HARMONIC MAPPINGS OF SPHERES

by

R. T. Smith

Thesis presented for the degree of

Doctor of Philosophy

University of Warwick
July 1972

BEST COPY

AVAILABLE

Variable print quality

BEST COPY AVAILABLE

TEXT IN ORIGINAL IS CLOSE TO THE EDGE OF THE PAGE

Abstract

This thesis is addressed to the following fundamental problem: given a homotopy class of maps between compact Riemannian manifolds N and M, is there a harmonic representative of that class? Eells and Sampson have given a general existence theorem for the case that M has no positive sectional curvatures [ES]. Otherwise, very little is known. Certainly no counterexample has ever been established.

The most important contributions of this dissertation are two: firstly, we have a direct construction technique for producing some essential harmonic maps between Euclidean spheres. Topologically, this consists simply of joining two harmonic polynomial mappings (e.g., the Hopf fibrations). Analytically, however, this method has a novel physical motivation: we study the equation of motion of an exotic pendulum driven by a gravity which changes sign. If this system has an exceptional trajectory of the right sort, it defines a harmonic map of spheres. One consequence of our theorem is that $\pi_n(S^n)$ is represented by harmonic maps for $n = 1,\dots,7$. Finally, the rudiments of an equivariant theory of harmonic maps having been set out earlier, we find that our examples can also be put in this framework.

The second significant result which arose from this study is a strong candidate for a counterexample: suppose S^n is stretched to a length b in one direction to make an ellipsoid $E^n(b)$. Then if $n \geq 3$ and b is large enough, there is <u>no</u> harmonic stretching (of degree one) of S^n onto $E^n(b)$. However, if $b = 1$ the identity is such a harmonic map, so it certainly appears that the existence of a harmonic representative in a homotopy class can depend upon the metric.

We also examine here a large collection of examples of harmonic maps of spheres which are defined by harmonic polynomials and orthogonal multiplications. The last chapter takes up the study of the Morse theory of a harmonic map: amongst several pleasing results, we have an example of a simple map whose index and degeneracy can be made arbitrarily large by equally simple changes in the metrics.

TABLE OF CONTENTS

Abstract i

Contents iii

Introduction 1

Chapter 1. QUALITATIVE THEORY OF HARMONIC MAPPINGS 6

Section 1. Energy and Tension 6

Section 2. Harmonic Fibrations 8

Section 3. Equivariant Harmonic Maps 12

Chapter 2. HARMONIC POLYNOMIAL MAPPINGS 20

Section 1. Spherical Harmonics 20

Section 2. Orthogonal Multiplications 25

Section 3. Mappings Associated to Orthogonal Multiplications 35

Chapter 3. JOINING POLYNOMIAL MAPPINGS: THE PENDULUM EQUATION 43

Section 1. Main Theorem, Derivation of Equation 43

Section 2. Existence 46

Section 3. Asymptotic Behavior 57

Section 4. Derivative Estimates 61

Section 5. The First Suspension 67

Section 6. Applications 68

Section 7. The Hopf Construction; Non-Conservation of Energy 74

Chapter 4. MAPPINGS OF ELLIPSOIDS AND TORI 78

Section 1. Maps of Ellipsoids into Spheres 78

Section 2. Maps of Ellipsoids into Ellipsoids 81

Section 3. Mappings of Tori 93

Chapter 5. THE SECOND VARIATION OF THE ENERGY 97

Section 1. Generalities 97

Section 2. The Identity Map 102

Section 3. Harmonic Fibrations 110

Section 4. Holomorphic Mappings of Kahler Manifolds 112

Section 5. Maps of Spheres 117

Appendix. A MORSE LEMMA FOR COMPACT OPERATORS 120

References 125

Introduction

A fundamental problem in Riemannian geometry posed by Eells and Sampson in [ES] is the following: given a homotopy class of maps between compact Riemannian manifolds N and M, is there a harmonic representative of that class? As the theory of harmonic mappings simultaneously generalizes the classical Dirichlet problem and the theory of geodesics, the importance of this question is self-evident. On the other hand, this problem is neither linear nor one-dimensional and is therefore much more poorly understood than its origins.

For example, there is no known case when the answer to the above question has been proved to be negative (with emphasis on the condition that M be compact). On the other hand, the basic affirmative result in the theory is this: if M has no positive sectional curvatures, then every homotopy class can be harmonically represented [ES]. Unfortunately, this condition excludes the most topologically interesting compact manifolds, so our question is still wide open.

The object of this thesis is to enrich the theory of harmonic mappings by providing vast quantities of examples, together with a strong candidate for a homotopy class with no harmonic maps. The title reflects the fact that most of our constructions take place on the Euclidean sphere; certainly everything in the thesis is at least motivated by the desire to study harmonic maps of spheres, as shall be indicated. Before explaining the contents more fully, however, we should remark that two other basic questions are raised and partially answered by the examples. Firstly, what can a harmonic map look like? For instance, what can one say about the topological behavior and differential topology of a harmonic map? One could also try to formulate an equidistribution theory for harmonic maps. Secondly, where should one look for the answer to the fundamental problem posed earlier? In other words, a) can there ever be an all powerful analytical technique (heat equation, variational calculus, or whatever) which will produce harmonic maps on command, or b) if an obstruction exists, will it be a homotopy property, or will it more properly be a function of the metrics? The example alluded to above suggests that a homotopy class may be representable for one choice of metric and not another.

Let us now outline the contents, and in so doing try to give substance to the foregoing extravagant remarks. In the first chapter, then, are collected some elementary results about the behavior of tension field which will bear significantly on the work to come. Also included is a longer section on equivariant harmonic mappings, modelled after the treatment of minimal submanifolds by Hsiang and Lawson in [HL]. The basic new problem solved here is to find reasonable conditions on an equivariant map such that the question of being harmonic reduces to a problem on the orbit space (Theorem 1.3.5). It turns out that some of the maps between spheres constructed later can be interpreted in just this way.

The second chapter deals with maps between spheres defined by homogeneous harmonic polynomials. Having made the basic observation that such maps are harmonic, we exhibit examples. Most are inessential, but they are all geometrically interesting in that they help give a picture of what a harmonic map can look like. Certainly these polynomial mappings form a natural testing ground for any proposed equidistribution theory.

Although the totality of such examples is too diffuse to be catalogued here, a few illustrations are in order: the polynomial maps of homogeneity two are of particular interest, the Hopf fibrations being examples. Some new examples of this type are constructed; these factor over various projective spaces to yield harmonic embeddings of the latter in Euclidean spheres. The orthogonal multiplications define some interesting maps of products of spheres; the complex tensor product in particular gives some harmonic maps which are closely tied to the Segré embeddings of projective spaces. We also find that there is a natural correspondence between such multiplications and certain geodesic spheres in Stiefel manifolds.

The third chapter is really the core of the thesis. A method is developed here for producing some essential harmonic maps between spheres. The basic idea is to join two harmonic polynomial maps so as to make the result harmonic (Theorem 3.1.1) This problem is feasible because it can be reduced to studying an ordinary differential equation for the join parameter. We are able to study this equation successfully because under the right transformation of variables it becomes the equation of motion of an exotic pendulum.

This pendulum is driven by a gravity which changes sign, and the problem before us in physical terms is to make it stand on end at $t = \infty$ and hang straight down at $t = -\infty$. A physical argument sets out the program for doing this, which is completed in Section 3.2 under the assumption that the equation is under-damped. This accomplished, a rather lengthy effort is required to analyze the asymptotic behavior of the machine and prove global regularity of the map.

A typical corollary of our main theorem is that $\pi_n(S^n) = Z$ can be represented by harmonic maps for $n = 1, \ldots, 7$. This was previously known only for $n = 1,2$ and for special reasons.

The dimension restriction arises from the under-damping condition mentioned above, and its significance has not been determined.

Chapter 4 pursues the above method into the domain of ellipsoids. The most striking result here falls on the negative side: suppose S^n is stretched to a length b in one direction to make an ellipsoid $E^n(b)$. The conclusion is that if $n \geq 3$ and b is large enough, there is no harmonic one-parameter stretching (of degree one) of S^n onto $E^n(b)$ (Theorem 4.2.4). Since the identity is such a harmonic map for b = 1, it certainly appears that the existence of a harmonic representative in a homotopy class can depend upon the metric. Note furthermore that if there is a harmonic map of degree one in the former case, it cannot be found by any reasonably general analytic technique. For if such a technique is fed symmetrical initial data, it will preserve that symmetry throughout the deformation; ergo the result cannot be harmonic by our theorem.

This chapter also contains an ODE construction of an interesting harmonic map of the flat torus to S^2. This map has folds, behavior which is prohibited for complex analytic mappings.

The last chapter takes up the study of the second variation of the energy. The original motivation was to compute the Morse index of our maps of spheres, but this turned out to be nearly impossible. However, the Morse theory of certain very simple harmonic maps is definitely instructive. For example, the second variation operator at the identity map happens to have a long history in the theory of transformation groups; however, its role had always been rather mysterious, and it is nice to have here a unified interpretation of its significance. Using standard methods, we are also able to deduce some interesting relations between the index of the identity, the Lie algebra $\underline{c}$ of conformal vector fields, and the spectrum of the manifold.

An even more interesting (and perhaps alarming) phenomenon can be noticed when we study the Morse theory of the projection in a trivial bundle. There are examples of such maps whose index and degeneracy can be made arbitrarily large by modifying the metric on one factor. In other words, the Morse theory of such a map can be completely pathological (Example 5.3.4).

On the other hand, the behavior of holomorphic mappings between compact Kahler manifolds is correspondingly good in this regard. Such a map always has index 0, and in certain cases one can verify that there are no degeneracies.

This thesis has benefitted immensely from the valuable ideas and invaluable guidance of my official and unofficial supervisors David Elworthy and James Eells. I am also indebted to Larry Markus and David Chillingworth for their insight on differential equations at a crucial juncture in this research.

Finally, I would like to thank the Marshall Scholarship Commission and the National Science Foundation for their generosity, and my wife for her tolerance.

Chapter 1

QUALITATIVE THEORY OF HARMONIC MAPPINGS

Section 1. Energy and Tension

Let us first recapitulate the well-known variational characterization of harmonic maps. A more self-contained presentation (in local coordinates) is available in the original reference [ES]. Let N and M be smooth Riemannian manifolds without boundary, equipped with their Levi-Civita connections. Assume that N is compact and oriented and, to be reasonable, that M is complete. If $f: N \to M$ is a smooth map, the energy of f is defined

$$E(f) = \tfrac{1}{2}\int_N |df|^2 * 1$$

Here df is considered as a section of the bundle of linear maps $L(TN, f^*TM)$, which inherits a metric from those of N and M. Thus at any point $x \in N$, $|df|^2(x) = \Sigma |df(e_i)|^2 (x)$, where e_i is an orthonormal basis for T_xN. The form $*1$ is the canonical volume on N associated to the metric and orientation, and will make no further explicit appearance.

Suppose now that f_t is a smooth one-parameter variation of f, with $f_o = f$. Then f_t defines a map $F : N \times \mathbb{R} \to M$, and

$$E(f_t) = \tfrac{1}{2}\int_N \langle dF(-), dF(-)\rangle$$

where (-) denotes arguments in TN. An invariant theory of pull-back connections, as expounded in, say, Eliasson [E1], yields a quick calculation

$$\begin{aligned}\frac{\partial E}{\partial t} &= \int_N \langle \nabla dF\left(\frac{\partial}{\partial t}, -\right), dF(-)\rangle \\ &= \int_N \langle \nabla dF\left(-, \frac{\partial}{\partial t}\right), dF(-)\rangle \\ &= \int_N \langle \nabla \frac{\partial F}{\partial t}(-), dF(-)\rangle\end{aligned}$$

Hence if $w = \left.\frac{\partial f}{\partial t}\right|_{t=o}$

$$\left.\frac{\partial E}{\partial t}\right|_{t=o} = \int_N \langle \nabla w, df \rangle$$

In general, the adjoint of a connection ∇ is a generalized divergence (see Eliasson [E2]), so that

$$\int_N \langle \nabla w, df \rangle = \int_N - \langle w, \mathrm{Div}\, df \rangle$$

where $\mathrm{Div}\, df = \mathrm{Trace}\, \nabla df = \Sigma \nabla df(e_i, e_i)$. This latter quantity is generally denoted $\tau(f)$, the tension field of f. The map f is said to be harmonic if $\tau(f) = 0$; equivalently, if f is a critical point of E.

In general, the tension field is a vector field along f which points in the direction of decreasing energy. Let us examine just two of its elementary properties. Firstly, it is useful to contrast the idea of a harmonic map with the more rigid concept of a totally geodesic map; i.e., a smooth map f such that $\nabla df = 0$. Such maps are characterized by the property that they carry geodesics to geodesics, as may be seen by appeal to normal coordinates. In these coordinates such maps are linear, whereas harmonic mappings merely satisfy Laplace's equation $\Delta f = 0$ at the origin. One may perhaps think of harmonic maps as preserving geodesics on average. This requirement is more flexible, as illustrated by the following observation due to Eells.

Lemma 1.1.1 Suppose $f : X \times Y \to M$ is harmonic with respect to each variable separately; that is, for each $x \in X$, $f^x : Y \to M$ is harmonic, and vice versa. Then f is harmonic.

Proof In an obvious sense we have $\tau(f) = \tau_1(f) + \tau_2(f)$. The assumption of the lemma is that $\tau_1(f) = \tau_2(f) = 0$.

Example 1.1.2 Let G be a compact Lie group with bi-invariant metric, and let $\mu: G \times G \to G$ be multiplication. Then μ is harmonic. In fact, μ is an isometry in each variable separately. However, μ is not geodesic unless G is abelian.

A number of similar examples of harmonic maps of products of spheres can be found in Chapter II.

Another property of the operator τ which will be very useful later is the following:

Lemma 1.1.3 [ES] Let $N \subset M$ be a submanifold with the induced metric, and let $f: X \to N$ be a smooth map. Let $F: X \to M$ be the map induced by inclusion. Then $\tau(f)$ is the orthogonal projection onto TN of $\tau(F)$. In particular, f is harmonic $\Leftrightarrow \tau(F) \perp N$.

Example 1.1.4 a) Suppose $N = S^{n-1} \subset \mathbb{R}^n = M$. Then if $f: X \to S^{n-1}$ and $F: X \to \mathbb{R}^n$ is the induced map, we find that f is harmonic $\Leftrightarrow \Delta F \propto F$. Here Δ is the Laplacian for functions on X and $\propto$ means "proportional to". In fact, a short calculation shows that $\Delta F = |dF|^2 F$ must hold if f is harmonic.

b) Suppose f as above is defined by eigenfunctions of Δ_X; i.e., $\Delta F = \lambda F$. Then f is harmonic. Furthermore, it therefore follows that $|df|^2 = \lambda$; this does not, however, mean that f is an immersion. Such examples form the subject of Chapter II.

Section 2. Harmonic Fibrations

Suppose that E and B are Riemannian manifolds and that $\pi: E \to B$ is a differentiable fibre bundle. Then there is a canonical splitting $TE = V \oplus H$, where V consists of the tangents to the fibres, or vertical vectors, and H is the ortho-complement of V (horizontal vectors). We will say π is <u>Riemannian</u> if

$d\pi|H_x$ is an isometry for all x.

It is a theorem of Hermann [HR] that if π is Riemannian, the unique horizontal lift of a geodesic starting at $\pi(x)$ is a geodesic starting at x. Using this fact, Eells and Sampson characterized the harmonic Riemannian fibrations:

<u>Lemma 1.2.1</u> [ES] A Riemannian fibration π is harmonic $\Leftrightarrow$ all fibres of π are minimal submanifolds of E.

Such a map π will be called a <u>harmonic fibration</u>. (The general relationship between minimal submanifolds and harmonic maps is discussed in the next section).

In this section we want to see how harmonic fibrations behave under composition. In general, given a composite map $g \circ f$ we find that

$$\tau(g \circ f) = dg(\tau(f)) + \text{Trace } \nabla dg\ (df, df)$$

Hence [ES], if f is harmonic and g is totally geodesic, then $g \circ f$ is harmonic. If, furthermore, df has nice properties, we can do better. For example:

<u>Lemma 1.2.2</u> If π: E $\to$ B is a harmonic fibration and f : B $\to$ X is a smooth map, then f is harmonic if and only if $f \circ \pi$ is harmonic.

<u>Proof</u> From the above, $\tau(f \circ \pi) = \text{Trace } \nabla df(d\pi, d\pi)$. Now form an orthonormal basis for TE from bases for V and H; as $d\pi$ annihilates the former and transforms the latter into an ortho-normal basis for TB, it follows that $\tau(f \circ \pi)(x) = \tau(f)(\pi(x))$. Hence the lemma.

<u>Example 1.2.3</u> Suppose that μ: G x M $\to$ M is a smooth action of a compact Lie group G by isometries of M. For $x \in M$, let G_x be the orbit through x. Suppose further that there is a fixed metric on G for which π_x: G $\to G_x$ is Riemannian (up to a scalar

multiple) for each x. Then μ is harmonic if and only if π_x is harmonic for each x and each G_x is a minimal submanifold of M.

The proof is quite simple. Since G acts by isometries, it follows as in Lemma 1.1.1. that μ is harmonic if and only if the composition $G \to G_x \to M$ is harmonic for each x. If π_x is harmonic and $i_x: G_x \to M$ is harmonic, $i_x \circ \pi_x$ is harmonic by Lemma 1.2.2. Conversely, if $i_x \circ \pi_x$ is harmonic, then π_x is harmonic by Lemma 1.1.3. Hence i_x is harmonic.

Application 1.2.4 Let SO(n+1) be given its natural metric. Then $SO(n+1) \times S^n \to S^n$ is harmonic, whereas $SO(p) \times S^n \to S^n$, for $p \leq n$, is not. In the latter case there are non-minimal orbits.

Composition from the other direction does not work so nicely. For example, let $h: S^3 \to S^2$ be the Hopf map. Although h is a harmonic fibration with geodesic fibres, h is not totally geodesic. To see this, take two orthogonal vectors in $S^3 \subset \mathbb{C}^2$ as follows: $v_1 = (1,0)$, $v_2 = \frac{1}{\sqrt{2}}(i,1)$. Then $\gamma(t) = \cos(t)\, v_1 + \sin(t) v_2$ is a geodesic in S^3; however, using the formula $h(z,w) = (|z|^2 - |w|^2, 2z\bar{w})$, one can check that $h \circ \gamma$ does not describe a geodesic in S^2.

Therefore Proposition C, p. 132 in [ES] is incorrect; this proposition states that if the harmonic fibration π has geodesic fibres, then $f : X \to E$ harmonic implies $\pi \circ f$ harmonic; in particular, such a map π must be totally geodesic. The proof rested on the assumption that one could choose coordinates on E which simultaneously were normal coordinates and gave a trivialization of the bundle. In general, one can choose bundle coordinates about a point which are normal in the horizontal and vertical directions, but not throughout the neighbourhood.

It appears that the two corollaries of the proposition mentioned above, on page 133 of [ES], are false as well.

Suppose now that we are given a diagram

$$\begin{array}{ccc} E_o & \xrightarrow{f} & E \\ \pi_o \downarrow & & \downarrow \pi \\ B_o & \xrightarrow[\bar{f}]{} & B \end{array}$$

where f is a bundle map and π_o, π are harmonic fibrations. Note that by the previous lemma $\bar{f}$ is harmonic $\Leftrightarrow \pi \circ f$ is harmonic

The next lemma may be considered a replacement for Proposition C above.

<u>Lemma 1.2.5</u> Suppose that f is <u>horizontal</u>; i.e. $df(H_o) \subset H$. Then if either a) the fibres of π are geodesic

b) for each $x \in B_o$, the map of fibres

$F_{o_x} \rightarrow F_{\bar{f}(x)}$ is a Riemannian fibration,

it follows that $\pi \circ f$ is harmonic $\Leftrightarrow \tau(f)$ is vertical. In particular, f harmonic implies $\pi \circ f$ harmonic.

<u>Proof</u> We have $\tau(\pi \circ f) = d\pi(\tau(f)) + \text{Trace}\, \nabla d\pi(df, df)$.

The object is therefore to show that either (a) or (b) forces the second term to vanish. As usual, form an orthonormal basis for TE_o from bases for V_o and H_o. As $df(H_o) \subset H$, note first that if $v \in H$, $\nabla d\pi(v,v) = 0$. This is because π takes the geodesic determined by v to a geodesic in B; this can also be verified in local coordinates (cf.[ES, p. 127]). As f is a bundle map, $df(V_o) \subset V$. If the fibres of π are geodesic, then the same reasoning shows that $\nabla d\pi(w,w) = 0$ for $w \in V$. This takes care of case (a).

For case (b), we find that the relevant term becomes

$\Sigma\nabla d\pi(e_i,e_i)$, where (e_i) is an ortho-normal basis for V. As $\nabla d\pi$ is zero on H x H, this term is just $\tau(\pi)$. This establishes the lemma.

Remark If f is the pullback of $\bar{f}$, then f is horizontal and (b) is satisfied. (cf. the end of the next section).

Section 3. Equivariant Harmonic Maps

The development in this section is motivated by a paper of Hsiang and Lawson on minimal submanifolds of G-spaces [HL]. The idea is to examine the general theorems in their first chapter and see in what sense "minimal submanifold" can be replaced by "harmonic map". Given this program, a certain amount of original work is still required to carry it out successfully; this centers around studying a certain decomposition of the energy of an equivariant map, and finding reasonable conditions under which this decomposition is useful. The interest of the authors in [HL] was of course directed towards the volume rather than the energy.

It should be emphasized that only generalities are treated here; this section is therefore in the nature of an exposition of an interesting domain for further study rather than such a study itself. On the other hand, we will see later that some of the important examples in this thesis fall naturally in the category of equivariant harmonic maps (cf. Chapter III, Section 6) so the collection of applications is certainly not vacuous.

Let us briefly recall the relationship between minimal submanifolds and harmonic maps. If N is an oriented manifold and M is a Riemannian manifold, an immersion $f\colon N \to M$ is said to be minimal if f is a critical point of the volume functional, $V(f) = \int_N * 1_f$. Here $*1_f$ is the volume form on N canonically associated to the metric $f^* g_M$. (If N is non-compact, we consider only compactly supported variations of f). Note that any original metric on N is irrelevant, as only induced metrics are considered. However, suppose that N is given the induced metric f^*g_M (so that f is now an isometric immersion) Then it can be shown that f is minimal if and only if f is harmonic as a map of Riemannian manifolds [ES]. Therefore the notion of harmonic map is more general than that of minimal immersion, in that every minimal immersion is harmonic in the appropriate metric.

To proceed with the equivariant theory, then, let N and M be Riemannian manifolds and let G_1, G_2 be Lie groups which act smoothly by isometries on N and M respectively. A map $f : N \to M$ is said to be equivariant with respect to a homomorphism $\varphi: G_1 \to G_2$ if $f(g_1x) = \varphi(g_1)f(x)$ for all $g_1 \in G_1$, $x \in N$. The following is obvious:

Lemma 1.3.1 If f is an equivariant map, so is the tension field $\tau(f)$; i.e., $\tau(f)(g_1x) = \varphi(g_1)_* \tau(f)(x)$.

A straightforward generalization of Theorem 1 in [HL] now yields

Theorem 1.3.2 Let $f: N \to M$ be an equivariant map. Then f is harmonic if and only if $E(f)$ is stationary with respect to all (compactly supported) equivariant variations.

Our object is to use this theorem to reduce the question of whether f is harmonic to a problem concerning the orbit map $\bar{f}: N/G_1 \to M/G_2$. One conclusion will be that this is not always possible; this contrasts with the case of minimal immersions, where such a reduction can always be made [HL; Theorem 2].

From here on we develop the machinery for some sufficient conditions.

Suppose that the groups G_1 and G_2 are compact and connected. Hence the principal orbits form an open dense set in N, say N^*, and the map $\pi : N^* \to N^*/G_1$ is a smooth fibre bundle. Assign N^*/G_1 the unique metric under which π becomes a Riemannian fibration. (These points are discussed in more detail in [HL]). Let $V : N^*/G_1 \to \mathbb{R}$ be the volume function; i.e, $V(\bar{x})$ is the volume of the orbit $\bar{x} \subset N$. It is well known that if ψ is a function on N^* which is constant on orbits, then

$$\int_{N^*} \psi = \int_{N^*/G_1} \psi V$$

To get a good decomposition of the energy of an equivariant map, we need a further condition on the group actions; namely, require that if $\bar{x}$ and $\bar{y}$ are orbits in N and M and $f: \bar{x} \to \bar{y}$ is an equivariant map of these transitive G-spaces, then $|df|^2(x)$ is a constant, independent of $x \in \bar{x}$ and f, and depending only on $\bar{x}$ and $\bar{y}$. Since it turns out that the homomorphism $\varphi : G_1 \to G_2$ plays a key role here, let us say that φ is <u>orbit-energy preserving</u> (with respect to the group actions) if the above condition is satisfied. Denote by $\gamma(\bar{x},\bar{y})$ the <u>orbit-energy function</u> in this case.

<u>Lemma 1.3.3</u> Suppose G_1 and G_2 can be given bi-invariant metrics so that (up to scalar factors) the following maps are Riemannian fibrations:

a) for each $x \in N$, $\pi_x : G_1 \to G_{1x}$ = the orbit through x

b) for each $y \in M$, $\pi_y : G_2 \to G_{2y}$

c) $\varphi : G_1 \to G_2$

the scale factors in (a) and (b) being allowed to vary from orbit to orbit. Then φ is orbit energy preserving.

<u>Proof</u> Let $f: \bar{x} \to \bar{y}$ be a φ-invariant map of orbits. Via (a) and (b) we may assume (up to scale factors) that f is of the form

$$f : G_1/H_1 \to G_2/H_2$$

(where these manifolcs have their natural metrics), and satisfies $f(g'g\, H_1) = \varphi(g')\, f(gH)$. In particular, if $f(H_1) = aH_2$, then $f(gH_1) = \varphi(g)\, aH_2$ for all g. Hence we have the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{F} & G_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ G_1/H_1 & \xrightarrow[f]{} & G_2/H_2 \end{array}$$

where $F(g) = \varphi(g)a$. If R_a denotes right translation by a, it follows that

$$|df|^2(x) = |\, d\pi_2 \circ dR_a \circ d\varphi|^2(\tilde{x})$$

for any $\tilde{x}$ such that $\pi_1\tilde{x} = x$. This is because, firstly, π_1 is Riemannian, so that

$$|df|^2(x) = \sum_j |d(\pi_2 \circ F)(v_j)|^2\ (\tilde{x})$$

where v_j is an orthonormal basis for the horizontal space at $\tilde{x}$; secondly, F is a bundle map, so that $d(\pi_2 \circ F)(w) = 0$ if w is a vertical vector. But now as φ is Riemannian and R_a is assumed to be an isometry of G_2, we find

$$|df|^2 = |d\pi_2|^2$$

which is a constant, as π_2 is Riemannian.

<u>Remark 1.3.4</u> The essential non-metric requirement of the lemma is that the homomorphism φ be surjective. For an easy counter-example in the case φ is not onto, let $\varphi : S^1 \to SO(3)$ be the inclusion of S^1 as the subgroup of rotations in the $\mathbb{R}^2$ -plane

Let S^1 act on itself and let $SO(3)$ act on S^2. Define equivariant maps $f,g : S^1 \to S^2$ as follows: f is the inclusion of S^1 as the great circle in the $\mathbb{R}^2$-plane, and let g be a parallel non-geodesic circle. Then f is harmonic whereas g is not. However, f and g are indistinguishable at the orbit level. Note also that the energies of f and g are not the same.

It is therefore quite obvious that in general the orbit space to study is not M/G_2 but $M/\varphi G_1)$.

Now assume that $f: N \to M$ is a φ-equivariant map, and that φ is orbit energy preserving. At any $x \in N$ we can write

$$|df(x)|^2 = |df(x)^G|^2 + |df(x)^{\perp}|^2$$

where df^G refers to derivatives in the orbit directions and $df^{\perp}$ to normal derivatives. If $\bar{f}$: $N/G_1 \to M/G_2$ is the orbit map, then by assumption

$$|df(x)^G|^2 = \Upsilon\,(\bar{x},\bar{f}(\bar{x}))$$

where Υ is the orbit energy function. For normal derivatives, assume that x and $f(x)$ lie in principal orbits, so that $\bar{f}$ is differentiable at $\bar{x}$. Let us write

$$|df(x)\;|^2 = |d\bar{f}(\bar{x})|^2 + |\text{Skew } df(x)|^2$$

Here $|\text{Skew } df|^2$ measures the extent to which df takes vectors normal to the orbit into vectors which are no longer normal. Precisely, if π_{G_2} is projection onto the tangent space to the orbit in M, then

$$|\text{Skew } df(x)|^2 = |\pi_{G_2}(f(x)) \circ df^{\perp}(x)|^2$$

Since the normal vectors are precisely the horizontal vectors with respect to the projections π_1 and π_2 (cf. Section 2), then f is horizontal if and only if Skew $df = 0$. Under the assumption that $f(N^*) \subset M^*$, we get

$$\int_{N^*} |df|^2 = \int_{N^*/G_1} (|d\bar{f}|^2 + \gamma(\bar{x},\bar{f}(\bar{x})))\, V(\bar{x}) \quad + \int_{N^*} |\text{Skew } df|^2$$

Suppose now that f_t is an equivariant variation of f which differs from f only in a compact subset of N^*. Then $f_t(N^*) \subset M^*$ for small t, and the above formula is valid for such f_t. Finally, assume that γ is smooth on $N^*/G_1 \times M^*/G_2$, and that the map f is a critical point of the functional $\int_{N^*} |\text{Skew } df|^2$. This latter assumption removes the last obstacle to reducing the problem on N^* to a problem on N^*/G_1; it is satisfied, clearly, if f is horizontal. Hence if $w = \frac{\partial f}{\partial t}\big|_{t=o}$

$$\frac{d}{dt} \int_{N^*} |df_t|^2 \big|_{t=o} = \int_{N^*/G_1} (2\langle \bar{\nabla}\bar{w}, d\bar{f}\rangle + \langle \operatorname{grad}\gamma(\bar{x},\bar{f}(\bar{x})),(0,\bar{w})\rangle V(\bar{x})$$

$$= \int_{N^*/G_1} \langle -2\bar{\tau}(\bar{f}) + \operatorname{grad}_{M^*/G_2} \gamma(\bar{x},\bar{f}(\bar{x})),\bar{w}\rangle\, V\,(\bar{x})$$

where $\bar{\tau}\,(\bar{f}) = \text{Trace } \bar{\nabla}_{\bar{f}}\, (V d\bar{f})/V$. Since f is harmonic $\Leftrightarrow$ $f|N^*$ is harmonic (as N^* is dense in N), and since there is a one to one correspondence between equivariant variation fields w upstairs and variation fields $\tilde{w}$ downstairs, an application of Theorem 1.3.2 yields

<u>Theorem 1.3.5</u> Suppose that $\varphi: G_1 \to G_2$ is orbit-energy preserving with respect to the actions of G_1 and G_2 on N and M, and that the orbit energy γ is smooth away from the singular sets. If $f: N \to M$ is C^2, equivariant, and horizontal, with $f(N^*) \subset M^*$, then it follows that f is harmonic $\Leftrightarrow$ $\bar{f}|N^*/G_1$ satisfies

$$(1.3.6) \qquad \bar{\tau}(\bar{f}) - \tfrac{1}{2} \operatorname{grad}_{M^*/G_2} \gamma(\bar{f}) = 0$$

Corollary 1.3.7 If the functions γ and V are constant, and $f : N \to M$ is equivariant and horizontal, then f is harmonic $\Leftrightarrow$ $\bar{f}$ is harmonic.

Equation 1.3.6 will be called the equivariant tension equation. Examples will be seen later in which it is solved and sufficient regularity at the singular set demonstrated to give harmonic maps upstairs. For the time being, let's just mention an application of the Corollary.

Example 1.3.8 Suppose G acts on M so that all orbits are isometric and the map $\pi_x : G \to G_x$ is a Riemannian fibration for each $x \in M$ (with respect to a fixed bi-invariant metric on G). Then $M^* = M$ and $\pi : M \to M/G$ is a harmonic fibration (all orbits have the same volume and are therefore minimal [HL, Thm. 2]). If $f: X \to M/G$ is a smooth map, then there is the diagram

$$\begin{array}{ccc} f^*M & \xrightarrow{f^*} & M \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & M/G \end{array}$$

Assuming X is also Riemannian, there is a natural pull-back metric for f^*M; i.e., that induced by the inclusion $f^*M = \{(x,m) : f(x) = \pi(m)\} \subset X \times M$. It is straightforward to check that $f^*\pi$ is a Riemannian fibration and that f^* is horizontal. The G action on M also pulls back to an action on f^*M which is as nice as the original, and f^* is equivariant with respect to id_G. Furthermore, $X \cong f^*M/G$. It follows by the Corollary (and Lemma 1.3.3) that f is harmonic $\Leftrightarrow$ f^* is harmonic. Note also that $f^*\pi$ is harmonic, by the same reasoning as for π; however, our conclusion is stronger than that in Lemma 1.2.5.

This example is a natural generalization of Theorem 4 in [HL], where the map f is assumed to be an isometric immersion. In other words, X is a minimal submanifold of M/G if and only if $\pi^{-1}(X)$ is minimal in M.

As a conclusion to this section, let us give a simple example to show that unless the equivardant map f is horizontal, we cannot get any information from the orbit map $\overline{f}$.

Example 1.3.9 Let S^1 act on $S^1 \times S^1$ by multiplication in the first factor. A map $f: S^1 \times S^1 \to S^1 \times S^1$ of the form $f(x,y) = (g(y)x, y)$, where $g : S^1 \to S^1$, is equivariant with respect to the identity on S^1; furthermore, all such maps f are the identity on the orbit level, and id_{S^1} is orbit-energy preserving. However, it is clear that some such maps f are harmonic while others are not.

Chapter 2

HARMONIC POLYNOMIAL MAPPINGS

Section 1. Spherical Harmonics

Recall that a spherical harmonic of degree k is a homogeneous harmonic polynomial of degree k defined on $\mathbb{R}^n$. If f is such a polynomial, then $f|S^{n-1}$ satisfies

$$\Delta_{S^{n-1}}(f|S^{n-1}) = (-\Delta_{\mathbb{R}^n} f)|\, S^{n-1} + \frac{\partial^2 f}{\partial r^2}|S^{n-1} + (n-1)\frac{\partial f}{\partial r}|S^{n-1}$$

$$= k(k + n - 2) \quad f|S^{n-1}$$

which is a standard computation. In other words, $f|S^{n-1}$ is an eigenfunction of $\Delta_{S^{n-1}}$. It is a theorem that all eigenfunctions of $\Delta_{S^{n-1}}$ arise this way (see Berger [B, p. 159 ff]). Each eigenspace V^λ has the usual L^2 inner product; it was observed by do Carmo and Wallach [DW] that if $f_1 \ldots f_p$ is an orthonormal basis for V^λ, then $F = (f_1 \ldots f_p)$ defines a minimal immersion of S^{n-1} in a higher dimensional sphere.

For the purposes of studying harmonic mappings, the most useful observation is this one:

Lemma 2.1.1 Suppose $P : S^n \to S^m$ is defined by harmonic polynomials of homogeneity k. Then P is harmonic.

Proof As $\Delta_{S^n} P = \lambda P$, this is just Example 1.1.4.

Are there examples other than the minimal immersions? The answer is definitely yes, as will be seen in the ensuing collection of second order polynomial mappings.

Example 2.1.2 Let $F : R^n \times R^n \to R^m$ be an orthogonal multiplication; in other words, F is bilinear and $|F(x,y)| = |x||y|$. Applying the Hopf construction to F yields a polynomial map

$$H : S^{2n-1} \to S^m$$

defined by $H(x,y) = (|x|^2 - |y|^2, 2F(x,y))$. Then H is homogeneous of degree 2; furthermore, as there is the same number of squares in $|x|^2$ as in $|y|^2$, H is harmonic. Finally, H takes values in S^m because $|H(x,y)|^2 = (|x|^2 + |y|^2)^2$.

If F is taken to be multiplication of complex numbers, quaternions, and Cayley numbers successively, we see that the Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ are all harmonic.

Orthogonal multiplications will be discussed more fully in the next section. Before proceeding with more examples, let us mention an interesting theoretical problem which already presents itself: given a general eigenspace of spherical harmonics, V^λ, we know from do Carmo and Wallach that if $f_1 \ldots f_p$ is an orthonormal basis, then in particular $\Sigma f_i^{\,2}(x) = c$, a constant. Suppose on the other hand that some subset $f_1 \ldots f_q$ also defines a map of spheres; i.e., $\sum_{i=1}^{q} f_i^2(x) = c_1$. Hence $(f_{q+1} \ldots f_p)$ also must have constant norm and define a harmonic map of spheres. One can also ask whether V^λ breaks up any further in this manner.

<u>Problem</u>: Classify the decompositions of V^λ into norm-preserving summands.

On the other hand, such decompositions do not give all possible examples:

<u>Example 2.1.3</u> There are essentially two orthogonal multiplications of $\mathbb{R}^2 \times \mathbb{R}^2$; namely, complex multiplication $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and the real tensor product $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$. By the Hopf construction we get harmonic polynomial maps $S^3 \to S^2$ and $S^3 \to S^4$ respectively. In the first case, the three polynomials defining the map have the same L^2 length and are mutually orthogonal. In the second case the components have different L^2 norms; therefore the latter map does not fit into the classification proposed above.

Let $h : S^3 \to S^2$ be the Hopf map, defined by an orthonormal set of 3 spherical harmonics. According to the formula in Berger [B], the entire space of second order harmonics is nine-dimensional; hence there must be a complementary mapping $h^\perp : S^3 \to S^5$. Here it is:

<u>Example 2.1.4</u> If points in S^3 are pairs (z,w) of complex numbers, define $h^\perp : S^3 \to S^5$ by

$$h^\perp(z,w) = (z_1^2 - z_2^2,\ 2z_1z_2,\ w_1^2 - w_2^2,\ 2w_1w_2,\ \sqrt{2}\ z\bar{w})$$

Note that we have chosen to assign the polynomial $z\bar{w}$ to $h^\perp$ and zw to h; as these roles could easily be reversed, it is clear that decompositions of eigenspaces will not generally be unique.

As $h^\perp$ is quadratic there is a natural factorization

$$\begin{tikzcd} S^3 \arrow[r, "h^\perp"] \arrow[d, "\pi"'] & S^5 \\ \mathbb{R}P^3 \arrow[ur, "\bar{h}^\perp"'] & \end{tikzcd}$$

We claim that $\bar{h}^\perp$ is a <u>harmonic embedding</u>. It is clear enough that $\bar{h}^\perp$ is harmonic, as h is harmonic and π is a Riemannian covering (a special case of a harmonic fibration). The proof that $\bar{h}^\perp$ is an embedding can be read off from the next example.

<u>Remark 2.1.5</u> a) $h^\perp$ is not isometric, so it would appear that the image is not a minimal submanifold.

b) $h^\perp$ is a special case of the following generality: if $f: S^p \to S^q$ and $g: S^r \to S^s$ are second order harmonic polynomial maps, and $F: \mathbb{R}^{p+1} \times \mathbb{R}^{r+1} \to \mathbb{R}^n$ is an orthogonal multiplication, then there is a harmonic polynomial map $h: S^{p+r+1} \to S^{q+s+n+1}$ defined by

$$h(x,y) = (f(x),\ g(y),\ \sqrt{2}\ \ F(x,y))$$

The norm condition is satisfied since by homogeneity, $|f(x)|^2 = |x|^4$ and similarly for g. In the case at hand, f

and g are given by the Hopf construction applied to real multiplication, and F is complex multiplication. This suggests:

Example 2.1.6 Define a map $f: S^7 \to S^9$ as follows: if points of S^7 are pairs (z,w) of quaternions, and each quaternion z has complex components (z_1,z_2) (i.e. $z = z_1 + z_2 j$), then let

$$f(z,w) = (|z_1|^2 - |z_2|^2,\ 2z_1\bar{z}_2,\ |w_1|^2 - |w_2|^2,\ 2w_1\bar{w}_2,\ \sqrt{2}\ \bar{z}w)$$

We claim there is a factorization

$$\begin{tikzcd} S^7 \arrow[r, "f"] \arrow[d, "\pi"'] & S^9 \\ \mathbb{C}P^3 \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

in such a way that $\bar{f}$ is a harmonic embedding. We will show first of all that $f(z,w) = f(x,y) \Leftrightarrow (z,w) = \lambda(x,y)$ for some $\lambda \in S^1$. The action of S^1 is the usual one and is consistent with quaternionic multiplication; i.e. $\lambda x = (\lambda x_1, \lambda x_2)$.

So suppose $f(z,w) = f(x,y)$. As with the Hopf map $S^3 \to S^2$, this forces $z = \lambda_1 x$, $w = \lambda_2 y$ for $\lambda_1, \lambda_2 \in S^1$. Also, $x\, \bar{\lambda}_1 \lambda_2 y = xy$, so that as long as $x \neq 0$, $y \neq 0$, $\bar{\lambda}_1\lambda_2 = 1$ and $\lambda_1 = \lambda_2$. In case, say, $x = 0$, we are done already. As the converse is trivial, $\bar{f}$ is well defined and one-to-one.

Since $\mathbb{C}P^3$ is compact, we now need only that $\bar{f}$ is an immersion. It suffices to show that if $df(\eta) = 0$, then η is a vertical vector. So suppose $\eta = (u,v)$ is tangent to S^7 at (z,w), with $z \neq 0$, $w \neq 0$, and that $df(\eta) = 0$. Since the extended Hopf map $\mathbb{R}^4 \to \mathbb{R}^3$ is a nice submersion away from 0, it follows that u is tangent to the circle through z. Similarly for v and w. In other words, u is a real multiple of iz and v is a real multiple of iw. It remains to show these real numbers are equal. However, another consequence of $df(\eta) = 0$

is that $\bar{z}v + \bar{u}w = 0$. From this relation the desired conclusion follows easily. In case $z = 0$, the same equation forces $u = 0$ and we proceed as before.

That $\bar{f}$ is harmonic follows from the fact that f is harmonic and Lemma 1.2.2.

The previous examples can be generalized as follows: if S^{4n-1} is the unit sphere in the space of n quaternionic variables, there is a diagram

$$\begin{tikzcd} S^{4n-1} \arrow[r, "f"] \arrow[d, "\pi"'] & S^{2n^2+n-1} \\ \mathbb{C}P^{2n-1} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where $\bar{f}$ is again a harmonic embedding. The definition of f is

$$f(x_1 \ldots x_n) = (|x_{11}|^2 - |x_{12}|^2, 2x_{11}\bar{x}_{12}, \ldots, |x_{n1}|^2 - |x_{n2}|^2, 2x_{n1}\bar{x}_{n2}; \sqrt{2}\{\bar{x}_j x_k : j < k\})$$

A similar construction is valid in the real-complex case, yielding

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[d, "\pi"'] & S^{n^2+n-1} \\ \mathbb{R}P^{2n-1} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where $\bar{f}$ is a harmonic embedding.

Let us give a final instance of the beautiful mappings which can be found by merely looking around.

<u>Example 2.1.7</u> One of the standard minimal immersions is the Veronese mapping $S^2 \to S^4$, which defines an embedding of $\mathbb{R}P^2$ as the Veronese surface in S^4. Using the same formula, with complex variables substituted for real ones, we obtain

$$\begin{tikzcd} S^5 \arrow[r, "f"] \arrow[d, "\pi"'] & S^7 \\ CP^2 \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

where $\overline{f}$ (you guessed it) is a harmonic embedding. The formula is in fact

$$f(x,y,z) = (x\overline{y}, x\overline{z}, y\overline{z}, \tfrac{1}{2}(|x|^2 - |z|^2), \frac{1}{2\sqrt{3}}(|x|^2 + |y|^2 - 2|z|^2))$$

The verification that $\overline{f}$ is an embedding proceeds along the same lines as before and is omitted. The quaternionic analogue of this example is also valid.

Section 2. Orthogonal Multiplications

The orthogonal multiplications are quite useful in constructing examples of harmonic maps. One possibility has already been mentioned as Example 2.1.2. Before plunging ahead, let us agree to abbreviate the phrase "an orthogonal multiplication $F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$" to "an $F(n,m;p)$" where convenient.

Recall that in 2.1.2 we needed $n = m$ in order to make use of an $F(n,m;p)$. However, they can all be used in the following way:

Lemma 2.2.1 Given an $F(n,m;p)$, the induced map $F: S^{n-1} \times S^{m-1} \to S^{p-1}$ is harmonic.

Proof With respect to each variable separately, F is a geodesic embedding; hence F is harmonic by Lemma 1.1.1.

A nice generalization of this lemma is the next observation.

Lemma 2.2.2 Suppose $F: S^n \times S^m \to S^p$ is a homogeneous harmonic polynomial in each variable separately. Then F is harmonic.

Example 2.2.3 Think of S^3 as the unit quaternions and of S^2

as quaternions whose real part is 0. Then there is a map $F : S^2 \times S^3 \to S^2$ defined by $F(x,y) = yx\bar{y}$. A short computation shows that if x is fixed, $F^x : S^3 \to S^2$ is a form of the Hopf map and is a harmonic polynomial. Thus F is harmonic. It is known [BS] that applying the Hopf construction to F (in the topologist's sense) yields the generator of $\pi_6(S^3) = Z_{12}$.

We remark that the orthogonal multiplications pose an interesting classification problem: the only thing known at present seems to be that there exists an $F(k,n;n)$ if and only if $\mathbb{R}^n$ can be given the structure of an (ungraded) C_{k-1} module, where C_k is the k^{th} Clifford algebra. The dimensions k and n for which this can occur have been completely determined (see Husemoller [HU]). In this section the general problem is merely given a more precise formulation.

To begin with, note that if we have an $F(n,m;p)$, then there must also be an $F(rn, sm; rsp)$ for all positive integers r,s: if the first multiplication is written vw, the second can be written

$$(v_1, \ldots , v_r)(w_1, \ldots ,w_s) = \{v_i w_j\}$$

To check the norms, we have

$$\begin{aligned} |\{v_i w_j\}|^2 &= \sum_{ij} |v_i|^2 |w_j|^2 \\ &= \sum_i |v_i|^2 \sum_j |w_j|^2 \\ &= |v|^2 \, |w|^2 \end{aligned}$$

This operation may be viewed as a sort of tensor product associated to F (cf. Examples 2.2.6 - 8); more generally, we could operate on each pair (v_i, w_j) with a different F_{ij}. At any rate, as far as classification is concerned, we clearly need:

Definition 2.2.4 A multiplication $F(n,m;p)$ is said to be left reducible if there is an orthogonal splitting $\mathbb{R}^n = V \oplus W$ such that $F(V,\mathbb{R}^m) \perp F(W,\mathbb{R}^m)$; otherwise, F is left irreducible.

Right reducibility is defined similarly. F is irreducible if it is left and right irreducible.

Problem: For what integers is there an irreducible $F(n,m;p)$? It is clear that any multiplication can be decomposed into a matrix of irreducible components in an obvious (but perhaps non-canonical) way: simply perform right and left reductions alternately until the process terminates. Therefore an answer to the above restricted question would provide the most natural solution to the general classification problem; hence some more manageable characterizations of irreducibility are needed, and we have nothing to offer here.

As the Clifford multiplications $F(k,n;n)$ furnish a large supply of orthogonal multiplications, one might also ask if there are any examples of an irreducible $F(n,m;p)$ which does not arise by restriction from an $F(n,p;p)$ (or $F(p,m;p)$). In other words, this is an extension problem. Again, we have no suggestions to make here. We remark, however, that the non-singular multiplications in exotic dimensions constructed by Adem, Lam, and others [A] are not orthogonal.

Another reasonable sounding project would be to attempt a finer classification along these lines: given n,m, and p, we ask how many $F(n,m;p)$ there are, up to orthogonal substitutions in all three Euclidean spaces. However, this problem does not seem to be solved even in cases where the answer should be obvious:

Example 2.2.5 Let us exhibit two non-isomorphic multiplications

$F(5,8;8)$. For F_1 we take Cayley multiplication restricted in the first variable to any 5-plane. F_2 will be defined by writing down four anti-commuting complex structures on $\mathbb{R}^8 = \mathbb{H}^2$, say $e_1 \dots e_4$; setting $e_0 = I$, we get $F_2(5,8;8)$ by linear extension in the first variable (cf. Husemoller [HU]).

Therefore set

$$e_1(x,y) = (iy, ix)$$

$$e_2(x,y) = (jy, jx)$$

$$e_3(x,y) = (ky, kx)$$

$$e_4(x,y) = (-y, x)$$

This construction, due to Vranceanu [V], gives four orthogonal vector fields on S^7; he shows furthermore that another orthogonal field cannot be added to this system. Hence F_2 does not extend, whereas F_1 obviously does. Therefore F_1 and F_2 are not isomorphic (in the sense mentioned above).

Added in Proof This example is incorrect. The structure of $\mathbb{R}^8$ as an ungraded C_4 module is unique, and isomorphisms between structures can be assumed isometries [M, p. 143]. Thus Vranceanu's theorem is also incorrect.

Putting theoretical considerations aside, we continue to hunt for interesting harmonic maps. As it happens, the tensor product discussed earlier produces a series of charming examples.

Example 2.2.6 Let us take first the real tensor product, which is an $F(n,m;nm)$; here $(v,w) \mapsto v \otimes w \equiv (v_i w_j)$. There results a diagram of harmonic mappings:

$$\begin{tikzcd}
S^{n-1} \times S^{m-1} \arrow[rr, "f"] \arrow[d] \arrow[dr] & & S^{nm-1} \arrow[d] \\
& K_{n,m} \arrow[ur] \arrow[dl] \arrow[dr] & \\
\mathbb{RP}^{n-1} \times \mathbb{RP}^{m-1} \arrow[rr, "\overline{f}"'] & & \mathbb{RP}^{nm-1}
\end{tikzcd}$$

Here f is an isometric (hence minimal) immersion whose image lies in no hyperplane; i.e., f is <u>full</u> in the terminology of [DW]. The map $\overline{f}$ is a harmonic embedding of $\mathbb{RP}^{k-1} \times \mathbb{RP}^{m-1}$ as the minimal algebraic submanifold of $n \times m$ matrices each of whose 2×2 sub-determinants vanishes. The manifold $K_{n,m}$ is the quotient $S^{n-1} \times S^{m-1}/S^o$, and all other maps are the evident ones.

We first prove f is isometric; i.e., that df at any point sends an orthonormal basis to an orthonormal set. Since f is induced from F, which is an isometry in each variable separately, it suffices to show that if y_1 and y_2 are tangent to S^{n-1} and S^{m-1} respectively, then $\langle df(y_1), df(y_2)\rangle = 0$. Therefore let $x = (v,w)$, $y_1 = (u,o)$, $y_2 = (o,t)$, with $\langle u,v\rangle = \langle w,t\rangle = 0$. Then $df_x(y_1) = dF_x(y_1) = (u_i w_j)$, and $df_x(y_2) = (v_i t_j)$. Hence

$$\langle df(y_1), df(y_2)\rangle = \sum_{ij} u_i w_j v_i t_j = \sum_j w_j z_j \sum_i u_i v_i = 0$$

It will be clear from the next example, however, that this conclusion (i.e. f isometric) is not valid for general multiplications.

The algebraic statement can be proved this way: it is clear that if a matrix A is given by $(A_{ij}) = (v_i w_j)$, then

$$\begin{vmatrix} A_{ij} & A_{ik} \\ A_{\ell j} & A_{\ell k} \end{vmatrix} = \begin{vmatrix} v_i w_j & v_i w_k \\ v_\ell w_j & v_\ell w_k \end{vmatrix} = 0 \quad \forall ijk\ell$$

Conversely, we need that if A is a matrix satisfying these conditions, then $A = (v_i w_j)$ for some vectors v and w. The proof is by induction: as the case $n = m = 1$ is trivial, assume it is true for $(n-1,m)$. Then there is a $(v_1 \ldots v_{n-1})$ and a $(w_1 \ldots w_m)$ such that $A_{ij} = v_i w_j$ for $i \leq n-1$, $j \leq m$. We must choose v_n to satisfy $v_n w_i = A_{ni}$ $\forall i$. In other words, we must verify $A_{ni}/w_i = A_{nj}/w_j$ for any i,j such that $w_i \neq 0$, $w_j \neq 0$ and simultaneously check that $A_{ni} = 0$ if $w_i = 0$. There are two cases:

a) $\forall m \leq n-1$, $v_m = 0$. In this case put $v_n = 1$ and $w_i = A_{ni}$ $\forall i$ (throwing away the former w).

b) $\exists m \leq n-1$ such that $v_m \neq 0$. Suppose that $w_i \neq 0$, $w_j \neq 0$. Then by assumption $A_{mi} A_{nj} - A_{ni} A_{mj} = v_m w_i A_{nj} - A_{ni} v_m w_j = 0$.

Hence $A_{ni}/w_i = A_{nj}/w_j$. Now suppose $w_i = 0$. If there is some $w_j \neq 0$, $A_{ni} = 0$ by the same formula. If $w = 0$, again put $v_n = 1$ and $w_i = A_{ni}$ $\forall i$.

The induction on m is similar.

Finally, we should check that $\bar{f}$ is one-to-one. In other words, if $(v_i w_j) = \pm(x_i y_j)$, then $v = \pm x$ and $w = \pm y$. As a similar statement is proved in the next example, the verification is omitted. Note also that if $v \otimes w = x \otimes y$, then $(v,w) = \pm(x,y)$. Hence $K_{n,m}$ is embedded as a minimal submanifold of S^{nm-1}.

<u>Special Case</u>: Let $n = m = 2$. It is easy to see that $K_{2,2}$ is a torus and is embedded as a copy of the Clifford torus in S^3. The Clifford torus may therefore be characterized as those 2×2 real matrices which are singular and of norm 1.

<u>Example 2.2.7</u> The <u>complex tensor product</u> $(z,w) \mapsto z \otimes w$ gives a similar diagram of harmonic maps:

$$\begin{tikzcd}
S^{2n-1} \times S^{2m-1} \arrow[rr, "f"] \arrow[dd, "\pi_1\times\pi_2"'] \arrow[dr, "\alpha"] & & S^{2nm-1} \arrow[dd] \\
& K^{\mathbb{C}}_{n,m} \arrow[ur, "\delta"'] \arrow[dl, "\beta"'] \arrow[dr, "\gamma"] & \\
\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} \arrow[rr, "\bar{f}"'] & & \mathbb{C}P^{nm-1}
\end{tikzcd}$$

Here $\bar{f}$ is an isometric holomorphic embedding and is known as a <u>Segré embedding</u>, after its originator. As before, $\mathbb{C}P^{n-1}\times\mathbb{C}P^{m-1}$ is embedded as the minimal algebraic submanifold of $n \times m$ complex matrices each of whose 2×2 subdeterminants vanishes. The manifold $K^{\mathbb{C}}_{n,m}$ is the quotient $S^{2n-1} \times S^{2m-1}/S^1$, where S^1 acts by $\lambda(z,w) = (\lambda z,\bar{\lambda}w)$. $K^{\mathbb{C}}_{n,m}$ is harmonically embedded (non-isometrically) as an algebraic minimal submanifold of S^{2nm-1}.

Note that the maps $\pi,\pi_1 \times \pi_2,\alpha$ and β are all harmonic fibrations, with geodesic fibres. The first three are harmonic fibrations by definition of the metrics on their images; similarly β is Riemannian and $\beta \circ \alpha = \pi_1 \times \pi_2$ is harmonic implies β is harmonic by Lemma 1.2.2. By Lemma 1.2.2 we also see that $\gamma = \bar{f} \circ \beta$ is harmonic, because $\bar{f}$ is a holomorphic map of Kahler manifolds and is hence harmonic [ES]. Similarly $\delta \circ \alpha = f$ is harmonic, so δ is harmonic. Thus all maps are harmonic. In fact, as $\delta(K^{\mathbb{C}}_{n,m})$ is minimal, δ is also harmonic with respect to the pull back metric on $K^{\mathbb{C}}_{n,m}$.

So now assume $f(z,w) = f(x,y)$; that is, $z_i w_j = x_i y_j \ \forall i,j$. Then $\forall i$ such that $x_i \neq 0$, $y_j = x_i^{-1} z_i w_j \ \forall j$. That is, $y = \lambda w$ for a scalar λ of norm 1. Similarly, we find $x = z\bar{\lambda}$, with $\lambda = w_j y_j^{-1}$ for all j such that $y_j \neq 0$. In conclusion, $f(z,w) = f(x,y)$ if and only if $(x,y) = (z\bar{\lambda},\lambda w)$ for some scalar λ of norm 1. (Note that this computation works as well for quaternions.) Therefore δ is injective. That δ is an immersion (but not isometric) follows in a manner similar to the calculation below.

Let us check that $\bar{f}$ is isometric. To do this, we will show that if γ is horizontal with respect to $\pi_1 \times \pi_2$, then $df(\gamma)$ is horizontal with respect to π and $|df(\gamma)| = |\gamma|$. Suppose this can be verified at some point $x = (x_1, x_2)$ in $S^{2n-1} \times S^{2m-1}$. Then given any other point $y = (y_1, y_2)$, choose unitary operators U_1, U_2 such that $U_i x_i = y_i$. It is clear that $f \circ (U_1 \times U_2) = U_1 \otimes U_2 \circ f$, and similarly for the induced maps of projective spaces. As $U_1 \times U_2$ and $U_1 \otimes U_2$ are isometries (and induce isometries), the desired condition must also hold at y.

Therefore choose $x = (e_k, e_\ell)$. A typical vertical vector is $(\lambda i e_k, \mu i e_\ell)$; horizontal vectors are thus of the form (z,w), with $z_k = w_\ell = 0$. Thus for a horizontal vector

$$
\begin{aligned}
[df_x(z,w)]_{ij} &= w_j & & i = k,\ j \neq \ell \\
&= z_i & & i \neq k,\ j = \ell \\
&= 0 & & \text{otherwise}
\end{aligned}
$$

In particular, the (k,ℓ) component is 0, so $df(z,w)$ is horizontal with respect to π; it is also evident that $|df_x(z,w)|^2 = |z|^2 + |w|^2$, which concludes the proof that $\bar{f}$ is isometric. That $\bar{f}$ is a holomorphic embedding is easily verified, and the algebraic characterization is valid as in the real case.

It is well known that complex submanifolds of Kahler manifolds are Kahler and minimal. Hence $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ is minimal in $\mathbb{C}P^{nm-1}$; therefore $\delta(K^{\mathbb{C}}_{n,m})$ is minimal in S^{2nm-1}, as it is the inverse image of a minimal submanifold under π (see e.g. Example 1.3.8).

Example 2.2.8 Repeating this process for the quaternions gives a diagram

$$\begin{tikzcd}
S^{4n-1} \times S^{4m-1} \arrow[rr, "f"] \arrow[d, "\pi_1 \times \pi_2"'] \arrow[dr, "\alpha"] & & S^{4nm-1} \arrow[d, "\pi"] \\
\mathbb{H}P^{n-1} \times \mathbb{H}P^{m-1} & K^{\mathbb{H}}_{n,m} \arrow[l, "\beta"] \arrow[ur, "\delta"] \arrow[r, "\gamma"] & \mathbb{H}P^{nm-1}
\end{tikzcd}$$

We are missing $\bar{f}$ because it is not well defined. The manifold $K^{\mathbb{H}}_{n,m}$ is defined as the quotient $S^{4n-1} \times S^{4m-1}/S^3$ (cf. the previous example). All maps are still harmonic, except perhaps γ: before we used the fact that $\gamma = \bar{f} \circ \beta$.

In fact, γ is not generally harmonic. To demonstrate this, we will give an example in which $\pi \circ f$ is not harmonic; therefore $\gamma \circ \alpha$ is not harmonic, so γ is not harmonic by Lemma 1.2.2. The example is the composition

$$S^3 \times S^7 \xrightarrow{f} S^7 \xrightarrow{\pi} S^4 = \mathbb{H}P^1$$

which is the polynomial

$$H(q, (x,y)) = (|q|^2 (|x|^2 - |y|^2), 2qxy\bar{q})$$

For fixed q, H is a harmonic polynomial in (x,y). Hence, as in Lemma 1.1.1, if for some fixed (x,y) the induced map $S^3 \to S^4$ is not harmonic, we can conclude that H is not harmonic.

Choose $(x,y) = (1/\sqrt{2}, (1-j)/2)$, and represent $q = z + wj$ for complex z and w. The second term in H is like

$$(z,w) \mapsto |z|^2 + |w|^2 + z\bar{w} - w\bar{z} - (z^2 + w^2)j$$

However in general we have

<u>Lemma 2.2.9</u> Suppose $H : S^n \to S^m$ is a homogeneous polynomial; then H is harmonic if and only if at each point $x \in S^n$ $\Delta H(x)$ is proportional to $H(x)$ (where Δ is the Euclidean Laplacian).

<u>Proof</u> This is immediate from Example 1.1.4 (a) and the first formula in this chapter.

Applying this lemma to the polynomial written above, we see immediately that H cannot be harmonic. This concludes Example 2.2.8.

To end this section, let us write down some maps which behave a little differently from those considered so far. Applying the Hopf construction to the real tensor product $\mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^{p^2}$ gives a diagram

$$\begin{tikzcd} S^{2p-1} \arrow[r, "h"] \arrow[d] & S^{p^2} \\ \mathbb{R}P^{2p-1} \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

Operating similarly on the complex tensor product $(z,w) \to z \otimes \bar{w}$ yields harmonic maps:

$$\begin{tikzcd} S^{4p-1} \arrow[r, "h"] \arrow[d] & S^{2p^2} \\ \mathbb{C}P^{2p-1} \arrow[ur, "\bar{h}"'] & \end{tikzcd}$$

If the arguments upstairs are (x,y), then in each case $\bar{h}$ is an immersion away from the planes $x = 0$ and $y = 0$, and is singular at these planes if $p > 1$. In the cases $p = 1$, h is $z \to z^2$ on S^1 and the Hopf map $S^3 \to S^2$, respectively.

Section 3. Mappings Associated to Orthogonal Multiplications

In this section we study the geometry of maps into Stiefel and Grassmann manifolds which are canonically associated to any orthogonal multiplication. The main results are:
a) the existence of a one-to-one correspondence between the multiplications and nicely embedded spheres in Stiefel manifolds; and b) the fact that the (harmonic) fibration $V_{n,p} \to G_{n,p}$ is not totally geodesic. The import of the last statement for the theory of harmonic maps is that most (but not all) of the canonical maps between spheres and Grassmanns are not harmonic.

Let us first consider $V_{n,p}$, the Stiefel manifold of p-frames in n-space. Assume that the metric on $V_{n,p}$ is normalized so that the natural embedding

$$V_{n,p} \to S^{n-1} \times \ldots \times S^{n-1} \qquad \text{(p times)}$$

is isometric. Our object is now to prove:

__Proposition 2.3.1__ There is a 1-1 correspondence between multiplications $F(k,p;n)$ and p-isometric embeddings $f\colon S^{k-1} \to V_{n,p}$ which satisfy $f(-x_o) = -f(x_o)$ for some point x_o in S^{k-1}.

__Proof__ The correspondence is given trivially in one direction as follows: given a multiplication $F\colon \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n$, let $[u_1 \ldots u_p]$ be the standard frame for $\mathbb{R}^p$ and let $f(x) = [F(x,u_1), \ldots, F(x,u_p)]$. It is clear that f satisfies the conditions above.

__Remark 2.3.2__ It is clear also that if γ is a geodesic in S^{k-1}, then each component of $f \circ \gamma$ is a geodesic in S^{n-1}. Hence $f \circ \gamma$ is a geodesic in $V_{n,p}$. The embedding f is therefore __totally geodesic__ (and hence harmonic).

To prove the converse we need

Lemma 2.3.3 Suppose a map $f = (f_1 \ldots f_p): S^{k-1} \to V_{n,p}$ is given, satisfying this condition: if $e_o \perp e_1$ in S^{k-1}, then for all i,j

$$\langle f_i(e_o), f_j(e_1)\rangle + \langle f_i(e_1), f_j(e_o)\rangle = 0$$

Then f defines an $F(k,p;n)$ in a natural way.

Proof Define $F: \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n$ to be the bilinear extension of $(e_i, u_j) \to f_j(e_i)$. Suppose u is any vector in S^{p-1}. Then $F(e_i,u)$ is a unit vector by construction. To prove the lemma, it therefore suffices to show that $F(e_i,u) \perp F(e_j,u), i \neq j$. But if $u = \Sigma a_k u_k$

$$\langle F(e_i,u), F(e_j,u)\rangle = \sum_{k\ell} a_k a_\ell \langle F(e_i,u_k), F(e_j,u_\ell)\rangle$$

$$= \sum_{k=\ell} a_k^2 \langle F(e_i,u_k), F(e_j,u_k)\rangle$$

$$+ \sum_{k\neq\ell} a_k a_\ell \langle F(e_i,u_k), F(e_j,u_\ell)\rangle$$

$$= \sum_k a_k^2 \langle f_k(e_i), f_k(e_j)\rangle + \sum_{k\neq\ell} a_k a_\ell \langle f_k(e_i), f_\ell(e_j)\rangle$$

Each term in the first sum is 0, and those of the second cancel in pairs. This proves the lemma.

Now suppose we are given $f: S^{k-1} \to V_{n,p}$ as in the statement of the proposition. We are going to show that if γ is any geodesic in S^{k-1}, then $f_i \circ \gamma$ is a geodesic in S^{n-1} for each i. It will furthermore become clear that f_i preserves arc length (cf. Remark 2.3.2)

We are given $x_o \in S^{k-1}$ such that $f(-x_o) = -f(x_o)$. Therefore first let γ be a geodesic of unit velocity between x_o and $-x_o$. Hence $f \circ \gamma$ is a path from $f(x_o)$ to $-f(x_o)$ of length $p\pi$. This is clearly the distance between $f(x_o)$ and $-f(x_o)$ in

$V_{n,p}$, and can only be achieved by a path which is of length π on each S^{n-1} (between two antipodal points). Hence $f_i \circ \gamma$ is a geodesic on S^{n-1} of unit velocity for each i. If this argument is repeated on following γ back to x_o, it follows that $f(-p) = -f(p)$ for all p on γ (since this is true for each f_i).

Now note that since S^{k-1} can be filled out by such geodesics γ, we get $f(-p) = -f(p)$ for all $p \in S^{k-1}$. Finally, this allows us to repeat the above argument with respect to <u>any</u> geodesic in S^{k-1}, and thus get the desired conclusion.

The last thing we need for the proposition is a property of orthogonal geodesics:

<u>Lemma 2.3.4</u> Let $\gamma_1(t)$, $\gamma_2(t)$ be unit velocity geodesics in S^n satisfying $\langle\gamma_1(t),\gamma_2(t)\rangle \equiv 0$. Then

$$\langle\gamma_1(t),\gamma_2(t+\pi/2)\rangle + \langle\gamma_1(t+\pi/2),\gamma_2(t)\rangle \equiv 0$$

Proof: Without loss of generality we may assume

$$\gamma_1(t) = \cos t\, e_1 + \sin t\, e_2$$
$$\gamma_2(t) = \cos t\, v + \sin t\, u$$

where

$$v = ae_2 + be_3$$
$$u = -ae_1 + ce_4$$

The lemma follows by direct calculation.

To prove 2.3.1, now apply 2.3.3 as follows: if $e_o \perp e_1$ in S^{k-1}, let γ be the unit velocity geodesic in the plane of e_o and e_1. Given integers i and j, let $\gamma_1 = f_i \circ \gamma$ and $\gamma_2 = f_j \circ \gamma$; Then γ_1 and γ_2 are unit velocity geodesics on S^{n-1} by the arguments above. The condition of Lemma 2.3.3 is immediately satisfied if $i = j$; if $i \neq j$, γ_1 and γ_2 are orthogonal (since f_i is a coordinate in a Stiefel manifold), so 2.3.4 applies.

This proves Proposition 2.3.1.

For the second part of our study, we need Wong's characterization of geodesics in Grassmann manifolds:

Theorem 2.3.5 [?G] $\Gamma_t \subset G_{n,r}$ is a geodesic if and only if the one-parameter family of r-planes satisfies

a) all pairs of r-planes in Γ have common angle 2-planes

b) the r angles between two nearby r-planes are proportional to a fixed set of non-negative constants.

To interpret: the cosine of an angle between two r-planes A and B is a stationary value of $\langle u, \pi_B u\rangle$, where u runs through A and π_B is orthogonal projection onto B. If u is such a vector, then the pair u, $\pi_B u$ determines an angle 2-plane (perhaps degenerate).

Therefore in statement (a) above we allow the possibility that for some pairs of points on a geodesic a full angle 2-plane may degenerate (e.g. if both points are the same point).

Definition 2.3.6 Let us say that two planes A and B are semi-normal if the only angles occurring between them are 0 and $\pi/2$ (i.e., all angle 2-planes are degenerate).

Note that any $F(2,k:n)$ determines a closed geodesic in $V_{n,k}$, and by projection a smooth closed path in $G_{n,k}$, denoted γ_F.

Proposition 2.3.7 The path γ_F is a geodesic in $G_{n,k}$ iff the planes $F(e_1, \mathbb{R}^k)$ and $F(e_2, \mathbb{R}^k)$ are semi-normal for any orthonormal vectors e_1, e_2.

Proof Let us abbreviate $F(x,y)$ to xy, and denote $F(e_i, \mathbb{R}^k)$ by P_i. Finally, let π be orthogonal projection of $\mathbb{R}^n$ onto $P_1 = F(e_1, \mathbb{R}^k)$. With this notation we make the following

Assertion: Suppose e_2u and $\pi(e_2u)$ define an angle 2-plane between P_1 and P_2, for some $u \in \mathbb{R}^k$. Then if e is any other vector on the circle, eu and $\pi(eu)$ define an angle 2-plane between P_1 and $P = F(e, \mathbb{R}^k)$.

To see this, note first that $\langle e_2u, \pi(e_2u)\rangle$ is stationary at u iff

$$\langle e_2u, \pi(e_2v)\rangle + \langle e_2v, \pi(e_2u)\rangle = 0$$

for all $v \perp u$ (it only makes sense to vary in directions perpendicular to u). Since F is orthogonal, this is equivalent to saying e_2u and $\pi(e_2u)$ define an angle 2-plane between P_1 and P_2. Since π is self adjoint, this condition reduces to

$$\langle e_2u, \pi(e_2v)\rangle = 0$$

for all $v \perp u$. The assertion therefore consists of demonstrating that if this condition holds, then for any t

$$\langle(\cos t\ e_1 + \sin t\ e_2)u, \pi((\cos t\ e_1 + \sin t\ e_2)v)\rangle = 0$$

for all $v \perp u$. By assumption, it suffices to show that

$$\langle e_1u, \pi(e_1v)\rangle = 0$$

and

$$\langle e_1u, \pi(e_2v)\rangle + \langle e_2u, \pi(e_1v)\rangle = 0$$

As π is self adjoint and is the identity on P_1, we simply need

$$\langle e_1u, e_1 v\rangle = 0 = \langle e_1u, e_2 v\rangle + \langle e_2u, e_1 v\rangle$$

for all $v \perp u$. The first is immediate, as F is orthogonal. The second equality is also a characteristic of orthogonal multiplications: just expand the identity $\langle(e_1 + e_2)u, (e_1 + e_2)v\rangle = 0$. Hence the assertion.

Now to the proposition. Suppose first that γ_F is a geodesic, and that e_2u and $\pi(e_2u)$ define an angle 2-plane between P_1 and P_2. We must show that $\pi(e_2u) = e_2u$ or $\pi(e_2u) = 0$. By the assertion, eu and $\pi(eu)$ define an angle 2-plane between P_1 and P_e for all

$e \in S^1$. If γ_F is a geodesic, it follows that Span $(eu,\pi(eu))$ is constant (except for degeneracies). However, eu sweeps out a 2-plane of its own as e traverses the circle, and therefore $\pi(eu)$ must lie in this plane Q, spanned by e_1u and e_2u, for all e. In particular, $\pi(e_2u) \in Q$.

There are two cases:

i) $e_2u \in P_1$; then $\pi(e_2u) = e_2u$

ii) $e_2u \notin P_1$; then $\pi(e_2u) \in Q \cap P_1$ which is one-dimensional and spanned by e_1u. However, $\langle\pi(e_2u),e_1u\rangle = \langle e_2u,e_1u\rangle = 0$, hence $\pi(e_2u) = 0$.

Conversely, suppose that P_1 and P_2 are semi-normal. We will show that P_1 and $P_e = F(e, \mathbb{R}^k)$ have common angle 2-planes for all e, from which it will be clear that γ_F is a geodesic.

Suppose first that $\pi(e_2u) = 0$; then by the assertion Q_e = Span $(eu, \pi(eu))$ is an angle 2-plane for each e, and we claim that Q_e is constant (except for degeneracies). In fact, since $\pi(eu) = \pi(\cos t\ e_1u + \sin t\ e_2u) = \cos t\ e_1u$, we see that $Q_e = Q$ = Span (e_1u,e_2u).

The other possibility is that $\pi(e_2u) = e_2u$. The way to handle this case, however, is not to fix our attention on a particular u, but instead to consider the entire intersection $P_1 \cap P_2 = V$. The problem with respect to this type of degeneracy is to demonstrate that $P_1 \cap P_e = V$ for all $e \neq \pm e_1$.

So suppose $v \in V$ is a unit vector, $v = e_1u = e_2x$, and we are given $e = \cos t\ e_1 + \sin t\ e_2$. Then if $w = \cos t\, u + \sin t\, x$,

$$ew = v + \cos t \sin t\ (e_1x + e_2u)$$

But as $x \perp u$ (since $e_1u = e_2x$),

$$\begin{aligned}\langle e_1u, e_2x\rangle + \langle e_2u, e_1x\rangle &= 0\\ &= 1 + \langle e_2u,e_1x\rangle\end{aligned}$$

hence $e_2 u = -e_1 x$. Therefore $ew = v$. This shows that $P_1 \cap P_e \supset V$ for all e. The opposite inclusion (for $e \neq \pm e_1$) follows similarly.

It is therefore easy to see that γ_F is a geodesic. This finishes the proposition.

<u>Example 2.3.8</u> We exhibit an $F(2,2;4)$ such that γ_F is not a geodesic in $G_{4,2}$. The first factor $\mathbb{R}^2$ will be the plane in the quaternions spanned by $e_1 = 1$ and $e_2 = ai + bj$, where a and b are both non-zero. The second factor is $\mathbb{C}$ and F is quaternionic multiplication. Then in the notation above we have $P_1 = \mathbb{C}$ and $P_2 = \mathrm{Span}\,(ai + bj, -a + bk)$. These planes are not semi-normal, because there is no non-zero vector in P_1 which is orthogonal to P_2. By the proposition, γ_F is not a geodesic.

<u>Corollary 2.3.9</u> In general

a) The harmonic fibration $V_{n,k} \to G_{n,k}$ is not totally geodesic.

b) Given an $F(p,k;n)$, the induced map $S^{p-1} \to G_{n,k}$ is not harmonic.

Lest this picture appear too bleak, however, note that if $F: \mathbb{C}^p \times \mathbb{C}^k \to \mathbb{C}^n$ is a complex bilinear multiplication, then we obtain a diagram of harmonic maps:

$$\begin{tikzcd} S^{2p-1} \arrow[r, "f"] \arrow[d, "\pi"'] & V_{n,k}^{\mathbb{C}} \arrow[d] \\ \mathbb{C}P^{p-1} \arrow[r, "\bar{f}"'] & G_{n,k}^{\mathbb{C}} \end{tikzcd}$$

As $\bar{f}$ is holomorphic (hence harmonic), $\bar{f} \circ \pi : S^{2p-1} \to G_{n,k}^{\mathbb{C}}$ is harmonic.

It can be shown that certain special real multiplications induce harmonic maps as well, by applying 2.3.7 in a suitable way.

Remark 2.3.10 The homotopy properties of the canonical mappings discussed in this section can be quite interesting. For example, from Proposition 2.3.1 we see that there is a correspondence between multiplications $F(k,n;n)$ and geodesic $(k-1)$-spheres in $O(n)$. It is known that every element of the stable homotopy of the orthogonal group can be so represented [ABS]. There should also be some nice connections between general $F(k,p;n)$ and the homotopy of the Stiefel varieties.

Chapter 3

JOINING POLYNOMIAL MAPS: THE PENDULUM EQUATION

A remark to the reader is in order concerning the presentation in this chapter and the succeeding one. This chapter contains a rather detailed analysis of a particular differential equation and certain of its solutions. A variety of similar but more complicated equations appear in Chapter IV. The treatment there tends to follow the same pattern, but for the sake of brevity concentrates only on crucial differences. This seemed preferable to a cumbersome attempt at a unified presentation, but certainly requires the critical reader of Chapter 4 to be familiar with the details of Sections 1 - 5 in Chapter 3.

The first 4 sections of this chapter are concerned with the proof of the main theorem. A special case and a related problem are considered separately in Sections 5 and 7. The applications are in Section 6.

Section 1. Main Theorem: Derivation of the Equation

Given 2 maps of spheres $f: S^{p-1} \to S^{q-1}$ and $g: S^{r-1} \to S^{s-1}$, recall that the (non-reduced) join $f * g$ of f and g is a map

$$f * g : S^{p+r-1} \to S^{q+s-1}$$

defined in Euclidean coordinates by

$$f * g(x,y) = (|x|f(x/|x|),\ |y|g(y/|y|))$$

The principal result of this chapter is

Theorem 3.1.1 Let $f: S^{p-1} \to S^{q-1}$ and $g : S^{r-1} \to S^{s-1}$ be homogeneous harmonic polynomials of degree ℓ, k respectively. If the damping conditions are satisfied, namely,

$$k > \theta(r-2)$$

$$\ell > \theta(p-2)$$

where $\theta = \frac{1}{2}(\sqrt{2} - 1)$, then there is a harmonic representative (in homotopy) of $f * g$.

Note that if $f: S^{p-1} \to S^{p-1}$ is the identity, then $f * g \simeq \Sigma^p g$, the pth suspension of g(cf. Toda [T] or by inspection). Hence

<u>Corollary 3.1.2</u> Let $g: S^{r-1} \to S^{s-1}$ be a harmonic polynomial map of homogeneity $k > \theta(r-2)$. Then the i^{th} suspension of g can be harmonically realized for $i = 1,\ldots,6$.

Further applications of these two results are studied in detail in Section 6.

The map we construct to represent the join of f and g has the form

$$f * g\,(x,y) = (\sin\alpha(t) f(x/|x|),\ \cos\alpha(t)\ g(y/|y|))$$
$$(x,y) \in \mathbb{R}^p \times \mathbb{R}^r \setminus (0)$$

where $$t = \log\,(|x|/|y|) \in (-\infty,\infty)$$

and α is a function to be determined. In all that follows, we will assume $r,p \geq 2$. The case $r = 1$ corresponds to constructing the first suspension of f and is treated separately in Section 5. We will also assume that f and g are non-constant, and therefore correspond to non-zero eigenvalues of the Laplacian on their respective domains.

Our first job is to reduce the question of whether this map is harmonic to an ordinary differential equation for α. Let Δ_S denote the Laplacian on $S = S^{p+r-1}$ and let $\Delta = \Delta_{\mathbb{R}}$ be the Laplacian on $\mathbb{R}^{p+r}$. By Example 1.1.4 $f * g|S$ is harmonic iff $\Delta_S(f * g|S)$ is proportional to $f * g$ at every point. Since $f * g$ by definition is constant in radial directions we can calculate as in Chapter 2, Section 1

$$-\Delta_S(f * g|S) = \Delta_{\mathbb{R}}(f * g)|S$$

Let us carry out such a computation on the first term in $f * g$:

$$\Delta(\sin\alpha(t)\ f(x/|x|)) = \Delta(\sin\alpha)f + 2\Sigma\ \frac{\partial\sin\alpha}{\partial x_i}\ \frac{\partial f}{\partial x_i}$$

$$+ \sin\alpha\ \Delta f$$

Here as usual arguments are suppressed. Consider the last term: if $\beta : \mathbb{R}^p \setminus (0) \to S^{p-1}$ is the harmonic retraction $x \to x/|x|$, we have the composition

$$\mathbb{R}^p\setminus(0) \xrightarrow{\beta} S^{p-1} \xrightarrow{f} \mathbb{R}^q$$

Hence $\Delta(f \circ \beta) = \text{Trace}\ \nabla df(d\beta, d\beta)$, and as β is something akin to a harmonic fibration we find

$$\Delta(f \circ \beta) = -|x|^{-2}\Delta_{S^{p-1}} f$$

$$= -|x|^{-2}\lambda_1 f$$

where $\qquad \lambda_1 = \ell(\ell + p-2)$

Now note that the function t satisfies

$$\frac{\partial t}{\partial x_i} = \frac{x_i}{|x|^2}$$

$$\frac{\partial t}{\partial y_j} = \frac{-y_j}{|y|^2}$$

Hence $\qquad \frac{\partial}{\partial x_i} \sin\alpha(t) = \cos\alpha\,\dot{\alpha}\frac{x_i}{|x|^2}$

and as $\Sigma x_i \frac{\partial}{\partial x_i}(f(x/|x|)) = \frac{\partial}{\partial r} f(x/|x|) = 0$, the second term in the above expression for $\Delta(\sin\alpha\ f)$ vanishes. After a straightforward computation on the first term we are led to

$$\Delta(\sin\alpha\ f) = \left[\frac{-\lambda_1\sin\alpha}{|x|^2} + \frac{\cos\alpha\ \ddot{\alpha} - \sin\alpha\ \dot{\alpha}^2}{|x|^2\ |y|^2} + \cos\alpha\,\dot{\alpha}\left(\frac{p-2}{|x|^2} - \frac{r-2}{|y|^2}\right)\right] f$$

Similarly

$$\Delta(\cos\alpha\ g) = \left[\frac{-\lambda_2\cos\alpha}{|y|^2} - \frac{\sin\alpha\ \ddot{\alpha} + \cos\alpha\ \dot{\alpha}^2}{|x|^2\ \ |y|^2} + \sin\alpha\ \dot{\alpha}\left(\frac{r-2}{|y|^2} - \frac{p-2}{|x|^2}\right)\right] g$$

But as $f * g = (\sin\alpha\, f, \cos\alpha\, g)$, the requirement that $\Delta(f * g)$

be proportional to $f * g$ is fulfilled if there is compatibility between the above expressions, namely

$$(\cot\alpha + \tan\alpha)\,[\ddot\alpha + \dot\alpha((p-2)|y|^2 - (r-2)|x|^2)] + \lambda_2|x|^2 - \lambda_1|y|^2 = 0$$

If we substitute $|x|^2 = \sin^2\tan^{-1}e^t = e^t/(e^t+e^{-t})$

$|y|^2 = e^{-t}/(e^t+e^{-t})$

the equation for α is

(3.1.3)

$$\ddot\alpha(t) + (e^t+e^{-t})^{-1}[((p-2)e^{-t} - (r-2)e^t)\quad \dot\alpha(t) + (\lambda_2 e^t - \lambda_1 e^{-t})$$

$$\sin\alpha(t)\cos\alpha(t)] \quad = 0$$

with $\lambda_1 = \ell(\ell+p-2)$

$\lambda_2 = k(k+r-2)$

The damping conditions can now be expressed in the more familiar form

(3.1.4) $\quad (p-2)^2 < 4\lambda_1$

$(r-2)^2 < 4\lambda_2$

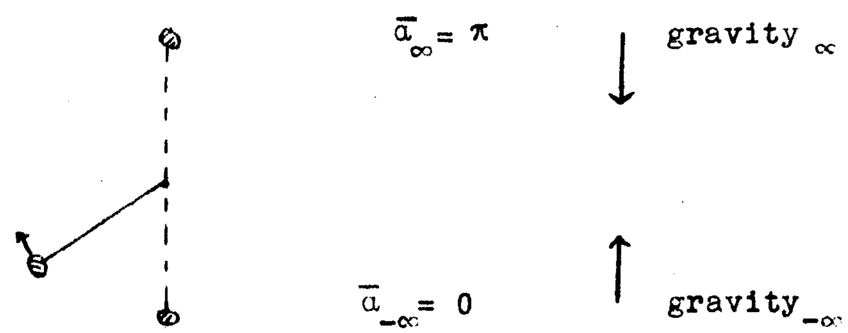
In section 2 we will demonstrate the existence of a special sort of solution to 3.1.3; from this it will be obvious that the map we have constructed is homotopic to the usual join of f and g. Sections 3 and 4 are devoted to the regularity problem; in other words, to showing that this map is smooth at $x = 0$ and $y = 0$.

Section 2. Existence

The object of this section is

Proposition 3.2.1 Subject to (3.1.4), there is a solution α of (3.1.3) which is strictly increasing and which is asymptotic to 0 and $\pi/2$ at $-\infty$ and ∞ respectively.

Before proceeding to the details, let us motivate and outline our methods: thinking of t as a time parameter, we see that 3.1.3 is the equation of motion of some sort of pendulum. Notice that "gravity" is positive for $t \gg 0$ and negative for $t \ll 0$, so that one can look for an exceptional trajectory in which $\alpha_\infty = \pi/2$ and $\alpha_{-\infty} = 0$. (The physical analogy is actually more accurate if we set $\bar{\alpha} = 2\alpha$ and replace $\sin\alpha\cos\alpha$ by $(\sin\bar{\alpha})/2$. However, the equation is more easily analyzed in its present state). The picture to keep in mind (for the function $\bar{\alpha}$) is that of a pendulum standing on end at $t = \infty$ and hanging straight down at $t = -\infty$:



Existence will be established in the following intuitive way. Fix t_o to be the time when gravity vanishes and manipulate the initial conditions α_o and $\dot{\alpha}_o$. For a given $\alpha_o \in (0,\pi/2)$, throw the pendulum just hard enough $(\dot{\alpha}_o = \dot{\alpha}_o^+(\alpha_o))$ that $\alpha(t) \sim \pi/2$ as $t \to \infty$; similarly, choose $\dot{\alpha}_o^-(\alpha_o)$ to get $\alpha(t) \sim 0$ as $t \to -\infty$ Then $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ are continuous in α_o. Furthermore, $\dot{\alpha}_o^+ \to 0$ as $\alpha_o \to \pi/2$ and $\dot{\alpha}_o^- \to 0$ as $\alpha_o \to 0$.

With regard to behavior at the opposite ends of the interval, our intuition fails us: it would seem obvious that $\dot{\alpha}_o^+$ is bounded away from 0 for α near 0, and likewise for $\dot{\alpha}_o^-$ when α_o is near $\pi/2$. However, this step requires essential use of the fact that our system is underdamped for large $|t|$; that is, we require inequalities 3.1.4.

Having done this, it follows at once by continuity considerations that there is some α_o with $\dot{\alpha}_o^+(\alpha_o) = \dot{\alpha}_o^-(\alpha_o)$. The solution with these initial data will satisfy the proposition.

Let us now make this argument precise. The first thing to remark is that the solution $\alpha(\alpha_o,\dot{\alpha}_o)$ and its derivatives are continuous functions of the initial conditions $\alpha_o,\dot{\alpha}_o$. That is, we can make two solutions α_1 and α_2 close in C^r over any compact time interval by making their initial data close. This is a standard consequence of the smoothness of the coefficients in 3.1.3 [CL], and we use this fact throughout.

We can now define the functions $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ more carefully.

<u>Definition 3.2.2</u> If $\alpha_o \in (0,\pi/2)$, let $A^+(\alpha_o)$ be the collection of $\dot{\alpha}_o \in \mathbb{R}$ such that $\alpha(\alpha_o,\dot{\alpha}_o)$ increases monotonically to $\pi/2$ in finite time as t increases from t_o.

Similarly, let $A^-(\alpha_o)$ be the set of $\dot{\alpha}_o$ such that $\alpha(\alpha_o,\dot{\alpha}_o)$ decreases monotonically to 0 in finite time as t decreases from t_o. Let

$$\dot{\alpha}_o^+(\alpha_o) = \inf A^+(\alpha_o)$$

$$\dot{\alpha}_o^-(\alpha_o) = \inf A^-(\alpha_o)$$

<u>Lemma 3.2.3</u> The functions $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ are well defined.

<u>Proof</u> Since $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ are defined symmetrically, we will give a proof only for $\dot{\alpha}_o^+$. It therefore suffices to show that A^+ is always non-empty, since this set is clearly bounded below by 0.

So given $\alpha_o \in (0,\pi/2)$ we must find $\dot{\alpha}_o$ such that $\alpha(\alpha_o,\dot{\alpha}_o)$ increases monotonically to $\pi/2$ in forward time. By inspection of Equation 3.1.3, it is clear that there are positive constants c_1, c_2 such that

$$\ddot{\alpha}(s) \geq -c_1 - c_2\,\dot{\alpha}(s)$$

as long as $\dot{\alpha}(s) \geq 0$. Hence if $\dot{\alpha}(s) \geq 0$ and $\alpha(s) \leq \pi/2$ on some interval $[t_o, t]$, with $\alpha(t_o) = \alpha_o$, we find by integrating that

$$\dot{\alpha}(t) \geq \alpha_o - c_1(t-t_o) - c_2\pi/2$$

Therefore given any $t > t_o$, choose $\dot{\alpha}_o > 0$ such that the above expression is $\geq (\pi/2 - \alpha_o)/(t - t_o) > 0$. Hence $\dot{\alpha}(s) \geq (\pi/2 - \alpha_o)/(t - t_o)$ for all $s \geq t_o$ (as long as α stays under $\pi/2$). Therefore by integrating again it follows that α increases monotonically to $\pi/2$ by time t. This proves the lemma.

<u>Lemma 3.2.4</u> $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ are strictly positive on $(0, \pi/2)$.

<u>Proof</u> Given $\alpha_o \in (0,\pi/2)$, suppose $\dot{\alpha}_o = 0$. Then from Equation 3.1.3, as gravity vanishes at t_o, $\ddot{\alpha}(t_o) = 0$ and

$$\dddot{\alpha}(t_o) = \sin\alpha_o \cos\alpha_o\,(-2(\lambda_1 + \lambda_2)/(e^{t_o} + e^{-t_o})^2) < 0$$

Hence $\alpha(t)$ initially decreases to some $\alpha(t_1) < \alpha_o$. Hence any function near α_o will take a value $< \alpha_o$ at t_1, without first reaching $\pi/2$. Therefore there cannot be elements $\dot{\alpha}_o$ in $A^+(\alpha_o)$ arbitrarily close to 0; this forces $\dot{\alpha}_o^+(\alpha_o) > 0$. Similarly, $\dot{\alpha}_o^-(\alpha_o) > 0$. Hence the lemma.

<u>Lemma 3.2.5</u> $A^+(\alpha_o)$ and $A^-(\alpha_o)$ are open sets for all $\alpha_o \in (0, \pi/2)$.

<u>Proof</u> As usual, it suffices by symmetry to show $A^+(\alpha_o)$ is open. Suppose $\dot{\alpha}_o \in A^+(\alpha_o)$ and $\alpha(\alpha_o,\dot{\alpha}_o)$ arrives at $\pi/2$ for the first time at time t. Then $\dot{\alpha}(t) > 0$. Certainly $\dot{\alpha}(t) \geq 0$, and if $\dot{\alpha}(t) = 0$ and $\alpha(t) = \pi/2$, it follows by the uniqueness theorem for OLE's that $\alpha \equiv \pi/2$, which is a contradiction. Hence α must increase past $\pi/2$ with positive derivative.

We claim also that $\dot{\alpha}(s) > 0$ for $t_o < s < t$. For if $\dot{\alpha}(s) = 0$, $\ddot{\alpha}(s) < 0$ follows from the equation (as gravity is positive for $s > t_0$). This would force α to decrease past s, which is not allowed. Finally, by (the proof of) Lemma 3.2.4, $\dot{\alpha}(t_0) > 0$. Hence $\dot{\alpha} > 0$ on $[t_0, t + \varepsilon]$ for some $\varepsilon > 0$, and thus any function C^1 close enough to α will increase monotonically to $\pi/2$ on this interval. Therefore points near $\dot{\alpha}_0$ are in $A^+(\alpha_0)$, so $A^+(\alpha_0)$ is open.

We are now in a position to prove the first lemma of interest in our program.

<u>Lemma 3.2.6</u> For any $\alpha_0 \in (0, \pi/2)$, $\alpha(\alpha_0, \dot{\alpha}_0^+(\alpha_0))$ is strictly increasing for $t \geq t_0$ and is asymptotic to $\pi/2$ as $t \to \infty$. Similarly, $\alpha(\alpha_0, \dot{\alpha}_0^-(\alpha_0))$ is strictly increasing for $t \leq t_0$ and is asymptotic to 0 as $t \to -\infty$.

<u>Proof</u> We give the proof for $\alpha(\alpha_0, \dot{\alpha}_0^+)$ only. It suffices to demonstrate two things: first, that $\dot{\alpha} > 0$ on $[t_0, \infty)$, and second, that $\alpha < \pi/2$ on $[t_0, \infty)$. These together show that α increases to some asymptotic value α_∞, with $0 < \alpha_\infty \leq \pi/2$. But than as $t \to \infty$, $\dot{\alpha}(t)$, $\ddot{\alpha}(t) \to 0$, so Equation 3.1.3 tells us that the only possible choice for α_∞ is $\pi/2$.

So assume that one or the other of the above conditions is violated. Then one goes wrong first, for we have noted that by the uniqueness theorem they cannot go wrong simultaneously. On the other hand, neither goes wrong at $t = t_0$, by Lemma 3.2.4. But now if, say, $\dot{\alpha}(t) = 0$ for $t > t_0$ (for the first time), and $0 < \alpha(t) < \pi/2$, then $\ddot{\alpha}(t) < 0$ and α decreases past t. But by assumption that $\dot{\alpha}_0 = \dot{\alpha}_0^+$, there are functions arbitrarily close to α on $[t_0, t + \varepsilon]$ which are strictly increasing or go past $\pi/2$ on this interval. This gives a contradiction.

So now assume that $\alpha(t) = \pi/2$ and $\dot{\alpha} > 0$ on $[t_0, t]$. This says that $\dot{\alpha}_0^+(\alpha_0) \in A^+(\alpha_0)$, which is not possible since $A^+(\alpha_0)$ is open (Lemma 3.2.5) and $\dot{\alpha}_0^+ = \inf A^+$. Thus neither condition can be violated on $[t_0, \infty)$, so the lemma follows.

A key ingredient in succeeding proofs is the following second order comparison theorem [CL, pp. 208-211]. Although it is a theorem about linear equations, we will be able to adapt to our use.

Theorem 3.2.7 [CL] Let p_i' and g_i be continuous on $[a,b]$, $i = 1,2$, and let

$$0 < p_2(t) \leq p_1(t)$$
$$g_2(t) \geq g_1(t)$$

Let L_i be the operator

$$L_i u = (p_i u')' + g_i u$$

If f_i is a solution of $L_i f_i = 0$, let $w_i = \tan^{-1}(f_i/p_i f_i')$
Then if $w_2(a) \geq w_1(a)$, $w_2(t) \geq w_1(t)$ for all $t \in [a,b]$.

Remark The significance of the function w is that it essentially measures angle in the phase portrait of the solution.

The procedure for applying this theorem to our equation is to write 3.1.3 as

$$L\alpha = (p\alpha')' + g_\alpha \alpha = 0 \tag{3.2.8}$$

where

$$p(t) = \exp \int_{t_o}^{t} - h(s)ds$$

$$h(s) = (e^s + e^{-s})^{-1}((p-2)e^s - (p-2)e^{-s})$$

$$g_\alpha(t) = \frac{\sin\alpha(t) \cos\alpha(t)}{\alpha(t)} (e^t + e^{-t})^{-1}(\lambda_2 e^t - \lambda_1 e^{-t})p(t)$$

Of course, g_α depends on the solution α we are interested in comparing, so a slight amount of care will be needed to avoid circular arguments.

A specific case where the comparison theorem is needed is

Lemma 3.2.8 For any $\alpha_o \in (0,\pi/2)$, $\dot\alpha_o^+(\alpha_o)$ is the unique initial derivative for which we get a solution of the desired form in forward time. More precisely, if $0 < \dot\alpha_o < \dot\alpha_o^+$, then the associated solution $\alpha(\alpha_o,\dot\alpha_o)$ must eventually start to decrease before reaching $\pi/2$. If we take $\dot{\alpha}_o > \dot{\alpha}_o^+$, then in fact $\dot{\alpha}_o \in A^+(\alpha_o)$.

Similarly, $\dot{\alpha}_o^-(\alpha_o)$ gives the unique solution of the desired form in backward time.

Proof Take $\dot{\alpha}_o < \dot{\alpha}_o^+(\alpha_o)$ and let $\alpha_1 = \alpha(\alpha_o, \dot{\alpha}_o^+(\alpha_o))$, $\alpha_2 = \alpha(\alpha_o, \dot{\alpha}_o)$. Suppose we are given a time interval on which $\alpha_1 \geq \alpha_2$ and both lie in $(0,\pi/2)$: then as the function $\sin\theta\cos\theta/\theta$ is decreasing on $[0,\pi/2]$, we have (in the notation of 3.2.8) that $g_{\alpha_1} \leq g_{\alpha_2}$ on this interval. Further, at time t_o we have $\alpha_1 = \alpha_2$ and $\alpha_1' > \alpha_2'$ by assumption. Hence $w_2(t_o) > w_1(t_o)$ (in the notation of 3.2.7), and by the comparison theorem $w_2(t) \geq w_1(t)$ on this given interval.

Suppose we add the requirement that $\alpha_1' > 0$ on this interval. Then $w_2 \geq w_1$ implies $\alpha_2/\alpha_2' \geq \alpha_1/\alpha_1'$ which forces $\alpha_1' \geq \alpha_2'$. Finally, we choose $[t_o,t]$ to be the maximal interval on which all of the above conditions are satisfied (allowing perhaps $\alpha_1'(t) = 0$). But since initially $\alpha_1' > \alpha_2'$, forcing $\alpha_1 > \alpha_2$ to hold for a while, it is clear that what must happen first is $\alpha_2'(t) = 0$ (since $\alpha_1'(t) = 0$ cannot occur by Lemma 3.2.6, and other possibilities lead immediately to contradictions).

Hence $\ddot{\alpha}_2(t) < 0$ and α_2 decreases past t without first reaching $\pi/2$. Of course, there is the possibility that the maximal interval is $[t_o,\infty)$; in this case, $\alpha_2(t) \leq \alpha_1(t) < \pi/2$ and $\alpha_2'(t) > 0$ for all $t \geq t_o$. Hence α_2 attains an asymptotic limit $\alpha_{2,\infty}$. However, we would also have $\alpha_1' > \alpha_2'$, so that $\alpha_1 - \alpha_2$ is positive and non decreasing on $[t_o,\infty)$. This forces $\alpha_{2,\infty} < \pi/2$, which is not possible. This proves half of the lemma.

Suppose on the other hand we take $\dot{\alpha}_o > \dot{\alpha}_o^+(\alpha_o)$. To show that $\dot{\alpha}_o \in A^+(\alpha_o)$, we reverse the roles in the previous argument and set $\alpha_1 = \alpha(\alpha_o, \dot{\alpha}_o)$, $\alpha_2 = \alpha(\alpha_o, \alpha_o^+(\dot{\alpha}_o))$; i.e., α_1 is the solution on top. Then the argument proceeds as before to the point where we take the maximal interval $[t_o, \bar{t}]$. Applying Lemma 3.2.6 to α_2, we see that what must happen first is $\alpha_1(t) = \pi/2$, and thus $\dot{\alpha}_o \in A^+(\alpha_o)$. It is also clear that $t = \infty$ cannot occur, because α_1 is forced from below by α_2 and $\alpha_2 \to \pi/2$.

Since the case of $\dot{\alpha}_o^-$ is symmetrical, this finishes the lemma.

We can now continue with our program as follows:

<u>Lemma 3.2.9</u> $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$ are continuous.

This is almost immediate from the preceding. Suppose $\alpha_{o_n} \to \alpha_o$ but $\dot{\alpha}_o^+(\alpha_{o_n}) \leq \dot{\alpha}_o^+(\alpha_o) - \varepsilon$ for some $\varepsilon > 0$. Let $\dot{\alpha}_o = \dot{\alpha}_o^+(\alpha_o) - \varepsilon/2$. Then 3.2.8 tells us that:

a) $\alpha(\alpha_{o_n}, \dot{\alpha}_o)$ increases monotonically to $\pi/2$ in finite time for all n.

b) $\alpha(\alpha_o, \dot{\alpha}_o)$ will eventually decrease before reaching $\pi/2$.

However, these statements become contradictory as $\alpha_{o_n} \to \alpha_o$.

The case $\dot{\alpha}_o^+(\alpha_{o_n}) \geq \dot{\alpha}_o^+(\alpha_o) + \varepsilon$ is similar. Here we use again the fact that a solution which increases to $\pi/2$ will actually go a small way beyond, with positive derivative throughout. (cf. Lemma 3.2.5). Hence the lemma.

<u>Lemma 3.2.10</u> $\dot{\alpha}_o^+(\alpha_o) \to 0$ as $\alpha_o \to \pi/2$.

$\dot{\alpha}_o^-(\alpha_o) \to 0$ as $\alpha_o \to 0$.

<u>Proof</u> This much is obvious. For example, refer to estimates in 3.2.3.

For the final step in the proof, we at last make use of assumptions 3.1.4. The main tool is again the comparison theorem.

<u>Lemma 3.2.11</u> $\dot{\alpha}_o^+(\alpha_o)$ is bounded away from 0 for α_o near 0

$\dot{\alpha}_o^-(\alpha_o)$ is bounded away from 0 for α_o near $\pi/2$.

<u>Proof</u> As usual, we give a proof for $\dot{\alpha}_o^+(\alpha_o)$. The proof is by contradiction, so assume there is a sequence $\alpha_{o_n} \to 0$ such that $\dot{\alpha}_o^+(\alpha_{o_n}) \to 0$. Hence there is also a sequence $\dot{\alpha}_{o_n} \to 0$ such that $\dot{\alpha}_{o_n} \in A^+(\alpha_{o_n})$. We will show that for some n, the solution $\alpha(\alpha_{o_n}, \dot{\alpha}_{o_n})$ eventually decreases before reaching $\pi/2$.

As before, we will study solutions α of the linearized equation 3.2.8; α will be compared with solutions β of the linear equation

$$\ddot{\beta} - m_1 \dot{\beta} + m_2\beta = 0$$

where m_1, m_2 are constants near $r - 2$ and λ_2, respectively, to be determined. Observe that this equation looks like 3.1.3 for large t. Furthermore, when $m_1^2 - 4m_2 < 0$, a solution β will consist of an exponential times an oscillatory term with frequency $w = (4m_2 - m_1^2)^{\frac{1}{2}}/2$ and period $\tau = 2\pi/w$. Under assumptions 3.1.4 we will choose m_1, m_2 near enough to $r - 2, \lambda_2$ so that τ is no larger than some *a priori* bound τ_o. For example, let τ_o be twice the period associated to the numbers $r - 2, \lambda_2$ themselves.

The important thing for us is the fact that any solution β will then satisfy $\dot{\beta} = 0$ at some point of any interval of length τ_o. Using the comparison theorem, we will show that one of the solutions $\alpha(\alpha_{o_n}, \dot{\alpha}_{o_n})$ must do likewise.

Recall that for α we will study Equation 3.2.8; however, write it as $L_2\alpha = (p_2\alpha')' + g_2\alpha = 0$

Likewise for β $\quad L_1\beta = (p_1\beta')' + g_1\beta = 0$

where we have set $\quad p_2(t) = \exp\int_{t_1}^{t} - h(s)ds$

$$p_1(t) = \exp\int_{t_1}^{t} - m_1 ds$$

and $t_1 > t_0$ is to be chosen. Here $g_1(t) = m_2\, p_1(t)$ and other quantities are as in 3.2.8. To choose t_1, define a function

$$\mu(t_1) = \exp\int_{t_1}^{t_1+\tau_0} (h(t) - h(t_1))dt$$

Then as $h(t)$ is increasing, $\mu(t_1) > 1$ and $\mu(t_1) \to 1$ as $t_1 \to \infty$. Therefore choose t_1 so large that there is an $\varepsilon > 0$ with the following properties:

i) if $\quad m_1 = h(t_1)$

$$m_2 = (e^{t_1} + e^{-t_1})^{-1}(\lambda_2 e^{t_1} - \lambda_1 e^{-t_1})(1 - \varepsilon)$$

then $\quad \tau(m_1, m_2) \leq \tau_0$.

ii) there is some $\varepsilon' > 0$ such that $(1-\varepsilon)\mu(t_1) < 1 - \varepsilon'$

Now choose $\delta > 0$ such that $0 \leq \alpha \leq \delta \to \sin\alpha\cos\alpha/\alpha \geq 1 - \varepsilon'$.

Finally, noting that $\alpha_0 = \dot{\alpha}_0 = 0$ forces $\alpha \equiv 0$, choose from our original sequence some $\alpha(\alpha_{o_n}, \dot{\alpha}_{o_n})$ satisfying $\alpha \leq \delta$ on $[t_0, t_2]$, where $t_2 = t_1 + \tau_0$ We take this α to be our solution of $L_2\alpha = 0$ and for β we take the solution of $L_1\beta = 0$ with initial data at t_1 equal to those of α at t_1.

Let us verify the hypotheses of the comparison theorem for the interval $[t_1, t_2]$. For $t \in [t_1, t_2]$, $h(t) \geq h(t_1) = m_1$, so clearly $p_2(t) \leq p_1(t)$. To check that $g_2(t) \geq g_1(t)$, observe that

$$g_2(t)/p_2(t) = \frac{\sin\alpha(t)\cos\alpha(t)}{\alpha(t)} (e^t+e^{-t})^{-1}(\lambda_2 e^t - \lambda_1 e^{-t})$$

$$\geq (1-\varepsilon') (e^{t_1}+e^{-t_1})^{-1}(\lambda_2 e^{t_1} - \lambda_1 e^{-t_1})$$

$$> (1-\varepsilon)\mu(t_1)(e^{t_1}+e^{-t_1})^{-1}(\lambda_2 e^{t_1} - \lambda_1 e^{-t_1})$$

$$= m_2\mu(t_1)$$

$$= m_2 \exp\int_{t_1}^{t_1+\tau_0} (h(t) - h(t_1))\, dt$$

$$\geq m_2 \exp\int_{t_1}^{t} (h(t) - m_1)\, dt$$

$$= m_2\, p_1(t)/p_2(t)$$

$$= g_1(t)/p_2(t)$$

Finally, in the notation of 3.2.7, we have $w_1(t_1) = w_2(t_2)$ by construction, so $w_2(t) \geq w_1(t)$ for all $t \in [t_1,t_2]$.

But $\dot\beta$ has a zero on $[t_1,t_2]$, so by taking the first such (at t_3, say), we know that β is positive on $[t_1,t_3]$ and $w_1(t_3) = \pi/2$. Hence $w_2(t_3) \geq \pi/2$, and as $w_2(t_1) < \pi/2$ by assumption, w_2 takes the value $\pi/2$ on $[t_1,t_3]$. That is, $\dot\alpha = 0$ somewhere on this interval and hence by the usual argument must decrease. Of course, α has not already reached $\pi/2$ by the assumption that α was small on $[t_1,t_2]$. This gives the contradiction we were after, and finishes the proof of Lemma 3.2.11.

These lemmas complete the program for this section, and together form a proof of Proposition 3.2.1.

<u>Remark 3.2.12</u> In some cases the conclusion of 3.2.1 is valid even though 3.1.4 is not satisfied. For example, we can always join two identity maps to get another identity map; in this case the function α is determined explicitly.

Section 3. Asymptotic Behavior

In this section we estimate the behavior at ∞ of the exceptional solution α which is guaranteed by Proposition 3.2.1. These estimates will be used in Section 4 to prove differentiability of the mapping of spheres which is defined in terms of α.

Throughout this section we will use the notation

$$h(t) = (e^t + e^{-t})^{-1}((r-2)e^t - (p-2)e^{-t})$$

$$g(t) = (e^t + e^{-t})^{-1}(\lambda_2 e^t - \lambda_1 e^{-t})$$

so that Equation 3.1.3 has the form

(3.3.1) $\quad \ddot{\alpha}(t) = h(t)\dot{\alpha}(t) - g(t) \sin\alpha(t) \cos\alpha(t)$

Recall also that $\quad \lambda_1 = \ell(\ell+p-2)$

$$\lambda_2 = k(k-r-2)$$

when ℓ and k are the degrees of the homogeneous polynomials to be joined.

<u>Lemma 3.3.2</u> Eventually $\dot{\alpha}(t) \leq (k + 0(e^{-2t})) \cos\alpha(t)$

<u>Remark 3.3.3</u> Such a statement will mean that there is a t_1 suitably large and a function $0(e^{-2t})$ defined for $t \geq t_1$ such that the above holds for all $t \geq t_1$. By $0(e^{-2t})$ we mean simply that $|e^{2t}\, 0(e^{-2t})|$ is bounded.

<u>Proof</u> Choose $t_1 > t_o$ such that $\alpha(t_1) \geq \pi/4$ and such that $h(t_1)$ and $g(t_1)$ are close to their asymptotic values of $r - 2$ and λ_2 respectively. For $t \geq t_1$ let $k(t)$ be the solution near k of the equation

$$(k(k+r-2) - k(t)^2)/k(t) = h(t) = (r-2) - 0(e^{-2t})$$

One can easily check that $k(t) \leq k + 0(e^{-2t})$. We propose to show that

$$\dot{\alpha}(t) \leq k(t) \cos\alpha(t) \qquad \forall t \geq t_1$$

Given any such t, the idea is to compare α with the solution β of the first order problem

$$(*) \quad \dot\beta(s) = k(t)\cos\beta(s)$$
$$\beta(t) = \alpha(t)$$

It is clear that β is monotone asymptotic to $\pi/2$ as $s \to \infty$.

For $s \geq t$ we can also estimate

$$\begin{aligned}\ddot\beta(s) &= -k^2(t)\sin\beta(s)\cos\beta(s)\\ &= (\lambda_2 - k^2(t))\sin\beta(s)\cos\beta(s) - \lambda_2\sin\beta(s)\cos\beta(s)\\ &= \frac{\lambda_2 - k^2(t)}{k(t)}\sin\beta(s)\dot\beta(s) - \lambda_2\sin\beta(s)\cos\beta(s)\\ &= h(t)\sin\beta(s)\dot\beta(s) - \lambda_2\sin\beta(s)\cos\beta(s)\\ &< h(s)\dot\beta(s) - \lambda_2\sin\beta(s)\cos\beta(s)\end{aligned}$$

Now suppose that $\dot\alpha(t) > \dot\beta(t) = k(t)\cos\alpha(t)$. Then by the above $\ddot\alpha(t) > \ddot\beta(t)$. Hence let $t_2 > t$ be the first time past t for which we have $\alpha(t_2)=\beta(t_2)$, $\dot\alpha(t_2)=\dot\beta(t_2)$, or $\ddot\alpha(t_2)=\ddot\beta(t_2)$. But if $\ddot\alpha > \ddot\beta$ and $\dot\alpha > \dot\beta$ on $[t,t_2)$, with $\alpha(t) = \beta(t)$, then certainly $\dot\alpha(t_2) > \dot\beta(t_2)$ and $\alpha(t_2) > \beta(t_2)$. Hence the only possibility is $\ddot\alpha(t_2) = \ddot\beta(t_2)$.

However, we assumed $\alpha(t_1) \geq \pi/4$; hence $\alpha(t_2) > \beta(t_2) > \pi/4$, forcing $\sin\alpha\cos\alpha < \sin\beta\cos\beta$ at t_2. Applying this with our estimate on $\ddot\beta$ shows that $\ddot\beta(t_2) = \ddot\alpha(t_2)$ is impossible also. Hence $t_2 = \infty$, and $\dot\alpha - \dot\beta$ is non-decreasing on $[t,\infty)$. But this is a contradiction, as $\dot\alpha(t) > \dot\beta(t)$, and both $\dot\alpha$ and $\dot\beta$ must tend to 0 as $t \to \infty$.

Hence we must have $\dot\alpha(t) \leq k(t)\cos\alpha(t)$ $\forall t \geq t_1$, which proves the lemma.

<u>Lemma 3.3.4</u> Eventually $\alpha(t) \geq (k - 0(e^{-2t}))\sin\alpha(t)\cos\alpha(t)$

<u>Proof</u> . Choose t_1 as before and this time let $k(t)$ be the solution near k of

$$(g(t) - k(t)^2)/k(t) = r-2$$

where recall $g(t) = k(k+r-2) - 0(e^{-2t})$

It follows that $k(t) \geq k - 0(e^{-2t})$

Let us show $\dot{\alpha}(t) \geq k(t)\cos\alpha(t)\sin\alpha(t) \quad \forall\, t \geq t_1.$

The method is comparison with the solution β of

$$(**) \qquad \dot{\beta}(s) = k(t)\cos\beta(s)\sin\beta(s)$$
$$\beta(t) = \alpha(t)$$

Note that β is monotone asymptotic to $\pi/2$ as $t \to \infty$. Also

$$\ddot{\beta}(s) = k^2(t)(\cos^2\beta(s) - \sin^2\beta(s))\cos\beta(s)\sin\beta(s)$$
$$\geq -k^2(t)\cos\beta(s)\sin\beta(s)$$
$$= \dot{\beta}(s)(g(t) - k^2(t))/k(t) - g(t)\cos\beta(s)\sin\beta(s)$$
$$= (r-2)\dot{\beta}(s) - g(t)\cos\beta(s)\sin\beta(s)$$
$$> h(s)\dot{\beta}(s) - g(s)\cos\beta(s)\sin\beta(s)$$

From this point an argument quite parallel to that in Lemma 3.3.2 finishes the proof.

If we set $k^-(s) = k - 0(e^{-2s})$, then we have shown

$$\dot{\alpha}(s) \geq k^-(s)\sin\alpha(s)\cos\alpha(s)$$
$$\geq k^-(t)\sin\alpha(s)\cos\alpha(s)$$

for all $s \geq t \geq t_1$. Hence α lies above the solution of (**) for $s \geq t$. This gives

$$\cos\alpha(s) \leq \cos\beta(s) \qquad \forall\, s \geq t$$

But (**) has an explicit solution, namely

$$\tan\beta(s) = \tan\alpha(t)\exp(k^-(t)(s-t))$$

Hence there is a constant $b_t > 0$ such that for all $s \geq t$

$$\cos\beta(s) \leq b_t\exp(-k^-(t)s)$$

This gives (3.3.5) $\quad \cos\alpha(s) \leq b_t\exp(-k^-(t)s)$

for all $s \geq t$.

Similarly we have from Lemma 3.3.2, with $k^+(t) = k + 0(e^{-2t})$

$$\alpha(s) \leq k^+(s)\cos\alpha(s) \leq k^+(t)\cos\alpha(s)$$

for all $s \geq t \geq t_1$. Hence α lies under the solution of (*), for $s \geq t$, giving

$$\cos\alpha(s) \geq \cos\beta(s)$$

The explicit solution of (*) is

$$\sec\beta(s) + \tan\beta(s) = (\sec\alpha(t) + \tan\alpha(t))\exp(k^+(t)(s-t))$$

hence $\qquad \cos\beta(s) \geq c_t \exp(-k^+(t)s)$

and (3.3.6) $\qquad \cos\alpha(s) \geq c_t \exp(-k^+(t)s)$

for all $s \geq t$. Unfortunately estimates 3.3.5 - 6 are not quite good enough, so we refine them further:

<u>Lemma 3.3.7</u> Eventually $\quad c_1 e^{-kt} \leq \cos\alpha(t) \leq c_2 e^{-kt}$

<u>Proof</u> Define $f(t) = \cos\alpha(t)$. To get the second half of the inequality, note first that

$$\begin{aligned} \dot f &= -\sin\alpha\,\dot\alpha \\ &\leq -\sin\alpha\, k^-\sin\alpha\cos\alpha \qquad (3.3.4) \\ &= -(1-f^2)k^- f \\ &\leq -(1-f)k^- f = -k^- f + k^- f^2 \end{aligned}$$

Then if $k^-(t_1)$ is reasonably close to k, we can use (3.3.5) to write

$$\cos^2\alpha(t) \leq b^2 e^{-\mu kt} \qquad \forall\, t \geq t_1$$

where $\mu > 1$. Hence

$$\dot f(t) \leq -k^-(t)f(t) + c_o e^{-\mu kt}$$

Therefore by a trivial first order comparison theorem it suffices to estimate a solution of

$$\dot f(t) = -k^-(t)f(t) + c_o e^{-\mu kt}$$

given some initial value at t_1. A solution of the homogeneous part will take the form

$$\begin{aligned} f_H(t) &= c' \exp\left(-\int_o^t k^-(s)\,ds\right) \\ &= c' \exp(-kt)\exp\left(\int_o^t 0(e^{-2s})\,ds\right) \\ &\leq c'' e^{-kt} \end{aligned}$$

A particular solution of the inhomogeneous equation will look like

$$\begin{aligned} f_I(t) &= f_H(t)\int_{t_1}^{t} \exp\left(\int_o^s k^-(x)\,dx\right)\exp(-\mu ks)\,ds \\ &\leq f_H(t)\int_{t_1}^{t} e^{ks}\, e^{-\mu ks}\,ds \\ &\leq bf_H(t) \end{aligned}$$

since as $\mu > 1$ the integral is uniformly bounded. Hence the general solution to such an equation will verify an inequality of the desired type.

For the other half of the inequality, we find similarly from 3.3.2 that

$$\dot{f} \geq -\sin\alpha\, k^+ f \geq -k^+ f$$

The desired conclusion is immediate. This finishes the lemma.

By a similar procedure we can analyze the behavior of α near $t = -\infty$. The results are

Lemma 3.3.8 a) Eventually $\dot{\alpha}(t) \leq (\ell + 0(e^{2t}))\sin\alpha(t)$

b) Eventually $\dot{\alpha}(t) \geq (\ell - 0(e^{2t}))\sin\alpha(t)\cos\alpha(t)$

c) Eventually $b_1 e^{\ell t} \leq \sin\alpha(t) \leq b_2 e^{\ell t}$

Section 4. Derivative Estimates

Having obtained our function α, we want to use it to define a harmonic mapping of spheres. The formula was

$$(f * g)(x,y) = (\sin\alpha(t)\ f(x/|x|),\ \cos\alpha(t)\ g(y/|y|))$$

with $$t = \log(|x|/|y|)$$

Since $\alpha(t)$ traverses $(0,\pi/2)$ monotonically with the correct limits at $t = \pm\infty$, we can conclude that $f * g : \mathbb{R}^{p+r}\setminus(0) \to S^{q+s-1}$ is continuous and that the restriction to S^{p+r-1} represents the join of the original maps f and g. Moreover, $f * g$ is smooth away from the planes $x = 0$ and $y = 0$. What we show in this section is

Proposition 3.4.1 The first and second partial derivatives of $f * g$ extend to continuous functions on $\mathbb{R}^{p+r} \setminus (0)$.

<u>Corollary 3.4.2</u> $f * g$ is of class C^2 on $\mathbb{R}^{p+r} - (0)$.

<u>Corollary 3.4.3</u> $f * g|S^{p+r-1} : S^{p+r-1} \to S^{q+s-1}$ is analytic and harmonic.

The deduction of 3.4.2 from 3.4.1 is elementary and standard. Corollary 3.4.2 then tells us that $f * g|S^{p+r-1}$ is C^2. But by construction this map is harmonic on the dense set $(x \neq 0, y \neq 0)$. As the tension field is continuous everywhere, it must therefore be zero everywhere. Thus $f * g$ is a C^2 harmonic map of analytic Riemannian manifolds and hence is analytic [ES].

Let us verify the proposition for the function H defined by

$$H(x,y) = \sin\alpha(t)\ f(x/|x|)$$

When studying derivatives of H near possible problem points, we treat two cases:

<u>Case I</u> x near 0, y bounded away from 0

<u>Case II</u> y near 0, x bounded away from 0

<u>Case I</u>: Break down H as follows

$$H(x,y) = R^{\frac{1}{2}}(x,y)\ f(x)$$

$$R(x,y) = \sin^2\alpha(t)/|x|^{2\ell}$$

Here we use the homogeneity of f to write $f(x/|x|) = f(x)/|x|^{\ell}$. Furthermore put

$$R = r \circ \rho$$

where $$\rho(x,y) = (|x|^2, |y|^2) \equiv (u,v)$$

and $$r(u,v) = \sin^2\alpha(\tfrac{1}{2}\log(u/v))/u^{\ell}$$

We now estimate successively the derivatives of r,R, and H.

<u>Lemma 3.4.4</u> For u near 0 and v bounded away from 0, all first and second derivatives of r are uniformly bounded except $\partial^2 r/\partial u^2$, which is at worst $O(1/u)$.

<u>Proof</u> We will assume throughout that the ratio u/v is small

enough that Lemma 3.3.8 applies. To convert the latter estimates (in terms of t, the argument of α) to estimates in terms of u/v, we will use the relation

$$e^{2t} = u/v$$

Now compute $\quad \dfrac{\partial r}{\partial u} = \sin\alpha(\cos\alpha\,\dot{\alpha} - \ell\sin\alpha)/u^{\ell+1}$

We claim $\quad |\cos\alpha\,\dot{\alpha} - \ell\sin\alpha| = 0(e^{(\ell+2)t})$

Certainly from 3.3.8 it is immediate that

$$\sin\alpha = 0(e^{\ell t})$$

$$\dot{\alpha} = 0(e^{\ell t})$$

Hence

$$\begin{aligned} 1-\cos\alpha &= 1 - (1 - \sin^2\alpha)^{\frac{1}{2}} \\ &\leq \sin^2\alpha = 0(e^{2\ell t}) \end{aligned}$$

Therefore $\cos\alpha\,\dot{\alpha}$ in the above expression may be replaced by $\dot{\alpha}$ with error $(1-\cos\alpha)\dot{\alpha} = 0(e^{3\ell t})$. As $\ell \geq 1$, this is permissible. Now apply 3.3.8 (a) - (b) together to find

$$\begin{aligned} |\dot{\alpha} - \ell\sin\alpha| &\leq \sin\alpha\,|\ell + 0(e^{2t}) - \cos\alpha(\ell - 0(e^{2t}))| \\ &= \ell\sin\alpha\,|1 - \cos\alpha| + 0(e^{(\ell+2)t}) \\ &= 0(e^{(\ell+2)t}) \end{aligned}$$

Hence $\quad \dfrac{\partial r}{\partial u} = 0(e^{2(\ell+1)t})/u^{\ell+1} = 0(1)$

Now consider

$$\frac{\partial^2 r}{\partial u^2} = [\sin\alpha(-\sin\alpha\,\dot{\alpha}^2 + \cos\alpha\,\ddot{\alpha} - \ell\cos\alpha\,\dot{\alpha})$$
$$+ (\cos\alpha\,\dot{\alpha} - 2(\ell+1)\sin\alpha)(\cos\alpha\,\dot{\alpha} - \ell\sin\alpha)]/2u^{\ell+2}$$

Our object is $\dfrac{\partial^2 r}{\partial u^2} = 0(1/u)$. The above estimates apply to the second term to give such a bound. In the first term we have the expression $\sin^2\alpha\,\dot{\alpha}^2 = 0(u^{2\ell})$, so its contribution is again satisfactory. For the remainder, substitute for $\ddot{\alpha}$ from the differential equation 3.3.1:

$$\ddot{\alpha} = -(p-2)\dot{\alpha} + \lambda_1\sin\alpha + 0(e^{(\ell+2)t})$$

The error is negligible for our purposes, so we have left to study

$$|\ddot{\alpha} - \ell\dot{\alpha}| \approx |\ell(\ell + p - 2)\sin\alpha - (\ell + p - 2)\dot{\alpha}|$$

But as above, $|\dot{\alpha} - \ell\sin\alpha| = 0(e^{(\ell+2)t})$, so we are done.

A few more similar calculations suffice to finish the lemma.

We now estimate derivatives of R for x near 0 and y bounded away from 0.

<u>Lemma 3.4.5</u> a) As $x \to 0$, $\frac{\partial R}{\partial x_i}$ and $\frac{\partial^2 R}{\partial x_i \partial y_j} \to 0$, while other first and second derivatives remain bounded.

b) Similarly for $R^{\frac{1}{2}}$.

c) $R^{\frac{1}{2}}$ and its first y-derivatives extend to continuous functions for $x = 0$, $y \neq 0$.

<u>Proof</u> a) $R(x,y) = r \circ \rho(x,y)$, where $\rho(x,y) = (|x|^2, |y|^2)$.

Hence

$$\frac{\partial R}{\partial x_i} = \frac{\partial r}{\partial u} \cdot 2x_i$$

$$\frac{\partial^2 R}{\partial x_i \partial y_j} = \frac{\partial^2 r}{\partial u \partial v} \cdot 4y_j x_i$$

$$\frac{\partial^2 R}{\partial x_i \partial x_j} = \frac{\partial^2 r}{\partial u^2} \cdot 4x_i x_j + 2\delta_{ij} \frac{\partial r}{\partial u}$$

The desired conclusions are immediate from Lemma 3.4.2.

b) It suffices to show that $R^{\frac{1}{2}}$ is bounded away from 0 as $x \to 0$, with y bounded away from 0. But by Lemma 3.3.8 (c),

$$\begin{aligned} R^{\frac{1}{2}} &= \sin\alpha(t)/|x|^\ell \\ &\geq b_1 e^{\ell t}/|x|^\ell = b_1/|y|^\ell \end{aligned}$$

c) By Lemma 3.4.2 we see that the differentials of r and $\frac{\partial r}{\partial v}$ are uniformly bounded for u near 0 and v bounded away from 0. Hence both these functions extend to functions which are locally Lipschitz on the set $(u \geq 0, v > 0)$. Therefore the conclusion of (c) holds for R, and hence for $R^{\frac{1}{2}}$.

It is now easy enough to finish Case I:

<u>Lemma 3.4.5</u> H extends to a C^2 function for $y \neq 0$.

<u>Proof</u>

$$H(x,y) = R^{\frac{1}{2}}(x,y)\, f(x)$$

By homogeneity, $f(0) = 0$. Hence

$$\frac{\partial H}{\partial x_i} = \frac{\partial R^{\frac{1}{2}}}{\partial x_i} f + R^{\frac{1}{2}} \frac{\partial f}{\partial x_i}$$

is continuous at $x = 0$ by Lemma 3.4.3. Also,

$$\frac{\partial^2 H}{\partial x_i \partial x_j} = \frac{\partial^2 R^{\frac{1}{2}}}{\partial x_i \partial x_j} f + \frac{\partial R^{\frac{1}{2}}}{\partial x_i} \frac{\partial f}{\partial x_j} + \frac{\partial R^{\frac{1}{2}}}{\partial x_j} \frac{\partial f}{\partial x_i} + R^{\frac{1}{2}} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

which is again continuous at $x = 0$ by 3.4.3, since the first three terms are 0. Other derivatives are treated similarly.

<u>Case II</u> For x bounded away from 0 we can clearly forget about f and concentrate on

$$R(x,y) = \sin \alpha(\log(|x|/|y|))$$

As usual let $R = r \circ \rho$ (where ρ is as before) and $r(u,v) = \sin \alpha(\frac{1}{2}\log(u/v))$. We now rely on Lemmas 3.3.2, 3.3.4, and 3.3.7. Note first that $\partial r/\partial u = \cos \alpha\, \dot{\alpha}/2u$ tends to 0 as $v \to 0$. Similarly for $\partial^2 r/\partial u^2$. Furthermore,

$$\frac{\partial^2 r}{\partial u \partial v} = -(\cos \alpha\, \ddot{\alpha} - \sin \alpha\, \dot{\alpha}^2)/4uv$$

$$= 0(e^{-2kt})/v = 0(1)$$

Hence immediately $\frac{\partial R}{\partial x_i}$, $\frac{\partial^2 R}{\partial x_i \partial x_j}$, $\frac{\partial^2 R}{\partial x_i \partial y_j}$ all tend to zero as $y \to 0$. Suppose we could also show that $\partial r/\partial v$ and $\partial^2 r/\partial v^2$ were bounded. Then $\partial r/\partial v$ would extend to a continuous function, showing that

$$\frac{\partial R}{\partial y_j} = \frac{\partial r}{\partial v} 2y_j \to 0 \qquad \text{as } y \to 0$$

and

$$\frac{\partial^2 R}{\partial y_i \partial y_j} = \frac{\partial^2 r}{\partial v^2} \cdot 4y_i y_j + \frac{\partial r}{\partial v} 2\delta_{ij}$$

is continuous at $y = 0$. In other words, this would finish Case II.

But $\partial r/\partial v = -\cos\alpha\,\dot{\alpha}/2v$ is clearly bounded, and

$$\begin{aligned}\partial^2 r/\partial v^2 &= (\cos\alpha\,\ddot{\alpha} - \sin\alpha\,\dot{\alpha}^2)/4v^2 \\ &= O(e^{-2kt})/v^2 \\ &= O(v^{k-2})\end{aligned}$$

is bounded if $k \geq 2$. However, if $k = 1$ we can write

$$\ddot{\alpha} = (r-2)\dot{\alpha} - (r-1)\cos\alpha + O(e^{-3t})$$

which is a bounded error when multiplied by $\cos\alpha$. Hence we consider

$$\begin{aligned}\cos\alpha\,\ddot{\alpha} - \sin\alpha\,\dot{\alpha}^2 &\approx \cos\alpha((r-2)\dot{\alpha} - (r-1)\cos\alpha) - \sin\alpha\,\dot{\alpha}^2 \\ &\approx \cos\alpha\,\dot{\alpha} - \sin\alpha\,\dot{\alpha}^2\end{aligned}$$

by Lemmas 3.3.2 and 3.3.4. Here $\approx$ means an error $O(e^{-4t})$ has been dropped. Finally, since $1 - \sin\alpha = O(e^{-2t})$, the entire remainder is $O(e^{-4t})$. Therefore $\partial^2 r/\partial v^2$ is bounded if $k = 1$.

This finishes Case II. The differentiability of the second coordinate of $f * g$ is handled similarly. This concludes the proof of Proposition 3.4.1, and therefore establishes Theorem 3.1.1.

<u>Remark 3.4.7</u> Having worked this hard to prove regularity, one might well ask if there is not a more general regularity principle. I do not know any smoothness theorem which applies, but perhaps it might be possible in the setting of equivariant maps to prove a companion piece to Theorem 1.3.5.

Section 5. The First Suspension

Although the first suspension of a harmonic polynomial map is the same thing as joining to the identity on S^o, the parametization one achieves in this way is not the natural symmetric one. Therefore we present this case separately. The benefits to be gained are not merely aesthetic, however; symmetry greatly simplifies the existence proof and will allow it to be extended to cover some mappings of ellipsoids in Chapter 4.

Therefore suppose $f: S^{p-1} \to S^{q-1}$ is a harmonic polynomial map of homogeneity ℓ. We will look for a harmonic representative of the suspension of f in the form

$$\Sigma f : S^p \to S^q$$

$$(x,y) \to (\sin\alpha(t) f(x/|x|),\ \cos\alpha(t)) \tag{3.5.1}$$

$$(x,y) \in \mathbb{R}^p \times \mathbb{R}$$

$$t = \log(|x|/r + y) \in (-\infty,\infty)$$

$$r = (|x|^2 + y^2)^{\frac{1}{2}}$$

It is readily verified that $t(x,y) = -t(x,-y)$. A moment's computation also yields

$$\partial t/\partial x_i = y x_i / r|x|^2$$

$$\partial t/\partial y = -1/r$$

As Σf is invariant under radial dilation, the spherical and Euclidean Laplacians coincide on f; the calculation of the latter proceeds as in Section 1, with the result

$$\Delta(\sin\alpha f) = (\cos\alpha\,\ddot{\alpha} - \sin\alpha\,\dot{\alpha}^2 - (p-2)\tanh(t)\cos\alpha\dot{\alpha} - \lambda\sin\alpha) f/|x|^2 \tag{3.5.2}$$

$$\Delta(\cos\alpha) = (-\sin\alpha\,\ddot{\alpha} - \cos\alpha\,\dot{\alpha}^2 + (p-2)\tanh(t)\sin\alpha\,\dot{\alpha})/|x|^2$$

with $\lambda = \ell(\ell+p-2)$

The condition that Σf be harmonic is determined as in Section 1; it is

$$\ddot{\alpha}(t) - (p-2)\tanh(t)\dot{\alpha}(t) - \lambda\sin\alpha(t)\cos\alpha(t) = 0 \tag{3.5.3}$$

Our object is clearly to find a monotone solution α such that $\alpha_{\infty} = \pi$ and $\alpha_{-\infty} = 0$. By symmetry, it suffices to set $\alpha(0) = \pi/2$ and $\dot{\alpha}(0) = \inf A^{+}(0)$, where $A^{+}(0)$ is the collection of initial derivatives which are "too large". If the damping condition is satisfied, it follows as in Lemma 3.2.11 that $\dot{\alpha}(0) > 0$. Then the same reasoning as in Lemma 3.2.6 shows that the associated solution α satisfies $\alpha_{\infty} = \pi$; we can then use the symmetry of the equation to deduce $\alpha_{-\infty} = 0$.

The only possible problem arises in proving the regularity of Σf. The procedure is entirely analogous to that in Sections 3 and 4, however. By symmetry; it furthermore suffices to prove smoothness at $x = 0$, $y = 1$. This avoids puzzling over what happens at $y = -1$, where the function t looks peculiar. The required calculations are naturally quite lengthy, and are omitted.

Section 6. Applications

In this section we list the examples of essential harmonic maps between spheres which can be obtained from Theorem 3.1.1. That these maps are naturally equivariant is also illustrated in a particular case, for which the equivariant tension equation is computed and shown to be equivalent to the original Equation 3.1.3. Some further applications to homotopy groups of Lie groups are indicated.

It is first helpful to have

List 3.6.1 The presently known essential homogeneous harmonic polynomial mappings of spheres are

a) the isometries, of degree 1.

b) $d_k : S^1 \to S^1$, the complex polynomial $z \to z^k$ of degree $|k|$.

c) the Hopf fibrations $h : S^3 \to S^2$
$\eta : S^7 \to S^4$
$\sigma : S^{15} \to S^8$

which are all of homogeneity 2.

The Hopf maps were shown to be harmonic in Example 2.1.2. We will also need.

Lemma 3.6.2. The Hopf maps are harmonic fibrations.

Proof As the fibres are all geodesic spheres, it suffices to check the Riemannian condition. This can be verified from the formulae, but in the case of the Cayley numbers the calculations required are long and tricky.

Eells has pointed out to me, however, that the bundle σ may be characterized alternatively as

$$\begin{array}{ccccc} Spin(8)/Spin(7) & \to & Spin(9)/Spin(7) & \to & Spin(9)/Spin(8) \\ S^7 & \to & S^{15} & \to & S^8 \end{array}$$

which is homogeneous and evidently Riemannian. Similar descriptions of h and η can easily be given (cf. Example 3.6.11). These remarks for us will constitute a proof of the Lemma.

It should be noted, however, that with the usual Euclidean metrics on the spheres these maps are "twice" Riemannian; i.e., horizontal vectors get their lengths multiplied by 2.

In this context we also mention some nice polynomial maps which are not harmonic

Example 3.6.3 Let $f: S^3 \to S^3$ be defined in terms of quaternions by $f(q) = q^2$. Then f is not harmonic, for in coordinates

$$f(x) = (x_1^2 - x_2^2 - x_3^2 - x_4^2, F(x))$$

where F is a second order homogeneous harmonic polynomial. Hence $\Delta f = (-4,0)$ and we can apply Lemma 2.2.9.

This is also an example of a more general fact: if G is a non-abelian Lie group and $f_1, f_2 : M \to G$ are harmonic maps, we cannot expect that the product $f_1 f_2 : M \to G$ is harmonic. Of course, if G is abelian, then multiplication $\mu : G \times G \to G$ will be totally geodesic, and thus $f_1 f_2$ will be harmonic.

This example furthermore indicates that the polynomial maps of degree $k : S^{2n-1} \to S^{2n-1}$ constructed by R.Wood [WD] are unlikely to be harmonic: the function above is his starting point.

Now for some harmonic maps.

<u>Example 3.6.4</u> It is well known that $\pi_n(S^n) = Z$ and that the classes correspond to suspensions of $d_k: S^1 \to S^1$. Hence by Corollary 3.1.2 we find that $\pi_n(S^n)$ is representable for $n = 1,\ldots,7$.

We remark that for $n = 2$ our construction gives the map $z \mapsto z^k$ on the Riemann sphere; this example also appears in [ES].

<u>Example 3.6.5</u>. Recall that $\pi_3(S^2) = Z$, generated by the Hopf map h. Maps $S^3 \to S^2$ are thus classified by their <u>Hopf invariant</u>. Let f_k: $S^2 \to S^2$ be our harmonic map of degree k; then it is known that $f_k \circ h$ has Hopf invariant k^2 (see Husemoller[HU,p.198]). This map is harmonic by Lemma 1.2.2. By similar compositions with $-h$ we obtain representatives for all elements of Hopf invariant $\pm k^2$.

Composition from the other direction in an attempt to get a harmonic map of Hopf invariant k does not work; i.e. given one of our harmonic $f_k: S^3 \to S^3$, the composition $h \circ f_k$ is not harmonic. In fact, our direct construction methods (applied now to the Hopf construction rather than the join) will always fail to give the general element of $\pi_3(S^2)$. The reason for this is found in Section 7.

Example 3.6.6 $\pi_{n+1}(S^n) = Z_2$ for $n \geq 3$, generated by the iterated suspension of h. By Corollary 3.1.2 we obtain a harmonic generator for this group for $n = 3, \ldots, 8$. (For the homotopy theory, the general reference is Toda [T].)

Note that by a theorem of R. Wood, all polynomial maps of S^4 to S^3 are constant [WD]. Hence it is certainly necessary to move to the transcendental domain to find a harmonic representation of $\pi_4(S^3)$.

Example 3.6.7 $\pi_7(S^5) = Z_2$, generated by $\Sigma^3 h \circ \Sigma^4 h$ [T]. However, the latter map is homotopic to $h * h$ [T, p.25]. Hence this generator is harmonically represented by Theorem 3.1.1.

Example 3.6.8 $\pi_7(S^4) = Z \oplus Z_{12}$, with Z generated by η. We have harmonic maps of every degree $f_k : S^4 \to S^4$, hence $f_k \circ \eta: S^7 \to S^4$ has Hopf invariant k^2 and is harmonic.

Example 3.6.9 $\pi_{n+3}(S^n) = Z_{24}$ for $n \geq 5$, generated by suspensions of η. We have a harmonic representative of that generator for $n = 5, \ldots, 10$. In fact, by taking $\eta * d_k \simeq \Sigma^2\eta \circ \Sigma^8 d_k$ one obtains all elements of $\pi_9(S^6)$.

Since $\pi_{11}(S^7) = 0$, $\eta * h = 0$. However, $\eta * \eta$ generates $\pi_{15}(S^9) = Z_2$.

Example 3.6.10 Let $\Sigma h: S^4 \to S^3$ be the essential harmonic map of Example 3.6.6. Then $\Sigma h \circ \eta = 1$ in $\pi_7(S^3) = Z_2$, and is harmonic by Lemma 1.2.2. We remark that this group is not stable, and is the only such group we can represent completely.

Another essential harmonic composition is $\Sigma\eta \circ \sigma : S^{15} \to S^5$.

Unfortunately we cannot make harmonic suspensions of σ by the method of this chapter on account of the dimension restrictions in Theorem 3.1.1. However, we will see in the next chapter that such a suspension can be done if the receiving sphere is given an ellipsoidal metric.

We promised in Chapter 1, Section 3 to show how the above constructions can be given a natural equivariant interpretation. Let us illustrate this in a particular case.

Example 3.6.11 The join $\eta * d_k: S^9 \to S^6$ may be thought of in this way: let $S^9 = S^7 * S^1$ and $S^6 = S^4 * S^1$. Then $Sp(2) \times S^1$ acts by isometries on both spaces with a one-parameter orbit space $[0,\pi/2]$ in each case. In fact, since $\eta : S^7 \to S^4$ is just

$$\eta : Sp(2)/Sp(1) \to Sp(2)/Sp(1) \times Sp(1)$$

the map $\eta * d_k$ is clearly equivariant with respect to the homomorphism

$$\varphi = id_{Sp(2)} \times d_k$$

of $Sp(2) \times S^1$ onto itself. Applying Lemma 1.3.3. we see that the conditions of Theorem 1.3.5 are satisfied. Therefore our map is harmonic iff the orbit map $\bar{f} : [0, \pi/2] \to [0,\pi/2]$ satisfies the equivariant tension equation (1.3.6).

To calculate (1.3.6) we need to know the volume function V and the orbit-energy function γ. Firstly, note that the orbit corresponding to θ in S^9 is just $S^7(\sin\theta) \times S^1(\cos\theta)$, so that

$$V(\theta) = c \sin^7(\theta)\cos\theta$$

It is clear that γ breaks up as $\gamma(d_k) + \gamma(\eta)$ in a natural way. Furthermore, it is obvious that

$$\gamma(d_k)(\theta,\bar{f}(\theta)) = k^2\cos^2\bar{f}(\theta)/\cos^2\theta$$

As for $\gamma(\eta)$, note that as $\eta : S^7 \to S^4$ is twice a Riemannian fibration, $|d\eta|^2 = 16$. Hence

$$\gamma(\eta)(\theta,\overline{f}(\theta)) = 16 \sin^2\overline{f}(\theta)/\sin^2\theta$$

Taking the gradient with respect to the second argument in γ then gives

$$\tfrac{1}{2}\,\mathrm{grad}_{\overline{f}}\ \gamma(\theta,\overline{f}(\theta)) = \sin \overline{f}(\theta) \cos \overline{f}(\theta)\ (16 \csc^2\theta - k^2\sec^2\theta)$$

Equation (1.3.6) now looks like

$$\overline{f}''(\theta) + \overline{f}'(\theta)(7 \cot\theta - \tan\theta) + \sin \overline{f}(\theta) \cos \overline{f}(\theta)(k^2\sec^2\theta - 16\csc^2\theta) = 0$$

If one now puts $t = \log \tan\theta$ and $\alpha(t) = \overline{f}(\theta)$ we get

$$\ddot{\alpha}(t) + (e^t + e^{-t})^{-1}(6e^{-t}\dot{\alpha}(t) + \sin\alpha(t)\cos\alpha(t)(k^2e^t - 16\, e^{-t})) = 0$$

which is exactly (3.1.3) in this case.

Another nice set of applications comes when we look at homotopy groups of certain Lie groups. Although we cannot expect to exponentiate or multiply harmonic maps, as noted in Example 3.6.3, another approach is available. If $f: S^m \to G$ is a totally geodesic generator for $\pi_m(G)$ and we have harmonic representatives h_k of $\pi_m(S^m)$, then compositions $f \circ h_k$ are harmonic representatives of the subgroup generated by f.

<u>Example 3.6.12</u> The geodesic inclusion of S^3 in $Sp(m)$ or in $SU(n)$, $n \geq 2$, generates π_3 of these groups, which is Z in each case [HU, p.93]. Hence there are harmonic representatives of $\pi_3(Sp(m))$ and $\pi_3(SU(m))$ $\forall m$.

The Riemannian covering $\pi: S^3 \to SO(3)$ generates $\pi_3(SO(3)) = Z$, hence this group is represented. By including $SO(3)$ in $SO(4)$ we also get one Z component of $\pi_3(SO(4)) = Z \oplus Z$. The other factor is given by the inclusion of S^3, which acts on $\mathbb{R}^4$ by quaternionic multiplication. Hence the subgroups $\{(k,0)\}$ and $\{(0,j)\}$ are represented harmonically. Although we can add such elements by multiplying representatives in the group, the result is not harmonic.

Example 3.6.13 $\pi_5(SO(6)) = Z$ is generated by the characteristic map of the principal $SO(6)$ bundle $\pi: SO(7) \to S^6$ [S]. It is not hard to see from Steenrod's description of the map (and comparison with some calculations in Husemoller [HU, p. 86 ff]) that the characteristic map $\xi: S^5 \to SO(6)$ is in fact almost the geodesic symmetry mapping. That is, up to multiplication by a constant transformation, $\xi(x)$ is the unique isometry which reverses the geodesics through x. It is a general property of symmetric spaces that the analogous map $\xi: G/K \to G$ is totally geodesic. (For example, this can be easily seen from the explicit description of ξ available in Helgsson [HE]). We conclude that $\pi_5(SO(6))$ is represented by harmonic maps.

Remark 3.6.14 The general problem of realizing homotopy groups of symmetric spaces by geodesic spheres has been studied by A.Fomenko[F]. By combining his work with ours a number of additional examples can be added to this collection.

Section 7. The Hopf Construction: Non-Conservation of Energy

So far we have devoted our efforts to studying the join of two harmonic polynomial maps. It is reasonable to ask what other topological constructions can be treated in a similar manner, and the answer seems to be "very few". For example, the reduced join and Whitehead product offer little hope of smooth, harmonic, one-parameter representation. However, the Hopf construction does offer some hope, as well as an interesting obstacle: this barrier may be interpreted physically as non-conservation of energy in a variable gravity system.(Proposition 3.7.6).

Recall that if we are given a map $F : S^{p-1} \times S^{r-1} \to S^q$, the Hopf construction applied to F yields a map $H(F) : S^{p+r-1} \to S^{q+1}$ which may be defined by

$$(x,y) \mapsto (|x|^2-|y|^2,\ 2|x||y|\ F(\tfrac{x}{|x|}, \tfrac{y}{|y|}))$$

for $(x,y) \in R^p \times R^r$. Now suppose that in each variable separately F is a homogeneous harmonic polynomial, of homogeneity k in y and ℓ in x. Then we look for a harmonic representative of H(F) in the form

$$(x,y) \to (\cos\alpha(t),\ \sin\alpha(t)\ F(x/|x|,\ y/|y|))$$

where $t = \log(|x|/|y|)$ as usual. The function α should now run from 0 to π. By performing a calculation like that in Section 1, one sees that the equation α must satisfy is

$$(3.7.1)\quad \ddot{\alpha} = \frac{1}{e^t+e^{-t}}\,(((p-2)e^t - (r-2)e^{-t})\dot{\alpha} + (\lambda_1 e^{-t}+\lambda_2 e^t)\sin\alpha\cos\alpha)$$

$$\lambda_1 = \ell(\ell+p-2)$$
$$\lambda_2 = k(k+r-2)$$

Note that the difference between this equation and (3.1.3) is that the gravity always has the same sign. As we shall see, this makes life more difficult. The most general theorem we can state is:

<u>Theorem 3.7.2</u> Suppose $F : S^{r-1} \times S^{r-1} \to S^q$ is a harmonic polynomial of homogeneity k in each variable separately, with $k > \theta(r-2)$ satisfied. Then there is a harmonic representative of H(F).

<u>Proof</u> Under these assumptions we can find a solution of (3.7.1) which satisfies $\alpha_0 = \pi/2$, $\alpha_\infty = \pi$, $\alpha_{-\infty} = 0$. For in this case (3.7.1) becomes

$$(3.7.3)\qquad \ddot{\alpha} = (r-2)\tanh(t)\dot{\alpha} + \lambda\sin\alpha\cos\alpha$$

As we did in Section 5, note that if α is a solution with $\alpha(0) = \pi/2$; then $\alpha(t) + \alpha(-t) = \pi$. It therefore suffices to find an initial derivative $\dot{\alpha}_0$ for which $\alpha_\infty = \pi$. The usual arguments finish the proof.

Remark 3.7.4 Unfortunately this theorem does not produce any new examples of essential harmonic maps. The only good candidate for F is $F : S^1 \times S^1 \to S^1$ given by $(z,w) \to z^k w^k$. Then $H(F): S^3 \to S^2$ has Hopf invariant k^2, and we easily check that we have simply reproduced our previous Example 3.6.5.

Rather more interesting is the fact that these methods can be proved to fail in the most important case. Namely, let $F : S^1 \times S^1 \to S^1$ be the map $(z,w) \to z^\ell w^k$ with $\ell \neq k$. Note that $H(F) : S^3 \to S^2$ has Hopf invariant $k\ell$. The relevant equation is

$$\ddot{\alpha} = \left(\frac{k^2 e^t + \ell^2 e^{-t}}{e^t + e^{-t}}\right) \sin\alpha \cos\alpha \tag{3.7.5}$$

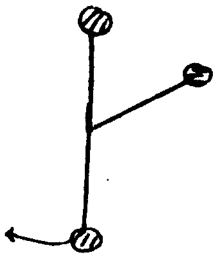
Proposition 3.7.6 If $k^2 \neq \ell^2$, then there is no solution α of the above equation with $\alpha_\infty = \pi$ and $\alpha_{-\infty} = 0$.

Proof The physical reasoning is this: suppose $k^2 < \ell^2$ and write

$$\ddot{\alpha} = f(t) \sin\alpha \cos\alpha$$

where $f(t) > 0$ and is strictly decreasing. Suppose $\alpha(t_0) = \pi/2$ and $\dot{\alpha}(t_0)$ is just large enough that $\alpha_\infty = \pi$. Then as gravity is stronger for $t < t_0$ than it is for $t > t_0$, the kinetic energy at time t_0 will not be sufficient (in backward time) to enable α to reach 0. A picture of this situation (with $\bar{\alpha} = 2\alpha$) is given below. The function $\bar{\alpha}$ must run out of steam at some time $t < t_0$, and then continue to make bounded oscillations for all time less than t.

$\bar{\alpha}_{\infty} = 2\pi$



$\bar{\alpha}(t) > 0$

$\dot{\bar{\alpha}}(t) = 0$

$\bar{\alpha}(t_o) = \pi$

$\dot{\bar{\alpha}}(t_o) > 0$

The energy estimates required to give a precise proof of these statements are well known to physicists and dynamical systems personnel and will not be reproduced here. Therefore we have shown that some elements of $\pi_3(S^2)$ definitely cannot be harmonically represented by our methods. Hence the existence or non-existence of a harmonic map of Hopf invariant 2 is for us the most interesting unsolved question of this type.

Chapter 4

MAPPINGS OF ELLIPSOIDS AND TORI

In geometry, an ellipsoid and a sphere are two entirely different animals. It is therefore not surprising that even the simplest ellipsoidal perturbation of the usual spherical metric produces interesting complications for our methods. We first study maps of ellipsoids into spheres and discover that the situation is substantially messier but essentially unchanged from Chapter III. A map of a sphere into an ellipsoid, however, poses some new problems altogether. The only case we can study effectively is that of the first suspension. It turns out here that it is a positive advantage to map into a short, fat ellipsoid, whereas for maps into long, thin ellipsoids we are led to a non-existence theorem in dimensions ≥ 3. Finally, we observe that all ellipsoids of revolution in $\mathbb{R}^3$ are conformally equivalent to S^2.

Some examples of harmonic maps of tori are also included because a) they succumb to ordinary differential equation methods, and b) their geometric behavior is instructive.

Section 1. Maps of Ellipsoids into Spheres

Most ellipsoids we consider will have the form

$$E = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^r = \mathbb{R}^n : b|x|^2 + |y|^2 = 1\}$$

where b is some positive constant. The first thing we need to know is how to compute the Laplacian of a function defined on E. More specifically, if f is defined on $\mathbb{R}^n$ and $i: E \to \mathbb{R}^n$ is the inclusion

$$(4.1.1)\qquad -\Delta_E(f|E) = \tau(f\circ i)$$
$$= \operatorname{Trace}\nabla df(di,di) + df(\tau(i))$$
$$= \tau(f) - \partial^2 f/\partial\eta^2 + df(\tau(i))$$

where $\partial/\partial\eta$ denotes normal differentiation. The term $\tau(f)$ is just the Euclidean Laplacian of f, and $\tau(i)$ is the mean normal curvature of the embedding i. For a sphere of dimension $n - 1$ it is well known that this vector is always an inward normal vector of length $n-1$. For an ellipsoid of the above type, the normal direction is given by $\eta = (bx,y)$. Furthermore

<u>Lemma 4.1.2</u> The mean normal curvature of E at (x,y) is an inward normal vector of length

$$L = b/(b^2|x|^2 + |y|^2) + (r-1) + b(p-1)$$

All that we actually use is that the obvious fact that the mean curvature is a smooth normal field on E, so the proof of the lemma is omitted.

Now suppose that we are given two harmonic polynomial maps $f\colon S^{p-1}\to S^{q-1}$ and $g\colon S^{r-1}\to S^{s-1}$. The procedure will be to join f and g so that the domain is an ellipsoid of the type discussed above and so that the resulting map is harmonic. Of course, the pair (x,y) in the definition of E will correspond to the domains of f and g.

The derivation of the equation for the join parameter $\alpha(t)$ proceeds in a fashion entirely analogous to that in Section 1, Chapter 3. There is, however, an additional complication arising from the fact that the map $f * g$ defined on R^n is of course not constant in directions normal to E; hence all terms in (4.1.1) contribute. At any rate, the net result is

$$(4.1.3)\qquad \ddot{\alpha}(t) - h(t)\dot{\alpha}(t) + g(t)\sin\alpha(t)\cos\alpha(t) = 0$$

where h and g behave more or less as before:

$$h_\infty = r-2$$
$$h_{-\infty} = -(p-2)$$
$$g_\infty = k(k+r-2)$$
$$g_{-\infty} = -\ell(\ell+p-2)$$

Furthermore, $|h(t) - h_\infty| = 0(e^{-2t})$ and so forth. Finally, there is a unique t_o with $g(t_o) = 0$ and $\partial g/\partial t > 0$ at t_o. With this information we can prove

Theorem 4.1.4 Let $f: S^{p-1} \to S^{q-1}$ and $g : S^{r-1} \to S^{s-1}$ be harmonic polynomials of homogeneity ℓ and k, respectively. For any $b > 0$ let E be the ellipsoid

$$E = \{(x,y) \in \mathbb{R}^p \times \mathbb{R}^r : b|x|^2 + |y|^2 = 1\}$$

If the damping conditions are satisfied (as in Theorem 3.1.1), there is a harmonic representative of the join of f and g with domain E

$$f * g : E \to S^{q+s-1}$$

Proof As before, our problem is to study solutions of (4.1.3). The only obstacle to blanket application of the arguments in Sections 2,3,4 of Chapter 3 is that the functions h and g may not be monotonic. Let us outline the few modifications which must be made in our former proof at each stage.

A) Existence In Lemma 3.2.11 we used the monotonicity of h to say $h(t_1) \leqslant h(t)$ for all $t \geqslant t_1$. However, if we are considering the behavior of solutions on $[t_1,\infty)$ for large t_1, we can always define

$$h^-(t) = \inf \{h(s) : s \geqslant t\}$$

It follows that $|h^-(t) - h_\infty| = 0(e^{-2t})$. Substituting $h^-(t_1)$ and $g^-(t_1)$ for $h(t_1)$ and $g(t_1)$ in Lemma 3.2.11, we see that the proof now goes through.

B) Asymptotic behavior For large t also define

$$h^+(t) = \sup \{ h(s) : s \geqslant t \}$$

and similarly for $g^+(t)$. For such t we let $k^+(t)$ and $k^-(t)$ be the solutions of

$$(g^+(t) - k^+(t)^2)/k^+(t) = h^-(t)$$
$$(g^-(t) - k^-(t)^2)/k^-(t) = h^+(t)$$

respectively. Then $k^+(t) < k + O(e^{-2t})$ and is decreasing, whereas $k^-(t) > k - O(e^{-2t})$ and is increasing. The lemmas we need are now proved essentially as before.

C) Regularity. Given the usual asymptotic estimates the proof proceeds as before.

Section 2. Maps of Ellipsoids into Ellipsoids

As all our examples have to do with suspensions, let us choose some terminology:

Definition 4.2.1 If $f\colon S^{p-1} \to S^{q-1}$ is a map, then a one-parameter suspension of f is a map Σf of the form

$$\Sigma f : S^p \to S^q$$
$$(\cos\theta\, x, \sin\theta) \to (\cos\alpha(\theta)\, f(x), \sin\alpha(\theta))$$

for continuous $\alpha\colon [-\pi/2,\pi/2] \to [-\pi/2,\pi/2]$ preserving end points.

The ellipsoids we study will naturally be compatible with the program of suspension:

Definition 4.2.2 For any $b > 0$, let

$$E^n(b) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : b|x|^2 + |y|^2 = b\}$$

It should thus be clear what we mean by a one-parameter suspension $\Sigma f\colon S^p \to E^q(b)$. Note also that $E^2(b)$ is a typical ellipsoid of revolution in $\mathbb{R}^3$. The following theorems illustrate a substantial difference between dimension 2 and dimensions ≥ 3.

<u>Theorem 4.2.3</u> Any two ellipsoids of revolution E_1 and E_2 in $\mathbb{R}^3$ are conformally equivalent. In particular the induced complex structure on either is the same as that of S^2. Hence for each k there is a harmonic map of degree k : $E_1 \to E_2$ which is a one parameter suspension of $z \to z^k$ on S^1.

<u>Theorem 4.2.4</u> Let I : $S^{n-1} \to S^{n-1}$ be the identity. Then for large b and $n \geq 3$ there is no harmonic one-parameter suspension ΣI: $S^n \to E^n(b)$.

<u>Theorem 4.2.5</u> For $n \leq 6$ there is always a harmonic one-parameter diffeomorphism $\Sigma I : E^n(b) \to S^n$.

Theorem 4.2.4 is that mentioned in the Introduction, and is the basis for our conjecture that there is no harmonic map of degree one in these cases. Putting 4.2.4 and 4.2.5 together we find that we have produced a collection of harmonic diffeomorphisms whose inverses are certainly not harmonic. These are the only harmonic diffeomorphisms I know which have this property.

Before giving proofs of these results, let us first derive the basic equation to be studied: if f: $S^{n-1} \to S^{q-1}$ is a harmonic polynomial map of degree ℓ, a one-parameter suspension Σf : $S^n \to E^q(b)$ can be assumed to have the form

$$(x,y) \to (\cos\alpha(t)\ f(x/|x|),\ \sqrt{b}\ \sin\alpha(t))$$

with $$t = \log\ (|x|/y + 1)$$

As in Section 3.5 we find for the Euclidean Laplacians

$$\Delta(\cos\alpha\ f) = (-\sin\alpha\ \ddot{\alpha} - \cos\alpha\ \dot{\alpha}^2 - (n-2)y\ \sin\alpha - \lambda\cos\alpha)f/|x|^2$$

$$\Delta(\sqrt{b}\ \sin\alpha) = (\cos\alpha\ \ddot{\alpha} - \sin\alpha\ \dot{\alpha}^2 + (n-2)y\ \cos\alpha)\sqrt{b}/|x|^2$$

Since the normal to $E^q(b)$ at the image point is spanned by $\eta = (\sqrt{b}\ \cos\alpha\ f,\ \sin\alpha)$, the requirement that $\Delta(\Sigma f)$ be proportioned to η becomes

$$(4.2.6) \qquad \ddot{\alpha} - (n-2)\tanh(t)\dot{\alpha} - \left(\frac{(b-1)\dot{\alpha}^2 - \lambda}{\varphi(\alpha)}\right)\sin_\alpha \cos_\alpha = 0$$

where
$$\varphi(\alpha) = b\cos^2_\alpha + \sin^2_\alpha$$
$$\lambda = \ell(\ell+n-2)$$

Note the appearance of a term involving $\dot{\alpha}^2$. The grouping in the last term of the equation is the natural and convenient one: the entire quantity in parentheses will often be thought of as a perturbation of the eigenvalue λ.

Proof of 4.2.3 We will demonstrate that any ellipsoid of revolution E is conformally equivalent to the Euclidean sphere S^2 by a one-parameter stretching

$$(x,y) \to (\cos\alpha(t)\ x/|x|, \sqrt{b}\ \sin\alpha(t))$$

Such a map will be a holomorphic equivalence of S^2 and E. Further, we know that any degree on S^2 can be represented by a holomorphic or anti-holomorphic map, so by compositions we obtain holomorphic or anti-holomorphic maps of degree k between our ellipsoids. As these manifolds are Kahler, such maps are harmonic.

The easiest way to derive the condition for our map to be conformal is to write it in the form

$$(\cos\theta\, e^{iu}, \sin\theta) \overset{f}{\mapsto} (\cos\alpha(t)\ e^{iu}, \sqrt{b}\ \sin\alpha(t))$$

where
$$t(\theta) = \log(\cos\theta/1 + \sin\theta)$$

The metric on S^2 is

$$g = \cos^2\theta\, du^2 + d\theta^2$$

and on E is

$$\bar{g} = \cos^2\alpha\ du^2 + (\sin^2\alpha + b\cos^2\alpha)\ d\alpha^2$$

As df preserves the orthogonality of $\partial/\partial u$ and $\partial/\partial\theta$, the condition for conformality is that unit vectors in these directions be uniformly dilated. Since

$$df(\partial/\partial u) = \partial/\partial u$$
$$df(\partial/\partial\theta) = -\dot{\alpha}(t)\ \sec\theta\ \partial/\partial\alpha$$

this condition is the autonomous equation

$$(4.2.7) \qquad \dot{\alpha} = \cos\alpha/(\sin^2\alpha + b\cos^2\alpha)^{\frac{1}{2}}$$

Note that upon differentiating again we recover the condition for f to be harmonic, namely

$$(4.2.8) \qquad \ddot{\alpha} = \left(\frac{(b-1)\dot{\alpha}^2 - 1}{\varphi(\alpha)}\right) \sin\alpha\cos\alpha$$

Using Equation 4.2.7, we see that if we assign α_0 to be any value in $(-\pi/2,\pi/2)$, then integration yields a solution which is asymptotic to $\pm\pi/2$ at $\pm\infty$.

To establish the asymptotic behavior of α, we study the second order equation (4.2.8). This has the advantage that the methods will carry over to other mappings into ellipsoids, where the equation considered will not have "first integrals". More generally, again in the interest of later reference, let us assume we are constructing a 1-parameter suspension of $z \to z^k$ on S^1, so that (4.2.8) becomes

$$(4.2.9) \qquad \ddot{\alpha} = \left(\frac{(b-1)\dot{\alpha}^2 - k^2}{\varphi(\alpha)}\right) \sin\alpha\cos\alpha$$

$$\varphi(\alpha) = b\cos^2\alpha + \sin^2\alpha$$

For large t we define

$$k^-(t)^2 = \inf_{s>t} \frac{k^2 - (b-1)\dot{\alpha}^2(s)}{\varphi(\alpha(s))}$$

$$k^+(t)^2 = \sup_{s>t} \frac{k^2 - (b-1)\dot{\alpha}^2(s)}{\varphi(\alpha(s))}$$

Note that $\dot{\alpha}$ is decreasing after α passes 0; for then $\sin\alpha\cos\alpha > 0$ and hence from (4.2.9) if $\ddot{\alpha}(t) > 0$ for some such t,

$$(b-1)\dot{\alpha}^2(t) - k^2 > 0$$

Therefore if $\dot{\alpha}$ increases past t, this inequality will continue to be satisfied, and conversely. This is clearly not possible for a monotone solution asymptotic to $\pi/2$. Hence $\dot{\alpha}$ decreases after α passes 0, and in particular $k^-(t)$ is well defined. The procedure in Section 3.3 now shows that eventually

$$\dot{\alpha}(t) \geq k^-(t)\cos\alpha(t)\sin\alpha(t)$$

$$\dot{\alpha}(t) \leq k^+(t)\cos\alpha(t)$$

We would also like to show that eventually

$$c_1 e^{-kt} \leq \cos\alpha(t) \leq c_2 e^{-kt}$$

An inspection of the methods in Section 3.3 shows that the only ingredient lacking is

$$\int_0^\infty (k^+(t) - k)\,dt < \infty$$

and similarly for k^-. (Previously, of course, we knew <u>a priori</u> that $k^+(t) = k + O(e^{-2t})$). It suffices, then, that

$$\int_0^\infty (k^+(t)^2 - k^2)\,dt < \infty$$

and for this it is not hard to see that

$$\int \dot{\alpha}^2(t)\,dt < \infty$$

and $$\int |1 - \varphi(\alpha(t))|\,dt = \int |(b-1)\cos^2\alpha(t)|\,dt < \infty$$

are sufficient. However, it is obvious that $\int|\dot{\alpha}(t)| < \infty$, since α is monotonic and bounded, and therefore $\int \dot{\alpha}^2(t) < \infty$. Furthermore, we know that eventually

$$\cos^2\alpha \sin^2\alpha < \dot{\alpha}^2/(k^-)^2$$

so that $\int \cos^2\alpha(t) < \infty$ follows also.

Having established the exponential decay of $\cos\alpha$, we still need for our regularity proof that

$$k^2 - k^+(t)^2 = 0(e^{-2t})$$

and likewise for k^-. However, we can now say

$$\dot{\alpha}^2(t) \leq k^+(t)^2 \cos^2\alpha(t) \leq ce^{-2kt}$$

which gives the required estimate on $k^+(t)^2$ by inspection.

From this point the regularity proof proceeds as before, substituting for $\ddot{\alpha}$ from (4.2.9) where necessary. The proof that the differential of our conformal map has maximal rank at the poles is contained in the proof of Theorem 4.2.5. This finishes Theorem 4.2.3.

<u>Proof of Theorem 4.2.4</u> The equation governing the suspension parameter in this case is

$$(4.2.10) \qquad \ddot{\alpha} = (n-2)\tanh(t)\dot{\alpha} + \left(\frac{(b-1)\dot{\alpha}^2 - \lambda}{\varphi(\alpha)}\right) \sin\alpha \cos\alpha$$

where $\qquad \varphi(\alpha) = \sin^2\alpha + b\cos^2\alpha$

and n is assumed ≥ 3. We will show that for b large enough, any solution which passes through 0 with positive derivative is unbounded. Note that this immediately gives us the theorem: certainly if α is unbounded it cannot define a one-parameter suspension. Furthermore, there is no loss of generality in assuming there is some t_o with $\alpha(t_o) = 0$ (as α traverses the interval $[-\pi/2, \pi/2]$), and since we require for a one-parameter suspension that $\alpha_{-\infty} = -\pi/2$, $\alpha_\infty = \pi/2$, there will be at least one such t_o with $\dot{\alpha}(t_o) > 0$. ($\dot{\alpha}(t_o) = 0$ is impossible by uniqueness).

In the proof, we will further assume that $t_o \geq 0$. For if $t_o < 0$, define $\beta(t) = -\alpha(-t)$. Then β also satisfies (4.2.10), $\beta(-t_o) = 0$ and $\dot{\beta}(-t_o) > 0$. Of course, β is unbounded iff α is unbounded.

The proof is now by a series of comparisons, at each stage giving conditions on the largeness of b. It will be evident that there is no logical difficulty in not choosing b *a priori*.

Comparison 0. It suffices to show that at some time $t > t_o$, $\dot{\alpha}^2(t) > \lambda/(b-1)$ and $\dot{\alpha}(t) > 0$. For then $\dot{\alpha}^2(s) \geq \lambda/(b-1)$ for all $s \geq t$. This is because Equation (4.2.10) shows that if $\dot{\alpha}^2(s)$ drops to this value, then $\ddot{\alpha}(s) > 0$.

Let us therefore assume that $\lambda/(b-1)$ is reasonably small. The object of the ensuing comparisons will be to nurture $\dot{\alpha}^2$ to this value.

Consider the function $\varphi(\alpha) = \sin^2\alpha + b\cos^2\alpha$. Choose b so that $\varepsilon = \lambda/\varphi(\pi/4)$ is very small. Now make

Comparison 1. Let $\alpha_1 = \alpha$, the solution in question, and let α_2 be the solution of

$$\ddot{\alpha}_2 = (n-2)\tanh(t)\dot{\alpha}_2 - \varepsilon\alpha_2$$

with the same initial data at t_o as α. Then as long as $0 \leq \alpha_1 \leq \pi/4$ and $\dot{\alpha}_2 > 0$, we have for $t \geq t_o$

$$(!) \qquad \alpha_2/\dot{\alpha}_2 \geq \alpha_1/\dot{\alpha}_1$$

Proof We show subsequently that the left side is always finite. In this event, (!) is a direct application of the Comparison Theorem (3.2.7). In the notation of that theorem we have

$$p_1 = p_2 = \exp\left(-\int_{t_o}^{t} (n-2)\tanh(s)ds\right)$$

$$g_2 = \varepsilon p_2$$

$$g_1 = \left(\frac{\lambda-(b-1)\dot{\alpha}^2}{\varphi(\alpha)}\right) \quad \frac{\sin\alpha\cos\alpha}{\alpha} \; p_1$$

$$\leq \frac{\lambda \sin\alpha \cos\alpha}{\varphi(\alpha)\alpha} \quad p_1$$

Hence we can verify that $g_1 \leq g_2$ by noting that $\sin\alpha \cos\alpha/\alpha < 1$ and that $\varphi(\alpha)$ decreases on $[0, \pi/4]$.

For the functions w_i, we have $w_1(t_o) = w_2(t_o) = 0$, hence by the theorem

$$w_2 = \tan^{-1}(\alpha_2/p_2\dot{\alpha}_2) \geq w_1 = \tan^{-1}(\alpha_1/p_1\dot{\alpha}_1)$$

which gives the desired conclusion as long as $\dot{\alpha}_2 \neq 0$.

Now choose $0 < t_n < 1$ such that $t_n \tanh(t_n) < 1/(n-2)$. If $t_o \geq t_n$, proceed to Comparison 3. Otherwise go to

Comparison 2. Let α_1 and α_2 be the solutions of

$$\ddot{\alpha}_1 = (n-2)\tanh(t)\dot{\alpha}_1 - \varepsilon\alpha_1$$

and

$$\ddot{\alpha}_2 = -\varepsilon_1\alpha_2$$

with the same initial data as α at t_o. Here

$$\varepsilon_1 = \varepsilon/\Lambda$$

$$\Lambda = \exp\left(-\int_{t_o}^{t_n} (n-2)\tanh(s)\,ds\right)$$

$$\geq \exp\left(-\int_0^1 (n-2)\tanh(s)\,ds\right)$$

Note in particular that ε_1 is small if ε is small, independent of t_o, t_n.

To apply the comparison theorem, rewrite the second equation as

$$\Lambda\ddot{\alpha}_2 = -\varepsilon\alpha_2$$

Then

$$p_1(t) = \exp\left(-\int_{t_o}^{t} (n-2)\tanh(s)\,ds\right)$$

$$p_2(t) = \Lambda \leq p_1(t) \qquad t_o \leq t \leq t_n$$

$$g_1(t) = \varepsilon p_1(t)$$

$$g_2(t) = \varepsilon \geq g_1(t)$$

For the functions w_i we again have $w_1(t_o) = w_2(t_o) = 0$.
Hence at $t = t_n$, $w_1 \leq w_2$, $p_1 = p_2$ and thus

$$\alpha_1/\dot{\alpha}_1 \leq \alpha_2/\dot{\alpha}_2$$
$$= \sin[\theta(t_n-t_o)]/\theta \cos[\theta(t_n-t_o)]$$
$$\leq \sin(\theta t_n)/\theta \cos(\theta t_n)$$

where $\theta = \sqrt{\varepsilon_1}$

By l'Hospital's rule the last ratio approaches t_n as $\theta \to 0$.
Since we have assumed $t_n \tanh(t_n) < 1/(n-2)$, we may now require that θ be small enough that

$$\alpha_1/\dot{\alpha}_1 < 1/(n-2)\tanh(t_n)$$

holds at $t = t_n$. This is Comparison 2.

<u>Comparison 3</u>. In this step we require that ε be small enough that an over-damping condition hold, namely

$$T^2 - 4\varepsilon > 0$$
$$T = (n-2)\tanh(t_n)$$

Let β be the solution of

$$\ddot{\beta} = (n-2)\tanh(t)\dot{\beta} - \varepsilon\beta$$

with $\beta(t_o) = 0$

and $\dot{\beta}(t_o) = \dot{\alpha}(t_o)$

as usual. We claim there is some $t_1 > t_n$ such that

$$\beta(t_1)/\dot{\beta}(t_1) < 1/T$$

In fact, if $t_o > t_n$, let $t_1 = t_o$, and if $t_o < t_n$ we set $t_1 = t_n$ and use the conclusion of Comparison 2. Now let

$$W = \tfrac{1}{2}(T + (T^2 - 4\varepsilon)^{\frac{1}{2}}) < T$$

W satisfies $W = T - \varepsilon/W$

Our object is to show

$$(!!) \qquad \beta(s)/\dot{\beta}(s) < 1/W \qquad \forall s \geq t_1$$

So suppose (!!) fails to hold for the first time at time $t > t_1$. Then on $[t_1, t)$ we have

$$\begin{aligned} \ddot{\beta}(s) &\geq T\dot{\beta}(s) - \varepsilon\beta(s) \\ &> (T - \varepsilon/W)\,\dot{\beta}(s) \end{aligned}$$

So that

$$\begin{aligned} \dot{\beta}(t) &> \dot{\beta}(t_1) + (T - \varepsilon/W)(\beta(t) - \beta(t_1)) \\ &> T\beta(t_1) + (T - \varepsilon/W)(\beta(t) - \beta(t_1)) \\ &= W\beta(t) + (\varepsilon/W)\beta(t_1) \\ &\geq W\beta(t) \end{aligned}$$

Hence (!!) holds $s \geq t_1$. This is Comparison 3.

The proof of the theorem is now finished, via the following remarks:

a) Having chosen b large, any such solution α must increase monotonically to at least $\pi/4$. This is a straightforward consequence of Comparisons 1-3.

b) By perhaps choosing b even larger, we can guarantee that $\dot{\alpha}^2$ reaches the value $\lambda/(b-1)$ by the time α reaches $\pi/4$. For if this time is t, we will have either

$$\alpha(t)/\dot{\alpha}(t) < 1/T$$

from Comparisons 1 and 2, or at worst

$$\alpha(t)/\dot{\alpha}(t) < 1/W$$

from 1 and 3, depending on whether $t \leq t_n$ or $t > t_n$. Substituting $\alpha(t) = \pi/4$, we get an estimate on $\dot{\alpha}(t)$ which may be assumed sufficient. Therefore by Comparison 0 the theorem is proved.

<u>Proof of Theorem 4.2.5</u>: This is just an application of Theorem 4.1.4. If f and g are the identities on S^{n-1} and S^o respectively, then there is a harmonic one-parameter suspension

$$f*g = \Sigma I : E^n(b) \to S^n$$

for $n \leq 6$. We only have to check that ΣI is a diffeomorphism.

We will suppose ΣI is given in the natural symmetric form (cf. Section 3.5). That is

$$(x,y) \to (\cos\alpha(t)\ x/|x|,\ \sin\alpha(t))$$
$$t = \log(|x|/y+1)$$

Here α ranges monotonically ($\dot{\alpha}(t) > 0\ \ \forall t$) from $-\pi/2$ to $\pi/2$. It is thus more or less clear that ΣI is a homeomorphism, and an immersion away from $x = 0$. Let us check that

$$\frac{\partial}{\partial x_1}(\Sigma I) \neq 0$$

at $(x,y) = (0,1)$. For the i^{th} coordinate we find

$$\frac{\partial}{\partial x_1}(\cos\alpha\ x_i/|x|) = [-\sin\alpha\ \dot{\alpha}\ y\ x_1^2 + \cos\alpha(|x|^2 - x_1^2)]\ /|x|^3$$

We know <u>a priori</u> that this expression is continuous at $x = 0$, so we evaluate the limit along a particular path with $x = x_1 e_1$. As

$$|x| = 2/(e^t + e^{-t})$$

we are led to consider

$$\frac{\partial}{\partial x_1}(\Sigma I)^1 = \lim_{t\to-\infty} \tfrac{1}{2}\dot{\alpha}(t)e^{-t}$$

But by Section 3.5 we can replace $\dot{\alpha}$ by $|\cos\alpha|$ and also use that

$$c_1 e^t \leq |\cos_\alpha(t)| \leq c_2 e^t$$

for t near $-\infty$. Hence the above limit is strictly positive. This concludes the proof of the theorem.

Having proved non-existence of certain harmonic maps into long, thin ellipsoids, let us now show how to actually improve the existence theory of Chapter 3 by mapping into a short, fat ellipsoid.

<u>Theorem 4.2.11</u> Let $f : S^{p-1} \to S^{q-1}$ be a harmonic polynomial map of homogeneity ℓ. Then if $b > 0$ is small, there is a harmonic one-parameter suspension

$$\Sigma f : S^p \to E^q(b)$$

<u>Remark 4.2.12</u> The point of the theorem is that if b is small, we do not need f to satisfy the damping condition. For example, this theorem would apply to the Hopf map $\sigma : S^{15} \to S^8$.

<u>Proof</u> Recall that the equation under consideration is

$$(4.2.13) \qquad \ddot{\alpha} = (p-2)\tanh(t)\dot{\alpha} + \left(\frac{(b-1)\dot{\alpha}^2 - \lambda}{\varphi(\alpha)}\right)\sin\alpha \cos\alpha$$

with $\qquad \varphi(\alpha) = \sin^2\alpha + b\cos^2\alpha$

Our object is to find a solution with $\alpha_{-\infty} = -\pi/2$ and $\alpha_\infty = \pi/2$. By symmetry, it suffices to set $\alpha(0) = 0$ and find some $\dot{\alpha}(0)$ such that $\alpha_\infty = \pi/2$. If we put $\dot{\alpha}(0) = \inf A^+(0)$ in the usual way, we need only check two points to see that the arguments of Section 3.2 can be applied:

a) $A^+(0)$ is non-empty. This point is no longer quite so obvious, but can be derived from a straightforward analysis of the differential inequality

$$\ddot{\alpha} \geq -c\dot{\alpha}^2$$

b) $\dot{\alpha}(0) > 0$. This is where it is necessary to choose b small. By doing so one can ensure that the term multiplying $\sin\alpha \cos\alpha$ is large and negative when α is near 0. Hence the equation is underdamped for small α, and we can essentially apply the arguments in Lemma 3.2.11.

This demonstrates existence. That the asymptotic behavior of α is correct follows from an appropriate mixture of the methods in Theorem 4.2.3 and the usual techniques of Section 3.3. Regularity of the map Σf defined by α follows as in Section 3.4. Hence the theorem.

Remark 4.2.13 It is natural to ask whether the above theorem could be extended to the general case of harmonic polynomial maps; thus if one of the polynomials failed to satisfy the damping condition, this difficulty could be removed by mapping into an appropriate ellipsoid.

However, a new non-trivial difficulty presents itself. Due to the asymmetry of the general case it would be necessary to reintroduce the functions $\dot{\alpha}_o^+$ and $\dot{\alpha}_o^-$. The problem is to verify that these functions are still continuous.

Section 3. Mappings of Tori

This section consists of two examples. In the first we construct some essential maps of the Euclidean torus to itself via harmonic equivariant theory. In the second we give some unusual inessential maps of the flat torus to the 2-sphere.

Theorem 4.3.1 Let T^2 be the torus of revolution in $\mathbb{R}^3$ obtained by rotating a circle of radius r_2 through a circle of radius $r_1 > r_2$. Then there is a harmonic map of T^2 to itself which wraps around j times in the short direction and k times in the long.

Proof By definition there is an action of S^1 on T^2 by isometries. Let θ be the angular variable in the direction of this action and let φ denote angle in the complementary direction. φ is therefore a coordinate on the orbit space, which is a circle. The volume function, normalized, is

$$V(\varphi) = r_1 + r_2 \cos \varphi$$

We look for a harmonic map of the form

$$(\varphi, \theta) \to (\alpha(\varphi), k\theta)$$

where α is required to vary between 0 and $2\pi j$ on the interval $[0,2\pi]$. Such a map is horizontal, and Theorem 1.3.5 applies. The orbit-energy is

$$\gamma(\varphi, \alpha(\varphi)) = k^2[(r_1 + r_2 \cos\alpha(\varphi))/(r_1 + r_2 \cos\varphi)]^2$$

and the equivariant tension equation is

$$\ddot{\alpha}(\varphi) - \frac{r_2 \sin\varphi \,\dot{\alpha}(\varphi)}{r_1 + r_2 \cos\varphi} + \frac{k^2(r_1+r_2\cos\alpha)r_2 \sin\alpha}{(r_1+r_2\cos\varphi)^2} = 0$$

Now make a change of variables

$$x(\varphi) = \int_0^{\varphi} \frac{1}{r_1+r_2\cos\varphi'} \, d\varphi'$$

and let

$$\alpha(\varphi) = \bar{\alpha}(x(\varphi))$$

with the result that $\bar{\alpha}(x)$ must satisfy the autonomous equation

$$\ddot{\bar{\alpha}} + r_2 k^2 \sin\bar{\alpha}\,(r_1+r_2\cos\bar{\alpha}) = 0$$

Let $t_o = x(2\pi)$. We are therefore looking for a solution $\bar{\alpha}$ satisfying

$$\bar{\alpha}(x + t_o) = \bar{\alpha}(x) + 2\pi j \qquad \forall x$$

The method of finding such a solution is essentially our standard one: we set $\bar{\alpha}(0) = 0$ and find $\dot{\bar{\alpha}}(0)$ such that $\bar{\alpha}(t_o/2j) = \pi$. This is possible: the term $r_1+r_2\cos\alpha$ is always positive, so the equation behaves like that of an ordinary pendulum. Hence we can make $\bar{\alpha}$ reach π in any specified finite time.

The required periodicity is therefore evident, and thus regularity is automatic. Hence the theorem.

<u>Theorem 4.3.2</u> Let T^2 be the flat torus. There are surjective harmonic maps $T^2 \to S^2$ of degree 0 which are neither open nor light. There are also harmonic maps $T^2 \to S^2$ whose images are proper closed subsets with interior.

<u>Remark 4.3.3</u> This shows that a harmonic map between compact surfaces need not behave anything like a complex analytic map. A non-constant analytic map must be open and light, by a theorem of Stoilow.

Simpler examples could of course be given if one did not ask that the image have interior.

<u>Proof</u> Choose angular coordinates (φ,θ) on T^2. Our maps $f : T^2 \to S^2$ will have the form

$$(\varphi,\theta) \to (\sin\alpha(\varphi)e^{i\theta},\ \cos\alpha(\varphi))$$

One easily verifies that Δf is proportional to f, i.e., f is harmonic, iff

$$\ddot{\alpha} = \sin\alpha\ \cos\alpha$$

Hence we simply look for solutions which are periodic. Again it is obvious that we can set $\alpha(0) = 0$ and choose $\dot{\alpha}(0)$ such that $\alpha(\pi/2) = \pi/2$. By symmetry we get the periodicity we need.

The map f is surjective and clearly has degree 0: the first half of the torus $(0 \leqslant \varphi \leqslant \pi)$ gets mapped onto S^2 with one orientation, while for the second half this orientation is reversed. Note also that the circles $\varphi = 0$ and $\varphi = \pi$ are mapped to points, so f is not light. Neither is it open.

Suppose instead we put $\alpha_0 = \pi/2$ and choose $\dot{\alpha}_0$ small. Then the system oscillates about $\pi/2$ with some period. This period depends on α_0 and gravity ($g = 1$ in this case). By modifying the metric on T^2 in one variable, however, the equation becomes

$$\ddot{\alpha} = c^2 \sin\alpha \cos\alpha$$

It is clear that by judicious choice of c and $\dot{\alpha}_0$ we can make the period 2π. The image of f in this case will be a closed band about the equator of S^2.

This concludes the theorem.

Chapter 5

THE SECOND VARIATION OF THE ENERGY

This chapter is concerned with the qualitative behaviour of the energy functional in the vicinity of a critical point, or harmonic map. For example, the second variation can tell us if a harmonic map is not a local minimum of the energy. If not, we would like to know in how many ways the energy can be decreased;ie., to compute the Morse index of the map. An associated qualitative problem is that of non-degeneracy: if the second variation is zero in some direction, does this indicate the presence of more harmonic maps? This question is also significant in Morse-theory.

It should be pointed out, however, that although the language of Morse theory is the natural one, the prospect of actually doing any Morse theory is distant. It is not clear what relationship, if any, should hold between the critical sets of the energy and the topology of the mapping space on which it is defined. The basic problem is of course the lack of a general existence theory. On the other hand, we have included an appendix which shows that locally the correct relation holds, under an appropriately weak non-degeneracy assumption.

Section 1. Generalities

If $f: N \to M$ is a harmonic map, then the second variation (or Hessian) of the energy at f is a symmetric bilinear form defined on the vector fields along f; i.e., on the sections of f^*TM. (If f is thought of as lying in a manifold of maps, such vector fields form the tangent space to the manifold at f). The Hessian H is defined as follows: given two fields along f, say v and w, choose a 2-parameter variation $f_{s,t}$ such that $\frac{\partial f}{\partial s}\Big|_{(s,t)=0} = v$ and $\frac{\partial f}{\partial t}\Big|_{(s,t)=0} = w$. Then

$$H(v,w) = \frac{\partial^2(E(f_{s,t}))}{\partial s\ \partial t}\Big|_{(s,t)=0}$$

Since f is a critical point of E, it follows as in finite dimensional Morse theory that H is well defined. This is also apparent from the computation below:

Proposition 5.1.1 If $f: N \to M$ is a harmonic map, the Hessian at f is given by

(5.1.2) $$H(v,w) = \int_N \langle\nabla_f v, \nabla_f w\rangle - \langle\rho_f(v), w\rangle$$

Remark Here ∇_f is the induced connection on f^*TM and ρ_f (which is like a Ricci tensor) is the trace of a bilinear form on TN. Explicitly,

$$\begin{aligned}\rho_f(v) &= \text{Trace } R_M(df, v)\, df \\ &= \sum_i R_M(df\cdot e_i, v)\, df\cdot e_i\end{aligned}$$

Here R_M is the curvature tensor on M and (e_i) is an orthonormal basis for TN at the point in question. The sign convention for R is that used in Milnor [M], namely, that

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

A good reference for connections and curvatures in pull-back bundles and in bundles of linear maps is Eliasson [El]. (His curvature convention opposite to ours). The ensuing calculation uses the formulae given there throughout; it also employs a method found in lecture notes by the same author.

Proof (5.1.1.) Choose a variation $f_{s,t}$ as above. Then $f_{s,t}$ defines a map $F: \mathbb{R}^2 \times N \to M$, and

$$E(f_{s,t}) = \tfrac{1}{2}\int_N \langle dF(-), dF(-)\rangle$$

where (-) will denote arguments in TN. Then as all Riemannian manifolds are assumed to carry their Levi-Civita connections, we have

$$\frac{\partial E}{\partial t} = \int_N \quad \langle \nabla dF\,(\tfrac{\partial}{\partial t},\ -),\ dF(-)\rangle$$

$$= \int_N \quad \langle \nabla dF(\ -,\ \tfrac{\partial}{\partial t}),\ dF(-)\rangle$$

$$\frac{\partial^2 E}{\partial s \partial t} \quad = \int_N \quad \langle \nabla^2 dF(\tfrac{\partial}{\partial s},-,\tfrac{\partial}{\partial t}), dF(-)\rangle + \langle \nabla dF(-,\tfrac{\partial}{\partial t}), \nabla dF(-,\tfrac{\partial}{\partial s})\rangle$$

Note that $\nabla^2 dF(\frac{\partial}{\partial s},-,\frac{\partial}{\partial t}) = \nabla^2 dF(-,\frac{\partial}{\partial s},\frac{\partial}{\partial t}) - (R_L(\frac{\partial}{\partial s},-)dF)\ \frac{\partial}{\partial t}$

where R_L is the curvature for the bundle $L(T(\mathbb{R}^2 \times N),\ F^*TM)$.

From [E1] we get

$$(R_L(\tfrac{\partial}{\partial s},-)dF)\ \frac{\partial}{\partial t} = R_{F^*TM}(\tfrac{\partial}{\partial s},-)\ \frac{\partial F}{\partial t} - dF(R_{\mathbb{R}^2 \times N}(\tfrac{\partial}{\partial s},-)\tfrac{\partial}{\partial t})$$

$$= R_M(\tfrac{\partial F}{\partial s},\ dF(-))\frac{\partial F}{\partial t}$$

Note also that $\nabla dF(\frac{\partial}{\partial s},\frac{\partial}{\partial t})$ is a field along F, so that at $s = t = 0$ we have

$$\int_N \ \langle \nabla^2 dF(-,\ \tfrac{\partial}{\partial s},\tfrac{\partial}{\partial t}),\ dF(-)\rangle = 0$$

as $f_{o,o}$ is harmonic. Thus we conclude

$$\left.\frac{\partial^2 E}{\partial s \partial t}\right|_{s=t=o} \qquad = \int_N \langle \nabla_f w,\ \nabla_f v\rangle\ -\ \langle R_M(v,df)w,df\rangle$$

In general, however, the form $\langle R(X,Y)Z,W\rangle$ is anti-symmetric in (X,Y) and in (Z,W). (See Milnor [M, p.53]). This proves the proposition.

An immediate corollary is that if all sectional curvatures of M are non-positive, then $H(v,v) \geq 0$ for all variations v; this indicates that every such harmonic map is a local minimum of the energy. In fact, a much stronger statement is true: if $f:N \to M$ is a harmonic map and M has non-positive sectional curvature, then f is an absolute minimum of the energy in its homotopy class. The proof is simply that any smooth map f_o in this class is smoothly homotopic to a harmonic map f_∞ via the heat equation (using Eells-Sampson [ES] and Hartman[H]). As energy decreases along a trajectory of the heat equation [ES], $E(f_\infty) \leq E(f_o)$. Furthermore, Hartman has shown that f_∞ must be homotopic to f through harmonic maps, and consequently that $E(f) = E(f_\infty)$. Hence $E(f)$ is an absolute minimum.

It is therefore reasonable to expect that in this case the harmonic maps are a deformation retract of the mapping space (as in finite dimensional Morse theory). Karen Uhlenbeck has claimed to have proved a statement to this effect [U2]. It is easy enough to see this if N and M are flat; all the higher order energies have only harmonic critical points in this case, and infinite dimensional Morse theory can be applied to a sufficiently high order energy.

By applying the divergence theorem, as in [E2], we see that the Hessian can also be written as

$$\begin{aligned} (5.1.3) \qquad H(v,w) &= \int_N \langle -\Delta_f v - \rho_f v, w \rangle \\ &= \int_N \langle J_f v, w \rangle \end{aligned}$$

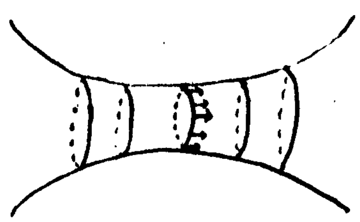
where $\Delta_f v = \text{Trace}\ (\nabla_f^2 v)$. Note that the second variation operator $J_f : C^\infty(f^*TM) \to C^\infty(f^*TM)$ is symmetric and elliptic. If the nullity of f (null (f)) is defined as the dimension of the space on which H is zero (i.e., those v for which $H(v,w) = 0$ for all w), and if the index of f is the dimension of the largest space on which H is negative definite, it is clear that

$$\begin{aligned} (5.1.4) \qquad \text{null}(f) &= \dim \ker J_f < \infty \\ \text{index}(f) &= \# \{ \text{eigenvalues } (J_f) < 0 \} < \infty \end{aligned}$$

Note that if the metrics on N and M are changed by (positive) scalar multiples, then so is J ; hence null(f) and index(f) are unchanged. However, we will see examples later in which these quantities can be altered dramatically by making less trivial changes in the metrics.

In analogy with the theory of geodesics, we will say v is a <u>Jacobi field</u> along f if $v \in \ker J_f$. Note that if f_t is a variation of f through harmonic maps, then $v = \left.\frac{\partial f}{\partial t}\right|_{t=0}$ is a Jacobi field along f. For clearly $J_f v = 0$ if and only if $H(v,w) = 0$ for all w. However, we can compute $H(v,w) = d^2E(v,w) = \frac{d}{dt}\,(dE(f_t)(w))_{t=0}$, and $dE(f_t) = 0$ by assumption. Hence $J_f v = 0$. The converse is not necessarily true, even when $f : S^1 \to M$ is a geodesic.

<u>Example 5.1.5</u>



The middle circle is a geodesic on the surface M, but no distinct parallel circle is a geodesic. If M is made quite flat along its middle ($R_M = 0$ there), the parallel field drawn along the geodesic will be a Jacobi field. It clearly does not arise from a variation through geodesics.

Regarding the nullity of a harmonic map, let us define the <u>killing nullity</u> as

$$\mathrm{Null}_k(f) = \dim \operatorname{span}\,(\underline{i}(M),\ df\,(\underline{i}(N)))$$

Here elements of $\underline{i}(M)$ (infinitesimal isometries) and $df(\underline{i}(N))$ are considered as variation fields along f. They are clearly Jacobi fields, as they arise from the composition of f with

1-parameter groups of isometries. It will be convenient to discount these trivial harmonic variations and define a reduced nullity

$$\mathrm{Null}_r(f) = \mathrm{Null}(f) - \mathrm{Null}_k(f)$$

Thus the qualitative problem of interest to us is this: if f is a harmonic map and $\mathrm{null}_r(f) \neq 0$, do all Jacobi fields arise from a variation of f through harmonic maps? Let us say that f is generate if this is the case.

Our second problem is to calculate index (f). Although this is usually impossible (cf. the last section of this chapter), the study of the index of certain simple maps is both feasible and rich in geometry.

Section 2. The Identity Map

If M is a closed oriented Riemannian manifold, then the identity map $id_M : M \to M$ is of course a harmonic map and we can study its index and nullity. One might expect that id_M is always a local minimum of the energy, and hence has index 0; that this is not the case was first observed in [ES]. Eells and Sampson constructed a one-parameter family of maps $f_t : S^n \to S^n$ with $f_o = id_{S^n}$ and $\lim_{t\to\infty} E(f_t) = 0$ for $n \geq 3$. We will return to interpret this example later and calculate index (id_{S^n}) exactly.

To begin with, note that a vector field along id_M is just a vector field on M. The second variation operator becomes

$$(5.2.1) \qquad Jv = -\Delta v - \rho(v)$$

where

$$\Delta v = \mathrm{Trace}\ (\nabla^2 v)$$

$$\rho(v) = \mathrm{Trace}\ R_M(-,v)-$$

Hence ρ is just the Ricci tensor, considered as a linear map. The operator Δ is not to be confused with the Hodge Laplacian, denoted Δ_H in this section. On one-forms (or vector fields), we have in fact the well-known formula [D]

(5.2.2) $$\Delta_H v = -\Delta v + \rho(v)$$

The operator J has also made its appearance in differential geometry before, for example in the work of Yano and Bochner on Curvature and Betti Numbers [BY]. The following integral formula is given by Yano [BY p. 57] and will be used later:

(5.2.3) $$\int_M \langle Jv, v\rangle = \int_M \tfrac{1}{2}|L_v g|^2 - \delta(v)^2$$

Here $L_v g$ is the Lie derivative of the metric w.r.t. v and δ is the divergence. Thus $L_v g = 0$ if and only if v is a Killing vector; from the formula we see that if $Jv = 0$ and $\delta(v) = 0$, then v is Killing. The converse follows by direct computation [BY].

J is also studied in Lichnerowicz [L2] with regard to the Lie algebra of infinitesimally conformal fields, denoted $\underline{c}$. The following observation can easily be derived from his work, but a self-contained proof is simpler.

Proposition 5.2.4 If M is closed and oriented of dimension ≥ 3, index $(id_M) \geq \dim(\underline{c}/\underline{i})$.

Proof Recall that a vector field v is in $\underline{c}$ if it satisfies

$$L_v g = (\tfrac{2}{n})\,\delta(v) g$$

Hence $$|L_v g|^2 = (\tfrac{2}{n})^2 \delta(v)^2\, |g|^2 = \tfrac{4}{n}\,\delta(v)^2$$

so that Yano's formula (5.2.3) gives

$$\int_M \langle Jv, v\rangle = \frac{2-n}{n} \int_M \delta(v)^2 < 0$$

if $\delta(v) \neq 0$. From the definition, however, it is clear that if $v \in \underline{c}$, then $v \in \underline{i}$ if and only if $\delta(v) = 0$. If we let V be the ortho-complement of $\underline{i}$ in $\underline{c}$ (with respect to the L^2 inner product), then it follows that J is negative definite on V. This proves the proposition.

Remarks 5.2.5 a) This gives an interpretation of the example of Eells and Sampson. The n-sphere carries n+1 linearly independent conformal fields which are orthogonal to the Killing fields. The maps f_t they constructed are essentially just the flow of one of these vector fields

b) It appears that in general index $(id_M) > \dim (\underline{c}/\underline{i})$. This will be discussed later (5.2.13).

c) It is easy to produce examples of manifolds for which $\underline{c}/\underline{i} \neq 0$, as pointed out in [N]. For if M has a one-parameter group of isometries, say f_t, choose a positive function ρ which is not invariant under the group. Define a new metric on M by setting $g^1 = \rho g$. The maps f_t are now conformal, but are no longer isometries.

Hence any compact homogeneous space of dimension ≥ 3 can be given a metric for which id_M is not a local minimum of the energy.

Lichnerowicz also studies J on compact Kahler manifolds and finds:

Proposition 5.2.6 [L1] Let M be a compact Kahler manifold and v a vector field on M with $Jv = 0$. Then v is an infinitesimal analytic transformation.

Since holomorphic maps of Kahler manifolds are harmonic [ES], this says that id_M is a generate critical point of the energy. Holomorphic maps are studied in more detail in a later section.

Although the operator J has appeared in the previously mentioned varied contexts, no unified interpretation of its significance was known. Yano and Nagano in fact devoted a paper to a study of the solutions of $Jv = 0$. [YN]. They labelled them "geodesic vector fields"; the reason for so doing is that the equation of an infinitesimally geodesic vector field (i.e. a field whose flow consists of totally geodesic maps) is just

$$\nabla^2 v + R(-,v) - = 0$$

Thus J is the trace of the above operator. For maps we have similarly that f is totally geodesic if $\nabla df = 0$ and harmonic if Trace $\nabla df = 0$. This rather strongly suggests that one should try to prove that the flow of a Jacobi field consists of harmonic maps. However, the next example shows that this is not the case.

<u>Example 5.2.7</u> Define a vector field on $\mathbb{R}^n$, for $n \geq 2$, by

$$v(x) = (\cos x_1 \cosh x_2, 0 \ldots 0)$$

Then v is harmonic and therefore a Jacobi field. The flow of v has the form

$$F(x,t) = (f_t(x), x_2, \ldots, x_n)$$

The map $f_t(x)$ can be defined as follows on the set $-\pi/2 < x_1 < \pi/2$ and extended periodically:

$$\log(\sec f_t(x) + \tan f_t(x)) = \log(\sec x_1 + \tan x_1) + t \cosh x_2$$

By differentiation one checks that $\dfrac{\partial f_t(x)}{\partial t} = \cos f_t(x) \cosh x_2$

so that f does give the flow of v.

We claim that for $t \neq 0$, $f_t(x)$ is not a harmonic function. To see this write

$$f_t(x) = \rho(\log(\sec x_1 + \tan x_1) + t \cosh x_2)$$

where ρ is a suitable inverse function for $\log(\sec\theta + \tan\theta)$.

Thus
$$\frac{\partial f}{\partial x_1} = \cos f \sec x_1$$

$$\frac{\partial^2 f}{\partial x_1^2} = -\sin f \cos f \sec^2 x_1 + \cos f \sec x_1 \tan x_1$$

$$\frac{\partial f}{\partial x_2} = \cos f \; t \sinh x_2$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\cos f \sin f \, (t \sinh x_2)^2 + \cos f \; t \cosh x_2$$

At $x_1 = x_2 = 0$, $\quad \Delta f = \cos f \left(\frac{t}{2} - \sin f\right)$

which is certainly non-zero for $t > 2$. Hence the flow is not harmonic.

Since our main interest is in closed manifolds, the above example is really telling us that the problem is not local. If one still hopes to prove that the flow of a Jacobi field on a closed manifold consists of harmonic maps, Stoke's theorem will almost certainly be needed at some stage. This is in contrast to the apparently related case of infinitesimally geodesic vector fields and totally geodesic maps, where the proof is completely local. At any rate, no counterexample is known.

As an antidote to the example, suppose that M is a <u>compact</u> flat manifold (e.g. a torus rather than Euclidean space). A Jacobi field satisfies $\Delta v = 0$. Therefore $0 = \int_M \langle \Delta v, v \rangle = -\int_M |\nabla v|^2$, so that $\nabla v = 0$. Hence v is a parallel vector field, and thus an infinitesimal isometry.

A related question is whether the harmonic diffeomorphisms of M form a group. On this line, do the Jacobi fields form a

Lie algebra? Yano and Nagano [YN] consider the latter problem on a general Einstein space of positive curvature (i.e., $\rho = cg$ for some $c > 0$). Their findings are inconclusive: the Lie bracket of two Jacobi fields will at worst lie in the Lie algebra of fields with divergence zero; both problems are open.

The assumption that M is Einstein is a natural one to make when studying infinitesimal transformations. If $\rho = cg$, then J and the Hodge Laplacian on vector fields (i.e., 1-forms) are related by

$$J = \Delta_H - 2cI \tag{5.2.8}$$

where I is the identity transformation. In particular, J and Δ_H have the same eigenfunctions. It is now not difficult to relate the index and nullity of id_M to the <u>spectrum</u> of M, which is the set of eigenvalues of Δ_H on <u>functions</u>. The following notation will be used to describe the distribution of these eigenvalues:

$$\lambda(r) = \# \{\text{eigenvalues } \lambda : 0 < \lambda < r\} \tag{5.2.9}$$

For multiplicities write (for $r > 0$)

$$m(r) = \text{multiplicity of } r \text{ as an eigenvalue} \tag{5.2.10}$$

and set $m(0) = 0$. Hence $\lambda(r)$ increases with r, whereas $m(r)$ is usually 0. The basic conclusion of this section is then

<u>Proposition 5.2.11</u> Let M be a closed oriented Einstein manifold, with $\rho = cg$ for some scalar c. Then

a) $\text{index } (id_M) = \lambda(2c)$

b) $\text{null } (id_M) = \dim(\underline{i}) + m(2c)$

c) $\text{null}_r (id_M) = m(2c)$

<u>Remark</u> The proof will hold no surprises for someone familiar with the references cited earlier.

Proof From the relation between J and Δ_H, we have immediately that

a_o) index (id_M) = #{ eigenvalues of Δ_H (on 1-forms) < 2c}

b_o) null (id_M) = multiplicity of 2c (as above)

So suppose v is an eigenfunction for Δ_H; $\Delta_H v = \lambda v$. Using the Hodge decomposition, we can always write

$$v = df + w$$

where f is a function and $w \in \ker(\delta)$. Then since df and w lie in orthogonal subspaces which are invariant under Δ_H, it follows that each must be an eigenfunction of Δ_H with eigenvalue λ. Now suppose $\lambda < 2c$. We claim that this forces $w = 0$. In fact, we simply apply Yano's formula (5.2.3) to w; $\lambda < 2c$ means that the left side is < 0 and $\delta(w) = 0$ says the right side is ≥ 0. Hence $w = 0$. As for the function f, we have $d(\Delta_H f) = \Delta_H df = \lambda df$, so that, up to a constant, $\Delta_H f = \lambda f$. Conversely, if f is a non-constant eigenfunction, then df is a non-zero eigenvector with the same eigenvalue. This gives (a).

The argument for (b) is similar: the contribution of the gradient fields is self-explanatory, and as they are orthogonal to ker (δ), we add the contributions of these subspaces. We have also noted earlier that $w \in \ker(\delta) \cap \ker J$ if and only if $w \in \underline{i}$. Hence (b), and (c) follows.

Example 5.2.12 Let $M = S^n$. S^n with its Euclidean metric is an Einstein space with $\rho = (n-1)g$. Recall that Spec (S^n) is $\{\lambda_k = k(k+n-1) : k \geq 0\}$. The first few are $0, n, 2(n+1), \ldots$. Thus n is the only non-0 eigenvalue which is $\leq 2(n-1)$. The eigenfunctions with $\lambda = n$ are the harmonic polynomials of degree 1; i.e., the linear forms on $\mathbb{R}^{n+1}$, of which there are n + 1 linearly independent ones.

Hence
$$\operatorname{index}(\mathrm{id}_{S^n}) = \begin{cases} n+1 & n \geq 3 \\ 0 & n = 1,2 \end{cases}$$
$$\mathrm{Null}_r(\mathrm{id}_{S^n}) = \begin{cases} 0 & n \neq 2 \\ 3 & n = 2 \end{cases}$$

We remark that the gradient of a linear form on S^n is an element of $\underline{c}$ (and not $\underline{i}$). Hence the connection between Prop. 5.2.4 and Prop. 5.2.11 is exact for spheres. Note also that the reduced nullity on S^2 comes from variations through conformal (i.e., holomorphic) maps. Hence id_{S^n} is a generate critical point for all n.

With regard to other Einstein spaces, it happens that Nagano has computed the spectrum of the classical compact irreducible symmetric spaces [N1]. His results are presented without proof, but using his table yields:

Proposition 5.2.13 Index $(\mathrm{id}_M) = 0$ for the classical compact irreducible symmetric spaces, with the following exceptions:

i) $M = S^n$, $n \geq 3$

ii) $M = Sp(p+q)/Sp(p) \times Sp(q)$, the quaternionic Grassmann.

iii) $M = SU(2m)/Sp(m)$ $m > 1$

Remarks a) the spaces in (ii) and (iii) satisfy $\underline{c} = \underline{i}$, by a theorem of Nagano [N2]. Hence the estimate in Proposition 5.2.4. is not sharp.

b) There seem to be one or two errors in Nagano's calculations for the real oriented Grassmanns. For example, if $M = SO(6)/SO(2) \times SO(4)$ (with metric normalized so that $\rho = \frac{1}{2}g$), the table says $\lambda = 15/16$ can occur. However, M is Hermitian symmetric (hence Kahler) [HE], so the smallest positive eigenvalue should be 1(i.e., index $(\mathrm{id}_M) = 0$).

Section 3. Harmonic Fibrations

Another simple example of a harmonic map is the projection in a trivial bundle $\pi: N \times M \to M$. Let N and M be closed manifolds, and let J_π be the second variation operator for fields along π. We can construct a large set of eigenvectors for J_π as follows: if J denotes the Jacobi operator for vector fields on M, let (v_j) be a complete orthogonal set of eigenvectors for J; $Jv_j = \lambda_j v_j$. Further, let (f_i) be a complete set of eigenfunctions for the Laplacian of N; $\Delta_N f_i = \mu_i f_i$.

<u>Proposition 5.3.1</u> The vector fields along π defined by $a_{ij}(n,m) = f_i(n) v_j(m)$ are eigenfunctions of J_π : $J_\pi(a_{ij}) = (\mu_i + \lambda_j)\, a_{ij}$.

<u>Proof</u> Recall that $J_\pi = -\Delta_\pi - \rho_\pi$. First compute

$$\nabla_\pi(f_i v_j) = df_i(\ )\, v_j + f_i \nabla v_j \circ d\pi$$

$$\nabla^2_\pi (f_i v_j) = \nabla df_i(\ ,\ )\, v_j + 2\, df_i(\ )\, \nabla v_j \circ d\pi(\ ) + f_i\, (\nabla^2 v_j\, (d\pi, d\pi) + \nabla v_j \circ \nabla d\pi)$$

To evaluate the trace of this thing, choose an orthonormal basis for $T(N \times M)$ by compounding such bases for TN and TM. Hence the middle term disappears, and as $\nabla d\pi = 0$ we find

$$\Delta_\pi(f_i v_j) = (-\Delta_N f_i) v_j + f_i\, \Delta v_j$$

Similarly we see that $\rho_\pi\, (f_i v_j) = f_i\, \rho_M(v_j)$. The proposition follows.

<u>Corollary 5.3.2</u> $$\text{index } (\pi) \geq \sum_{\lambda_j < 0} m_M(\lambda_j)\ \ \lambda_N(|\lambda_j|)$$

$$\text{null}_r\ (\pi) \geq \sum_{\lambda_j < 0} m_M(\lambda_j)\ m_N(|\lambda_j|)$$

Here $m_M(\lambda_j)$ is the multiplicity of λ_j as an eigenvalue of the Jacobi operator on M, and $\lambda_N\ (|\lambda_j|)$ is the number of eigenvalues of ∇_N which are less than $|\lambda_j|$, this time including 0.

Corollary 5.3.3 Let M be a closed Riemannian manifold for which index $(id_M) > 0$, and let N be any closed Riemannian manifold of dimension ≥ 1. If $\pi: N \times M \to M$ is the trivial bundle:

a) By making a scalar change of metric on N, we can make the Morse index of π arbitrarily large.

b) If further the multiplicities in the spectrum of N satisfy $\lim_{n\to\infty} m(\lambda_n) = \infty$, we can similarly make the reduced nullity of π arbitrarily large.

Proof If index $(id_m) > 0$, there is at least one $\lambda_j < 0$ in the previous Corollary. Now simply observe that if the metric on N is multiplied by c, the spectrum of N is multiplied by 1/c. For c large, we can therefore make arbitrarily many eigenvalues of Δ_N less than $|\lambda_j|$. This proves (a). For (b), we similarly make an eigenvalue of large multiplicity coincide with $|\lambda_j|$.

Example 5.3.4 Let $M = S^n$, for $n \geq 3$, and let $N = S^p$. If v is a conformal gradient field on S^n and f is a harmonic polynomial of homogeneity k on S^p, then by a scalar change of metric on S^p we can assume that $w(x,y) = f(x)v(y)$ is a Jacobi field for π. It seems quite unlikely that these fields arise from a variation of π through harmonic maps. Since the harmonic polynomials satisfy the condition in (b) of the above corollary, for $p \geq 2$, it appears that π can be given an arbitrarily large degeneracy as well as a massive index. Hence the innocuous map π is completely pathological from the standpoint of Morse theory.

Regarding more general harmonic fibrations $\pi: E \to B$, the best one can say at this stage is that index $(\pi) \geq$ index (id_B). The same calculation as given in the proposition shows that if v is a field on B with $Jv = \lambda v$, then $J_\pi(v \circ \pi) = \lambda(v \circ \pi)$

(although in this case $\nabla d\pi$ only disappears after Trace is applied.)

For example, the Hopf maps $S^7 \to S^4$ and $S^{15} \to S^8$ have positive index. Since index $(id_{S^2}) = 0$, the index of the Hopf map $S^3 \to S^2$ should be zero. It is not hard to see that the index of any trivial bundle projection over S^2 is zero.

Section 4. Holomorphic Mappings of Kahler Manifolds

Let us first describe a startling observation due to Lichnerowicz [L2]. If $f: N \to M$ is a map of Kahler manifolds (N compact), we can write

$$df = df_{(1,0)} + df_{(0,1)}$$

a decomposition into complex linear and conjugate linear parts. Accordingly there is a decomposition of the energy

$$E(f) = E'(f) + E''(f)$$

Hence f is holomorphic $\Leftrightarrow$ $E''(f) = 0$. Now consider the difference

$$K(f) = E'(f) - E''(f)$$

One can show that $K(f) = \int_N \langle \Phi_N, f^*\Phi_M \rangle$, where Φ_N and Φ_M are the fundamental 2-forms of the respective metrics. From this Lichnerowicz deduces that if f_u is a smooth homotopy, then $K(f_u)$ is constant!

Corollary 5.4.1 A holomorphic (or anti-holomorphic) map of Kahler manifolds gives an absolute minimum of the energy in its homotopy class.

Proof If f is holomorphic and g is homotopic to f, then $E(f) = K(f) = K(g) \leq E'(g) \leq E(g)$.

Hence the index of a holomorphic map is 0 in a strong sense.

Regarding the nullity, we have seen that any vector in the null space at the identity map is infinitesimally analytic. (Prop. 5.2.6). In general, a reasonable conjecture is that any Jacobi field along a holomorphic map arises from a variation through holomorphic maps. A proof will be given below in the special case that f is a conformal diffeomorphism of 2-manifolds.

Before proceeding, it is necessary to know something about conformal maps. Recall that $f: N \to M$ is said to be conformal if there is a positive function φ on N such that $f^* g_M = \varphi^2 g_N$, where g means metric. Equivalently, $\langle df(v), df(w)\rangle = \varphi^2 \langle v,w\rangle$ holds for all $v,w \in TN$. Now we need some generalizations of a few familiar notions.

<u>Definition 5.4.2</u> a) Let $f: N \to M$ be a smooth map and v a vector field along f. The <u>Lie derivative</u> of $f^* g_M$ with respect to v is the 2-form on N given by

$$L_v f^* g_M (\alpha,\beta) = \langle \nabla_f v(\alpha), df(\beta)\rangle + \langle \nabla_f v(\beta), df(\alpha)\rangle$$

b) Define the <u>divergence</u> (along f) of v to be

$$\mathrm{Div}_f v = \langle \nabla_f v, df\rangle$$

c) If dim (N) = n and f is a conformal diffeomorphism, say that v is <u>infinitesimally conformal</u> (along f) if

$$L_v f^* g_M = (\tfrac{2}{n})\ \mathrm{Div}_f(v)\ g_N$$

<u>Remark 5.4.3</u> It is not hard to see that these definitions reduce to the usual ones when $f = id_N$ and v is an ordinary vector field. Note also that:

a) if f is an isometry of N, $L_v f^* g_N = 0 \Leftrightarrow v = w \circ f$, where $w \in \underline{i}$ (N).

b) As $\mathrm{Div}_f v$ is just the first variation of the energy, it follows by definition that $\int_N \mathrm{Div}_f v = 0$ for all variations v if and only if f is harmonic. Harmonic maps may therefore be characterized as those smooth maps which preserve the divergence theorem.

c) If f is a conformal diffeomorphism, then v is infinitesimally conformal along f if and only if $v = w \circ f$, where $w \in \underline{c}(M)$.

Proof Given v, let $w(x) = v(f^{-1}(x))$. Then $\nabla w = \nabla_f v \circ df^{-1}$, and it follows that $\mathrm{Div}_f(v) = \varphi^2 \mathrm{Div}(w)$. The result is now obtained by direct computation.

Hence if v is infinitesimally conformal (and N is compact), then v arises from a variation of f through conformal diffeomorphisms.

Recall now that a conformal map of 2-manifolds is harmonic. Our basic result is:

Theorem 5.4.4 Let $f: N \to M$ be a conformal diffeomorphism of compact oriented 2-manifolds. Then f is a generate critical point of the energy.

Proof It suffices to show that if v is a Jacobi field along f, then v is a conformal field in the sense of Definition 5.4.2. For notational simplicity set

$$\Phi = \mathrm{Div}_f v$$
$$w(\alpha,\beta) = \langle \nabla_f v(\alpha),\ df(\beta) \rangle$$
$$W = L_v f^* g_M = w + w^*$$

Thus if we are given that

$$-\nabla_f v - \rho_f(v) = 0$$

we must show $\quad W = \Phi g_N$

Observe that $\quad |W - \Phi g_N|^2 = |W|^2 - 2\,\Phi\,\mathrm{Trace}\ W + 2\,\Phi^2$

$$|W|^2 = 2(|w|^2 + \langle w, w^* \rangle)$$
$$\mathrm{Trace}\ W = 2\ \mathrm{Trace}\ w = 2\Phi$$
$$|w|^2 = \sum_{ij} \langle \nabla_f v(e_i),\ df(e_j) \rangle^2$$

where the last sum is over an orthonormal basis as usual. Note, however, that $df(e_j) = \varphi \alpha_j$, where α_j is an orthonormal base at

$f(x)$. Thus

$$|w|^2 = \varphi^2 \sum_i |\nabla_f v(e_i)|^2 = \varphi^2 |\nabla_f v|^2$$

and hence $|W - \Phi g_N|^2 = 2(\varphi^2 |\nabla_f v|^2 + \langle w, w^* \rangle - \Phi^2)$

We would like to show

$$\int_N \frac{1}{2\varphi^2} |W - \Phi\, g_N|^2 = \int_N |\nabla_f v|^2 + \frac{1}{\varphi^2} (\langle w, w^* \rangle \quad - \Phi^2) = 0$$

From what is given, $\int_N |\nabla_f v|^2 - \langle \rho_f(v), v \rangle = 0$

Hence it suffices that $\int_N \frac{1}{\varphi^2} (\langle w, w^* \rangle - \Phi^2) + \langle \rho_f(v), v \rangle = 0$

To finish the theorem we need this fact:

<u>Lemma 5.4.5</u> If $f: N \to M$ is a conformal diffeomorphism of compact oriented 2-manifolds, with $f^* g_M = \varphi^2 g_N$, then for any field v along f we have

$$\int_N \frac{1}{\varphi^2} (\langle w, w^* \rangle \quad - \Phi^2) + \langle \rho_f(v), v \rangle = 0$$

where w, w^*, Φ are as defined in the proof of Theorem 5.4.4.

<u>Remark 5.4.6</u> This formula is a generalization of one of Yano [BY, p.50], which is valid for $f = id_N$ (but with dim (N) arbitrary). In his proof, the integrand is expressed as the difference of two divergences. The proof here is a suitable modification of this technique. The dimension restriction comes of course from the requirement that a conformal map be harmonic (and hence preserve the divergence theorem).

<u>Proof</u> Given v, define fields u, t on M by

$$u = v \circ f^{-1}$$

$$t = \nabla u(u)$$

Then $t \circ f$ is a field along f satisfying

$$\begin{aligned} \mathrm{Div}_f t &= \langle \nabla(\nabla u(u)) \circ df, df \rangle \\ &= \varphi^2 \langle (\nabla^2 u(-,u) + \nabla u \circ \nabla u, I \rangle \end{aligned}$$

where I is the identity map on TM. Note that here (and throughout) reference to composition with f is suppressed when possible. Continuing,

$$\mathrm{Div}_f\ t = \varphi^2 \langle \nabla^2 u(u,-) - R_M(-,u)u + \nabla_u \circ \nabla_u, I \rangle$$

$$= \varphi^2(\langle \nabla^2 u(u,-) + \nabla u \circ \nabla u,\ I\rangle + \langle \rho_M(u), u\rangle)$$

using the identity $\langle R(a,b)c,d\rangle = -\langle R(a,b)d,c\rangle$. Define a second field along f by

$$s = u\,\mathrm{Div}(u) \circ f$$

Hence

$$\begin{aligned}\mathrm{Div}_f s &= \langle \nabla_f(u\mathrm{Div}(u)), df\rangle \\ &= \mathrm{Div}(u)\langle \nabla_f u, df\rangle + \langle u\, d\mathrm{Div}(u), df\rangle\end{aligned}$$

However

$$\begin{aligned}d\mathrm{Div}(u) &= d(\langle \nabla u, I\rangle \circ f) = d\langle \nabla u, I\rangle \circ df \\ &= \langle \nabla^2 u(df,-), I\rangle\end{aligned}$$

since I is parallel. Hence

$$\begin{aligned}\langle u\, d\mathrm{Div}(u), df\rangle &= \sum_i \varphi^2\langle u\langle \nabla^2 u(a_i,-), I\rangle, a_i\rangle \\ &= \varphi^2 \langle \nabla^2 u(\sum_i \langle u, a_i\rangle a_i, -), I\rangle \\ &= \varphi^2\langle \nabla^2 u(u,-), I\rangle\end{aligned}$$

where (a_i) is an orthonormal basis for $T_{f(x)}M$. Finally, since

$\Phi = \langle \nabla_f(u \circ f), df\rangle = \varphi^2\ \mathrm{Div}(u)$, it follows that

$$\mathrm{Div}_f s - \mathrm{Div}_f t = \frac{\Phi^2}{\varphi^2} - \varphi^2(\langle \nabla u \circ \nabla u, I\rangle + \langle \rho_M(u), u\rangle)$$

As $\langle \rho_f(v), v\rangle = \varphi^2\langle \rho_M(u), u\rangle$, to prove the lemma it suffices that

$$\frac{1}{\varphi^2}\langle w, w^*\rangle = \varphi^2\langle \nabla u \circ \nabla u, I\rangle$$

But suppose that at a particular point x the linear map $\nabla_f v$ is represented by the matrix v^i_j with respect to orthonormal bases e_i and $\varphi^{-1} df(e_i)$ at x and f(x). Then

$$\begin{aligned}\langle w, w^*\rangle &= \sum_{ij} \langle \nabla_f v(e_j), df(e_i)\rangle \langle \nabla_f v(e_i), df(e_j)\rangle \\ &= \varphi^2 \sum_{ij} v^i_j\ v^j_i\end{aligned}$$

Also

$$\begin{aligned}\langle \nabla u \circ \nabla u, I\rangle &= \frac{1}{\varphi^2} \sum_i \langle \nabla_f v \circ df^{-1} \circ \nabla_f v \circ df^{-1}(df(e_i)),\ df(e_i)\rangle \\ &= \frac{1}{\varphi^2}\sum_{ij} v^i_j\ v^j_i\end{aligned}$$

This finishes the proof of the lemma, and also Theorem 5.4.4.

Example 5.4.7 The projective group of conformal transformations of S^2 is a critical manifold of index 0. The theorem says that it is a generate critical manifold, in the sense that every vector in the null space of the Hessian is tangent to the manifold.

In general, the theory of harmonic maps seems to present no pathologies in two dimensions.

Section 5. Maps of Spheres

An earlier chapter contains constructions of essential harmonic maps between spheres. It would therefore be nice to be able to compute the index of these maps; in particular, the maps of degree k from S^n to S^n should be studied. This is because they are the simplest and since one might look for an analogy with geodesics on S^n. Our results, however, are unfortunately inconclusive for two reasons: a) there is not sufficient explicit knowledge about the maps themselves; b) the computations required are incredibly cumbersome.

Nevertheless, let us at least summarize what can be done. Recall that a harmonic suspension of a polynomial map g takes the form

$$(x,y) \xrightarrow{f} (\sin \alpha(t)\ g(x/|x|),\ \cos \alpha(t)(y/|y|)$$
$$t = \log(|x|/|y|)$$

Experience with the identity map on S^n (and geodesics) indicates a possible choice for an "energy-decreasing vector field". Namely, let $w = \operatorname{grad} y_p$, where y_p is a coordinate function, and let $v = w \circ f$. In higher dimensions we expect that

$\langle J_f v, v\rangle < 0$ for all such fields v. (More generally, if h is a harmonic polynomial on the domain sphere of homogeneity less than that of g, there should be a similar relation for fields of the form hv.) Although v is not an eigenfunction of J_f, it can be shown that $\langle J_f v, v\rangle < 0$ holds at every point, provided that the dimension of the y variables is ≥ 5.

<u>Conclusion</u> If $\Sigma^k g$ is the k^{th} harmonic suspension of a polynomial map, as constructed in Chapter 3, then index $(\Sigma^k g) \geq k$ for $k = 5,6$.

We remark that if our knowledge of the behavior of the function α in the construction was more complete, the same statement should be verifiable for $k \geq 3$. Without further estimates on α, however, there is no hope of improving these results.

A more pleasant exercise is that of generalizing some Morse theory of geodesics on spheres to the case of harmonic polynomial maps. Precisely, suppose $\gamma: S^1 \to S^n$ is a geodesic which wraps S^1 around a great circle k-times. Then γ may be thought of as the composition of $z \to z^k$ on S^1 with an inclusion. The question for us is then: what can be said about the index and nullity if $z \to z^k$ is replaced by a general harmonic polynomial map?

Therefore let $f: S^n \to S^m$ be a harmonic polynomial map of homogeneity k. Then by geodesic inclusion we have a map $S^n \to S^{m+r}$ for all $r \geq 0$. Let w be a gradient field on S^{m+r} corresponding to one of the last r coordinate functions, and let g be a harmonic polynomial of homogeneity ℓ on S^n. Then $x \to g(x)\, w(f(x))$ is a field along f, and as $\nabla_f w = 0$ we find

$$\Delta_f(gw) = (-\Delta_{S^n} g)\, w = -\lambda_\ell gw$$

Also $\rho_f(w) = \Sigma\, R\, (df(e_i), w)\; df(e_i)$

However, in a space of constant sectional curvature c, we have [KN, p. 203]

$$R(X,Y)Z = c(\langle Z,X\rangle Y - \langle Z,Y\rangle X)$$

so that as $c = 1$ on S^{n+r} and $df(e_i) \perp w$,

$$\rho_f(w) = |df|^2 w = \lambda_k w$$

Here we have used the comments in Example 1.1.4. Hence

$$J_f(gw) = (\lambda_\ell - \lambda_k)\; gw$$

so that
$$\text{index } (f) \geq r(\lambda(\lambda_k) + 1)$$
$$\text{null } (f) \geq r\; m(\lambda_k)$$

where the latter quantities are as defined in 5.2.9 - 10. For the case of geodesics $f\colon S^1 \to S^{1+r}$, it is easy to see we have equality in both places, and that null $(f) = \text{null}_k(f)$. Here $m(\lambda_k) = 2$, and $\lambda(\lambda_k) = 2(k-1)$ for $k \geq 1$.

In general, things are not so simple. If $f\colon S^n \to S^n$ is the identity, for $n \geq 3$, then the above bound on the index (for $r = 0$) is not helpful. Furthermore, it is easy to construct examples where the nullity is not all Killing nullity. Suppose that (in the above notation) degree $g = k$, so that gw is a Jacobi field. As $w \perp df$, gw cannot be included in $df(\underline{i}(S^n))$. On the other hand, if there are $x, y \in S^n$ such that $f(x) = f(y)$ but $g(x) \neq g(y)$, then gw is not of the form $v \circ f$, where $v \in \underline{i}(S^{m+r})$. For example, one can take the Hopf map $S^3 \to S^2 \subset S^3$ and let g be any spherical harmonic of degree 2 on S^3 which is not invariant under the action of S^1. This seems to be another natural instance of a degenerate harmonic map.

Appendix

A Morse Lemma for Compact Operators

The general setting for infinite dimensional Morse theory is the following: A Hilbert manifold X and a smooth functional $f: X \to \mathbb{R}$ which satisfies Condition C and has non-degenerate critical points (or manifolds) [P]. Condition C roughly means that if you follow a gradient line of f you eventually reach a critical point. What we wish to examine here is the non-degeneracy condition for critical points; in particular we show that the usual condition can be relaxed when dealing with energy functionals.

Recall that for each critical point x of f there is a second variation operator $J_x : T_x X \to T_x X$ satisfying $\langle J_x v, u \rangle_x = d^2 f_x(v,u) \; \forall u,v \in T_x X$. We say that a critical point x is <u>non-degenerate</u> if J_x is an isomorphism. In infinite dimensions this condition is of course stronger than requiring that J_x be an injection, and in general is necessary to prove a Morse lemma. However, our conclusion is that the weaker condition is adequate for energy integrals.

<u>Definition A.1</u> A critical point $x \in X$ is <u>generate</u> if J_x is an injection.

<u>A.2</u> A critical manifold $G \subset X$ is generate if for each $x \in G$, J_x is an injection on $(T_x G)^{\perp}$.

<u>Remark A.3</u> Suppose f were the usual energy functional. If G is a generate critical manifold and $x \in G$, then x is generate in the sense of Chapter 5, Section 1. For if $J_x v = 0$, then $v \in T_x G$; i.e., v arises from a variation of x through harmonic maps. Conversely, suppose we were given a critical manifold G, each of whose points was generate in the sense that every solution v of $J_x v = 0$ arose from a variation of x through

elements of G. Then G would be a generate critical manifold.

We have given examples of such critical manifolds: $SO(n+1)$ for $n \geq 3$, and the projective group of conformal transformations of S^2.

In our examples, the operator J_x is generally smoothing, hence compact, so that ordinary non-degeneracy is out of the question.

We first need some terminology from the theory of manifolds of maps: if N and M are smooth finite dimensional Riemannian manifolds, with N compact, then $H^k(N,M)$ consists roughly of all maps $f:N \to M$ for which the k^{th} order energy is defined:

$$E_k(f) = \sum_{j=0}^{k-1} \int_N |\nabla^j df|^2$$

If $k > \dim N/2$, $H^k(N,M)$ is a Hilbert manifold and its elements are all continuous functions. Eliasson showed that E_k is a smooth function on H^k satisfying Condition C [E2]. Clearly E_j is then a smooth function on H^k for $1 \leq j \leq k$. The critical points of E_j are called polyharmonic maps; note that they are just the harmonic maps if $j = 1$. For our purposes, an energy functional will be one of the E_j, but could be considered more generally [E2]. We remark that it is unlikely that E_j will satisfy Condition C on H^k, for $j < k$.

We recall also that if $g \in C^\infty(N,M)$, then there is a natural chart about g for $H^k(N,M)$ in which the model space is $H^k(g^*TM)$. The latter is the Hilbert space of sections of g^*TM obtained by completing the smooth sections with respect to the inner product

$$\langle u,v \rangle_k = \sum_{j=0}^{k-1} \int_N \langle \nabla^j u, \nabla^j v \rangle$$

Note that we can also consider $\langle\ \rangle_0$ as a dual pairing $H^k \times H^{-k} \to \mathbb{R}$.

The natural chart mentioned above is given by the exponential map of M. The final fact we need is that in this chart the energy E_j takes the form

$$E_j\ (\xi) = |\nabla^j \xi|_o^2 + \int_N Q(\xi) \qquad \forall\, \xi \in H^k(g^* TM)$$

Here Q is a polynomial differential operator in lower derivatives; for details see [E2]. Our conclusion is the following, quite analogous to that given in Palais [P].

<u>Morse Lemma</u> Let x be a generate critical point of $f = E_j : H^k(N,M) \to \mathbb{R}$. Then about x there is a chart in which f takes the form

$$f(y) = f_o + |BP_1 y|^2 - |CP_2 y|^2$$

Here the norm is that of the model space $H^k = H^k(x^* TM)$, P_1, P_2 are complementary orthogonal projections of H^k, and B,C are linear injections on the relevant subspaces. B and C are compact if $j < k$, and isomorphisms of their domains otherwise.

This lemma is sufficient to do local Morse theory; for example, one can easily imitate the necessary handlebody construction in Palais [P]. Alternatively, it is immediate from the Lemma that a generate critical point is non-degenerate in the sense of K. Uhlenbeck [U1]; she has a general handlebody construction in this case.

<u>Corollary A.4</u> Generate critical points of energy functionals are isolated.

<u>Proof</u> Using the lemma, we verify that if $df(y) = 0$ then $y = 0$. For if $df(y)z = 0\ \forall z$ then

$$\langle BP_1 y,\ BP_1 z \rangle - \langle CP_2 y,\ CP_2 z \rangle = 0 \qquad \forall\, z$$

Putting $z = P_1 y$ and $P_2 y$ in turn, we get $P_1 y = P_2 y = 0$, as B and C are injections. Hence $y = 0$.

Proof of Morse Lemma:

We begin working in the natural chart mentioned earlier. Note that since

$$f(y) = |\nabla^j y|_0^2 + \int_N Q(y)$$

we have

$$d^2f(y)\ (u,v) = \langle \nabla^j u, \nabla^j v\rangle_0 + \int_N d^2Q(y)(u,v)$$

$$= \langle A(y)u,v\rangle_0$$

where $A(y)$ is a self-adjoint linear elliptic operator of order $2j$. Note also that we can write

$$\langle \alpha,\beta\rangle_k = \langle D\alpha,\beta\rangle_0 \qquad \forall\ \alpha,\beta\in H^k(x^*TM)$$

where D is another self-adjoint elliptic operator, of order $2k$; by construction, $D:H^r \to H^{r-2k}$ is an isomorphism $\forall r$. Finally, for the second variation operator itself, we have by definition

$$d^2f(y)(u,v) = \langle J(y)u,\ v\rangle_k$$

$$= \langle DJ(y)u,v\rangle_0$$

Therefore $J(y) = D^{-1}A(y)\colon H^k \to H^{k+2(k-j)}$ is a Fredholm map of index 0 for all y. On a neighborhood of x we may assume J is an injection, and hence an isomorphism.

From this point we can parallel the proof in Palais. Near x we have by Taylor's theorem

$$f(y) = f_0 + \langle T(y)y,y\rangle_k$$

where T is obtained by integrating J. Hence we can also assume that $T(y)\colon H^k \to H^{k+2(k-j)}$ is an $\cong$. Therefore define $R(y) : H^k \cong H^k$ by $R(y) = T(y)^{-1}T_0$ and let $S(y) = R(y)^{-\frac{1}{2}}$, which is a smooth operation near the identity. Then $T(y)R(y) = T_0$ and taking H^k adjoints yields $R^*(y)T(y) = T_0$; hence $S(y)^*T(y) = T(y)S(y)$. Palais shows that $\varphi(y) = S(y)y$ is a diffeomorphism near x; we also have

$$f(y) = f_o + \langle T_o\varphi(y),\ \varphi(y)\rangle_k$$

Hence with respect to our new chart φ^{-1} we get

$$f(v) = f_o + \langle T_o v, v\rangle_k$$

Now let $H^k = H_1^k \oplus H_2^k$ be the decomposition of H^k into positive and negative eigenspaces of T_o, with projections P_1 and P_2. (Note that T_o is a self adjoint operator, which is an $\cong$ if $j = k$ and compact if $j < k$). If we put $B = (T_o|H_1^k)^{\frac{1}{2}}$ and $C = (-T_o|H_2^k)^{\frac{1}{2}}$, the conclusion of the lemma follows.

Remark A5 There should be a generalization of the lemma to generate critical manifolds. The principal difficulty is that we needed a very special chart of $H^k(N,M)$ in order to apply elliptic operator theory to a local form of the energy. In general it might not be possible to find a chart which both straightened out a submanifold and left the energy in a reasonable form.

REFERENCES

[A] Adem, J. On Nonsingular Bilinear Maps, The Steenrod Algebra and its Applications, Springer Notes 168,1970.

[ABS] Atiyah, M., Bott, R. and Shapiro, A., Clifford Modules Topology 3, Supplement 1 (1964), 3-38.

[B] Berger, M. et al., Le Spectre d'une Variete Riemannienne, Springer Notes 194, 1971.

[BY] Bochner, S. and Yano, K. Curvature and Betti Numbers, Annals of Math. Studies, 1953.

[BS] Borel, A. and Serre, J-P., Groupes de Lie et Puissances Reduites de Steenrod, Am. J.Math. 75(1953), 409-448.

[CL] Coddington, E. and Levinson, N. Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

[D] De Rham, Georges, Varietes Differentiables, Hermann,1968.

[DW] Do Carmo, M. and Wallach, N.R., Minimal Immersions of Spheres into Spheres, Annals of Math. 93 (1971),43-62.

[ES] Eells, J. and Sampson, J.H., Harmonic Mappings of Riemannian Manifolds, Am. J. Math. 86(1964), 109-160.

[E1] Eliasson, H., On the Geometry of Manifolds of Maps, J. Diff. Geom. 1(1967), 169-194.

[E2] Eliasson, H., Variational Integrals in Fibre Bundles, AMS Proc. Symposia in Pure Math XVI, 67-89.

[F] Fomenko, A.T., Some Cases When Elements of Homotopy Groups of Homogeneous Spaces are realized by Totally Geodesic Spheres, Soviet Math Dokl. 11(1970), 219-222.

[H] Hartman, P., On Homotopic Harmonic Maps, Can. J.Math, 19 (1967), 673-687.

[HE] Helgason, S., Differential Geometry and Symmetric Spaces, Academic Press, 1962.

[HR] Hermann, R., A Sufficient Condition that a Map of Riemannian Manifolds be a Fibre Bundle, Proc. AMS, 11(1969), 236-242.

[HL] Hsiang, W.-Y. and Lawson, B., Minimal Submanifolds of Low Cohomogeneity, J. Diff. Geom., 5(1971), 1-38.

[HU] Husemoller, D., Fibre Bundles, McGraw-Hill, 1966.

[KN] Kobayashi, S. and Nomizu, K., Foundations of Differential Geometry, Vol. I, Interscience, 1969.

[L1] Lichnerowicz, A., Geometrie des Groupes de Transformations Dunod, 1958.

[L2] Lichnerowicz, A., Applications Harmoniques et Varietes Kahleriennes, Symposia Mathematica (III), 341-402.

[M] Milnor, J., Morse Theory, Annals of Math. Studies 51, 1970.

[N1] Nagano, T., On the Minimum Eigenvalues of the Laplacians in Riemannian Manifolds, Sci. Papers College Gen. Educ. Univ. Tokyo 11 (1961), 177-182.

[N2] Nagano, T., On Conformal Transformations of Riemannian Spaces, J. Math Soc. Japan, 10(1958), 79-93.

[P] Palais, R.S., Morse Theory on Hilbert Manifolds, Topology 2 (1963), 299-340.

[S] Steenrod, N., Topology of Fibre Bundles, Princeton Univ. Press, 1951.

[T] Toda, H., Composition Methods in Homotopy Groups of Spheres, Annals of Math. Studies 49, Princeton Univ. Press, 1962.

[U1] Uhlenbeck, K., Morse Theory on Banach Manifolds, Bull. AMS 76 (1970), 105-106.

[U2] Uhlenbeck, K., Harmonic Maps: A Direct Method in the Calculus of Variations, Bull. AMS 76 (1970), 1082-1087.

[V] Vranceanu, G., Sur les Vecteurs Tangents aux Spheres, Rev. Roum. Math., 10 (1965), 895-914.

[WG] Wong, Y.C., Differential Geometry of Grassmann Manifolds, Proc. Nat. Acad. Sci., 57 (1967), 589-594.

[WD] Wood, R., Polynomial Maps from Spheres to Spheres, Inventiones Math. 5 (1968), 163 - 168.

[YN] Yano, K. and Nagano, T., On Geodesic Vector Fields in a Compact Orientable Riemannian Space, Comment. Math. Helv. 35 (1961), 55-64.