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THE BEST CONSTANT OF L^p SOBOLEV INEQUALITY CORRESPONDING TO DIRICHLET-NEUMANN BOUNDARY VALUE PROBLEM

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Abstract. We have obtained the best constant of the following L^p Sobolev inequality

$$\sup_{0\le y\le 1}\left|u^{(j)}(y)\right| \le C\left(\int_0^1 \left|u^{(M)}(x)\right|^p dx\right)^{1/p},$$

where u is a function satisfying $u^{(M)} \in L^p(0,1)$, $u^{(2i)}(0) = 0$ $(0 \le i \le [(M-1)/2])$ and $u^{(2i+1)}(1) = 0$ $(0 \le i \le [(M-2)/2])$, where $u^{(i)}$ is the abbreviation of $(d/dx)^i u(x)$. In [9], the best constant of the above inequality was obtained for the case of $p = 2$ and $j = 0$. This paper extends the result of [9] under the conditions $p > 1$ and $0 \le j \le M-1$. The best constant is expressed by Bernoulli polynomials.

1. Introduction

For $M = 1, 2, 3, \cdots$, let us consider the following 1-dim Sobolev inequality:

$$\sup_{0\le y\le 1}|u(y)| \le C\left(\int_0^1 \left|u^{(M)}(x)\right|^2 dx\right)^{1/2},$$

where u is an element of Sobolev-Hilbert space $H(\mathrm{X}, M) = \{u | u^{(M)} \in L^2(0,1), u \text{ satisfies } A(\mathrm{X})\}$. Here the condition $A(\mathrm{X})$ assumes

$$\begin{array}{lll}
A(\mathrm{P}) & : & u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \qquad \int_0^1 u(x)dx = 0, \\
A(\mathrm{AP}) & : & u^{(i)}(1) + u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \\
A(\mathrm{C}) & : & u^{(i)}(0) = u^{(i)}(1) = 0 \quad (0 \le i \le M-1), \\
A(\mathrm{D}) & : & u^{(2i)}(0) = u^{(2i)}(1) = 0 \quad (0 \le i \le [\,(M-1)/2\,]), \\
A(\mathrm{N}) & : & u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0 \quad (0 \le i \le [\,(M-2)/2\,]), \\
 & & \int_0^1 u(x)dx = 0, \\
A(\mathrm{DN}) & : & u^{(2i)}(0) = 0 \quad (0 \le i \le [\,(M-1)/2\,]), \\
 & & u^{(2i+1)}(1) = 0 \quad (0 \le i \le [\,(M-2)/2\,]),
\end{array}$$

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Boundary condition of Sobolev space	$p = 2$	$1 < p < \infty$ (general case)
P (Periodic)	[10]	[3]
AP (Anti Periodic)	[10]	—
C (Clamped)	[8]	$M = 1, 2, 3$ [7]
D (Dirichlet)	[10]	$M = 2m$ [4], $M = 1, 3, 5$ [5]
N (Neumann)	[10]	[6]
DN (Dirichlet-Neumann)	[10]	this paper

TABLE 1. Various boundary conditions and best constants.

etc., and $u^{(i)}$ denotes i-th derivative in a distributional sense. It should be noted that if $M = 1$ the boundary conditions for u in $A(\mathrm{N})$ and for u on $x = 1$ in $A(\mathrm{DN})$ are not required. In our previous work, we obtained the best constant of the following L^p Sobolev inequality (1.1) in some boundary conditions:

$$\sup_{0 \le y \le 1} |u(y)| \le C \left(\int_0^1 \left| u^{(M)}(x) \right|^p dx \right)^{1/p}, \tag{1.1}$$

where u is an element of $W(\mathrm{X}, M, p) = \{u | u^{(M)} \in L^p(0,1), u \text{ satisfies } A(\mathrm{X})\}$. From this table, we see that the difficulty in obtaining the best constant seems to increase in the case of $p \ne 2$. Here, we would like to stress that each result in the case of $p \ne 2$ was obtained through a different method. The unified approach (maximizing the diagonal value of reproducing kernels; see [8, 10]) as in the case of $p = 2$ does not exist in the case of $p \ne 2$.

This paper studies the best constant of the following j-th L^p Sobolev inequality:

$$\sup_{0 \le y \le 1} \left| u^{(j)}(y) \right| \le C \left(\int_0^1 \left| u^{(M)}(x) \right|^p dx \right)^{1/p}, \tag{1.2}$$

where $u \in W(\mathrm{DN}, M, p)$ for any fixed $p > 1$ and $0 \le j \le M - 1$. To see that the inequality (1.2) itself is valid, we express u as follows; see [1, Th.VIII.2]

$$u^{(M-1)}(y) = \begin{cases} \displaystyle\int_0^y u^{(M)}(x) dx & (M = 2n - 1), \\ \displaystyle -\int_y^1 u^{(M)}(x) dx & (M = 2n), \end{cases}$$

where the boundary conditions $u^{(M-1)}(0) = 0$ $(M = 2n-1)$ and $u^{(M-1)}(1) = 0$ $(M = 2n)$ are used. Applying Hölder inequality, we obtain (1.2) for $j = M - 1$, and similar argument leads (1.2). Now, to state the result, let us introduce Bernoulli polynomials $b_j(x)$ defined by

$$\begin{cases} b_0(x)=1, \\ b_j'(x)=b_{j-1}(x), \quad \int_0^1 b_j(x)dx=0 \qquad (j=1,2,3,\cdots). \end{cases}$$

Hence, we have

$$b_1(x)=x-\frac{1}{2},\ b_2(x)=\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12},\ b_3(x)=\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12},\ \cdots .$$

The main result is as follows:

Theorem 1.1. *Let $M,m,n=1,2,3,\cdots$ be integers, $j=0,1,\cdots,M-1$ and $l=0,1,2,\cdots$. Then, the best constant $C(M,j,q)$ of (1.2) is given by*

$$(1.3)\quad C(M,j,q)=$$
$$\begin{cases} 2^{2(M-j)-1}\left(\int_0^1 \left|(-1)^{M+1}b_{M-j}\left(\frac{1-x}{4}\right)+b_{M-j}\left(\frac{1+x}{4}\right)\right|^q dx\right)^{1/q} \\ (j=2l), \\ 2^{2(M-j)-1}\left(\int_0^1 \left|b_{M-j}\left(\frac{x}{4}\right)+(-1)^M b_{M-j}\left(\frac{2-x}{4}\right)\right|^q dx\right)^{1/q} \\ (j=2l+1), \end{cases}$$

where q satisfies $1/p+1/q=1$. Moreover, the equality of (1.2) attained by

$$(1.4)\quad u(x)=$$
$$\begin{cases} (-1)^l\int_0^1 \partial_z G(n;x,z)\Big(\partial_z G(n-l;z,1)\Big)^{q-1}dz \\ (M=2n-1,\ j=2l), \\ (-1)^l\int_0^1 \partial_z G(n;x,z)\Big(\partial_y\partial_z G(n-l;z,y)|_{y=0}\Big)^{q-1}dz \\ (M=2n-1,\ j=2l+1), \\ (-1)^l\int_0^1 G(n;x,z)\Big(G(n-l;z,1)\Big)^{q-1}dz \\ (M=2n,\ j=2l), \\ (-1)^l\int_0^1 G(n;x,z)\Big(\partial_y G(n-l;z,y)|_{y=0}\Big)^{q-1}dz \\ (M=2n,\ j=2l+1), \end{cases}$$

where

$$(1.5)\quad G(m;x,y)=(-1)^{m+1}4^{2m-1}\left[b_{2m}\left(\frac{|x-y|}{4}\right)-b_{2m}\left(\frac{x+y}{4}\right)+\right.$$

$$b_{2m}\left(\frac{1}{2}-\frac{x+y}{4}\right)-b_{2m}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\ \Big]$$

is Green function of the following Drichlet-Neumann boundary value problem

$$\text{BVP}\,(\text{DN}, m)$$
$$\begin{cases} (-1)^m u^{(2m)} = f(x) & (0 < x < 1), \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0 & (0 \le i \le m-1). \end{cases}$$

The following are concrete forms of $C(M, j, q)$ for small value of M.

$M\backslash j$	0	1	2	3
1	1	-	-	-
2	$(q+1)^{-1/q}$	1	-	-
3	$32\pi^{\frac{1}{2q}}\left(\frac{2^{-6q-1}\Gamma(q+1)}{\Gamma\left(q+\frac{3}{2}\right)}\right)^{\frac{1}{q}}$	$(q+1)^{-1/q}$	1	-
4	$\frac{1}{2}\left(\frac{{}_2F_1\left(-q,\frac{q+1}{2};\frac{q+3}{2};\frac{1}{3}\right)}{q+1}\right)^{\frac{1}{q}}$	$32\pi^{\frac{3}{2q}}\left(-\frac{2^{-6q-1}\csc(\pi q)}{\Gamma(-q)\Gamma\left(q+\frac{3}{2}\right)}\right)^{\frac{1}{q}}$	$(q+1)^{-1/q}$	1

TABLE 2. $C(M, j, q)$ for $M = 1, 2, 3, 4$, where ${}_2F_1$ is Gaussian hypergeometric function; see [2, Section 5.5].

2. LEMMAS

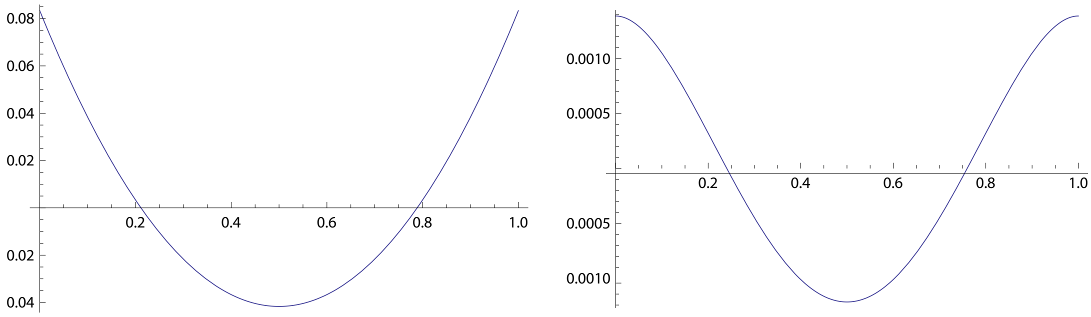
In this section, lemmas necessary for proving Theorem 1.1 are enumerated. We assume that $M, m, n = 1, 2, 3, \cdots$, $j = 0, 1, \cdots, M-1$ and $l = 0, 1, 2, \cdots$. First, we prepare the lemma concerning the properties of Bernoulli polynomials.

Lemma 2.1. $u(x) = (-1)^{m+1} b_{2m}(x)$ *satisfies the following properties:*

$$\max_{0\le x\le 1} u(x) = u(0) = u(1) > 0, \qquad \min_{0\le x\le 1} u(x) = u(1/2) < 0,$$
$$u'(x) < 0 \quad (0 < x < 1/2) \quad\ > 0 \quad (1/2 < x < 1),$$
$$\max_{0\le x\le 1} |\,u(x)\,| = u(0) = u(1).$$

Proof. See; [2, Section 9.5]. Figure 1 shows the graphs of u for $m = 1$ and 2. In fact, we show the case of $m = 1$. For $u(x) = b_2(x) = x^2/2 - x/2 + 1/12$, we have $u(0) = u(1) = 1/12$ and $u(1/2) = -1/24$. Since $u'(x) = b_1(x) = x - 1/2 \ < 0$ $(0 < x < 1/2)$, > 0 $(1/2 < x < 1)$, we have $\max_{0\le x\le 1} |u(x)| = u(0) = u(1)$. □

Next, we introduce lemmas for $G(m; x, y)$.



(i) (ii)

FIGURE 1. The graphs of (i) $u(x) = b_2(x)$ $(m = 1)$ and (ii) $u(x) = -b_4(x)$ $(m = 2)$.

Lemma 2.2. *For any bounded continuous function* $f(x)$ $(0 < x < 1)$, BVP(DN,m) *has a unique classical solution* $u(x)$ $(0 < x < 1)$ *expressed as*

$$u(x) = \int_0^1 G(m; x, y) f(y) dy,$$

where $G(m; x, y)$ $(0 < x, y < 1)$ *is given by* (1.5).

Proof. See; Yamagishi [9, Theorem 3.1]. □

Lemma 2.3. *For any* $u \in W(\mathrm{DN}, m, p)$ *and for any fixed* y $(0 \le y \le 1)$, *the following reproducing relation holds:*

$$u(y) = \int_0^1 u^{(m)}(x) \partial_x^m G(m; x, y) dx. \tag{2.1}$$

Proof. See; Yamagishi [9, Theorem 5.1] (although the proof of Theorem 5.1 is written in the case of $u \in H(\mathrm{DN}, m)$, it still applies to our case of $u \in W(\mathrm{DN}, m, p)$ without modification). □

Lemma 2.4. *The following relations hold in* $0 < x, y < 1$ *and* $x \neq y$.

(1) $\partial_x^2 G(m; x, y) = \partial_y^2 G(m; x, y) = -G(m-1; x, y)$.

(2) $\partial_y^j \partial_x^M G(M; x, y) =$

$$\begin{cases} (-1)^{n-1+l}\, \partial_x G(n-l; x, y) & (M = 2n-1, \quad j = 2l), \\ (-1)^{n-1+l}\, \partial_y \partial_x G(n-l; x, y) & (M = 2n-1, \quad j = 2l+1), \\ (-1)^{n+l}\, G(n-l; x, y) & (M = 2n, \quad j = 2l), \\ (-1)^{n+l}\, \partial_y G(n-l; x, y) & (M = 2n, \quad j = 2l+1). \end{cases}$$

Proof. Differentiating $G(m; x, y)$ with respect to x twice, we have

(2.2) $\partial_x G(m; x, y) =$

$$(-1)^{m+1}4^{2m-2}\left[\operatorname{sgn}(x-y)\,b_{2m-1}\left(\frac{|x-y|}{4}\right)-b_{2m-1}\left(\frac{x+y}{4}\right)-\right.$$
$$\left.b_{2m-1}\left(\frac{1}{2}-\frac{x+y}{4}\right)+\operatorname{sgn}(x-y)\,b_{2m-1}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\ \right],$$
$$\partial_x^2 G(m;x,y)=$$
$$-(-1)^m 4^{2(m-1)-1}\left[\,b_{2(m-1)}\left(\frac{|x-y|}{4}\right)-b_{2(m-1)}\left(\frac{x+y}{4}\right)+\right.$$
$$\left.b_{2(m-1)}\left(\frac{1}{2}-\frac{x+y}{4}\right)-b_{2(m-1)}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\ \right]=$$
$$-\ G(m-1;x,y).$$

Since $G(m,x,y)=G(m;y,x)$, we have $\partial_y^2 G(m;x,y)=\partial_y^2 G(m;y,x)=-G(m-1;y,x)=-G(m-1;x,y)$. So, (1) is obtained. Using (1), we have the following relation:

$$\partial_y^j\partial_x^M G(M;x,y)=$$
$$\begin{cases}\partial_y^{2l}\partial_x^{2(n-1)+1}G(2n-1;x,y)=\\ (-1)^{n-1+l}\,\partial_x G(2n-1-(n-1)-l;x,y),\\[2mm] \partial_y^{2l+1}\partial_x^{2(n-1)+1}G(2n-1;x,y)=\\ (-1)^{n-1+l}\,\partial_y\partial_x G(2n-1-(n-1)-l;x,y),\\[2mm] \partial_y^{2l}\partial_x^{2n}G(2n;x,y)=(-1)^{n+l}\,G(2n-n-l;x,y),\\[2mm] \partial_y^{2l+1}\partial_x^{2n}G(2n;x,y)=(-1)^{n+l}\,\partial_y G(2n-n-l;x,y).\end{cases}$$

This shows (2). □

Lemma 2.5. *The following relations hold.*

$$G(m;x,y)\ >\ 0\qquad(0<x,y<1).$$
$$\partial_x G(m;x,y),\quad \partial_y G(m;x,y),\quad \partial_y\partial_x G(m;x,y)\ >\ 0$$
$$(0<x,y<1,\quad x\neq y).$$
$$\partial_x^2 G(m;x,y),\quad \partial_y^2 G(m;x,y)\ <\ 0\qquad(0<x,y<1,\quad x\neq y).$$

Proof. Considering Lemma 2.1 and

$$0\le\frac{|x-y|}{4}<\frac{x+y}{4}<\frac{1}{2},\quad 0<\frac{1}{2}-\frac{x+y}{4}<\frac{1}{2}-\frac{|x-y|}{4}\le\frac{1}{2},$$

we have

$$(-1)^{m+1}\left[\,b_{2m}\left(\frac{|x-y|}{4}\right)-b_{2m}\left(\frac{x+y}{4}\right)\ \right]\ >\ 0,$$

$$(-1)^{m+1}\left[\, b_{2m}\left(\frac{1}{2}-\frac{x+y}{4}\right)-b_{2m}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\,\right] > 0.$$

Hence we have $G(m;x,y)>0$. Moreover, from Lemma 2.4 (1), we have

$$\partial_x^2 G(m;x,y)=\partial_y^2 G(m;x,y)=-G(m-1;x,y)<0 \quad (0<x,y<1, \quad x\neq y). \tag{2.3}$$

For any fixed y $(0\le y\le 1)$, since $\partial_x G(m;x,y)\big|_{x=1}=0$ and (2.3), we have $\partial_x G(m;x,y)>0$ $(0<x<1,\ x\neq y)$. Similarly, for any fixed x $(0\le x\le 1)$, since $\partial_y G(m;x,y)\big|_{y=1}=0$ and (2.3), we have $\partial_y G(m;x,y)>0$ $(0<y<1,\ x\neq y)$. From (2.2) and Lemma 2.4 (2), we have

$$\begin{aligned}\partial_y\partial_x G(m;x,y)&= \\ &(-1)^{m+1}4^{2m-3}\left[\,-b_{2m-2}\left(\frac{|x-y|}{4}\right)-b_{2m-2}\left(\frac{x+y}{4}\right)+\right. \\ &\left. b_{2m-2}\left(\frac{1}{2}-\frac{x+y}{4}\right)+b_{2m-2}\left(\frac{1}{2}-\frac{|x-y|}{4}\right)\,\right], \\ \partial_y^2\partial_x G(m;x,y)&=-\,\partial_x G(m-1;x,y)<0 \quad (0<x,y<1,\ x\neq y).\end{aligned} \tag{2.4}$$

For any fixed x $(0\le x\le 1)$, from $\partial_y\partial_x G(m;x,y)\big|_{y=1}=0$ and (2.4), we have $\partial_y\partial_x G(m;x,y)>0$ $(0<y<1,\ x\neq y)$. □

Using these lemmas, we can prove the following lemma.

Lemma 2.6. *Let us define*

$$g(M,j;y)=\int_0^1 \left|\partial_y^j\partial_x^M G(M;x,y)\right|^q dx \qquad (0\le y\le 1).$$

Then, it holds that

$$\max_{0\le y\le 1} g(M,j;y)=\begin{cases} g(M,j;1) & (j=2l), \\ g(M,j;0) & (j=2l+1).\end{cases}$$

Proof. By Lemma 2.4 (2) and 2.5, we obtain:

$$g(2n-1,2l;y)=\int_0^1 \Big(\partial_x G(n-l;x,y)\Big)^q dx > 0,$$

$$g'(2n-1,2l;y)=$$

$$q\int_0^1 \Big(\partial_x G(n-l;x,y)\Big)^{q-1} dx\, \partial_y\partial_x G(n-l;x,y) > 0,$$

$$g(2n-1,2l+1;y)=\int_0^1 \Big(\partial_y\partial_x G(n-l;x,y)\Big)^q dx > 0,$$

$$g'(2n-1,2l+1;y)=$$

$$-q\int_0^1 \Big(\partial_y\partial_x G(n-l;x,y)\Big)^{q-1} dx\,\partial_x G(n-l-1;x,y) \,<\, 0,$$

$$g(2n,2l;y) = \int_0^1 \Big(G(n-l;x,y)\Big)^{q} dx \,>\, 0,$$

$$g'(2n,2l;y) = q\int_0^1 \Big(G(n-l;x,y)\Big)^{q-1} dx\,\partial_y G(n-l;x,y) \,>\, 0,$$

$$g(2n,2l+1;y) = \int_0^1 \Big(\partial_y G(n-l;x,y)\Big)^{q} dx \,>\, 0,$$

$$g'(2n,2l+1;y) =$$

$$-q\int_0^1 \Big(\partial_y G(n-l;x,y)\Big)^{q-1} dx\,G(n-l-1;x,y) \,<\, 0,$$

where g' stands for the derivative with respect to y. Thus, if j is even, $g(M,j,y)$ takes its maximum at $y=1$, else at $y=0$. □

Next lemma is necessary for the construction of the function, which attains the best constant. Note the difference from BVP(DN,m) (it is the odd order differential equation and boundary conditions at $x=1$ are slightly different).

Lemma 2.7. *For any bounded continuous function $f(x)$ $(0<x<1)$, the following boundary value problem:*

BVP (DN', m)

$$\begin{cases} (-1)^{m-1}u^{(2m-1)} = f(x) & (0<x<1), \\ u^{(2i)}(0) = 0 & (0\le i\le m-1), \\ u^{(2i+1)}(1) = 0 & (0\le i\le m-2), \end{cases}$$

has a unique classical solution $u(x)$ $(0<x<1)$ expressed as

$$u(x) = \int_0^1 \partial_y G(m;x,y) f(y)dy. \tag{2.5}$$

Proof. Integrating the equation $f(y) = (-1)^{m-1}u^{(2m-1)}(y)$ with respect to y $(0<y<x)$ and noting $u^{(2m-2)}(0)=0$, we have

$$\int_0^x f(y)dy = (-1)^{m-1}\int_0^x u^{(2m-1)}(y)dy = (-1)^{m-1}u^{(2(m-1))}(x).$$

Since u satisfies $u^{(2i)}(0) = u^{(2i+1)}(1) = 0$ for $(0\le i\le m-2)$, u is a solution of BVP(DN,$m-1$). Thus, from Lemma 2.2, we have the solution formula

$$u(x) = \int_0^1 G(m-1;x,z)\int_0^z f(y)dydz =$$

$$\int_0^1 \int_y^1 G(m-1;x,z)dzf(y)dy.$$

Using Lemma 2.4 (1), we have $-\partial_z^2 G(m;x,z) = G(m-1;x,z)$. So, we obtain

$$u(x) = \int_0^1 \int_y^1 \Big(-\partial_z^2 G(m;x,z)\Big) dzf(y)dy = \int_0^1 \partial_y G(m;x,y)f(y)dy,$$

where we use $\partial_z G(m;x,z)\big|_{z=1} = 0$. The uniqueness easily follows from the expression (2.5). $\square$

3. Proof of Theorem 1.1

Proof of Theorem 1.1 Let $u \in W(\mathrm{DN}, M, p)$. Differentiating (2.1) in Lemma 2.3 with respect to y $(0 \le y \le 1)$, j $(j = 0, 1, \cdots, M-1)$ times, we obtain

$$u^{(j)}(y) = \int_0^1 u^{(M)}(x)\, \partial_y^j \partial_x^M G(M;x,y)dx.$$

Applying Hölder inequality, we have

$$\Big| u^{(j)}(y) \Big| \le \left(\int_0^1 \big| \, \partial_y^j \partial_x^M \, G(M;x,y) \,\big|^q \, dx \right)^{1/q} \left(\int_0^1 \Big| u^{(M)}(x) \Big|^p \, dx \right)^{1/p}.$$

Taking the supremum of the above relation and using Lemma 2.6, we have

$$\sup_{0 \le y \le 1} \Big| u^{(j)}(y) \Big| \le \tag{3.1}$$
$$\begin{cases} \left(\int_0^1 \Big| \partial_y^j \partial_x^M \, G(M;x,y)\big|_{y=1} \Big|^q \, dx \right)^{1/q} \left(\int_0^1 \Big| u^{(M)}(x) \Big|^p \, dx \right)^{1/p} \\ (j = 2l), \\ \left(\int_0^1 \Big| \partial_y^j \partial_x^M \, G(M;x,y)\big|_{y=0} \Big|^q \, dx \right)^{1/q} \left(\int_0^1 \Big| u^{(M)}(x) \Big|^p \, dx \right)^{1/p} \\ (j = 2l+1). \end{cases}$$

The equality holds if u satisfies

$$u^{(M)}(x) = \tag{3.2}$$
$$\begin{cases} \mathrm{sgn}\Big(\partial_y^j \partial_x^M G(M;x,y)\big|_{y=1} \Big) \Big| \partial_y^j \partial_x^M G(M;x,y)\big|_{y=1} \Big|^{q-1} & (j = 2l), \\ \mathrm{sgn}\Big(\partial_y^j \partial_x^M G(M;x,y)\big|_{y=0} \Big) \Big| \partial_y^j \partial_x^M G(M;x,y)\big|_{y=0} \Big|^{q-1} & (j = 2l+1). \end{cases}$$

Therefore, if the equality holds, the best constant is

$$C(M,j,q) = \tag{3.3}$$
$$\begin{cases} \left(\int_0^1 \left| \partial_y^j \partial_x^M \, G(M;x,y)\big|_{y=1} \right|^q dx \right)^{1/q} & (j=2l), \\ \left(\int_0^1 \left| \partial_y^j \partial_x^M \, G(M;x,y)\big|_{y=0} \right|^q dx \right)^{1/q} & (j=2l+1). \end{cases}$$

The concrete expression of (3.3) is (1.3). Finally, we see that the equality of (3.1) is attained by (1.4). From the equation (3.2) and Lemma 2.4 (2), we have

$$u^{(2n-1)}(x) =$$
$$(-1)^{n-1+l} \begin{cases} \left(\partial_x G(n-l;x,y)\big|_{y=1} \right)^{q-1} & (M=2n-1,\ j=2l), \\ \left(\partial_y \partial_x G(n-l;x,y)|_{y=0} \right)^{q-1} & (M=2n-1,\ j=2l+1), \end{cases}$$
$$u^{(2n)}(x) =$$
$$(-1)^{n+l} \begin{cases} \left(G(n-l;x,y)\big|_{y=1} \right)^{q-1} & (M=2n,\ j=2l), \\ \left(\partial_y G(n-l;x,y)|_{y=0} \right)^{q-1} & (M=2n,\ j=2l+1). \end{cases}$$

Since u satisfies boundary condition A(DN), we have the following boundary value problems:

$$\begin{cases} (-1)^{n-1} u^{(2n-1)} = \\ (-1)^l \begin{cases} \left(\partial_x G(n-l;x,y)\big|_{y=1} \right)^{q-1} & (M=2n-1,\ j=2l), \\ \left(\partial_y \partial_x G(n-l;x,y)|_{y=0} \right)^{q-1} & (M=2n-1,\ j=2l+1), \end{cases} \\ u^{(2i)}(0) = 0 \ (0 \le i \le n-1), \quad u^{(2i+1)}(1) = 0 \ (0 \le i \le n-2), \end{cases} \tag{3.4}$$

$$\begin{cases} (-1)^{n} u^{(2n)} = \\ (-1)^l \begin{cases} \left(G(n-l;x,y)\big|_{y=1} \right)^{q-1} & (M=2n,\ j=2l), \\ \left(\partial_y G(n-l;x,y)|_{y=0} \right)^{q-1} & (M=2n,\ j=2l+1), \end{cases} \\ u^{(2i)}(0) = u^{(2i+1)}(1) = 0 \quad (0 \le i \le n-1). \end{cases} \tag{3.5}$$

Thus u in (3.4) is the solution of BVP(DN$'$,n) and u in (3.5) is the solution of BVP(DN,n). So, by Lemma 2.7 and Lemma 2.2, we have (1.4). □

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