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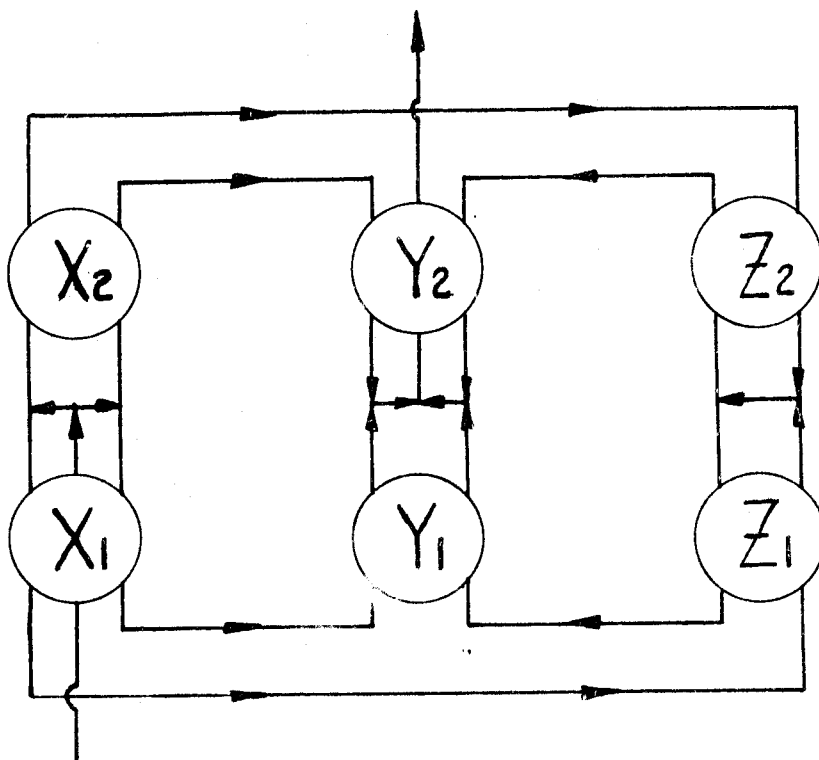For more information, please contact the WRAP Team at: publications@warwick.ac.uk

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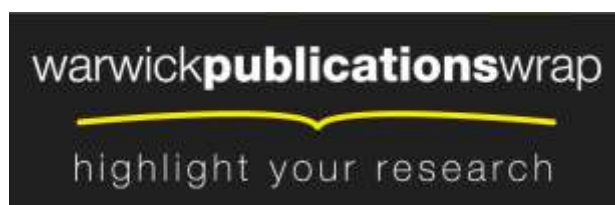
# The University of Warwick

# THEORY OF

# COMPUTATION

# REPORT

A Complete Natural Deduction System
for the
Relational Calculus

WILLIAM W.WADGE

## 0. Introduction

A relational calculus is a formal system in which relation is the fundamental concept. The simplest relational calculus, that of ordinary binary relations, was introduced by Tarski in [4]. Tarski's system is essentially an algebra in which the operations are the usual Boolean operations (on sets of ordered pairs) together with the two special operations <u>converse</u> (denoted by $\smile$) and composition (denoted by ";" or by juxtaposition). For any binary relations P and Q

$$\breve{P} = \{ <x,y> \mid <y,x> \in P \}$$
$$P;Q = \{ <x,y> \mid <x,v> \in P \text{ and } <v,z> \in Q \text{ for some } v \}.$$

The unusual feature of the relational calculi is the manner in which it is possible to make assertions about relations without referring to points. For example, the inclusion $R;R \subseteq R$ holds iff R is transitive, and $R = \breve{R}$ iff R is symmetric. In a sense the relational calculi are to the concept of relation as category theory is to the concept of function.

Interest in relational calculi has been spurred recently by the search for suitable mathematical systems in which to express the semantics of programming languages. Given a machine of the appropriate type we can associate with each program or part of a program the binary relation which characterizes its input-output behaviour: as de Roever says in [3,p.1]

> "the collection of all pairs [consisting] of an initial state of
> the memory, for which the program terminates, and its corresponding
> final state of the memory."

For example, if relations $R_1$ and $R_2$ represent programs $P_1$ and $P_2$ respectively, then the relation $R_1;R_2$ represents the program "first $P_1$ then $P_2$", and the equation

$$(R_1;R_2);U = U$$

(where U is the universal relation) asserts that the resulting program halts for every initial state.

For this approach to work it is necessary to consider not just simple binary relations (between points and points) but also typed binary relations (between sequences of points and sequences of points). For example, the system described by Hitchcock and Park in [1] has for each natural numbers $n$ and $m$ relations of type $n \to m$ which under a standard interpretation would denote subsets of $D^n \times D^m$.

Certain difficulties arise when it comes to constructing proof systems for relational calculi. The usual approach is to axiomatize the notion of a relational algebra. Tarski gave as his axioms the usual Boolean identities plus

T1: $(X;Y);Z = X;(Y;Z)$

T2: $\breve{\breve{X}} = X$

T3: $(X;Y)\breve{} = \breve{Y};\breve{X}$

T4: $X;E = X$

T5: $(X;Y) \cap Z = \Omega \rightarrow (Y;\breve{Z}) \cap \breve{X} = \Omega$

where $U$ is the universal relation, $\Omega$ is its complement and $E$ is equality. De Roever's axioms are basically a typed version of Tarski's with some added rules for projections. Neither system has any non-logical rules of inference, i.e. any rules other than substitution, replacement, modus ponens etc. A model of these axioms is a _relation_ algebra, and a relation **algebra** is _standard iff_ it is isomorphic to an algebra obtained by taking a collection of relations on some domain and interpreting the relation constants and operations in the usual way (there are nonstandard relation algebras).

The trouble with at least Tarski's axioms is that they are incomplete: Lyndon in [2] showed that the inclusion

$$X_1 X_2 \cap Y_1 Y_2 \cap Z_1 Z_2 \subseteq X_1 (\breve{X_1} Y_1 \cap X_2 \breve{Y_2} \cap (\breve{X_1} Z_1 \cap X_2 \breve{Z_2})(\breve{Z_1} Y_1 \cap Z_2 \breve{Y_2})) Y_2 ,$$

though true in every standard relation algebra, does not follow from Tarski's axioms. The theory of standard relation algebras is nevertheless axiomatizable simply because it is enumerable: every formula $\varphi$ of the relational calculus

can be translated into a formula $\varphi'$ of the predicate calculus in such a way that models of $\varphi'$ correspond to standard models of $\varphi$ and vice versa. The difficulty lies in finding a sufficiently elegant axiomatization. It is possible that de Roever's axioms are enough, for for he has proved Lyndon's formula by using projections, but the question has so far not been resolved.

The most serious drawback of the algebraic approach is the tendency of proofs of even simple assertions to be complex and unnatural. Consider, for example, the proof on [3,p37] of the important lemma

$$X;Y \cap Z = X;(\check{X};Z \cap Y) \cap Z .$$

the proof is nontrivial, uses the somewhat cryptic axiom T5 and has as a substep the verification of

$$(X;\overline{\check{X};Z}) \cap Z = \Omega .$$

In this work we attempt to remedy this defect by constructing a system of <u>natural</u> <u>deduction</u> for the relational calculus (typed or untyped) which is complete for standard relation algebras, i.e. in which a formula is provable iff it is true in every standard relation algebra. The system is natural in the sense that rules used in proving an assertion are determined by the structure of the assertion. The only disadvantage of the method is that it expands the formalism by adding point variables.

## 1. <u>Informal</u> <u>Explanation</u>

To understand how the system works, consider how one might prove the lemma given above in a nonalgebraic manner. The equality is actually two inclusions, the most difficult being

$$X;Y \cap Z \subseteq X;(\check{X};Z \cap Y) \cap Z .$$

In terms of points, we must show that for any $a$ and $b$, if $aX;Y\cap Zb$ (i.e. if $\langle a,b \rangle \in X;Y \cap Z$ ) then $aX;(\check{X};Z \cap Y )\cap Zb$. The assertion that $aX;Y\cap Zb$ is equivalent to the assertion that $aX;Yb$ and that $aZb$, and the assertion that $aX;Yb$ is in turn equivalent to the assertion that $aXc$ and that $cYb$ for some $c$.

Thus we must derive $aX;(\breve{X};Z \cap Y) \cap Zb$ from the assumption that $aXc$, $cYb$ and aZb hold for some c. Now the former assertion is equivalent to the assertion that both $aX;(\breve{X};Z \cap Y)b$ and $aZb$ and since aZb is an assumption, we need only derive $aX;(\breve{X};Z \cap Y)b$. This is in turn equivalent to $aXv$ and $v\breve{X};Z \cap Yb$ for some v. Since we have aXc as an assumption it will be enough to prove $c\breve{X};Z \cap Yb$. Next we break $c\breve{X};Z \cap Yb$ down into cYb (an assumption) and $c\breve{X};Zb$ which in turn follows from $c\breve{X}w$ and wZb for any w. But both aXc and aZb are assumptions and so the proof is complete.

The important point is that in formalizing this type of proof it is not necessary to add the quantifiers and logical connectives of first order logic. For example, although in connection with ";" we used the phrases "for some" and "for any" in fact only the following two rules are needed: (1) to derive aR;Sv derive aRv and vSb for some v, and (2) to to derive a result from some assumptions plus aR;Sb derive the result from the same assumptions plus aRv and vSb for any variable v not appearing in the result or in the other assumptions.

## 2. Formalism

Our formal system consists of (1) a language whose formulas are of the form xRy with R a relation expression and x and y point variables; (2) a collection of interpretations corresponding to standard relation algebras; and (3) a relation "⊢" of deducibility generated by a set of natural deduction rules. The completeness result states that a formula F can be derived from a set $\Gamma$ of formulas iff every interpretation which makes every formula in $\Gamma$ true also makes F true.

## 2.1 Syntax

The alphabet of the formal language contains the symbols

(i) a, b, c,...          a set V of point variables;

(ii) A, B, C,...         relation variables;

(iii) ∩, U, E            relation constants;

(iv) U, ∩, ;, $^{-}$, $^{\smile}$     relation operations.

A <u>term</u>, for example "A;B∩C", is an expression formed in the usual way from the relation constants, variables and expressions. A <u>formula</u>, for example "dA;B∩Ca", is a word of the form xRy with R a term and x and y point variables. The negation of the formula xRy is the formula x$\bar{R}$y. Strictly speaking the set of formulas depends on the set V of point variables and at times we will indicate this dependence by referring to "V-formulas". Note that the symbol "⊑" is not in the alphabet; instead, we define "R ⊆ S" to be the formula "a$\overline{U;(R∩\bar{S});U}$a".

## 2.2 Semantics

For any nonvoid set D and any set V of point variables a (V,D)-interpretation $\mathcal{A}$ is a map which assigns to each term a subset of D×D and to each point variable an element of D. The map $\mathcal{A}$ must be consistent with the usual interpretation of the relation operations and constants, e.g. $\mathcal{A}$(R∩S) must be the intersection of $\mathcal{A}$(R) and $\mathcal{A}$(S), and $\mathcal{A}$(Ω) must be the empty set. Note that an interpretation is therefore determined by its restriction to the point and relation variables.

If $\mathcal{A}$ is an interpretation, x and y are point variables and R is a term, then $\mathcal{A} \models$ xRy ($\mathcal{A}$ <u>satisfies</u> xRy, or xRy is <u>true</u> in $\mathcal{A}$) iff <$\mathcal{A}$(x),$\mathcal{A}$(y)> is in $\mathcal{A}$(R). If Γ is a set of formulas and F is a formula then $\mathcal{A} \models$ Γ iff $\mathcal{A} \models$ G for every formula G in Γ, and Γ $\models$ F iff $\mathcal{A} \models$ Γ implies $\mathcal{A} \models$ F for every interpretation $\mathcal{A}$.

## 2.3 <u>Rules of Inference</u>

The rules of inference of our system are the introduction and elimination rules given on the next page together with the rule F ⊢F. In these rules F and G are arbitrary formulas; x, y and v are arbitrary variables; Γ is an arbitrary set of formulas and R and S are arbitrary terms. In (;E) "v" is assumed not to occur in any formula in Γ and in (⊑I) "x" and "y" are assumed not to occur in any formula in Γ. Formally speaking the relation "⊢" between sets of formulas and formulas is the least one for which these rules are valid.

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## Rules of Inference

<div>

(AI) $\dfrac{\Gamma \vdash F}{\Gamma,G \vdash F}$

(UI) $\dfrac{xRy}{xR\cup Sy} \qquad \dfrac{xSy}{xR\cup Sy}$

(∩I) $\dfrac{xRy \quad xSy}{xR\cap Sy}$

(⁻I) $\dfrac{\Gamma,xRy \vdash a\Omega a}{\Gamma \vdash x\bar{R}y}$

(ΩI) $\dfrac{xRy \quad x\bar{R}y}{a\Omega a}$

(UI) $\dfrac{}{x\bar{U}y}$

(⌣I) $\dfrac{xRy}{y\breve{R}x}$

(;I) $\dfrac{xRv \quad vSy}{xR;Sy}$

(⊆I) $\dfrac{\Gamma,xRy \vdash xSy}{\Gamma \vdash R \subseteq S}$

(EI) $\dfrac{}{x\bar{E}x}$

</div>

<div>

(AE) $\dfrac{\Gamma \vdash G \quad \Gamma,G \vdash F}{\Gamma \vdash F}$

(UE) $\dfrac{\Gamma,xRy \vdash F \quad \Gamma,xSy \vdash F \quad \Gamma \vdash xR\cup Sy}{\Gamma \vdash F}$

(∩E) $\dfrac{xR\cap Sy}{xRy} \qquad \dfrac{xR\cap Sy}{xSy}$

(⁻E) $\dfrac{\Gamma,xRy \vdash F \quad \Gamma,x\bar{R}y \vdash F}{\Gamma \vdash F}$

(ΩE) $\dfrac{a\Omega a}{F}$

(UE) $\dfrac{\Gamma,xUy \vdash F}{\Gamma \vdash F}$

( E) $\dfrac{y\breve{R}x}{xRy}$

(;E) $\dfrac{\Gamma,xRv,vSy \vdash F \quad \Gamma \vdash xR;Sy}{\Gamma \vdash F}$

(⊆E) $\dfrac{R \subseteq S \quad xRy}{xSy}$

(EE) $\dfrac{xRv \quad yEv}{xRy}$

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## 3. Examples

For our first example we verify Lyndon's inclusion (see p2). The proof illustrates the manner in which a natural deduction system allows one to work backward from the statement of a problem. It is for this reason that natural deduction systems are easier to mechanize than axiomatic-algebraic ones.

For convenience let

$$\psi_1 = \breve{X}_1 Z_1 \cap X_2 \breve{Z}_2$$
$$\psi_2 = \breve{Z}_1 Y_1 \cap Z_2 \breve{Y}_2$$
$$\varphi = \breve{X}_1 Y_1 \cap X_2 \breve{Y}_2 \cap \psi_1 \psi_2$$

so that we wish to show

$$X_1 X_2 \cap Y_1 Y_2 \cap Z_1 Z_2 \subseteq X_1 \varphi Y_2$$

(note we have omitted ";"s).. Then by $(\subseteq)$ it is enough to show that

$$aX_1X_2 \cap Y_1Y_2 \cap Z_1Z_2 b \vdash aX_1\varphi Y_2 b .$$

It follows from the $(\cap E)$ and the (AE) rules that it is enough to establish

$$aX_1X_2 b, \quad aY_1Y_2 b, \quad aZ_1Z_2 b \vdash aX_1\varphi Y_2 b$$

and it follows from the $(;E)$ rule in turn that it is enough to show

$$aX_1 p, \quad pX_2 b, \quad aY_1 q, \quad qY_2 b, \quad aZ_1 r, \quad rZ_2 b \vdash aX_1\varphi Y_2 b .$$

Let us call the set of formulas on the left hand side $\Gamma$. Then we will have

$$\Gamma \vdash aX_1\varphi Y_2 b$$

iff from $\Gamma$ we can derive $aX_1 v$, $v\varphi w$ and $wY_2 b$ for some $v$ and $w$. The fact that $aX_1 p$ and $qY_2 b$ are in $\Gamma$ suggest $v=p$ and $q=w$. Thus it remains to derive $p\varphi q$ from $\Gamma$. Since

$$\varphi = \breve{X}_1 Y_1 \cap X_2 \breve{Y}_2 \cap \psi_1 \psi_2$$

we must derive $p\breve{X}_1 Y_1 q$, $pX_2\breve{Y}_2 q$ and $p\psi_1\psi_2 q$ from $\Gamma$. The first two are immediate: for example, since $aX_1 p$ and $aY_1 q$ are in $\Gamma$, we have $p\breve{X}_1 a$ by $(\breve{\ }E)$ and then $p\breve{X}_1 Y_1 q$ by $(;I)$. To derive $p\psi_1\psi_2 q$ we need $p\psi_1 t$ and $t\psi_2 q$ for some $t$. From the definition of $\psi_1$ it follows we must derive $p\breve{X}_1 Z_1 t$ and $pX_2\breve{Z}_2 t$, and this follows easily if we take $t=r$. Since $r\psi_2 q$ also follows in the same way, the proof is complete.

For our second example we verify Tarski's fifth axiom, i.e. show that

$$X;Y \cap Z \subseteq \Omega \vdash Y;\breve{Z} \cap \breve{X} \subseteq \Omega.$$

The proof is presented in straight line form as in first order logic, each assertion being derived from those above and not to the left.

| (1) | $X;Y \cap Z \subseteq \Omega$ | assumption |
|-----|------|------|
| (2) | $aY;\breve{Z} \cap \breve{X} b$ | assumption |
| (3) | $aY;\breve{Z} b$ | $(\cap E)$ and (2) |
| (4) | $a\breve{X} b$ | " |
| (5) | $b X a$ | $(\breve{\ }E)$ |
| (7) | $aYv$ | assumption |
| (8) | $v\breve{Z} b$ | assumption |
| (9) | $bZv$ | $(\breve{\ }E)$ and (8) |
| (10) | $bX;Yv$ | $(;I)$ using (5) and (7) |
| (11) | $bX;Y \cap Zv$ | $(\cap I)$ using (9) and (10) |

(12)    $b \cap v$          $(\subseteq E)$ using (1) and (11)

(13)    $a \cap b$          $(\cap E)$ using (12)

(14)    $a \cap b$          $(;E)$ using (13) and (3) cancelling (7),(8)

(15)    $Y;Z \cap X \subseteq \cap$    $(\subseteq I)$ using (14) cancelling (2)

## 4. Completeness

In this section we show that $\vdash$ and $\models$ are equivalent, i.e. that
$\Gamma \models F$ iff $\Gamma \vdash F$ for all $\Gamma$ and F. The 'if' direction, soundness, is
straight forward.

**Theorem I.** For any set V of variables, any set $\Gamma$ of V-formulas and
any V-formula F:
$$\text{if } \Gamma \vdash F \text{ then } \Gamma \models F .$$

**Proof.** Since $\vdash$ was defined to be the least relation for which our rules
are valid, it is sufficient to verify that each of the rules is sound (this is
omitted).

Our proof of the 'only if' direction is an adaptation of the Henkin com-
pleteness proof for first order logic - we show that every consistent theory
has an interpretation. By a _V-theory_ (V a set of variables) we mean a set T
of V-formulas such that T $\vdash$ F implies F $\in$ T for every V-formula F. A
V-theory T is _consistent_ iff T does not contain the negation of any formula
in T, and T is _complete_ iff T contains the negation of every formula not
in T. Finally, T is _Henkin-complete_ (H-complete) iff for any variables x
and y and any terms R and S, if $xR;Sy \in T$ then there is a v in T
such that $xRv \in T$ and $vSy \in T$. The strategy of the proof is to show that
every consistent theory has a consistent, complete, H-complete extension, and
that every such theory has an interpretation.

The first step is to show that every consistent theory has a consistent
complete extension.

Theorem II. For any countable set  V  of variables and any V-theory  T:

if  T  is consistent there is a consistent complete V-theory  T'

extending  T.

Proof. The standard proof carries over. Let  $\langle F_i \mid i \in N \rangle$  be an enumeration

of all V-formulas. Define a sequence  $\langle T_i \mid i \in N \rangle$  of V-theories extending  T

by setting  $T_0 = T$  and  $T_{i+1} = T_i$  if  $\bar{F}_i \in T_i$ ,  otherwise  $T_{i+1} =$  the closure

of  $T \cup \{F_i\}$ .  It is easily verified that  $\cup\{T_i \mid i \in N\}$  is a consistent com-

plete extension of  T.

Next we show that every consistent theory has a consistent H-complete

extension.

Theorem III. For any countable set  V  of variables and any V-theory  T:

if  T  is consistent there is a countable set  V'  of variables ex-

tending  V'  and a consistent H-complete V'-theory  T'  extending  T.

Proof. Define a sequence  $\langle V_i \mid i \in N \rangle$  of sets of variables and a sequence

$\langle T_i \mid i \in N \rangle$  of theories, each  $T_i$  a  $V_i$-theory,  as follows:

(1)  $T_0 = T$ ;

(2)  for any  i,  for each formula in  $T_i$  of the form  xR;Sy  add to

$V_i$  a new variable  v  and add to  $T_i$  the formulas  xRv  and  vSy.  The

resulting set of variables is  $V_{i+1}$  and the closure under  $\vdash$  of the resulting

set of formulas is  $T_{i+1}$ .

Then the  $\cup\{V_i \mid i \in N\}$-theory  T'  $(= \cup\{T_i \mid i \in N\})$  is a consistent H-complete

extension of  T.

To see this note first that an easy induction on  i  shows that each  $T_i$

is consistent. In the case  i = 0  the result is immediate. Now suppose that

for some  i,  $T_i$  is consistent but  $T_{i+1}$  is not. Then  $T_{i+1} \vdash a \cap a$  and so

by the definition of  $T_{i+1}$  there is a finite set of the form

$$\{x_j R_j v_j,\ v_j S_j y_j \mid j \leq n\}$$

such that

$$T_i,\ x_0 R_0 v_0,\ v_0 S_0 y_0,\ \cdots,\ x_n R_n v_n,\ v_n S_n y_n \vdash a \cap a.$$

But then  n  applications of the (;E) rule give us  $T_i \vdash a\Omega a$  which contradicts the consistency of  $T_i$ .

Since each  $T_i$  is consistent,  T'  must be consistent.

Next, note that  T'  must be H-complete. For if  xR;Sy  is in  T' , it must be in  $T_i$  for some  i. But then there is a  v  in  $V_{i+1}$  such that  xRv  and  vSy  are in  $T_{i+1}$  and therefore in  T'.

Finally we put the last two theorems together to get the desired result on extensions.

Theorem IV. For any countable set  V  of variables and any V-theory  T:

if  T  is consistent there is a set  V'  of variables extending  V  and a consistent, complete, H-complete theory  T'  extending  T.

Proof. Define a sequence  $\langle T_i \mid i \in N \rangle$  of theories extending  T  as follows:

(1)  $T_0 = T$ ;

(2)  $T_{2i+1}$  is a complete consistent extension of  $T_{2i}$ ;

(3)  $T_{2i+2}$  is an H-complete consistent extension of  $T_{2i+1}$ .

It is easily verified that  $\cup \{ T_i \mid i \in N \}$  is a consistent, complete, H-complete extension of  T.

Theorem V. For any countable set  V  of variables and any V-theory  T:

if  T  is consistent, complete and H-complete then there is an interpretation  $\mathcal{A}$  such that  $\mathcal{A} \models F$  for every  F  in  T.

Proof. For any varibles  x  and  y  in  V  let  $x \cong y$  iff  $xEy \in T$ . It is easily verified that  $\cong$  is an equivalence relation. Therefore for any  v  in  V  let  $\tilde{v}$  be the equivalence class of  v,  and let  $\tilde{V} = \{ \tilde{v} \mid v \in V \}$ . Then for any variable  v  in  V  let  $\mathcal{A}(v) = \tilde{v}$ ,  and for any  V-formula  xRy  let  $\mathcal{A}(xRy) = \{ \langle \tilde{x}, \tilde{y} \rangle \mid xRy \in T \}$ . The map  $\mathcal{A}$  so defined is then an interpretation of  T.

We must first show that  $\mathcal{A}$  is in fact an interpretation, i.e. that  $\mathcal{A}$  is consistent with the intended meanings of the relation operations and

constants. We give only the proofs for ";" and "U", the others being similar.

(1) Union. We must show that $\mathcal{O}(R \cup S)$ is the union of $\mathcal{O}(R)$ and $\mathcal{O}(S)$ for any terms R and S. This amounts to showing that for any variables x and y in V, $xR \cup Sy \in T$ iff $xRy \in T$ or $xSy \in T$.

If either xRy or xSy is in T the (UI) rule gives us $xR \cup Sy \in T$.

Now suppose that $xR \cup Sy \in T$ but that neither xRy nor xSy is in T. Since T is complete, both $x\bar{R}y$ and $x\bar{S}y$ must be in T. But this means that $x\bar{R} \cap \bar{S}y \in T$ and therefore (omitting the proof) that $x\overline{\bar{R} \cap \bar{S}}y \in T$, which contradicts the consistency of T.

(1) Composition. We must show that $\langle \bar{x}, \bar{y} \rangle \in \mathcal{O}(R;S)$ iff $\langle \bar{x}, \bar{v} \rangle \in \mathcal{O}(R)$ and $\langle \bar{v}, \bar{y} \rangle \in \mathcal{O}(S)$ for some v. This amounts to showing that $xR;Sy \in T$ iff $xRv \in T$ and $vSy \in T$ for some v, and this is just the statement of H-completeness.

Finally, the fact that $\mathcal{O} \models F$ for every F in T is immediate.

Corollary. Every consistent set of formulas has an interpretation.

Proof. Extend the theory of the set to a consistent, complete, H-complete theory.

This gives us the main result.

Theorem V. For any countable set V of variables, any set $\Gamma$ of V-formulas and any V-formula F:

if $\Gamma \models F$ then $\Gamma \vdash F$.

Proof. Suppose that F does not follow from $\Gamma$. Then $\Gamma$ and the negation of F are consistent and so have an interpretation, impossible.

5. Typed Relational Calculi

This natural deduction system can be carried over quite easily to a typed system such as that of Hitchcock and Park. Formulas would be of the form $x_1 x_2 \ldots x_n R y_1 y_2 \ldots y_m$ with R a term of type $n \to m$. The rules of inference given on page 6 would remain valid if x, y and are interpreted as sequences of

variables of appropriate lengths.  The rules for the extra operations are completely obvious, e.g.

$$(|I) \quad \frac{x_0 \cdots x_j R y_0 \cdots y_k \quad v_0 \cdots v_m S w_0 \cdots w_n}{x_0 \cdots x_j v_0 \cdots v_m R | S y_0 \cdots y_k w_0 \cdots w_n}$$

$$(|E) \quad \frac{x_0 \cdots x_j v_0 \cdots v_m R | S y_0 \cdots y_k w_0 \cdots w_n}{x_0 \cdots x_j R y_0 \cdots y_k} \qquad \frac{x_0 \cdots x_j v_0 \cdots v_m R | S y_0 \cdots y_k w_0 \cdots w_n}{v_0 \cdots v_m S w_0 \cdots w_n}$$

The completeness proof for this typed system goes through with little change. The extra operations offer no difficulty because their rules are purely definitional, like the rules for $\smile$.

## References

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