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GROWTH OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Abdallah El Farissi and Benharrat Belaïdi

Abstract. This paper is devoted to studying the growth of solutions of the higher order nonhomogeneous linear differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+...+A_2f''+\left(D_1(z)+A_1(z)e^{P(z)}\right)f'$$

$$+\left(D_0(z)+A_0(z)e^{Q(z)}\right)f=F\ \ (k\geqslant 2),$$

where $P(z)$, $Q(z)$ are nonconstant polynomials such that $\deg P=\deg Q=n$ and $A_j(z)$ $(j=0,1,...,k-1)$, $F(z)$ are entire functions with $\max\{\rho(A_j)\ (j=0,1,...,k-1), \rho(D_j)\ (j=0,1)\}<n$. We also investigate the relationship between small functions and the solutions of the above equation.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory and the basic notions of the Wiman-Valiron as well (see [12, 13, 15]). In addition, we will use $\lambda(f)$ $(\lambda_2(f))$ and $\overline{\lambda}(f)$ $(\overline{\lambda}_2(f))$ to denote respectively the exponents (hyper-exponents) of convergence of the zero-sequence and the sequence of distinct zeros of f, $\rho(f)$ to denote the order of growth of a meromorphic function f and $\rho_2(f)$ to denote the hyper-order of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi)=o(T(r,f))$ as $r\to+\infty$ except possibly a set of r of finite linear measure, where $T(r, f)$ is the Nevanlinna characteristic function of f. If f is of infinite order and φ is of finite order, then clearly that $\varphi(z)$ is a small function with respect to $f(z)$. We also define

$$\overline{\lambda}(f-\varphi)=\limsup_{r\to+\infty}\frac{\log\overline{N}\left(r,\frac{1}{f-\varphi}\right)}{\log r}$$

and

$$\overline{\lambda}_2(f-\varphi)=\limsup_{r\to+\infty}\frac{\log\log\overline{N}\left(r,\frac{1}{f-\varphi}\right)}{\log r}$$

for any meromorphic function $\varphi(z)$.

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For the second order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of equation (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [8], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But equation (1.1) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [9], Ozawa [16], Amemiya-Ozawa [1] and Gundersen [10], Langley [14] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order.
In 2002, Z. X. Chen [6] considered the question: What conditions on $B(z)$ when $\rho(B) = 1$ will guarantee that every nontrivial solution of (1.1) has infinite order? He proved the following results, which improved results of Frei, Amemiya-Ozawa, Ozawa, Langley and Gundersen.

Theorem 1.1. *([6]) Let $A_j(z)\,(\not\equiv 0)$ $(j = 0, 1)$ and $D_j(z)$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j)\ (j = 0, 1), \rho(D_j)\ (j = 0, 1)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + (D_1(z) + A_1(z)e^{az})f' + \left(D_0(z) + A_0(z)e^{bz}\right)f = 0 \tag{1.2}$$

is of infinite order.

Setting $D_j \equiv 0$ $(j = 0, 1)$ in Theorem 1.1, we obtain the following result.

Theorem 1.2. *Let $A_j(z)\,(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\max\{\rho(A_j) : j = 0, 1\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. Then every solution $f \not\equiv 0$ of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \tag{1.3}$$

is of infinite order.

Theorem 1.3. *([6]) Let $A_j(z)\,(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\rho(A_j) < 1$ $(j = 0, 1)$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every solution $f \not\equiv 0$ of equation (1.3) is of infinite order.*

Very recently in [18], H. Y. Xu and T. B. Cao have investigated the growth of solutions of some higher order nonhomogeneous linear differential equations and have obtained the following result.

Theorem 1.4. *([18]) Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, ..., n)$ *are complex numbers,* $a_n b_n (a_n - b_n) \neq 0$. *Suppose that* $h_i(z)\,(2 \leqslant i \leqslant k-1)$ *are polynomials of degree no more* $n-1$ *in* z, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ *and* $H(z) \not\equiv 0$ *are entire functions with* $\max\{\rho(A_j)\ (j = 0, 1), \rho(H)\} < n$, *and* $\varphi(z)$ *is an entire function of finite order. Then every nontrivial solution* f *of the equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + ... + h_2 f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = H \tag{1.4}$$

satisfies $\rho(f) = \lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = \infty$ *and* $\rho_2(f) = \lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) \leqslant n$.

Remark 1. In the original statement of Theorem 1.4 (see [18]), the condition $H \not\equiv 0$ must be added. Indeed, if $H \equiv 0$, then the conclusions of Theorem 1.4 are false. For example the equation $f''' - f'' - 2e^z f' - e^{3z} f = 0$ possesses the solution $f(z) = e^{e^z}$ with $\rho(f) = \infty$ and $\lambda(f) = 0$.

It is natural to ask whether the polynomials $h_{k-1}(z), ..., h_2(z)$ in (1.4) can be replaced by entire functions of orders that are less than n. The main purpose of this paper is to study the growth and the oscillation of solutions of the linear differential equation

$$\begin{aligned} f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_2 f'' + \left(D_1(z) + A_1(z) e^{P(z)}\right) f' \\ + \left(D_0(z) + A_0(z) e^{Q(z)}\right) f = F \ \ (k \geqslant 2). \end{aligned} \tag{1.5}$$

We obtain the following results.

Theorem 1.5. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, ..., n)$ *are complex numbers,* $a_n b_n (a_n - b_n) \neq 0$. *Suppose that* $A_j(z)$ $(j = 0, 1, ..., k-1)$, $A_j(z) \not\equiv 0$ $(j = 0, 1)$, $D_j(z)$ $(j = 0, 1)$ *and* $F(z)$ *are entire functions with* $\max\{\rho(A_j)\ (j = 0, 1, ..., k-1), \rho(D_j)\ (j = 0, 1), \rho(F)\} < n$ *and let* $\varphi(z) \not\equiv 0$ *be an entire function of finite order. Then every solution* $f \not\equiv 0$ *of equation (1.5) satisfies*

$$\overline{\lambda}(f - \varphi) = \rho(f) = \infty, \ \overline{\lambda}_2(f - \varphi) = \rho_2(f) \leqslant n. \tag{1.6}$$

Furthermore if $F \not\equiv 0$, *then every solution* f *of equation (1.5) satisfies*

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty \tag{1.7}$$

and

$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f-\varphi) = \rho_2(f) \leqslant n. \tag{1.8}$$

Remark 2. The proof of Theorem 1.5 in which every solution f of equation (1.5) has infinite order is quite different from that in the proof of Theorem 1.4 (see [18]). The main ingredient in the proof is Lemma 2.9.

Remark 3. In [18], H. Y. Xu and T. B. Cao studied equation (1.5) and obtained the same result as in Theorem 1.5 but under restriction that the complex constants a_n, b_n satisfy $a_n b_n \neq 0$, $a_n b_n < 0$ and $A_j(z)$ $(j = 2, ..., k-1)$ are polynomials of degree no more $n-1$ in z.

Setting $D_j \equiv 0$ $(j = 0, 1)$ in Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0, 1, ..., n)$ *are complex numbers,* $a_n b_n (a_n - b_n) \neq 0$. *Suppose that* $A_j(z)$ $(j = 0, 1, ..., k-1)$, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ *and* $F(z)$ *are entire functions with* $\max\{\rho(A_j)\ (j = 0, 1, ..., k-1), \rho(F)\} < n$ *and let* $\varphi(z) \not\equiv 0$ *be an entire function of finite order. Then every solution* $f \not\equiv 0$ *of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_2 f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = F \quad (k \geqslant 2) \tag{1.9}$$

satisfies (1.6). Furthermore if $F \not\equiv 0$, *then every solution* f *of equation (1.9) satisfies (1.7) and (1.8).*

Remark 4. If $\rho(F) \geqslant n$, then equation (1.5) can possesses solution of finite order. For instance the equation

$$f''' - f'' + \left(e^{-z^n} - e^{z^{n-1}}\right) f' + e^{z^n} f = e^{z^n}$$

satisfies $\rho(F) = \rho\left(e^{z^n}\right) = n$ and has a finite order solution $f(z) = 1$.

Theorem 1.7. *Let* $P(z), Q(z), A_j(z)$ $(j = 0, 1, ..., k-1)$, $D_j(z)$ $(j = 0, 1)$ *and* $\varphi(z)$ *satisfy the additional hypotheses of Theorem 1.5, and let* $F(z)$ *be an entire function such that* $\rho(F) \geqslant n$. *Then every solution* f *of equation (1.5) satisfies (1.7) and (1.8) with at most one finite order solution* f_0. *For the exceptional solution* f_0 *we have, if* $\rho(F) > n$, *then* $\rho(f_0) = \rho(F)$ *and if* $\rho(F) = n$, *then* $\rho(f_0) \leqslant n$.

Corollary 1.8. *Let* $P(z)$, $Q(z)$, $A_j(z)$ $(j = 0, 1)$, $D_j(z)$ $(j = 0, 1)$ *and* $\varphi(z)$ *satisfy the additional hypotheses of Theorem 1.5, and let* $F(z)$ *be an entire function. Then the following statements hold:*

(i) If $\rho(F) < n$, then every solution $f \not\equiv 0$ of the equation

$$f'' + \left(D_1(z) + A_1(z)e^{P(z)}\right)f' + \left(D_0(z) + A_0(z)e^{Q(z)}\right)f = F \tag{1.10}$$

has infinite order and satisfies (1.6). Furthermore if $F \not\equiv 0$, then every solution f of equation (1.10) satisfies (1.7) and (1.8).
(ii) If $\rho(F) = n$, then every solution f of equation (1.10) has infinite order and satisfies (1.7) and (1.8), with at most one finite order solution f_0 satisfying $\rho(f_0) \leqslant n$.
(iii) If $\rho(F) > n$, then every solution f of equation (1.10) has infinite order and satisfies (1.7) and (1.8), with at most one finite order solution f_0 satisfying $\rho(f_0) = \rho(F)$.

2. Preliminary lemmas

Our proofs depend mainly upon the following lemmas. Before starting these lemmas, we recall the concept of the logarithmic density of subsets of $(1, +\infty)$. For $E \subset (1, +\infty)$, we define the logarithmic measure of a set E by

$$lm(E) = \int_1^{+\infty} \frac{\chi_E(t)}{t} dt,$$

where χ_E is the characteristic function of E. The upper logarithmic density and the lower logarithmic density of E are defined by

$$\overline{\log dens}(E) = \limsup_{r\to+\infty} \frac{lm(E \cap [1, r])}{\log r}$$

and

$$\underline{\log dens}(E) = \liminf_{r\to+\infty} \frac{lm(E \cap [1, r])}{\log r}.$$

Lemma 2.1. *([11]) Let f be a transcendental meromorphic function of finite order ρ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, ..., m$ and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) - E_1$, then there is a constant $R_1 = R_1(\psi) > 1$ such that for all z satisfying* $\arg z = \psi$ *and $|z| \geqslant R_1$ and for all $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2. *([6]) Let $P(z) = a_n z^n + ... + a_0$, $(a_n = \alpha + i\beta \neq 0)$ be a polynomial with degree $n \geqslant 1$ and $A(z)$ $(\not\equiv 0)$ be an entire function with $\rho(A) < n$.*

Set $f(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset [0,2\pi)$ *that has linear measure zero, such that if* $\theta \in [0,2\pi)\setminus(E_2 \cup E_3)$, *where* $E_3 = \{\theta \in [0,2\pi) : \delta(P,\theta) = 0\}$ *is a finite set, then for sufficiently large* $|z| = r$, *we have*
(i) If $\delta(P,\theta) > 0$, *then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} \leqslant |f(z)| \leqslant \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}. \tag{2.2}$$

(ii) If $\delta(P,\theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \leqslant |f(z)| \leqslant \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}. \tag{2.3}$$

Lemma 2.3. *([5]) Let* $A_0, A_1, ..., A_{k-1}$, $F \not\equiv 0$ *be finite order meromorphic functions. If* f *is a meromorphic solution with* $\rho(f) = +\infty$ *of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + ... + A_1 f' + A_0 f = F, \tag{2.4}$$

then $\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$.

Lemma 2.4. *([2]) Let* $A_0, A_1, ..., A_{k-1}$, $F \not\equiv 0$ *be finite order meromorphic functions. If* f *is a meromorphic solution with* $\rho(f) = \infty$ *and* $\rho_2(f) = \rho$ *of equation (2.4), then* $\overline{\lambda}(f) = \lambda(f) = \rho(f) = \infty$ *and* $\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho$.

Lemma 2.5. *([3]) Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0,1,...,n)$ *are complex numbers,* $a_n b_n \neq 0$ *such that* $\arg a_n \neq \arg b_n$ *or* $a_n = cb_n$ $(0 < c < 1)$. *We denote index sets by*

$$\begin{aligned} \Lambda_1 &= \{0,P\}, \\ \Lambda_2 &= \{0,P,Q,2P,P+Q\}. \end{aligned}$$

(i) If H_j $(j \in \Lambda_1)$ *and* $H_Q \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, *setting* $\Psi_1(z) = \sum_{j\in\Lambda_1} H_j(z)e^j$, *then* $\Psi_1(z) + H_Q e^Q \not\equiv 0$.
(ii) If H_j $(j \in \Lambda_2)$ *and* $H_{2Q} \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, *setting* $\Psi_2(z) = \sum_{j\in\Lambda_2} H_j(z)e^j$, *then* $\Psi_2(z) + H_{2Q}e^{2Q} \not\equiv 0$.

Lemma 2.6. *([4]) Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *and* $Q(z) = \sum_{i=0}^{n} b_i z^i$ *be nonconstant polynomials where* a_i, b_i $(i = 0,1,...,n)$ *are complex numbers,* $a_n b_n \neq 0$ *such that* $a_n = cb_n$ $(c > 1)$. *We denote index set by*

$$\Lambda_3 = \{0,Q\}.$$

If H_j $(j \in \Lambda_3)$ *and* $H_P \not\equiv 0$ *are all meromorphic functions of orders that are less than* n, *setting* $\Psi_3(z) = \sum_{j\in\Lambda_3} H_j(z)e^j$, *then* $\Psi_3(z) + H_P e^P \not\equiv 0$.

Lemma 2.7. *([7]) Let $f(z)$ be a transcendental entire function. Then there is a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z with $|z| = r \notin [0,1] \cup E_4$ at which $|f(z)| = M(r,f)$, we have*

$$\left| \frac{f(z)}{f^{(s)}(z)} \right| \leqslant 2r^s \quad (s \in \mathbf{N}). \tag{2.5}$$

Lemma 2.8. *([17]) Let $f(z)$ and $g(z)$ be two nonconstant entire functions with $\rho(g) < \rho(f) < +\infty$. Given $0 < 4\varepsilon < \rho(f) - \rho(g)$ and $0 < \delta < \frac{1}{8}$, there exists a set E_5 with $\overline{\log dens}(E_5) > 0$ such that*

$$\left| \frac{g(z)}{f(z)} \right| \leqslant \exp\left\{ -r^{\rho(f)-2\varepsilon} \right\} \tag{2.6}$$

for all z such that $|z| = r \in E_5$ is sufficiently large and that $|f(z)| \geqslant M(r,f)\, \upsilon_f(r)^{\delta - \frac{1}{8}}$.

Lemma 2.9. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ and $Q(z) = \sum_{i=0}^{n} b_i z^i$ be nonconstant polynomials where a_i, b_i $(i = 0, 1, ..., n)$ are complex numbers, $a_n b_n (a_n - b_n) \neq 0$. Suppose that $A_j(z)$ $(j = 0, 1, ..., k-1)$, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $D_j(z)$ $(j = 0, 1)$ are entire functions with $\max\{\rho(A_j)\,(j = 0, 1, ..., k-1), \rho(D_j)\,(j = 0, 1)\} < n$. We denote*

$$L_f = f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_2 f'' + \left(D_1(z) + A_1(z) e^{P(z)} \right) f'$$

$$+ \left(D_0(z) + A_0(z) e^{Q(z)} \right) f. \tag{2.7}$$

If $f \not\equiv 0$ is a finite order entire function, then we have

$$\rho(L_f) = \max\{\rho(f), n\}.$$

Proof. Let $f \not\equiv 0$ be a finite order entire function. First, if $f(z) \equiv C \neq 0$, then

$$L_f = \left(D_0(z) + A_0(z) e^{Q(z)} \right) C.$$

Hence $\rho(L_f) = n$ and Lemma 2.9 holds.
We suppose $f \not\equiv C$. Then, by (2.7), we have $\rho(L_f) \leqslant \max\{n, \rho(f)\}$.
(i) If $\rho(f) = \rho < n$, then $\rho(L_f) \leqslant n$. Suppose that $\rho(L_f) < n$. By (2.7), we have

$$f^{(k)} + A_{k-1} f^{(k-1)} + ... + A_2 f'' + \left(D_1(z) + A_1(z) e^{P(z)} \right) f'$$

$$+ \left(D_0(z) + A_0(z) e^{Q(z)} \right) f - L_f = 0$$

has the form of

$$\Psi_1(z)+H_Q e^{Q(z)} = f^{(k)}+A_{k-1}f^{(k-1)}+\dots+A_2 f''+D_1(z)f' +D_0(z)f-L_f+A_1(z)f'e^{P(z)}+A_0(z)fe^{Q(z)}=0 \tag{2.8}$$

or

$$\Psi_3(z)+H_P e^{P(z)} = f^{(k)}+A_{k-1}f^{(k-1)}+\dots+A_2 f''+D_1(z)f'+D_0(z)f -L_f+A_0(z)fe^{Q(z)}+A_1(z)f'e^{P(z)}=0 \tag{2.9}$$

and from (2.8) and (2.9) we obtain a contradiction by Lemma 2.5 (i) or Lemma 2.6. Then $\rho(L_f)=n$.
(ii) If $\rho(f)=\rho\geqslant n$, then $\rho(L_f)\leqslant\rho(f)$. Suppose that $\rho(L_f)<\rho(f)$. We can rewrite (2.7) as

$$\frac{L_f}{f}=\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\dots+A_2\frac{f''}{f}+\left(D_1(z)+A_1(z)e^{P(z)}\right)\frac{f'}{f} +D_0(z)+A_0(z)e^{Q(z)}. \tag{2.10}$$

We divide the proof on three cases.

Case 1. Suppose first that $\arg a_n\neq\arg b_n$. Set

$$\max\{\rho(A_j)\ \ (j=0,1,\dots,k-1),\rho(D_j)\,(j=0,1)\}=\beta<n.$$

Then, for any given ε $\left(0<\varepsilon<\min\left(n-\beta,\frac{\rho(f)-\rho(L_f)}{4}\right)\right)$, we have for sufficiently large r

$$\begin{aligned}|D_j(z)|&\leqslant\exp\left\{r^{\beta+\varepsilon}\right\}\quad(j=0,1),\\|A_j(z)|&\leqslant\exp\left\{r^{\beta+\varepsilon}\right\}\quad(j=0,1,\dots,k-1).\end{aligned} \tag{2.11}$$

By Lemma 2.8, we know that there exists a set E_5 with $\overline{\log dens}(E_5)>0$ such that

$$\left|\frac{L_f}{f}\right|\leqslant\exp\left\{-r^{\rho(f)-2\varepsilon}\right\}\leqslant 1 \tag{2.12}$$

for all z such that $|z|=r\in E_5$ is sufficiently large and that $|f(z)|\geqslant M(r,f)\,v_f(r)^{\delta-\frac{1}{8}}$. Also, by Lemma 2.1, for the above ε, there exists a set $E_1\subset[0,2\pi)$ that has linear measure zero, such that if $\theta\in[0,2\pi)-E_1$, then there is a constant $R_1=R_1(\theta)>1$ such that for all z satisfying $\arg z=\theta$ and $|z|\geqslant R_1$, we have

$$\left|\frac{f^{(i)}(z)}{f(z)}\right|\leqslant|z|^{i(\rho-1+\varepsilon)}\quad(i=1,\dots,k). \tag{2.13}$$

By Lemma 2.2, there exists a ray $\arg z = \theta \in [0, 2\pi) \backslash E_1 \cup E_2 \cup E_3$, $E_3 = \{\theta \in [0, 2\pi) : \delta(P(z), \theta) = 0 \text{ or } \delta(Q(z), \theta) = 0\} \subset [0, 2\pi)$, $E_1 \cup E_2$ having linear measure zero, E_3 being a finite set, such that

$$\delta(P(z), \theta) < 0, \ \delta(Q(z), \theta) > 0$$

and for any given ε $\left(0 < \varepsilon < \min\left(n - \beta, \frac{\rho(f) - \rho\left(L_f\right)}{4}\right)\right)$, by (2.11), (2.13), we have for sufficiently large $|z| = r$

$$\left|A_0 e^{Q(z)}\right| \geqslant \exp\left\{(1-\varepsilon)\,\delta(Q(z), \theta)\, r^n\right\}, \tag{2.14}$$

$$\begin{aligned} &\left|\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_2\frac{f''}{f} + D_0(z)\right| \\ &\leqslant \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right| + \dots + |A_2|\left|\frac{f''}{f}\right| + |D_0(z)| \\ &\leqslant r^{k(\rho-1+\varepsilon)} + r^{(k-1)(\rho-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\} \\ &+ \cdots + r^{2(\rho-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\} + \exp\left\{r^{\beta+\varepsilon}\right\} \\ &\leqslant k r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\left|\left(D_1(z) + A_1(z)e^{P(z)}\right)\frac{f'}{f}\right| \\ &\leqslant r^{\rho-1+\varepsilon}\left(\exp\left\{(1-\varepsilon)\,\delta(P(z), \theta)\, r^n\right\} + \exp\left\{r^{\beta+\varepsilon}\right\}\right) \\ &\leqslant r^{\rho-1+\varepsilon}\left(1 + \exp\left\{r^{\beta+\varepsilon}\right\}\right). \end{aligned} \tag{2.16}$$

By (2.10), (2.12) and (2.14)-(2.16), we have

$$\exp\left\{(1-\varepsilon)\,\delta(Q(z), \theta)\, r^n\right\} \leqslant \left|A_0 e^{Q(z)}\right| \leqslant K r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\},$$

where $K > 0$ is some real constant. This is a contradiction by $\beta + \varepsilon < n$. Hence $\rho(L_f) = \rho(f)$.

Case 2. Suppose now $a_n = cb_n$ $(0 < c < 1)$. Then for any ray $\arg z = \theta$, we have

$$\delta(P(z), \theta) = c\delta(Q(z), \theta).$$

Then, by Lemma 2.2, for any given ε $(0 < \varepsilon < \min(\frac{1-c}{2(1+c)}, n-\beta, \frac{\rho(f)-\rho\left(L_f\right)}{4}))$, there exist $E_j \subset [0, 2\pi)$ $(j = 1, 2, 3)$ such that E_1, E_2 having linear measure zero and E_3 being a finite set, where E_1, E_2 and E_3 are defined as in the Case 1 respectively. We take the ray $\arg z = \theta \in [0, 2\pi) \setminus E_1 \cup E_2 \cup E_3$ such that $\delta(Q(z), \theta) > 0$ and for sufficiently large $|z| = r$, we have (2.14), (2.15) and

$$\left|\left(D_1(z) + A_1(z) e^{P(z)}\right) \frac{f'}{f}\right| \tag{2.17}$$

$$\leqslant r^{\rho-1+\varepsilon}(\exp\left\{r^{\beta+\varepsilon}\right\} + \exp\{(1+\varepsilon) c\delta(Q(z), \theta) r^n\}).$$

Thus by (2.10), (2.12), (2.14), (2.15) and (2.17) we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(Q(z),\theta) r^n\} \\ &\leqslant \left|A_0 e^{Q(z)}\right| \\ &\leqslant k r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} \\ &+ r^{\rho-1+\varepsilon}\left(\exp\left\{r^{\beta+\varepsilon}\right\} + \exp\{(1+\varepsilon) c\delta(Q(z),\theta) r^n\}\right) + 1 \\ &\leqslant (k+1) r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\} + r^{\rho-1+\varepsilon} \exp\{(1+\varepsilon) c\delta(Q(z),\theta) r^n\} + 1. \end{aligned} \tag{2.18}$$

By ε $\left(0 < \varepsilon < \min\left(\frac{1-c}{2(1+c)}, n-\beta, \frac{\rho(f)-\rho(L_f)}{4}\right)\right)$, we have as $r \to +\infty$

$$\frac{r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\beta+\varepsilon}\right\}}{\exp\{(1-\varepsilon)\delta(Q(z),\theta) r^n\}} \to 0, \tag{2.19}$$

$$\frac{r^{\rho-1+\varepsilon} \exp\{(1+\varepsilon) c\delta(Q(z),\theta) r^n\}}{\exp\{(1-\varepsilon)\delta(Q(z),\theta) r^n\}} \to 0, \tag{2.20}$$

$$\frac{1}{\exp\{(1-\varepsilon)\delta(Q(z),\theta) r^n\}} \to 0. \tag{2.21}$$

By (2.18)-(2.21), we get $1 \leqslant 0$. This is a contradiction. Hence $\rho(L_f) = \rho(f)$.

Case 3. Finally, we suppose $a_n = c b_n$ $(c > 1)$. We can rewrite (2.7) as

$$\frac{L_f}{f}\frac{f}{f'} = \frac{f^{(k)}}{f'} + A_{k-1}\frac{f^{(k-1)}}{f'} + \cdots + A_2\frac{f''}{f'} + \left(D_0(z) + A_0(z) e^{Q(z)}\right)\frac{f}{f'} \tag{2.22}$$

$$+ D_1(z) + A_1(z) e^{P(z)}.$$

By Lemma 2.7, there is a set $E_4 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z with $|z| = r \notin [0,1] \cup E_4$ at which $|f(z)| = M(r, f)$, we have

$$\left|\frac{f(z)}{f'(z)}\right| \leqslant 2r. \tag{2.23}$$

By Lemma 2.8, for ε $\left(0<\varepsilon<\min(\frac{c-1}{2(c+1)}, n-\beta, \frac{\rho(f)-\rho\left(L_f\right)}{4})\right)$, we know that there exists a set E_5 with $\overline{\log dens}\,(E_5)>0$ such that

$$\left|\frac{L_f}{f}\right| \leqslant \exp\left\{-r^{\rho(f)-2\varepsilon}\right\} \leqslant 1 \tag{2.24}$$

for all z such that $|z|=r\in E_5$ is sufficiently large and that $|f(z)| \geqslant M(r,f)\,v_f(r)^{\delta-\frac{1}{8}}$. Since $E_4\subset(1,+\infty)$ has finite logarithmic measure and E_5 satisfies $\overline{\log dens}(E_5)>0$, we have $\overline{\log dens}(E_5-([0,1]\cup E_4))>0$. By (2.23) and (2.24), we have for sufficiently large $|z|=r$

$$\left|\frac{L_f}{f'}\right| = \left|\frac{L_f}{f}\frac{f}{f'}\right| \leqslant 2r\exp\left\{-r^{\rho(f)-2\varepsilon}\right\} \leqslant 2r. \tag{2.25}$$

For any ray $\arg z=\theta$, we have

$$\delta\left(P(z),\theta\right)=c\delta\left(Q(z),\theta\right).$$

By Lemma 2.2, there exists a ray $\arg z=\theta\in[0,2\pi)\setminus E_1\cup E_2\cup E_3$, $E_3=\{\theta\in[0,2\pi):\delta\left(P(z),\theta\right)=0$ or $\delta\left(Q(z),\theta\right)=0\}\subset[0,2\pi)$, $E_1\cup E_2$ having linear measure zero, E_3 being a finite set, such that

$$\delta\left(P(z),\theta\right)=c\delta\left(Q(z),\theta\right)>0$$

and by (2.11), (2.13) and (2.23) for sufficiently large $|z|=r$, we have

$$\left|A_1e^{P(z)}\right| \geqslant \exp\left\{(1-\varepsilon)\,c\delta\left(Q(z),\theta\right)r^n\right\}, \tag{2.26}$$

$$\begin{aligned}&\left|\left(D_0(z)+A_0(z)\,e^{Q(z)}\right)\frac{f}{f'}\right| \\ &\leqslant 2r\exp\left\{r^{\beta+\varepsilon}\right\}+2r\exp\left\{(1+\varepsilon)\,\delta\left(Q(z),\theta\right)r^n\right\},\end{aligned} \tag{2.27}$$

$$\begin{aligned}&\left|\frac{f^{(k)}}{f'}+A_{k-1}\frac{f^{(k-1)}}{f'}+\cdots+A_2\frac{f''}{f'}+D_1\right| \\ &\leqslant \left|\frac{f(z)}{f'(z)}\right|\left|\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\cdots+A_2\frac{f''}{f}\right|+|D_1| \\ &\leqslant \left|\frac{f(z)}{f'(z)}\right|\left(\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_2|\left|\frac{f''}{f}\right|\right)+|D_1|\end{aligned} \tag{2.28}$$

$$\leqslant 2r\left(k-1\right)r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\beta+\varepsilon}\right\}+\exp\left\{r^{\beta+\varepsilon}\right\}$$
$$\leqslant 2kr^{k(\rho-1+\varepsilon)+1}\exp\left\{r^{\beta+\varepsilon}\right\}.$$

By (2.22), (2.25) and (2.26)-(2.28), we have

$$\begin{aligned}(2.29)\qquad &\exp\left\{(1-\varepsilon)\,c\delta\left(Q(z),\theta\right)r^{n}\right\}\\ &\leqslant \left|A_1e^{P(z)}\right|\\ &\leqslant 2kr^{k(\rho-1+\varepsilon)+1}\exp\left\{r^{\beta+\varepsilon}\right\}+2r\exp\left\{r^{\beta+\varepsilon}\right\}\\ &+2r\exp\left\{(1+\varepsilon)\,\delta\left(Q(z),\theta\right)r^{n}\right\}+2r\\ &\leqslant 2\left(k+1\right)r^{k(\rho-1+\varepsilon)+1}\exp\left\{r^{\beta+\varepsilon}\right\}\\ &+2r\exp\left\{(1+\varepsilon)\,\delta\left(Q(z),\theta\right)r^{n}\right\}+2r.\end{aligned}$$

By $\varepsilon\left(0<\varepsilon<\min(\frac{c-1}{2(c+1)},n-\beta,\frac{\rho(f)-\rho(L_f)}{4})\right)$, we have as $r\to+\infty$

$$(2.30)\qquad \frac{r^{k(\rho-1+\varepsilon)+1}\exp\left\{r^{\beta+\varepsilon}\right\}}{\exp\left\{(1-\varepsilon)\,c\delta\left(Q(z),\theta\right)r^{n}\right\}}\to 0,$$

$$(2.31)\qquad \frac{2r\exp\left\{(1+\varepsilon)\,\delta\left(Q(z),\theta\right)r^{n}\right\}}{\exp\left\{(1-\varepsilon)\,c\delta\left(Q(z),\theta\right)r^{n}\right\}}\to 0,$$

$$(2.32)\qquad \frac{2r}{\exp\left\{(1-\varepsilon)\,c\delta\left(Q(z),\theta\right)r^{n}\right\}}\to 0.$$

By (2.29)-(2.32), we get $1\leqslant 0$. This is a contradiction. Hence $\rho\left(L_f\right)=\rho\left(f\right).$ □

By using Wiman-Valiron theory [13] (see also [18]), we easily obtain the following result which we omit the proof.

Lemma 2.10. *Let $A_0\left(z\right),...,A_{k-1}\left(z\right)$, $F\left(z\right)$ be entire functions of finite order. If f is a solution of the equation*

$$(2.33)\qquad f^{(k)}+A_{k-1}\left(z\right)f^{(k-1)}+...+A_1\left(z\right)f^{'}+A_0\left(z\right)f=F,$$

then $\rho_2\left(f\right)\leqslant\max\left\{\rho\left(A_0\right),...,\rho\left(A_{k-1}\right),\rho\left(F\right)\right\}.$

3. **Proof of Theorem 1.5**

Assume that $f \not\equiv 0$ is a solution of equation (1.5). We prove that f is of infinite order. We suppose the contrary $\rho(f) < \infty$. By Lemma 2.9, we have $n \leqslant \rho(L_f) = \rho(F) < n$ and this is a contradiction. Hence, every solution of equation (1.5) is of infinite order and by Lemma 2.10, we have $\rho_2(f) \leqslant n$. Suppose that $\varphi(z) \not\equiv 0$ is an entire function of finite order. Set $g = f - \varphi$, then $f = g + \varphi$ and by $\rho(\varphi) < \infty$ we have $\rho(f) = \rho(g) = \infty$ and $\rho_2(f) = \rho_2(g) \leqslant n$. Thus, g is a solution of the equation

$$g^{(k)} + A_{k-1}g^{(k-1)} + ... + A_2 g'' + \left(D_1 + A_1 e^{P(z)}\right) g' + \left(D_0 + A_0 e^{Q(z)}\right) g = H,$$

where

$$H = F - (\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_2\varphi'' + \left(D_1 + A_1 e^{P}\right)\varphi' + \left(D_0 + A_0 e^{Q}\right)\varphi).$$

By $\varphi(z) \not\equiv 0$ and $\rho(\varphi) < \infty$ we have $H \not\equiv 0$. Since $\rho(H) < \infty$, then by Lemma 2.3 and Lemma 2.4, we get

$$\overline{\lambda}(f - \varphi) = \rho(f - \varphi) = \rho(f) = \infty, \; \overline{\lambda}_2(f - \varphi) = \rho_2(f - \varphi) = \rho_2(f) \leqslant n.$$

Furthermore if $F \not\equiv 0$, then by f is an infinite order solution of equation (1.5), Lemma 2.3 and Lemma 2.4, we have

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty,$$

$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) = \rho_2(f) \leqslant n.$$

4. **Proof of Theorem 1.7**

Assume that f_0 is a solution of (1.5) with $\rho(f_0) = \rho < \infty$. If f_1 is another finite order solution of (1.5), then $\rho(f_1 - f_0) < \infty$, and $f_1 - f_0$ is a solution of the corresponding homogeneous equation of (1.5), but $\rho(f_1 - f_0) = \infty$ from Theorem 1.5 , this is a contradiction. Hence (1.5) has at most one finite order solution f_0 and all other solutions f_1 of (1.5) are of infinite order and satisfy (1.7) and (1.8). If $\rho(F) > n$, suppose there exists f_0 a solution of (1.5) with $\rho(f_0) < \infty$, then, we have $\rho(f_0) > n$ and by Lemma 2.9 we get $\rho(L_f) = \rho(f_0) = \rho(F)$. Suppose that $\rho(F) = n$, if there exists f_0 a solution of (1.5) with $\rho(f_0) < \infty$, then $\rho(f_0) \leqslant n$. Indeed, if we suppose that $\rho(f_0) > n$, then by Lemma 2.9 we get $\rho(L_f) = \rho(f_0) = \rho(F) > n$ and this is a contradiction.

5. **Proof of Corollary 1.8**

By using Theorem 1.5 and Theorem 1.7, we obtain Corollary 1.8.

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Abdallah El Farissi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)

e-mail address: elfarissi.abdallah@yahoo.fr

Benharrat Belaïdi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)
e-mail address: belaidibenharrat@yahoo.fr; belaidi@univ-mosta.dz

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