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{(\dot B^{-1}_{2,\infty})_{\rm h}(L^2_{\rm v})} \lesssim \|v^3\|_{\dot H^{\frac12+\frac2p}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{1-\frac2p}}. \tag{45}$$

Let us admit the above inequalities for the time being, and proceed the proof of Lemma 3.2. Firstly, in view of (43) and (44) for $p = \infty$, we deduce that

$$\begin{aligned} \Big|\int_{\mathbb{R}^3} \big(T^{\rm h} + \bar T^{\rm h}\big)(\partial_i v^3, \partial_i v^m)\partial_3 v^{\rm h}\, dx\Big| &\lesssim \big\|\big(T^{\rm h} + \bar T^{\rm h}\big)(\partial_i v^3, \partial_i v^m)\big\|_{\dot H^{1,0}} \|\partial_3 v^{\rm h}\|_{\dot H^{1,0}} \\ &\lesssim \|v^3\|_{\dot H^{\frac12}} \|\nabla_{\rm h} v^{\rm h}\|^2_{\dot H^1}. \end{aligned} \tag{46}$$

On the other hand, it follows from Lemma A.1 that

$$\begin{aligned} \|\Delta^{\rm h}_k\Delta^{\rm v}_\ell R^{\rm h} T^{\rm v}(\partial_i v^3, \partial_i v^m)\|_{L^2} &\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} \|\Delta^{\rm h}_{k'} S^{\rm v}_{\ell'-1}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})} \|\widetilde\Delta^{\rm h}_{k'}\widetilde\Delta^{\rm v}_{\ell'}\partial_i v^m\|_{L^2} \\ &\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} c_{k',\ell'} \log^{-1}\big(2^{k'}E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{\frac12,\frac12}} \\ &\lesssim 2^k \log^{-1}\big(2^k E + e\big) \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} c_{k',\ell'} \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1} \\ &\lesssim c_\ell 2^k \log^{-1}\big(2^k E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1}. \end{aligned}$$

Along the same line, we get, by a similar derivation of the above inequality, that

$$\begin{aligned} \|\Delta^{\rm h}_k\Delta^{\rm v}_\ell R^{\rm h} R^{\rm v}(\partial_i v^3, \partial_i v^m)\|_{L^2} &\lesssim 2^k 2^{\frac\ell2} \sum_{\substack{k'\geq k-3\\ \ell'\geq \ell-3}} \|\Delta^{\rm h}_{k'}\Delta^{\rm v}_{\ell'}\partial_i v^3\|_{L^2} \|\widetilde\Delta^{\rm h}_{k'}\widetilde\Delta^{\rm v}_{\ell'}\partial_i v^m\|_{L^2} \\ &\lesssim 2^k 2^{\frac\ell2} \sum_{\substack{k'\geq k-3\\ \ell'\geq \ell-3}} c^2_{k',\ell'} 2^{-\frac{\ell'}{2}} \log^{-1}\big(2^{k'}E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{\frac12,\frac12}} \\ &\lesssim c_\ell 2^k \log^{-1}\big(2^k E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1}, \end{aligned}$$

and

$$\begin{aligned} \|\Delta^{\rm h}_k\Delta^{\rm v}_\ell R^{\rm h} \bar T^{\rm v}(\partial_i v^3, \partial_i v^m)\|_{L^2} &\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} \|\Delta^{\rm h}_{k'}\Delta^{\rm v}_{\ell'}\partial_i v^3\|_{L^2} \|\widetilde\Delta^{\rm h}_{k'} S^{\rm v}_{\ell'-1}\partial_i v^m\|_{L^2_{\rm h}(L^\infty_{\rm v})} \\ &\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} c_{k',\ell'} \log^{-1}\big(2^{k'}E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^{\frac12}_{\rm h}(\dot B^{\frac12}_{2,1})_{\rm v}} \\ &\lesssim c_\ell 2^k \log^{-1}\big(2^k E + e\big) \|v^3\|_{\dot H^{\frac12}_{\log,E}} \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1}. \end{aligned}$$

This together with (45) for $p = \infty$ ensures that

$$\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell R^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{L^2} \lesssim c_\ell 2^k \min\Big(\|v^3\|_{\dot H^{\frac12}}, \log^{-1}\big(2^k E + e\big)\|v^3\|_{\dot H^{\frac12}_{\log,E}}\Big) \|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1}. \tag{47}$$

Now for any positive integer N to be fixed later on, we write

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} R^{\rm h}(\partial_i v^3,\partial_i v^m)\partial_3 v^{\rm h}\,dx\Big| &\leq I_N^1+I_N^2 \quad \text{with}\\
I_N^1 &= \sum_{\substack{k\leq N\\ \ell\in\mathbb{Z}}}\Big|\int_{\mathbb{R}^3}\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}(\partial_i v^3,\partial_i v^m)\widetilde{\Delta}_k^{\rm h}\widetilde{\Delta}_\ell^{\rm v}\partial_3 v^{\rm h}\,dx\Big|\\
I_N^2 &= \sum_{\substack{k\geq N\\ \ell\in\mathbb{Z}}}\Big|\int_{\mathbb{R}^3}\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}(\partial_i v^3,\partial_i v^m)\widetilde{\Delta}_k^{\rm h}\widetilde{\Delta}_\ell^{\rm v}\partial_3 v^{\rm h}\,dx\Big|.
\end{aligned}$$

By virtue of (47), we have

$$\begin{aligned}
I_N^1 &\lesssim \sum_{\substack{k\leq N\\ \ell\in\mathbb{Z}}}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}(\partial_i v^3,\partial_i v^m)\|_{L^2}\|\widetilde{\Delta}_k^{\rm h}\widetilde{\Delta}_\ell^{\rm v}\partial_3 v^{\rm h}\|_{L^2}\\
&\lesssim 2^N\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}\|\partial_3 v^{\rm h}\|_{L^2}\,,
\end{aligned}$$

and

$$\begin{aligned}
I_N^2 &\lesssim \sum_{\substack{k\geq N\\ \ell\in\mathbb{Z}}}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}(\partial_i v^3,\partial_i v^m)\|_{L^2}\|\widetilde{\Delta}_k^{\rm h}\widetilde{\Delta}_\ell^{\rm v}\partial_3 v^{\rm h}\|_{L^2}\\
&\lesssim \sum_{\substack{k\geq N\\ \ell\in\mathbb{Z}}}c_{k,\ell}c_\ell\log^{-1}\bigl(2^k\|v^3\|_{L^2}^2+e\bigr)\|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2\\
&\lesssim \Bigl(\sum_{k\geq N}\log^{-2}\bigl(2^k\|v^3\|_{L^2}^2+e\bigr)\Bigr)^{\frac12}\|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2\,.
\end{aligned}$$

Without loss of generality, we may assume that $2^N\|v^3\|_{L^2}^2\geq e$. Then there holds

$$\begin{aligned}
\sum_{k\geq N}\log^{-2}\bigl(2^k\|v^3\|_{L^2}^2+e\bigr) &\leq \int_N^\infty \log^{-2}\bigl(2^\tau\|v^3\|_{L^2}^2+e\bigr)\,d\tau\\
&\leq \frac{1}{\log 2}\frac{1}{\bigl(N\log 2+\log\|v^3\|_{L^2}^2\bigr)}\\
&\leq \frac{1}{\log 2}\,,
\end{aligned}$$

which implies that

$$I_N^2\lesssim \|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}\,.$$

Hence we achieve

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3}R^{\rm h}(\partial_i v^3,\partial_i v^m)\partial_3 v^m\,dx\Big| &\leq C\Bigl(2^N\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}\|\partial_3 v^{\rm h}\|_{L^2}+\|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2\Bigr)\\
&\leq C\Bigl(\|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2+2^{2N}\|v^3\|_{\dot H^{\frac12}}\|\partial_3 v^{\rm h}\|_{L^2}^2\Bigr)\,.
\end{aligned}$$

Choosing $2^NE_0\sim 1$ in the above inequality gives rise to

$$\Big|\int_{\mathbb{R}^3}R^{\rm h}(\partial_i v^3,\partial_i v^m)\partial_3 v^m\,dx\Big|\leq C\Bigl(\|v^3\|_{\dot H^{\frac12}_{\log,E}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2+\|v^3\|_{\dot H^{\frac12}}\frac{\|\partial_3 v^{\rm h}\|_{L^2}^2}{E_0^2}\Bigr)\,. \tag{48}$$

By summing (46) and (48), we conclude the proof of (19). □

Let us now present the proof (43), (**??**) and (45).

Proof of (43). Observing from Lemma A.1 that

$$\|S^{\rm h}_{k-1}S^{\rm v}_{\ell-1}\nabla_{\rm h}v^3\|_{L^\infty}\lesssim c_{k,\ell}2^{\frac{3k}{2}}2^{\frac{\ell}{2}}\|v^3\|_{\dot H^{\frac12,0}}.$$

Then due to $H^1\hookrightarrow H^{\frac12,\frac12}$ (see Lemma A.2) and considering the support properties to terms in $T^{\rm h}T^{\rm v}(\partial_i v^3,\partial_i v^m)$, we write

$$\begin{aligned}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell T^{\rm h}T^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}}\|S^{\rm h}_{k'-1}S^{\rm v}_{\ell'-1}\partial_i v^3\|_{L^\infty}\|\Delta^{\rm h}_{k'}\Delta^{\rm v}_{\ell'}\partial_i v^m\|_{L^2}\\ &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} c_{k',\ell'}2^{k'}\|v^3\|_{\dot H^{\frac12,0}}\|\partial_i v^m\|_{\dot H^{\frac12,\frac12}}\\ &\lesssim c_{k,\ell}2^k\|v^3\|_{\dot H^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}.\end{aligned}$$

When $p\in[2,\infty[$, It follows from Lemma A.2 that

$$\dot H^{\frac12+\frac2p}(\mathbb{R}^3)\hookrightarrow \dot H^{\frac2p}_{\rm h}\big(\dot B^{\frac12}_{2,1}\big)_{\rm v},$$

so that we have

$$\|S^{\rm h}_{k-1}S^{\rm v}_{\ell-1}\nabla_{\rm h}v^3\|_{L^\infty}\lesssim 2^{2k\left(1-\frac1p\right)}\|v^3\|_{\dot H^{\frac2p}_{\rm h}\left(\dot B^{\frac12}_{2,1}\right)_{\rm v}}\lesssim 2^{2k\left(1-\frac1p\right)}\|v^3\|_{\dot H^{\frac12+\frac2p}},$$

and

$$\begin{aligned}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell T^{\rm h}T^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} c_{k',\ell'}2^{k'}\|v^3\|_{\dot H^{\frac12+\frac2p}}\|\partial_i v^m\|_{\dot H^{1-\frac2p,0}}\\ &\lesssim c_{k,\ell}2^k\|v^3\|_{\dot H^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^{1-\frac2p}}.\end{aligned}$$

Similarly, since for $p\in[2,\infty]$, there holds

$$\|S^{\rm h}_k\Delta^{\rm v}_\ell\partial_i v^3\|_{L^\infty_{\rm h}(L^2_{\rm v})}\lesssim c_{k,\ell}2^{2k\left(1-\frac1p\right)}2^{-\frac{\ell}{2}}\|v^3\|_{\dot H^{\frac2p,\frac12}},\tag{49}$$

we get, by applying Lemma A.1, that

$$\begin{aligned}\|\Delta^{\rm h}_k\Delta^{\rm v}_\ell T^{\rm h}R^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim 2^{\frac{\ell}{2}}\sum_{\substack{|k'-k|\leq 4\\ \ell'\geq \ell-3}}\|S^{\rm h}_{k'-1}\Delta^{\rm v}_{\ell'}\partial_i v^3\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}\widetilde{\Delta}^{\rm v}_{\ell'}\partial_i v^m\|_{L^2}\\ &\lesssim 2^{\frac{\ell}{2}}\sum_{\substack{|k'-k|\leq 4\\ \ell'\geq \ell-3}} c_{k',\ell'}2^{k'}2^{-\frac{\ell'}{2}}\|v^3\|_{\dot H^{\frac12+\frac2p}}\|\partial_i v^m\|_{\dot H^{1-\frac2p,0}}\\ &\lesssim c_{k,\ell}2^k\|v^3\|_{\dot H^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^{1-\frac2p}}.\end{aligned}$$

Whereas due to (49) and

$$\|\Delta^{\rm h}_k S^{\rm v}_\ell\partial_i v^m\|_{L^2_{\rm h}(L^\infty_{\rm v})}\lesssim c_{k,\ell}2^{-k\left(1-\frac2p\right)}2^{\frac{\ell}{2}}\|\partial_i v^m\|_{\dot H^{1-\frac2p,0}},$$

we deduce that

$$
\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} T^{\rm h}\bar{T}^{\rm v}(\partial_{\rm h}v^3,\partial_{\rm h}v^{\rm h})\|_{L^2} &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} \|S^{\rm h}_{k'-1}\Delta^{\rm v}_{\ell'}\partial_i v^3\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}S^{\rm v}_{\ell'-1}\partial_i v^m\|_{L^2_{\rm h}(L^\infty_{\rm v})}\\
&\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} c_{k',\ell'}2^{k'}\|v^3\|_{\dot{H}^{\frac12+\frac2p}}\|\partial_i v^m\|_{\dot{H}^{1-\frac2p}}\\
&\lesssim c_{k,\ell}2^k\|v^3\|_{\dot{H}^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot{H}^{1-\frac2p}}.
\end{aligned}
$$

This leads to (43). □

Proof of (44). Note from Lemma A.1 that

$$
\begin{aligned}
&\|\Delta_k^{\rm h}S_\ell^{\rm v}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\lesssim c_{k,\ell}2^{\frac k2}2^{\frac\ell2}\|v^3\|_{\dot{H}^{\frac12,0}} \quad \text{and}\\
&\|S_k^{\rm h}\Delta_\ell^{\rm v}\partial_i v^m\|_{L^\infty_{\rm h}(L^2_{\rm v})}\lesssim c_{k,\ell}2^{\frac k2}2^{-\frac\ell2}\|\partial_i v^m\|_{\dot{H}^{\frac12,\frac12}},
\end{aligned}
\tag{50}
$$

we infer

$$
\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\bar{T}^{\rm h}T^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} \|\Delta^{\rm h}_{k'}S^{\rm v}_{\ell'-1}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|S^{\rm h}_{k'-1}\Delta^{\rm v}_{\ell'}\partial_i v^m\|_{L^\infty_{\rm h}(L^2_{\rm v})}\\
&\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} c_{k',\ell'}2^{k'}\|v^3\|_{\dot{H}^{\frac12,0}}\|\partial_i v^m\|_{\dot{H}^{\frac12,\frac12}}\\
&\lesssim c_{k,\ell}2^k\|v^3\|_{\dot{H}^{\frac12}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot{H}^1}.
\end{aligned}
$$

While for $p\in[2,\infty[$, we deduce from Lemma A.1 that

$$
\begin{aligned}
&\|\Delta_k^{\rm h}S_\ell^{\rm v}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\lesssim c_k 2^{k\left(1-\frac2p\right)}\|v^3\|_{\dot{H}^{\frac2p}_{\rm h}\left(\dot{B}^{\frac12}_{2,1}\right)_{\rm v}} \quad \text{and}\\
&\|S_k^{\rm h}\Delta_\ell^{\rm v}\partial_i v^m\|_{L^\infty_{\rm h}(L^2_{\rm v})}\lesssim c_{k,\ell}2^{\frac{2k}{p}}\|\partial_i v^m\|_{\dot{H}^{1-\frac2p,0}},
\end{aligned}
\tag{51}
$$

so that there holds

$$
\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\bar{T}^{\rm h}T^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} \|\Delta^{\rm h}_{k'}S^{\rm v}_{\ell'-1}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|S^{\rm h}_{k'-1}\Delta^{\rm v}_{\ell'}\partial_i v^m\|_{L^\infty_{\rm h}(L^2_{\rm v})}\\
&\lesssim \sum_{\substack{|k'-k|\leq 4\\ |\ell'-\ell|\leq 4}} c_{k',\ell'}2^{k'}\|v^3\|_{\dot{H}^{\frac12+\frac2p}}\|\partial_i v^m\|_{\dot{H}^{1-\frac2p}}\\
&\lesssim c_{k,\ell}2^k\|v^3\|_{\dot{H}^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot{H}^{1-\frac2p}}.
\end{aligned}
$$

While applying Lemma A.1 and (49) yields

$$
\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\bar{T}^{\rm h}R^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim 2^{\frac\ell2}\sum_{\substack{|k'-k|\leq 4\\ \ell'\geq\ell-3}} \|S^{\rm h}_{k'-1}\Delta^{\rm v}_{\ell'}\partial_i v^3\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\Delta^{\rm h}_{k'}\widetilde{\Delta}^{\rm v}_{\ell'}\partial_i v^m\|_{L^2}\\
&\lesssim 2^{\frac\ell2}\sum_{\substack{|k'-k|\leq 4\\ \ell'\geq\ell-3}} c_{k',\ell'}2^{k'}2^{-\frac{\ell'}{2}}\|v^3\|_{\dot{H}^{\frac2p,\frac12}}\|\partial_i v^m\|_{\dot{H}^{1-\frac2p,0}}\\
&\lesssim c_{k,\ell}2^k\|v^3\|_{\dot{H}^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot{H}^{1-\frac2p}}.
\end{aligned}
$$

Lemma A.1 also ensures that

$$
\begin{aligned}
\|S_k^{\mathrm{h}} S_\ell^{\mathrm{v}} \partial_i v^m\|_{L^\infty} & \lesssim \sum_{\substack{k' \leq k-1 \\ \ell' \leq \ell-1}} 2^{k'} 2^{\frac{\ell'}{2}} \|\Delta_{k'}^{\mathrm{h}} \Delta_{\ell'}^{\mathrm{v}} \partial_i v^m\|_{L^2} \\
& \lesssim \sum_{\substack{k' \leq k-1 \\ \ell' \leq \ell-1}} c_{k',\ell'} 2^{\frac{2k'}{p}} 2^{\frac{\ell'}{2}} \|\partial_i v^m\|_{\dot{H}^{1-\frac{2}{p},0}} \lesssim c_{k,\ell} 2^{\frac{2k}{p}} 2^{\frac{\ell}{2}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^{1-\frac{2}{p}}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\Delta_k^{\mathrm{h}} \Delta_\ell^{\mathrm{v}} \bar{T}^{\mathrm{h}} \bar{T}^{\mathrm{v}} (\partial_i v^3, \partial_i v^m)\|_{L^2} & \lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} \|\Delta_{k'}^{\mathrm{h}} \Delta_{\ell'}^{\mathrm{v}} \partial_i v^3\|_{L^2} \|S_{k'-1}^{\mathrm{h}} S_{\ell'-1}^{\mathrm{v}} \partial_i v^m\|_{L^\infty} \\
& \lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} c_{k',\ell'} 2^{k'} \|v^3\|_{\dot{H}^{\frac{2}{p},\frac{1}{2}}} \|\partial_i v^m\|_{\dot{H}^{1-\frac{2}{p},0}} \\
& \lesssim c_{k,\ell} 2^k \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^{1-\frac{2}{p}}}.
\end{aligned}
$$

Consequently, we conclude the proof of (44). □

Proof of (45). Thanks to (50), we deduce from Lemma A.1 that

$$
\begin{aligned}
\|\Delta_k^{\mathrm{h}} \Delta_\ell^{\mathrm{v}} R^{\mathrm{h}} T^{\mathrm{v}} (\partial_{\mathrm{h}} v^3, \partial_{\mathrm{h}} v^{\mathrm{h}})\|_{L^2} & \lesssim 2^k \sum_{\substack{k' \geq k-3 \\ |\ell'-\ell| \leq 4}} \|\Delta_{k'}^{\mathrm{h}} S_{\ell'-1}^{\mathrm{v}} \partial_{\mathrm{h}} v^3\|_{L^2_{\mathrm{h}}(L^\infty_{\mathrm{v}})} \|\widetilde{\Delta}_{k'}^{\mathrm{h}} \widetilde{\Delta}_{\ell'}^{\mathrm{v}} \partial_{\mathrm{h}} v^{\mathrm{h}}\|_{L^2} \\
& \lesssim 2^k \sum_{\substack{k' \geq k-3 \\ |\ell'-\ell| \leq 4}} c_{k',\ell'}^2 \|v^3\|_{\dot{H}^{\frac{1}{2},0}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^{\frac{1}{2},\frac{1}{2}}} \\
& \lesssim c_\ell 2^k \|v^3\|_{\dot{H}^{\frac{1}{2}}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^1}.
\end{aligned}
$$

Whereas for $p \in [2, \infty[$, by virtue of (52), we write

$$
\begin{aligned}
\|\Delta_k^{\mathrm{h}} \Delta_\ell^{\mathrm{v}} R^{\mathrm{h}} T^{\mathrm{v}} (\partial_i v^3, \partial_i v^m)\|_{L^2} & \lesssim 2^k \sum_{\substack{k' \geq k-3 \\ |\ell'-\ell| \leq 4}} c_{k'} c_{k',\ell'} \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_i v^m\|_{\dot{H}^{1-\frac{2}{p},0}} \\
& \lesssim c_\ell 2^k \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^{1-\frac{2}{p}}}.
\end{aligned}
\tag{52}
$$

Applying Lemma A.1 once again gives

$$
\begin{aligned}
\|\Delta_k^{\mathrm{h}} \Delta_\ell^{\mathrm{v}} R^{\mathrm{h}} R^{\mathrm{v}} (\partial_i v^3, \partial_i v^m)\|_{L^2} & \lesssim 2^k 2^{\frac{\ell}{2}} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} \|\Delta_{k'}^{\mathrm{h}} \Delta_{\ell'}^{\mathrm{v}} \partial_i v^3\|_{L^2} \|\widetilde{\Delta}_{k'}^{\mathrm{h}} \widetilde{\Delta}_{\ell'}^{\mathrm{v}} \partial_i v^m\|_{L^2} \\
& \lesssim 2^k 2^{\frac{\ell}{2}} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} c_{k',\ell'}^2 2^{-\frac{\ell'}{2}} \|v^3\|_{\dot{H}^{\frac{2}{p},\frac{1}{2}}} \|\partial_i v^m\|_{\dot{H}^{1-\frac{2}{p},0}} \\
& \lesssim c_\ell 2^k \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{\dot{H}^{1-\frac{2}{p}}},
\end{aligned}
$$

and

$$\begin{aligned}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}\bar{T}^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim 2^k\sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}}\|\Delta_{k'}^{\rm h}\Delta_{\ell'}^{\rm v}\partial_i v^3\|_{L^2}\|\widetilde{\Delta}_{k'}^{\rm h}S_{\ell'-1}^{\rm v}\partial_i v^m\|_{L^2_{\rm h}(L^\infty_{\rm v})}\\ &\lesssim 2^k\sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}}c_{k',\ell'}^2\|v^3\|_{\dot H^{\frac{2}{p},\frac12}}\|\partial_i v^m\|_{\dot H^{1-\frac{2}{p},0}}\\ &\lesssim c_\ell 2^k\|v^3\|_{\dot H^{\frac12+\frac2p}}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^{1-\frac2p}}.\end{aligned}$$

This gives rise to (45). □

4. Proof of Energy estimate

Lemma 4.1. *We have that*

$$I(v^{\rm h},v^3)\lesssim C\|v^3\|_{\dot H^{\frac32}}\log\Big(\|\nabla_{\rm h}v^{\rm h}\|_{L^2}^2+e\Big)\|\nabla_{\rm h}v\|_{L^2}^2+\frac12\|\nabla_{\rm h}v\|_{\dot H^1}^2+\|\partial_3 v^{\rm h}\|_{L^2}^2\,. \tag{53}$$

Let $\mathcal{E}_i(v)$ be given by (20). Then we deduce from the law of product that

$$\begin{aligned}|\mathcal{E}_1(v)| &\lesssim \|\partial_3 v^3\|_{\dot H^{\frac12}}\|(\nabla_{\rm h}v^{\rm h})^2\|_{\dot H^{-\frac12}}\\ &\lesssim \|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}.\end{aligned}$$

Along the same line, one has

$$\begin{aligned}|\mathcal{E}_2(v)| &\lesssim \|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}v^3\|_{\dot H^1};\\ |\mathcal{E}_4(v)| &\lesssim \|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v^3\|_{L^2}\|\nabla_{\rm h}v^3\|_{\dot H^1}.\end{aligned}$$

As a result, it comes out

$$\begin{aligned}|\mathcal{E}_1(v)|+|\mathcal{E}_2(v)|+|\mathcal{E}_4(v)| &\leq C\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v\|_{L^2}\|\nabla_{\rm h}v\|_{\dot H^1}\\ &\leq C\|v^3\|_{\dot H^{\frac32}}^2\|\nabla_{\rm h}v\|_{L^2}^2+\frac12\|\nabla_{\rm h}v\|_{\dot H^1}^2.\end{aligned}$$

Inserting the above estimate and (53) into (20) results in

$$\frac{d}{dt}\|\nabla_{\rm h}v\|_{L^2}^2+\|\nabla_{\rm h}v\|_{\dot H^1}^2\leq C\|v^3\|_{\dot H^{\frac32}}^2\log\Big(\|\nabla_{\rm h}v^{\rm h}\|_{L^2}^2+e\Big)\|\nabla_{\rm h}v\|_{L^2}^2+\|\partial_3 v^{\rm h}\|_{L^2}^2.$$

Integrating the above inequality over $[0,t]$ with $t\leq T^\star$, we obtain

$$\|\nabla_{\rm h}v(t)\|_{L^2}^2+\int_0^t\|\nabla_{\rm h}v\|_{\dot H^1}^2\,dt'\leq\|v_0\|_{L^2}^2+C\int_0^t\|v^3\|_{\dot H^{\frac32}}^2\log\Big(\|\nabla_{\rm h}v^{\rm h}\|_{L^2}^2+e\Big)\|\nabla_{\rm h}v\|_{L^2}^2\,dt'.$$

Then Osgood Lemma ensures that

$$\|\nabla_{\rm h}v(t)\|_{L^2}^2+\int_0^t\|\nabla_{\rm h}v\|_{\dot H^1}^2\,dt'\leq\|v_0\|_{L^2}^2\exp\Big(\exp\big(C\int_0^t\|v^3\|_{\dot H^{\frac32}}^2\,dt'\big)\Big). \tag{54}$$

Proof of Lemma 4.1. Taking $p=2$ in (43) and (44), we infer

$$\begin{aligned}|I(v^{\rm h},v^3)| &\lesssim\big\|(T^{\rm h}+\bar{T}^{\rm h})(\partial_i v^3,\partial_i v^m)\big\|_{\dot H^{-1,0}}\|\partial_3 v^m\|_{\dot H^{1,0}}\\ &\lesssim\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}v\|_{\dot H^1}.\end{aligned} \tag{55}$$

Whereas by taking $p=2$ in (45) results in

$$\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}(\partial_i v^3,\partial_i v^m)\|_{L^2}\lesssim c_\ell 2^k\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}. \tag{56}$$

Let us now deal with the estimate of $\displaystyle\int_{\mathbb{R}^3} R^{\rm h}(\partial_i v^3, \partial_i v^m)\partial_3 v^m\,dx$. Again we use (36) to write

$$\partial_3 v^{\rm h} = \partial_3 v^m_{\flat,1} + \partial_3 v^m_{\natural,1,\Lambda} + \partial_3 v^m_{\sharp,\Lambda}.$$

Let us study first low horizontal frequencies. Thanks to (56), we get, by a similar derivation of (39), that

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} R^{\rm h}(\partial_i v^3, \partial_i v^m)|\partial_3 v^m_{\flat,1}\,dx\Big| &\lesssim \sum_{k,\ell\in\mathbb{Z}^2} \|\Delta_k^{\rm h}\Delta_\ell^{\rm v} R^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{L^2}\|\widetilde{\Delta}_k^{\rm h}\widetilde{\Delta}_\ell^{\rm v}\partial_3 v^m_{\flat,1}\|_{L^2}\\
&\lesssim \sum_{\substack{2^k\leq 1\\ \ell\in\mathbb{Z}}} c_{k,\ell}c_\ell 2^k\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h} v^{\rm h}\|_{L^2}\|\partial_3 v^{\rm h}\|_{L^2}\\
&\lesssim \|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h} v^{\rm h}\|_{L^2}\|\partial_3 v^{\rm h}\|_{L^2}.
\end{aligned}\tag{57}$$

Whereas along the same line to the proof of (40), we obtain

$$\Big|\int_{\mathbb{R}^3} R^{\rm h}(\partial_i v^3, \partial_i v^m)|\partial_3 v^{\rm h}_{\natural,1,\Lambda}\,dx\Big| \leq C\log^{\frac12}\Lambda\|v^3\|_{\dot H^{\frac32}}\|\nabla_{\rm h} v^{\rm h}\|_{L^2}\|\nabla_{\rm h} v^{\rm h}\|_{\dot H^1}\,.\tag{58}$$

To handle the estimate of $\int_{\mathbb{R}^3} R^{\rm h}(\partial_i v^3, \partial_i v^m)|\partial_3 v^{\rm h}_{\natural,\Lambda}\,dx$, we claim that

$$\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} R^{\rm h}(\partial_i v^3, \partial_i v^m)\|_{L^2} \lesssim c_\ell 2^{\frac k2}\|\nabla_{\rm h} v\|_{\dot H^1}\|\nabla_{\rm h} v^{\rm h}\|_{L^2}.\tag{59}$$

Indeed note from Lemmas A.1 and A.2 that

$$\begin{aligned}
\|\Delta_k^{\rm h} S_\ell^{\rm v}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})} &\lesssim c_k 2^{-\frac k2}\|v^3\|_{\dot H^{\frac32}_{\rm h}(\dot B^{\frac12}_{2,1})_{\rm v}}\\
&\lesssim c_k 2^{-\frac k2}\|v^3\|_{\dot H^2} \lesssim c_k 2^{-\frac k2}\|\nabla_{\rm h} v\|_{\dot H^1},
\end{aligned}$$

where we used $\operatorname{div} v = 0$ so that $\partial_3^2 v^3 = -\operatorname{div}_{\rm h}\partial_3 v^{\rm h}$. Then along the same line to the proof of (45), we write

$$\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} R^{\rm h} T^{\rm v}(\partial_i v^3, \partial_i v^m)\|_{L^2} &\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} \|\Delta_{k'}^{\rm h} S_{\ell'-1}^{\rm v}\partial_i v^3\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\widetilde{\Delta}_{k'}^{\rm h}\widetilde{\Delta}_{\ell'}^{\rm v}\partial_i v^m\|_{L^2}\\
&\lesssim 2^k \sum_{\substack{k'\geq k-3\\ |\ell'-\ell|\leq 4}} c_{k'}c_{k',\ell'}2^{-\frac{k'}{2}}\|v^3\|_{\dot H^2}\|\partial_i v^m\|_{L^2}\\
&\lesssim c_\ell 2^{\frac k2}\|\nabla_{\rm h} v\|_{\dot H^1}\|\nabla_{\rm h} v^{\rm h}\|_{L^2}.
\end{aligned}$$

Whereas applying Lemma A.1 once again gives

$$\begin{aligned}
\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} R^{\rm h} R^{\rm v}(\partial_i v^3, \partial_i v^m)\|_{L^2} &\lesssim 2^k 2^{\frac\ell2}\sum_{\substack{k'\geq k-3\\ \ell'\geq \ell-3}} \|\Delta_{k'}^{\rm h}\Delta_{\ell'}^{\rm v}\partial_i v^3\|_{L^2}\|\widetilde{\Delta}_{k'}^{\rm h}\widetilde{\Delta}_{\ell'}^{\rm v}\partial_i v^m\|_{L^2}\\
&\lesssim 2^k 2^{\frac\ell2}\sum_{\substack{k'\geq k-3\\ \ell'\geq \ell-3}} c^2_{k',\ell'}2^{-\frac{k'}{2}}2^{-\frac{\ell'}{2}}\|v^3\|_{\dot H^{\frac32,\frac12}}\|\partial_i v^m\|_{L^2}\\
&\lesssim c_\ell 2^{\frac k2}\|v^3\|_{\dot H^2}\|\nabla_{\rm h} v^{\rm h}\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}R^{\rm h}\bar{T}^{\rm v}(\partial_i v^3,\partial_i v^m)\|_{L^2} &\lesssim 2^k\sum_{\substack{k'\geq k-3\\|\ell'-\ell|\leq 4}}\|\Delta_{k'}^{\rm h}\Delta_{\ell'}^{\rm v}\partial_i v^3\|_{L^2}\|\widetilde{\Delta}_{k'}^{\rm h}S_{\ell'-1}^{\rm v}\partial_i v^m\|_{L^2_{\rm h}(L^\infty_{\rm v})}\\ &\lesssim 2^k\sum_{\substack{k'\geq k-3\\|\ell'-\ell|\leq 4}}c_{k',\ell'}^2 2^{-\frac{k'}{2}}\|v^3\|_{\dot H^{\frac32,\frac12}}\|\partial_i v^m\|_{L^2}\\ &\lesssim c_\ell 2^{\frac{k}{2}}\|v^3\|_{\dot H^2}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}.\end{aligned}$$

This leads to (59) due to the fact that $\|v^3\|_{\dot H^2}\leq C\|\nabla_{\rm h}v\|_{\dot H^1}$.

Thanks to (59), we get, by a similar derivation of (41), that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}R^{\rm h}(\partial_i v^3,\partial_i v^m)|\partial_3 v^{\rm h}_{\natural,\Lambda}\,dx\Big| &\lesssim\Lambda^{-\frac12}\|\nabla_{\rm h}v\|_{\dot H^1}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}\partial_3 v^{\rm h}\|_{L^2}\\ &\lesssim\Lambda^{-\frac12}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}v\|_{\dot H^1}^2.\end{aligned}\tag{60}$$

By summing up (55) and (57), (58) and (60), we achieve

$$\begin{aligned}|I(v^{\rm h},v^3)| &\leq C\Big(\|v^3\|_{\dot H^{\frac32}}\big(\|\partial_3 v^{\rm h}\|_{L^2}+\log^{\frac12}\Lambda\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}\big)+\Lambda^{-\frac12}\|\nabla_{\rm h}v\|_{\dot H^1}^2\Big)\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\\ &\leq C\Big(\log\Lambda\|v^3\|_{\dot H^{\frac32}}^2\|\nabla_{\rm h}v^{\rm h}\|_{L^2}^2+\Lambda^{-\frac12}\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\|\nabla_{\rm h}v\|_{\dot H^1}^2\Big)\\ &\quad+\frac13\|\nabla_{\rm h}v^{\rm h}\|_{\dot H^1}^2+\|\partial_3 v^{\rm h}\|_{L^2}^2\,.\end{aligned}\tag{61}$$

Choosing Λ in the above inequality such that

$$\Lambda=\big(6C\|\nabla_{\rm h}v^{\rm h}\|_{L^2}\big)^2$$

gives rise to (53). □

Appendix A. Tool box on Functional spaces

Let us mention that, as in [5], [6], [7] and [9], the definitions of the function spaces we are going to work with require anisotropic dyadic decomposition of the Fourier variables. Let us first recall some basic facts on anisotropic Littlewood-Paley theory from [1]

$$\begin{aligned}&\Delta_j a=\mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}),\quad \Delta_k^{\rm h}a=\mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\rm h}|)\widehat{a}),\quad \Delta_\ell^{\rm v}a=\mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}),\\ &S_j a=\mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}),\quad S_k^{\rm h}a=\mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\rm h}|)\widehat{a}),\quad S_\ell^{\rm v}a=\mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}),\end{aligned}\tag{A.1}$$

where $\xi=(\xi_{\rm h},\xi_3)$ and $\xi_{\rm h}=(\xi_1,\xi_2)$, $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a,

$\widehat{a}$ denote the Fourier transform of the distribution a, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\begin{aligned}&\mathrm{Supp}\ \varphi\subset\Big\{\tau\in\mathbb{R}\,/\ \ \frac34\leq|\tau|\leq\frac83\Big\}\quad\text{and}\quad\forall\tau>0\,,\ \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1,\\ &\mathrm{Supp}\ \chi\subset\Big\{\tau\in\mathbb{R}\,/\ \ |\tau|\leq\frac43\Big\}\qquad\text{and}\quad\forall\tau\in\mathbb{R}\,,\ \chi(\tau)+\sum_{j\geq0}\varphi(2^{-j}\tau)=1.\end{aligned}$$

We first recall the definition of homogeneous Besov space:

Definition A.1. *Let (p,q,r) be in $[1,\infty]^3$ and s in $\mathbb{R}$. Let us consider u in $\mathcal{S}'_h(\mathbb{R}^d)$, which means that u is in $\mathcal{S}'(\mathbb{R}^d)$ and satisfies $\lim\limits_{j\to-\infty}\|S_j u\|_{L^\infty}=0$. We set*

$$\|u\|_{\dot B^s_{p,r}}\stackrel{\rm def}{=}\big\|\big(2^{js}\|\Delta_j u\|_{L^p}\big)_j\big\|_{\ell^r(\mathbb{Z})}.$$

- *For* $s < \frac{d}{p}$ *(or* $s = \frac{d}{p}$ *if* $r = 1$*), we define* $\dot{B}^s_{p,r}(\mathbb{R}^d) \stackrel{\text{def}}{=} \big\{u \in \mathcal{S}'_h(\mathbb{R}^d) \;\big|\; \|u\|_{\dot{B}^s_{p,r}} < \infty\big\}$.
- *If* $k \in \mathbb{N}$ *and if* $\frac{d}{p}+k \leq s < \frac{d}{p}+k+1$ *(or* $s = \frac{d}{p}+k+1$ *if* $r = 1$*), then we define* $\dot{B}^s_{p,r}(\mathbb{R}^d)$ *as the subset of* u *in* $\mathcal{S}'_h(\mathbb{R}^d)$ *such that* $\partial^\beta u$ *belongs to* $\dot{B}^{s-k}_{p,r}(\mathbb{R}^d)$ *whenever* $|\beta| = k$.

We remark that $\dot{B}^s_{2,2}$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s$.

Definition A.2. *Let us define the space* $\big(\dot{B}^{s_1}_{p_1,r_1}\big)_{\mathrm{h}}\big(\dot{B}^{s_2}_{p_2,r_2}\big)_{\mathrm{v}}$ *as the space of distribution in* $\mathcal{S}'_h$ *such that*

$$\|u\|_{\left(\dot{B}^{s_1}_{p_1,r_1}\right)_{\mathrm{h}}\left(\dot{B}^{s_2}_{p_2,r_2}\right)_{\mathrm{v}}} \stackrel{\text{def}}{=} \Big(\sum_{k\in\mathbb{Z}} 2^{r_1 k s_1}\Big(\sum_{\ell\in\mathbb{Z}} 2^{r_2\ell s_2}\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_\ell u\|^{r_2}_{L^{p_1}_{\mathrm{h}}L^{p_2}_{\mathrm{v}}}\Big)^{r_1/r_2}\Big)^{1/r_1}$$

is finite. When $p_1 = p_2 = p$, $r_1 = r_2 = r$, *we briefly denote* $\big(\dot{B}^{s_1}_{p,r}\big)_{\mathrm{h}}\big(\dot{B}^{s_2}_{p,r}\big)_{\mathrm{v}}$ *as* $\dot{B}^{s_1,s_2}_{p,r}$. *In particular, we shall denote* $\dot{B}^{s_1,s_2}_{2,2}$ *by* $\dot{H}^{s_1,s_2}$.

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [7, 9]:

Lemma A.1. *Let* $\mathcal{B}_h$ *(resp.* $\mathcal{B}_v$*) a ball of* $\mathbb{R}^2_h$ *(resp.* $\mathbb{R}_v$*), and* $\mathcal{C}_h$ *(resp.* $\mathcal{C}_v$*) a ring of* $\mathbb{R}^2_h$ *(resp.* $\mathbb{R}_v$*); let* $1 \leq p_2 \leq p_1 \leq \infty$ *and* $1 \leq q_2 \leq q_1 \leq \infty$. *Then there holds:*
If the support of $\widehat{a}$ *is included in* $2^k\mathcal{B}_h$, *then*

$$\|\partial^\alpha_{x_{\mathrm{h}}} a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))}\|a\|_{L^{p_2}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})}.$$

If the support of $\widehat{a}$ *is included in* $2^\ell\mathcal{B}_v$, *then*

$$\|\partial^\beta_{x_3} a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})} \lesssim 2^{\ell(\beta+(1/q_2-1/q_1))}\|a\|_{L^{p_1}_{\mathrm{h}}(L^{q_2}_{\mathrm{v}})}.$$

If the support of $\widehat{a}$ *is included in* $2^k\mathcal{C}_h$, *then*

$$\|a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})} \lesssim 2^{-kN}\sup_{|\alpha|=N}\|\partial^\alpha_{x_{\mathrm{h}}} a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})}.$$

If the support of $\widehat{a}$ *is included in* $2^\ell\mathcal{C}_v$, *then*

$$\|a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})} \lesssim 2^{-\ell N}\|\partial^N_{x_3} a\|_{L^{p_1}_{\mathrm{h}}(L^{q_1}_{\mathrm{v}})}.$$

Lemma A.2 (Lemma 4.3 of [6])**.** *For any* s *positive and any* θ *in* $]0, s[$, *we have*

$$\|f\|_{(\dot{B}^{s-\theta}_{p,q})_{\mathrm{h}}(\dot{B}^{\theta}_{p,1})_{\mathrm{v}}} \lesssim \|f\|_{\dot{B}^s_{p,q}}.$$

At the end of this section, let us recall the para-differential decomposition (Bony's decomposition) from [2]: let a and b be in $\mathcal{S}'(\mathbb{R}^3)$, then we have the following decomposition

$$\begin{gathered} ab = T(a,b) + \bar{T}(a,b) + R(a,b) \quad \text{with} \\ T(a,b) = \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b, \quad \bar{T}(a,b) = T(b,a), \quad R(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a\widetilde{\Delta}_j b, \end{gathered} \tag{A.2}$$

where $\widetilde{\Delta}_j b \stackrel{\text{def}}{=} \sum_{|j'-j|\leq 1}\Delta_{j'}b$. In order to study product laws between distributions in anisotropic Besov spaces, we shall also use Bony's decomposition in both horizontal variables and vertical variable.

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(J.-Y. Chemin) Laboratoire Jacques Louis Lions - UMR 7598, Université Pierre et Marie Curie, Boîte courrier 187, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail address: chemin@ann.jussieu.fr

(I. Gallagher) DMA, École normale suprieure, CNRS, PSL Research University, 75005 Paris, and UFR de mathématiques, Université Paris-Diderot, Sorbonne Paris-Cité, 75013 Paris, France.

E-mail address: gallagher@math.ens.fr

(P. Zhang) Academy of Mathematics & Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, The Chinese Academy of Sciences, CHINA

E-mail address: zp@amss.ac.cn