Math. J. Okayama Univ. **56** (2014), 157–169

STUDY OF A PARABOLIC PROBLEM IN A CONICAL DOMAIN

Boubaker-Khaled Sadallah

Abstract. In this paper we consider the heat equation with Dirichlet boundary conditions in a conical domain. We look for a sufficient condition on the lateral surface of the cone in order to have the optimal regularity of the solution in an anisotropic Sobolev space when the right hand side of the equation is in a Lebesgue space.

1. Intoduction

Let φ be a locally Lipschitz function $\varphi : (0,T] \to \mathbb{R}^+$ where $T>0$, such that

$$\begin{array}{rcl} \varphi(0) & = & 0, \\ \varphi(t) & > & 0, \ t \in (0,T], \end{array}$$

and $\Omega \subset \mathbb{R}^3$ be the open subset of conical type :

$$\Omega = \left\{ (t,x,y) \in \mathbb{R}^3 : 0 < t < T, 0 \leq \sqrt{x^2+y^2} < \varphi(t) \right\}.$$

Consider in Ω the parabolic problem

$$\left\{ \begin{array}{c} \partial_t u - \partial_x^2 u - \partial_y^2 u = f \in L^2(\Omega), \\ u_{|\partial\Omega \setminus D(T,\varphi(T))} = 0, \end{array} \right. \tag{1.1}$$

where $D(s,r)$ denotes the disc of radius r centred at $(s,0,0)$, and $L^2(\Omega)$ is the usual Lebesgue space on Ω. We look for the solution u in the anisotropic Sobolev space

$$H^{1,2}(\Omega) = \left\{ u \in H^1(\Omega) : \partial_x^2 u \in L^2(\Omega), \partial_{xy} u \in L^2(\Omega), \partial_y^2 u \in L^2(\Omega) \right\},$$

here, $H^1(\Omega)$ stands for the Sobolev space defined by

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \partial_t u \in L^2(\Omega), \partial_x u \in L^2(\Omega), \partial_y u \in L^2(\Omega) \right\}.$$

The space $H^{1,2}(\Omega)$ is equipped with the natural norm, that is

$$\|u\|_{H^{1,2}} = \sqrt{\|u\|_{H^1}^2 + \|\partial_x^2 u\|_{L^2}^2 + \left\|\partial_{xy} u\right\|_{L^2}^2 + \left\|\partial_y^2 u\right\|_{L^2}^2}.$$

Mathematics Subject Classification. 35K05, 35K20.

Key words and phrases. Key Words and Phrases. Heat equation, Parabolic equation, Nonregular domain, Cone.

The belonging of the solution u to $H^{1,2}(\Omega)$ depends on the function φ. Our aim is to find a sufficient condition on φ, as weak as possible in order to obtain a solution $u \in H^{1,2}(\Omega)$. Our main result is

Theorem 1. *Assume that* $\varphi(0) = 0$ *and*

$$\lim_{t\to 0}\varphi(t)\varphi'(t) = 0$$

or

$$\varphi'(t) \geq 0 \text{ a.e. in a neighborhood of } 0.$$

Then, for all $f \in L^2(\Omega)$, *Problem (1) admits a (unique) solution* $u \in H^{1,2}(\Omega)$.

For example, if $\varphi(t) = at^\alpha$, Problem (1) admits a unique solution for any positive constants a and α. The same result holds true if $\varphi(t) = t^\alpha \ln t$, with $\alpha > \frac{1}{2}$.

Observe that the main difficulty is due to the condition $\varphi(0) = 0$ (instead of $\varphi(0) > 0$) which gives a conical point at $(0,0,0)$ and then, it does not allow to transform the cone Ω into a usual cylindrical domain. This type of problems has been studied by Sadallah [14] in one space dimension for the parabolic operator $\partial_t u + (-1)^m \partial_x^{2m} u$ with $m \geq 1$.

Alkhutov [1] has treated, in some weighted Sobolev L^p-spaces, the heat equation in bounded and unbounded domains of paraboloid type. He has considered in [2] the case of the heat equation in a ball; this case corresponds here to $\varphi(t) = \sqrt{t(2R-t)}$ in the neighborhood of 0 (for a ball of radius R centred at $(R,0,0)$). It is clear that this function satisfies the hypothesis of Theorem 1. In [15] Sadallah has obtained the optimal regularity of the solution for any disc $\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 \leq \sqrt{t^2+x^2} < R\right\}$.

In the case of one space variable, the heat equation in non Hilbretian spaces has been considered in some works, for instance in Labbas *et al.* [10] and [11]. Berroug *et al.* [3] have treated the same equation with a singular domain in Hölder spaces, whereas Kheloufi *et al.* [7] have studied the case when the domain is cylindrical, not with respect to the time variable, but with respect to one of space variables.

Some authors have considered the singularities which appear in the solutions when the domain is non cylindrical as Kozlov and Maz'ya in [8] and [9]. We can find in Nazarov [13] some results about the solution of Neumann problem in a conical domains. In Degtyarev [5] a more general parabolic equation has been studied in domains with a conical point.

The method used here is to approach the domain Ω for T small enough by a sequence of subsomains $(\Omega_n)_{n\in\mathbb{N}}$ in which we can solve Problem (1). More precisely, when T is any positive real number, we divide Ω into two parts : $\Omega_{T'}$, with T' small enough,

$$\Omega_{T'} = \{(t, x, y) \in \Omega : 0 < t < T'\}$$

and

$$\Omega_{T'T} = \{(t, x, y) \in \Omega : T' < t < T\}.$$

So, we obtain two solutions $u_1 \in H^{1,2}(\Omega_{T'})$ in $\Omega_{T'}$ and $u_2 \in H^{1,2}(\Omega_{T'T})$ in $\Omega_{T'T}$. Finally, we prove that the function u defined by

$$u := \begin{cases} u_1 & \text{in } \Omega_{T'} \\ u_2 & \text{in } \Omega_{T'T} \end{cases}$$

is the solution of Problem (1) and has the optimal regularity, that is $u \in H^{1,2}(\Omega)$.

It is not difficult to prove the uniqueness of the solution. Indeed, if we consider the inner product $(\partial_t u - \partial_x^2 u - \partial_y^2 u, u)_{L^2}$ with $\partial_t u - \partial_x^2 u - \partial_y^2 u = 0$ and take into account the boundary conditions, we obtain

$$\begin{aligned} 0 &= (\partial_t u - \partial_x^2 u - \partial_y^2 u, u)_{L^2} \\ &= \frac{1}{2}\int_{D(T,\varphi(T))} u^2 dxdy + \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dtdxdy \end{aligned}$$

where $|\nabla u|^2 = (\partial_x u)^2 + (\partial_y u)^2$. Then $|\nabla u|^2 = 0$. This leads to $\partial_x u = \partial_y u = 0$ and therefore, $\partial_x^2 u = \partial_y^2 u = 0$. From the equation $\partial_t u - \partial_x^2 u - \partial_y^2 u = 0$ we deduce that $\partial_t u = 0$. Accordingly, the solution u is constant, but it is null on one part of the boundary. So, $u = 0$. This proves the uniqueness of the solution of Problem (1). This is why we will be interested in the sequel only by the question of the existence of the solution when $f \in L^2(\Omega)$.

Our work is motivated by the interest of researchers for many mathematical questions related to non-regular domains. In fact, some important applied problems reduce to the study of boundary-value problems for partial differential equations (Laplace equation, heat equation, Fokker-Planck equation, Chapman-Kolmogorov equation...) in domains with non-regular points on the boundary.

The plan of the paper is : In Section 2 we give a change of variables which transforms a subdomain of Ω into a cylindrical domain, so we confine ourselves to the neighborhood of the origin $(0, 0, 0)$. Section 3 is devoted to the proof of an estimate in order to get a subsequence of functions converging to the solution of Problem (1) in a subdomain of Ω. Finally, in Section 4, we complete the proof of Theorom 1.

2. Change of variables

Let $\varepsilon > 0$ be a real which we will choose small enough. The continuity of φ at 0 proves the existence of a real number $T' < T$ such that :

$$\forall t \in \left(0, T'\right), |\varphi(t)| \leq \varepsilon. \tag{2.1}$$

Consider a decreasing sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that $0 < t_n < T'$ for all $n \in \mathbb{N}$ and $\lim\limits_{t\to 0^+} t_n = 0$. We define the sequence of subdomains $(\Omega_n)_n$ of $\Omega_{T'}$ by

$$\Omega_n = \left\{(t, x, y) \in \Omega : t_n < t < T'\right\}.$$

The solution u of Problem (1) will be approached by the solutions u_n of the problems

$$\left\{\begin{array}{c} \partial_t u_n - \partial_x^2 u_n - \partial_y^2 u_n = f_n \in L^2(\Omega_n), \\ u_{n|\partial\Omega_n \setminus D(T', \varphi(T'))} = 0, \end{array}\right. \tag{2.2}$$

where $f_n = f_{|\Omega_n}$.

Then, we perform the change of variables :

$$\begin{array}{ccc} \Omega_n & \rightarrow & (t_n, T') \times B(0,1) \\ (t, x, y) & \longmapsto & (t, \dfrac{x}{\varphi(t)}, \dfrac{y}{\varphi(t)}) = (t, x', y') \end{array}$$

where $B(0,1) = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$. Problem (3) is transformed into the problem

$$\left\{\begin{array}{c} \partial_t v_n - \frac{1}{\varphi^2(t)} \Delta v_n - \frac{\varphi'(t)}{\varphi(t)} \left(x' \partial_{x'} v_n + y' \partial_{y'} v_n\right) \\ = g_n \in L^2((t_n, T') \times B(0,1)), \\ v_{n|\{t_n\}\times B(0,1)} = 0, \\ v_{n|(t_n,T')\times\partial B(0,1)} = 0, \end{array}\right. \tag{2.3}$$

here $f_n(t, x, y) = g_n(t, x', y')$, $u_n(t, x, y) = v_n(t, x', y')$ and $\Delta v_n = \partial_{x'}^2 v_n + \partial_{y'}^2 v_n$. It is easy to see that the operator

$$\frac{\varphi'(t)}{\varphi(t)} \left(x' \partial_{x'} + y' \partial_{y'}\right) : H^{1,2}((t_n, T') \times B(0,1)) \rightarrow L^2((t_n, T') \times B(0,1))$$

is compact and the problem

$$\left\{\begin{array}{c} \partial_t v_n - \frac{1}{\varphi^2(t)} \Delta v_n = g_n \in L^2((t_n, T') \times B(0,1)), \\ v_{n|\{t_n\}\times B(0,1)} = 0, \\ v_{n|(t_n,T')\times\partial B(0,1)} = 0, \end{array}\right.$$

admits a unique solution $v_n \in H^{1,2}$ in the cylindrical domain $(t_n, T') \times B(0,1)$. This proves that Problem (4) has a unique solution $v_n \in H^{1,2}((t_n, T') \times B(0,1))$. The previous change of variables shows that Problem (3) has also a unique solution $u_n \in H^{1,2}(\Omega_n)$ for all $n \in \mathbb{N}$. The goal of the following section is to establish an estimate concerning the sequence (u_n) which allows us to extract a subsequence converging to the solution of Problem (1).

3. An estimate

Consider the sequence (u_n) and look for a constant $C > 0$ independent of n satisfying the estimate

$$\|u_n\|_{H^{1,2}(\Omega_n)} \leq C \|f_n\|_{L^2(\Omega_n)} . \tag{3.1}$$

For this purpose we will estimate $\left\|\partial_t u_n - \partial_x^2 u_n - \partial_y^2 u_n\right\|_{L^2(\Omega_n)}^2$.

Lemma 2. *There exists a constant $C > 0$ independent of n such that for almost every $t \in (0, T')$*

$$\begin{aligned} \|u_n\|_{L^2(D(t,\varphi(t)))} &\leq C\varphi^2(t) \|\Delta u_n\|_{L^2(D(t,\varphi(t)))} , \\ \max\left(\|\partial_x u_n\|_{L^2(D(t,\varphi(t)))} , \|\partial_y u_n\|_{L^2(D(t,\varphi(t)))}\right) &\leq C\varphi(t) \|\Delta u_n\|_{L^2(D(t,\varphi(t)))} . \end{aligned}$$

Proof. Let H^2 and H_0^1 be the usual Sobolev spaces defined, for instance, in Lions-Magenes [12]. We know that the Laplace operator $\Delta : H^2(D(0,1)) \cap H_0^1(D(0,1)) \to L^2(D(0,1))$ is an isomprphism. Then, there exists a constant $C > 0$ such that

$$\|v\|_{H^2(D(0,1))} \leq C \|\Delta v\|_{L^2(D(0,1))} , \forall v \in H^2(D(0,1)). \tag{3.2}$$

Let $t_n < t < T'$. The change of variables

$$\begin{array}{rcl} D(0,1) & \to & D(t,\varphi(t)) \\ (x,y) & \longmapsto & (\varphi(t)x, \varphi(t)y) = (x', y') \end{array}$$

allows to write, thanks to (6) :

$$\begin{aligned} \|v\|_{L^2(D(0,1))} &= \frac{1}{\varphi(t)} \|u_n\|_{L^2(D(t,\varphi(t))} \\ &\leq C \|\Delta v\|_{L^2(D(0,1))} \\ &= C\varphi(t) \|\Delta u_n\|_{L^2(D(t,\varphi(t))} , \end{aligned}$$

with the notation $v(x,y) = u_n(x', y')$. Notice that $u_n \in H^2(D(t,\varphi(t))$ for almost every $t \in (t_n, T')$. Hence

$$\|u_n\|_{L^2(D(t,\varphi(t))} \leq C\varphi^2(t) \|\Delta u_n\|_{L^2(D(t,\varphi(t))} .$$

On the other hand, we have

$$\begin{aligned} \|\partial_{x'} u_n\|_{L^2(D(t,\varphi(t)))} &= \|\partial_x v\|_{L^2(D(0,1))} \\ &\leq C \|\Delta v\|_{L^2(D(0,1))} \end{aligned}$$

$$= \quad C\varphi(t) \left\| \Delta u_n \right\|_{L^2(D(t,\varphi(t))} .$$

The same result is true for $\left\| \partial_{y'} u_n \right\|_{L^2(D(t,\varphi(t)))}$. □

The following lemma (see Theorem 2.1, Lions-Magenes [12]) will justify some calculations in this section.

Lemma 3. *The space*

$$\left\{ u \in H^2\left((0,T) \times D(0,1)\right) : u_{|\{0\}\times D(0,1)} = 0, u_{|(0,T')\times \partial D(0,1)} = 0 \right\}$$

is dense in the space

$$\left\{ u \in H^{1,2}((0,T) \times D(0,1)) : u_{|\{0\}\times D(0,1)} = 0, u_{|(0,T')\times \partial D(0,1)} = 0 \right\}.$$

Now, consider the solution $u_n \in H^{1,2}(\Omega_n)$ of Problem (3) and let $\Delta u_n = \partial_x^2 u_n + \partial_y^2 u_n$. Expanding the inner product $\|f_n\|^2_{L^2(\Omega_n)} = (\partial_t u_n - \Delta u_n, \partial_t u_n - \Delta u_n)_{L^2(\Omega_n)}$ we obtain

$$\|f_n\|^2_{L^2(\Omega_n)} = \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\Delta u_n\|^2_{L^2(\Omega_n)} - 2(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)}. \tag{3.3}$$

We deduce the following result from Lemma 2 and Grisvard-Ioss [6] (see Theorem 2.2).

Lemma 4. *There exists a constant $C > 0$ independent of n such that*

$$\left\| \partial_{xy} u_n \right\|^2_{L^2(\Omega_n)} + \left\| \partial_x^2 u_n \right\|^2_{L^2(\Omega_n)} + \left\| \partial_y^2 u_n \right\|^2_{L^2(\Omega_n)} \leq C \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)} . \tag{3.4}$$

We have to estimate the term $(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)}$. For this end, we need to prove the result

Proposition 5. *One has*

$$\begin{aligned} &-2(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} \\ &= \int_0^{2\pi} \int_{t_n}^{T'} \left|\nabla u_n\right|^2 (t, \varphi(t)\cos\theta, \varphi(t)\sin\theta)\varphi(t)\varphi'(t) dt d\theta \\ &+ \int_{D(T',\varphi(T'))} \left|\nabla u_n\right|^2 (T', x, y) dx dy. \end{aligned}$$

Proof. To establish this relationship we perform the following parametrization of Γ_1 :

$$\begin{array}{rcl} (t_n, T') \times (0, 2\pi) & \rightarrow & \Gamma_1 \\ (t, \theta) & \mapsto & (t, \varphi(t)\cos\theta, \varphi(t)\sin\theta) = (t, x, y) \end{array}$$

and we denote by $\nu = \frac{1}{\sqrt{1+\varphi'^2(t)}}(-\varphi'(t), \cos\theta, \sin\theta) = (\nu_t, \nu_x, \nu_y)$ the outward unit normal vector to the part Γ_1 of $\partial\Omega_n$ defined by

$$\Gamma_1 = \left\{ (t, x, y) : \sqrt{x^2 + y^2} = \varphi(t) \right\}.$$

The other part of $\partial\Omega_n$ will be denoted by $\Gamma_{T'}$, i.e., $\Gamma_{T'} = D(T', \varphi(T'))$.We have

$$\begin{aligned}
(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} &= \int_{\Omega_n} \partial_t u_n.\Delta u_n dtdxdy \\
&= \int_{\partial\Omega_n} \partial_t u_n.(\nu_x \partial_x u_n + \nu_y \partial_y u_n) d\sigma \\
&\quad -\frac{1}{2}\int_{\Omega_n} \partial_t \left((\partial_x u_n)^2 + (\partial_y u_n)^2\right) dtdxdy \\
&= \int_{\Gamma_1} \partial_t u_n.(\cos\theta.\partial_x u_n + \sin\theta.\partial_y u_n) \frac{1}{\sqrt{1+\varphi'^2(t)}} d\sigma \\
&\quad +\frac{1}{2}\int_{\Gamma_1} |\nabla u_n|^2 \frac{\varphi'(t)}{\sqrt{1+\varphi'^2(t)}} d\sigma - \frac{1}{2}\int_{\Gamma_{T'}} |\nabla u_n|^2 \, dxdy.
\end{aligned}$$

The Dirichlet boundary condition on Γ_1 leads to

$$\begin{aligned}
\partial_t u_n &= -\varphi'(t) \left(\cos\theta.\partial_x u_n + \sin\theta.\partial_y u_n\right), \\
\sin\theta.\partial_x u_n &= \cos\theta.\partial_y u_n.
\end{aligned}$$

Taking into account these relationships we deduce

$$\begin{aligned}
(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} &= -\int_{\Gamma_1} (\cos\theta.\partial_x u_n + \sin\theta.\partial_y u_n)^2 \frac{\varphi'(t)}{\sqrt{1+\varphi'^2(t)}} d\sigma \\
&\quad +\frac{1}{2}\int_{\Gamma_1} |\nabla u_n|^2 \frac{\varphi'(t)}{\sqrt{1+\varphi'^2(t)}} d\sigma \\
&\quad -\frac{1}{2}\int_{\Gamma_{T'}} |\nabla u_n|^2 (T', x, y) dxdy \\
&= -\int_0^{2\pi}\int_{t_n}^{T'} (\cos\theta.\partial_x u_n + \sin\theta.\partial_y u_n)^2 \varphi(t)\varphi'(t) dtd\theta \\
&\quad +\frac{1}{2}\int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 \varphi(t)\varphi'(t) dtd\theta \\
&\quad -\frac{1}{2}\int_{D(T',\varphi(T'))} |\nabla u_n|^2 (T', x, y) dxdy \\
&= -\int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 \varphi(t)\varphi'(t) dtd\theta \\
&\quad +\frac{1}{2}\int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 \varphi(t)\varphi'(t) dtd\theta \\
&\quad -\frac{1}{2}\int_{D(T',\varphi(T'))} |\nabla u_n|^2 (T', x, y) dxdy
\end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2}\int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 \varphi(t)\varphi'(t)dtd\theta \\ &\quad -\frac{1}{2}\int_{D(T',\varphi(T'))} |\nabla u_n|^2 (T',x,y)dxdy. \end{aligned}$$

This ends the proof of the proposition. □

Remark 6. *Observe that the integral* $\int_{D(T',\varphi(T'))} |\nabla u_n|^2 (T',x,y)dxdy$ *which appears in Proposition 5 is non negative.This is a good sign for our estimate because we can deduce immediately*

$$\begin{aligned} \|f_n\|^2_{L^2(\Omega_n)} \geq & \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\Delta u_n\|^2_{L^2(\Omega_n)} \\ & + \int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 (t,\varphi(t)\cos\theta,\varphi(t)\sin\theta)\varphi(t)\varphi'(t)dtd\theta. \end{aligned}$$

So, in the sequel, we will attach no importance to this integral. On the other hand, if φ *is an increasing function in the interval* (t_n, T'), *then*

$$\int_0^{2\pi}\int_{t_n}^{T'} |\nabla u_n|^2 (t,\varphi(t)\cos\theta,\varphi(t)\sin\theta)\varphi(t)\varphi'(t)dtd\theta \geq 0.$$

Consequently,

$$\|f_n\|^2_{L^2(\Omega_n)} \geq \|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|\Delta u_n\|^2_{L^2(\Omega_n)}. \tag{3.5}$$

But, thanks to Lemma 2

$$\begin{aligned} \int_{t_n}^{T'} \|u_n\|^2_{L^2(D(t,\varphi(t)))} dt &\leq C^2 \int_{t_n}^{T'} \varphi^4(t) \|\Delta u_n\|^2_{L^2(D(t,\varphi(t)))} dt, \\ \int_{t_n}^{T'} \|\partial_x u_n\|^2_{L^2(D(t,\varphi(t)))} dt &\leq C^2 \int_{t_n}^{T'} \varphi^2(t) \|\Delta u_n\|^2_{L^2(D(t,\varphi(t)))} dt, \\ \int_{t_n}^{T'} \|\partial_y u_n\|^2_{L^2(D(t,\varphi(t)))} dt &\leq C^2 \int_{t_n}^{T'} \varphi^2(t) \|\Delta u_n\|^2_{L^2(D(t,\varphi(t)))} dt. \end{aligned}$$

Since φ *is bounded on* $(0,T)$, *there exists a constant* $C' > 0$ *such that*

$$\max\left(\|u_n\|^2_{L^2(\Omega_n)}, \|\partial_x u_n\|^2_{L^2(\Omega_n)}, \|\partial_y u_n\|^2_{L^2(\Omega_n)}\right) \leq C' \|\Delta u_n\|^2_{L^2(\Omega_n)}.$$

Taking into account (8) and (9), this proves (5). So, it remains to establish (5) under the hypothesis $\lim_{t\to 0^+} \varphi(t)\varphi'(t) = 0$. *For this purpose, we need the following proposition*

Proposition 7. *One has*

$$-2(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} = 2\int_{\Omega_n} \frac{\varphi'(t)}{\varphi(t)}(x\partial_x u_n + y\partial u_n)\Delta u_n dtdxdy$$

$$+\int_{D(T',\varphi(T'))} |\nabla u_n|^2 (T',x,y)dxdy.$$

Proof. For $t_n < t < T'$, denote by D_t the disc $D(t,\varphi(t))$ and consider the inner product $(\Delta u_n, x\partial_x u_n + y\partial u_n)_{L^2(D_t)}$. We have

$$\begin{aligned}(\Delta u_n, x\partial_x u_n + y\partial_y u_n)_{L^2(D_t)} &= \frac{1}{2}\int_{D_t}\left(x\partial_x(\partial_x u_n)^2 + y\partial_y(\partial_y u_n)^2\right)dxdy \\ &+\int_{\partial D_t}(x\nu_y + y\nu_x)\,\partial_x u_n\partial_y u_n d\sigma \\ &-\int_{D_t}(x\partial_y u_n\partial_{xy}u_n + y\partial_x u_n\partial_{xy}u_n)\,dxdy.\end{aligned}$$

Then

$$\begin{aligned}(\Delta u_n, x\partial_x u_n + y\partial_y u_n)_{L^2(D_t)} &= \frac{1}{2}\int_{\partial D_t}\left(x\nu_x(\partial_x u_n)^2 + y\nu_y(\partial_y u_n)^2\right)d\sigma \\ &-\frac{1}{2}\int_{D_t}|\nabla u_n|^2\,dxdy \\ &+2\int_{\partial D_t}\partial_x u_n\partial_y u_n\cos\theta\sin\theta d\sigma \\ &-\frac{1}{2}\int_{D_t}\left(x\partial_x(\partial_y u_n)^2 + y\partial_y(\partial_x u_n)^2\right)dxdy.\end{aligned}$$

The boundary condition $u(t,\varphi(t)\cos\theta,\varphi(t)\sin\theta) = 0$ leads to $\sin\theta\partial_x u_n = \cos\theta\partial_y u_n$. Consequently

$$\begin{aligned}(\Delta u_n, x\partial_x u_n + y\partial_y u_n)_{L^2(D_t)} &= \frac{\varphi^2(t)}{2}\int_0^{2\pi}(\cos\theta.\partial_x u_n)^2 + (\sin\theta.\partial_y u_n)^2)d\theta \\ &+\varphi^2(t)\int_0^{2\pi}(\sin\theta.\partial_x u_n)^2 + (\cos\theta.\partial_y u_n)^2)d\theta \\ &-\frac{1}{2}\int_{D_t}|\nabla u_n|^2\,dxdy \\ &-\frac{\varphi^2(t)}{2}\int_0^{2\pi}(\sin\theta.\partial_x u_n)^2 + (\cos\theta.\partial_y u_n)^2)d\theta \\ &+\frac{1}{2}\int_{D_t}|\nabla u_n|^2\,dxdy.\end{aligned}$$

So

$$(\Delta u_n, x\partial_x u_n + y\partial_y u_n)_{L^2(D_t)} = \frac{\varphi^2(t)}{2}\int_0^{2\pi}|\nabla u_n|^2\,d\theta,$$

and

$$2\int_{\Omega_n}\frac{\varphi'(t)}{\varphi(t)}(x\partial_x u_n+y\partial_y u_n)\Delta u_n dtdxdy=\int_{t_n}^{T'}\int_0^{2\pi}|\nabla u_n|^2\varphi(t)\varphi'(t)d\theta dt.$$

Finally, in virtue of Proposition 5, it follows

$$\begin{aligned}-2(\partial_t u_n,\Delta u_n)_{L^2(\Omega_n)} &= 2\int_{\Omega_n}\frac{\varphi'(t)}{\varphi(t)}(x\partial_x u_n+y\partial_y u_n)\Delta u_n dtdxdy\\ &\quad+\int_{D(T',\varphi(T'))}|\nabla u_n|^2(T',x,y)dxdy.\end{aligned}$$

Theorem 8. *Assume that* $\lim\limits_{t\to 0}\varphi(t)\varphi'(t)=0$ *or* $\varphi'(t)\geq 0$ *in a neighborhood of* 0*.Then, there exists a constant* $C>0$ *independent of* n *satisfying the estimate*

$$\|u_n\|_{H^{1,2}(\Omega_n)}\leq C\|f_n\|_{L^2(\Omega_n)}.$$

□

Proof. The case when $\varphi'(t)\geq 0$ in a neighborhood of 0 has been treated in Remark 6. Then, assume that $\lim\limits_{t\to 0}\varphi(t)\varphi'(t)=0$. We have

$$\begin{aligned}\left|\int_{\Omega_n}\frac{\varphi'(t)}{\varphi(t)}(x\partial_x u_n+y\partial_y u_n)\Delta u_n dtdxdy\right| &\leq \|\Delta u_n\|_{L^2(\Omega_n)}\left\|\frac{x.\varphi'(t)}{\varphi(t)}\partial_x u_n\right\|_{L^2(\Omega_n)}\\ &\quad+\|\Delta u_n\|_{L^2(\Omega_n)}\left\|\frac{y.\varphi'(t)}{\varphi(t)}\partial_y u_n\right\|_{L^2(\Omega_n)},\end{aligned}\tag{3.6}$$

but Lemma 2 yields

$$\begin{aligned}\left\|\frac{x.\varphi'(t)}{\varphi(t)}\partial_x u_n\right\|^2_{L^2(\Omega_n)} &= \int_{t_n}^{T'}\varphi'^2(t)\int_{D(t,\varphi(t))}\left(\frac{x}{\varphi(t)}\right)^2(\partial_x u_n)^2 dtdxdy\\ &\leq \int_{t_n}^{T'}\varphi'^2(t)\int_{D(t,\varphi(t))}(\partial_x u_n)^2 dtdxdy\\ &\leq C^2\int_{t_n}^{T'}(\varphi(t)\varphi'(t))^2\int_{D(t,\varphi(t))}(\Delta u_n)^2 dtdxdy.\end{aligned}$$

The hypothesis $\lim\limits_{t\to 0}\varphi(t)\varphi'(t)=0$ implies the existence of $0<\alpha<\frac{1}{2}$ such that $|\varphi(t)\varphi'(t)|<\frac{\alpha}{2C}$ for all $t\in(0,T')$ since T' is chosen small enough. So

$$\left\|\frac{x.\varphi'(t)}{\varphi(t)}\partial_x u_n\right\|^2_{L^2(\Omega_n)}\leq\frac{\alpha^2}{4}\|\Delta u_n\|^2_{L^2(\Omega_n)}.$$

It is clear that we have also

$$\left\| \frac{y.\varphi'(t)}{\varphi(t)} \partial_y u_n \right\|^2_{L^2(\Omega_n)} \leq \frac{\alpha^2}{4} \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)}.$$

Accordingly, Relationship (10) leads to

$$2 \left| \int_{\Omega_n} \frac{\varphi'(t)}{\varphi(t)} (x\partial_x u_n + y\partial_y u_n) \Delta u_n dtdxdy \right| \leq 2\alpha \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)}.$$

Therefore, Proposition 7 shows that

$$\begin{aligned} -2(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} &\geq -2 \left| \int_{\Omega_n} \frac{\varphi'(t)}{\varphi(t)} (x\partial_x u_n + y\partial_y u_n) \Delta u_n dtdxdy \right| \\ &\quad + \int_{D(T',\varphi(T'))} \left| \nabla u_n \right|^2 (T', x, y) dxdy \\ &\geq -2\alpha \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)}. \end{aligned}$$

Hence

$$\begin{aligned} \left\| f_n \right\|^2_{L^2(\Omega_n)} &= \left\| \partial_t u_n \right\|^2_{L^2(\Omega_n)} + \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)} - 2(\partial_t u_n, \Delta u_n)_{L^2(\Omega_n)} \\ &\geq \left\| \partial_t u_n \right\|^2_{L^2(\Omega_n)} + (1-2\alpha) \left\| \Delta u_n \right\|^2_{L^2(\Omega_n)}. \end{aligned}$$

Since $2\alpha < 1$, this inequality with Lemma 4 completes the proof. □

4. Proof of Theorom 1

From the previous theorem we deduce that $\left\| u_n \right\|_{H^{1,2}(\Omega_n)} \leq C \left\| f \right\|_{L^2(\Omega)}$ for all $n \in \mathbb{N}$ because $\left\| f_n \right\|_{L^2(\Omega_n)} \leq \left\| f \right\|_{L^2(\Omega)}$. For any function v defined in Ω_n, denote by $\tilde{v}$ the function

$$\tilde{v} = \begin{cases} v : \text{in } \Omega_n \\ 0 : \text{in } \Omega_{T'} \backslash \Omega_n. \end{cases}$$

It is then possible to extract a subsequence $\left(\tilde{u}_{n_j}\right)_j$ from $(\tilde{u}_n) \in L^2(\Omega_{T'})$ which converges weakly in $H^{1,2}(\Omega_{T'})$. Denote by u_1 the limit of $\left(\tilde{u}_{n_j}\right)_j$. Therefore, passing to the limit for $n_j \to +\infty$, we get

$$\begin{cases} \partial_t u_1 - \partial_x^2 u_1 - \partial_y^2 u_1 = f_{|\Omega_{T'}} \in L^2(\Omega_{T'}), \\ u_{1|\partial\Omega_{T'} \backslash D(T,\varphi(T))} = 0. \end{cases} \tag{4.1}$$

This proves Theorem 1 in the subdomain $\Omega_{T'}$.

Let T be any positive real number, $f \in L^2(\Omega)$, and $T' < T$ small enough in order to Problem 11 admits a solution $u_1 \in H^{1,2}(\Omega_{T'})$

We have to solve Problem (1) in Ω. We know (see Section 2) that the Cauchy-Dirichlet problem

$$\text{(4.2)} \qquad \begin{cases} \partial_t v - \partial_x^2 v - \partial_y^2 v = f_{|\Omega_{T'T}} \in L^2(\Omega_{T'T}), \\ v_{|\partial\Omega_{T'T} \backslash D(T,\varphi(T))} = 0. \end{cases}$$

has a unique solution $v \in H^{1,2}(\Omega_{T'T})$.

We need the following trace result (see Lions-Magenes [12])

Lemma 9. *For $v \in H^{1,2}(\Omega_{T'T})$, one has $v_{|D(T',\varphi(T'))} \in H^1(D(T',\varphi(T')))$. Moreover, for each $\psi \in H_0^1(D(T',\varphi(T')))$, there exists $w \in H^{1,2}(\Omega_{T'T})$ such that $w_{|D(T',\varphi(T'))} = \psi$ and $w_{|\partial\Omega_{T'T} \backslash D(T',\varphi(T'))} = 0$.*

Since $u_1 \in H^{1,2}(\Omega_{T'})$, from this lemma and the homogeneous Problem (12), we obtain

Proposition 10. *The problem*

$$\text{(4.3)} \qquad \begin{cases} \partial_t u_2 - \partial_x^2 u_2 - \partial_y^2 u_2 = f_{|\Omega_{T'T}} \in L^2(\Omega_{T'T}), \\ u_{2|D(T',\varphi(T'))} = u_{1|D(T',\varphi(T'))}, \\ u_{2|\partial\Omega_{T'T} \backslash D(T',\varphi(T')) \cup D(T,\varphi(T))} = 0. \end{cases}$$

admits a unique solution $u_2 \in H^{1,2}(\Omega_{T'T})$.

Now, define the function u in Ω by

$$u := \begin{cases} u_1 & \text{in } \Omega_{T'} \\ u_2 & \text{in } \Omega_{T'T} \end{cases}$$

where u_1 and u_2 are the solutions of Problem (11) and Problem (13) respectively. Observe that inasmuch as the boundary condition $u_{2|D(T',\varphi(T'))} = u_{1|D(T',\varphi(T'))}$ is satisfied, it follows that $\partial_x^2 u \in L^2(\Omega)$ and $\partial_y^2 u \in L^2(\Omega)$, so, we have also $\partial_{xy} u \in L^2(\Omega)$. Moreover, in view of the equations given in Problem (11) and Problem (13) we deduce that $\partial_t u \in L^2(\Omega)$ and $\partial_t u - \partial_x^2 u - \partial_y^2 u = f$.

Finally, the function $u \in H^{1,2}(\Omega)$ is a solution of Problem (1). This completes the proof of Theorem 1.

Remark 11. *This study can be extended to the case when the function φ depends also on an angle $\theta \in (0, 2\pi)$. For instance, the domain Ω may be defined by*

$$\Omega = \left\{ (t, x, y) \in \mathbb{R}^3 : 0 < t < T, 0 \le \theta \le 2\pi, 0 \le \sqrt{x^2 + y^2} < \varphi(t, \theta) \right\}$$

where φ is a Lipschitz function $\varphi : [0, T] \times [0, 2\pi] \to \mathbb{R}^+$ satisfying the condition $\varphi(0, \theta) = 0$ for $0 \le \theta \le 2\pi$ and $\varphi(t, 0) = \varphi(t, 2\pi)$ Then, a new condition on φ depending on θ will be appear. In particular, if $\varphi(t, \theta) = 0$ for $0 < t < T$ and $0 \le \theta \le \theta_0$ with $0 < \theta_0 < 2\pi$ the solution may contain some singularities, and no conditions on φ could avoid these singularities. This problem will be treated later.

References

[1] Yu. A Alkhutov, L^p-solvability of the Dirichlet problem for the heat equation in non-cylindrical domains, *Sbornik Math.*, 193, 9 (2002), 1243–1279.

[2] Yu. A.Alkhutov, L^p-estimates of solutions of the Dirichlet problem for the heat equation in a ball, *J. of Math. Sci.*, 142, 3 (2007), 2021–2032.

[3] T. Berroug, D. Hua , R. Labbas and B.-K. Sadallah, On a degenerate parabolic problem in Hölder spaces". *Appli. Math. and Computation*, 162, 2 (2005), 811-833.

[4] V. Besov, The continuation of function in L_p^1 and W_p^1 , *Proc. Steklov Inst. Math.* 89 (1967), 5–17.

[5] S.P. Degtyarev, The solvability of the first initial-boundary problem for parabolic and degenerate parabolic equations in domains with a conical point, *Sbornik Math.* 201, 7 (2010) 999–1028.

[6] P. Grisvard and G. Iooss, Problèmes aux limites unilatéraux dans des domaines non réguliers, *Journées Equations aux dérivées partielles*, (1976), 1-26. http://www.numdam.org.

[7] A. Kheloufi, R. Labbas, B.-K. Sadallah, On the resolution of a parabolic equation in a nonregular domain of $\mathbb{R}^3$, *Diff. Equa. and Appl.,* 2, 2 (2010), 251–263.

[8] V. A. Kozlov and V. G.Maz'ya, On singularities of a solution to the first boundary-value problem for the heat equation in domains with conical points, I, *Izv. Vyssh. Uchebn. Zaved.*, Ser. Mat., 2 (1987), 38–46.

[9] V. A. Kozlov and V. G.Maz'ya, On singularities of a solution to the first boundary-value problem for the heat equation in domains with conical points II, *Izv. Vyssh. Uchebn. Zaved.*, Ser. Mat., 3 (1987), 37–44.

[10] R. Labbas, A. Medeghri and B.-K. Sadallah, On a parabolic equation in a triangular domain, *Appl. Math. Comput.*, 2-3, 130 (2002), 511–523.

[11] R. Labbas, A. Medeghri and B.-K. Sadallah, An L^p-approach for the study of degenerate parabolic equations", *E. J. of Diff. Eq.*, 36 (2005), 1-20.

[12] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, 1,2, Springer-Verlag, New York-Heidelberg, 1972.

[13] A. I. Nazarov, L^p-estimates for a solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension, *J. of Math. Sci.*, 106, 3 (2001), 2989-3014.

[14] B.-K. Sadallah, Etude d'un problème $2m$-parabolique dans des domaines plans non rectangulaires, *Boll. Un. Mat. Ital.,* 5, 2-B (1983), 51–112.

[15] B.-K. Sadallah, Existence de la solution de l'équation de la chaleur dans un disque, *C. R. Acad. Sci. Paris,* Sér. I, Math. 327, 9 (1998), 813–816.

Boubaker-Khaled Sadallah
Lab. PDE & Hist. Maths
Ecole Normale Supérieure, Kouba,
Algiers, Algeria.
e-mail address: sadallah@ens-kouba.dz

(*Received August 19, 2011*)
(*Revised December 19, 2011*)