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PRIME, MAXIMAL AND PRIMITIVE IDEALS IN SOME SUBRINGS OF POLYNOMIAL RINGS

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Abstract. In this paper we describe prime, maximal and primitive ideals in some graded subrings of polynomial rings. As applications the corresponding radicals are determined.

Introduction

Throughout this paper rings are associative but do not necessarily have an identity element. The authors in [2], [3] and [7] described prime and maximal ideals in polynomial rings.

It is easy to see that every subring of $R[x]$ containing $R[x]x$ is of type $S + R[x]x$, where S is a subring of R. In this paper we consider graded subrings of a polynomial ring $R[x]$ of the following type:

$$T = S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n,$$

where S_0 is a subring and $S_1, ..., S_{n-1}$ are additive subgroups of R. We call them admissible subrings of $R[x]$. It is clear that T is an admissible subring of $R[x]$ if and only if there exist $n \in \mathbb{N}$ and additive subgroups S_i of R, $1 \leq i \leq n-1$, such that S_0 is a subring, S_i is an S_0-sub-bimodule of R and $T = S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$. Note that we have $S_m = R$ for every $m \geq n$.

In the rest of the paper we denote by T an admissible subring of $R[x]$. Let $\mathcal{F}$ be a special class of prime rings. First we give a description of the ideals P of T with $R[x]x^n \nsubseteq P$ and $T/P \in \mathcal{F}$. In particular, we prove that P is a prime ideal of T with $R[x]x^n \nsubseteq P$ if and only if there exists a prime ideal L of $R[x]$ with $R[x]x \nsubseteq L$ such that $L \cap T = P$. In the case $R[x]x^n \subseteq P$, P is a prime ideal of T if and only if $P = P \cap S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$, where $P \cap S_0$ is a prime ideal of S_0. We obtain similar results for primitive and maximal ideals of T (more precisely, ideals M of T such that T/M is either simple and prime or simple with an identity).

Extending a well-known terminology, an ideal I of T with $I \cap S_0 = 0$ is called an S_0-disjoint ideal.

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As we already said, in general, R do not necessarily have an identity element. So we have to take care with some argumentations. For example, if $f = a_k x^k + a_{k-1}x^{k-1} + ... + a_0 \in R[x]$, we denote by fx the polynomial $a_k x^{k+1} + a_{k-1}x^k + ... + a_0 x \in R[x]$ (note that, in general, $x \notin R[x]$). The following fact, easy to prove, will be used throughout the paper: if P is a prime ideal of $R[x]$ with $R[x]x \nsubseteq P$, then for some $f \in R[x]$ we have that $fx \in P$ if and only if $f \in P$.

Results

Recall that a class of rings $\mathcal{F}$ is said to be a special class if it satisfies the following three conditions ([1], Chapter 7):
(i) Every ring in the class $\mathcal{F}$ is a prime ring;
(ii) every non-zero ideal of a ring in $\mathcal{F}$ is itself a ring in $\mathcal{F}$;
(iii) if I is a ring in $\mathcal{F}$, and I is an ideal of a ring R, then R/I^* is in $\mathcal{F}$, where I^* is the annihilator of I in R, i.e., $I^* = \{x \in R \mid xI = Ix = 0\}$.

The classes of all prime rings, simple rings with identity and right (left) primitive rings, are special classes of rings.

We begin with the following.

Lemma 1. Let $B \subseteq A$ rings and $\mathcal{F}$ a special class of prime rings. Assume that there exists an ideal I of A with $I \subseteq B$. Then there exists an order preserving one-to-one correspondence, via contraction, between:

(i) The set of all ideals P of B such that $B/P \in \mathcal{F}$ and $I \nsubseteq P$.

(ii) The set of all ideals L of A such that $A/L \in \mathcal{F}$ and $I \nsubseteq L$.

Proof. Assume that P is an ideal of B such that $B/P \in \mathcal{F}$ and $I \nsubseteq P$. Put $J = P + AP + PA + APA$, a non-zero ideal of A and note that $J \cap B = P$. In fact, $I(J \cap B)I$ is an ideal of B which is contained in P. It follows that $J \cap B \subseteq P \subseteq J \cap B$.

By Zorn's lemma there exists an ideal L of A which is maximal with respect to the condition $L \cap B = P$ and clearly $I \nsubseteq L$. Since $\mathcal{F}$ is special class it follows that $I/P \cap I \simeq (I + P)/P \in \mathcal{F}$.

Thus $I/L \cap I = I/P \cap I$ is a non-zero ideal of A/L and the annihilator of $I/L \cap I$ in A/L is zero. In fact, for $(I/L \cap I)^* = \{x \in A/L \mid x(I/L \cap I) = (I/L \cap I)x = 0\}$ we have that $(I/L \cap I)^* = H/L$, where H is an ideal of A containing L with $H \cap B = P$. The maximality of L implies that $H = L$ and so $(I/L \cap I)^* = 0$. Since $I/L \cap I$ is a non-zero ideal of A/L it follows that $A/L \simeq (A/L)/(I/L \cap I)^* \in \mathcal{F}$.
Now we prove that L is unique. Assume that K is another ideal of A with $A/K \in \mathcal{F}$ and $K \cap B = P$. Then $LI \subseteq L \cap B = P = K \cap B \subseteq K$. Since $I \nsubseteq K$ it follows that $L \subseteq K$ and consequently $L = K$.

Conversely, assume that L is an ideal of A with $A/L \in \mathcal{F}$ and $I \nsubseteq L$. Note that $I/L \cap I$ is a non-zero ideal of A/L and also of $B/L \cap B$. Hence $I/L \cap I \in \mathcal{F}$ and we easily see that the annihilator of $I/L \cap I$ in $B/(L \cap B)$ is zero. Thus $B/L \cap B \in \mathcal{F}$. □

As an immediate consequence of the above lemma we have one of the main results of this paper. In the following primitive means either right or left primitive.

Theorem 2. *There is an order preserving one-to-one correspondence, via contraction, between:*

(*i*) *The set of all ideals L of $R[x]$ with $R[x]x \nsubseteq L$ such that $R[x]/L$ is a prime (resp. primitive, simple with identity) ring.*

(*ii*) *The set of all ideals P of T with $R[x]x^n \nsubseteq P$ such that T/P is a prime (resp. primitive, simple with identity) ring.*

Proof. The class of prime (resp. primitive, simples with identity) rings is a special class and $R[x]x^n$ is an ideal of T and of $R[x]$. From this remark and Lemma 1 the result follows. □

The next two propositions give a result corresponding to Theorem 2 for maximal ideals.

Proposition 3. Let L a maximal ideal of $R[x]$.

(i) Assume that L is not prime and that $M = L \cap T$ is a proper ideal of T. Then M is a maximal ideal of T.

(ii) Suppose that L is a prime ideal of $R[x]$ such that $R[x]x \nsubseteq L$. Then $M = L \cap T$ is a maximal ideal of T with $R[x]x^n \nsubseteq M$. Moreover, in this case $T/M \simeq R[x]/L$.

Proof. (i) Suppose that K is an ideal of T with $M = L \cap T \subseteq K$. Hence $K+L$ is an ideal of $R[x]$ since $(R[x])^2 \subseteq L \subseteq K+L$. Thus by the maximality of L we have that either $K = L \cap T$ or $K + L = R[x]$. If $K + L = R[x]$, then $T = (K + L) \cap T \subseteq (L \cap T) + K \subseteq K$. Consequently either $K = M$ or $K = T$, and so M is a maximal ideal of T.

(ii) By Theorem 2, $M = L \cap T$ is a prime ideal of T with $R[x]x^n \nsubseteq M$. Then $R[x] = L + R[x]x^n$ and it easily follows that $T = M + R[x]x^n$. Hence

$$T/M = (M + R[x]x^n)/M \simeq R[x]x^n/M \cap R[x]x^n =$$

$$R[x]x^n/L \cap R[x]x^n \simeq (L + R[x]x^n)/L = R[x]/L,$$

and the proof is complete. □

Proposition 4. Let M be a prime ideal of T with $R[x]x^n \nsubseteq M$. If M is a maximal ideal, then there exists a maximal ideal L of $R[x]$ with $R[x]x \nsubseteq L$ such that $L \cap T = M$. In this case $T/M \simeq R[x]/L$.

Proof. By Theorem 2 there exists a prime ideal L of $R[x]$ with $R[x]x \nsubseteq L$ such that $L \cap T = M$. We have that

$$T/M = (M + R[x]x^n)/M \simeq R[x]x^n/M \cap R[x]x^n = R[x]x^n/L \cap R[x]x^n.$$

Note that if $f \in L \cap R[x]x^n$, then $f = gx^n$ where $g \in R[x]$. Since $gR[x]x^n = fR[x] \subseteq L$ it follows that $g \in L$. Therefore $f \in Lx^n$ and so $L \cap R[x]x^n = Lx^n$. Assume that K an ideal of $R[x]$ with $L \subseteq K$. Then $Lx^n \subseteq Kx^n$ and so by the maximality of Lx^n in $R[x]x^n$ we have that either $Kx^n = Lx^n$ or $Kx^n = R[x]x^n$. Consequently, either $K = L$ or $K = R[x]$. Then L is a maximal ideal of $R[x]$. Finally $T/M \simeq R[x]x^n/L \cap R[x]x^n \simeq R[x]/L$. □

Putting together the above results we immediately have the following

Corollary 5. There is an order preserving one-to-one correspondence, via contraction, between:

(i) The set of all maximal prime ideals L of $R[x]$ with $R[x]x \nsubseteq L$.

(ii) The set of all maximal prime ideals M of T with $R[x]x^n \nsubseteq M$.

The prime ideals of T containing $R[x]x^n$ can also be easily characterized:

Theorem 6. *Assume that P is an ideal of T with $R[x]x^n \subseteq P$. Then T/P is a prime (resp. primitive, prime simple, simple with identity) ring if and only if $P = P \cap S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$ and $S_0/P \cap S_0$ is a prime (resp. primitive, prime simple, simple with identity) ring.*

Proof. For $n = 1$ the result is clear. So assume that $n \geq 2$ and suppose that T/P is prime with $R[x]x^n \subseteq P$. Clearly $(S_{n-1}x^{n-1} + R[x]x^n)^2 \subseteq R[x]x^n \subseteq P$ and so $S_{n-1}x^{n-1} + R[x]x^n \subseteq P$. Using induction we show that $S_jx^j \subseteq P$, for $1 \leq j \leq n-1$, and hence $P = P \cap S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$. Therefore $T/P \simeq S_0/P \cap S_0$ and the result follows for the class of prime rings. The other cases are similar. Finally, the converse is clear. □

The intersection of a prime ideal of $R[x]$ with T is not always a prime ideal of T. The following example shows this.

Example 7. Let $R = M_m(K)$ be the ring of $m \times m$ matrices over a field K and let S be the subring of all lower triangular matrices over K. Then $P = R[x]x$ is a prime ideal of $R[x]$ and $P \cap (S + R[x]x)$ is not prime ideal of $T = S + R[x]x$.

Recall that the prime radical $Nil_*(R)$ of a ring R is defined as the intersection of all prime ideals of R. It is well-known that $Nil_*(R[x]) = Nil_*(R)[x]$ ([6], Theorem 10.19).

Corollary 8. $Nil_*(T) = Nil_*(R[x]) \cap T = \sum_{0 \le i \le n-1} (Nil_*(R) \cap S_i)[x] + Nil_*(R)[x]x^n$.

Proof. Let I be a prime ideal of R. By Theorem 2, $I[x] \cap T$ is a prime ideal of T and so $Nil_*(T) \subseteq I[x] \cap T$. Thus $Nil_*(T) \subseteq Nil_*(R)[x] \cap T = Nil_*(R[x]) \cap T$.

For the other inclusion, let P be a prime ideal of T. If $R[x]x^n \nsubseteq P$, then there exists a prime ideal L of $R[x]$ with $L \cap T = P$. Consequently $Nil_*(R[x]) \cap T \subseteq L \cap T = P$. In the other case $R[x]x^n \subseteq P$ and we have that $P \cap S_0$ is a prime ideal of S_0. Take an ideal H of $R[x]$ which is maximal with respect to $H \cap S_0 \subseteq P \cap S_0$. Then H is a prime ideal of $R[x]$ and so $Nil_*(R) \cap S_0 \subseteq H \cap S_0 \subseteq P \cap S_0$ and the first equality follows. The second equality is clear. $\square$

Definition 9. A subring S of R is said to be an essential subring if $I \cap S \neq 0$ for every non-zero ideal I of R.

Example 10. If R is a ring with identity, then T is an essential subring of $R[x]$. In fact, if I is a non-zero ideal of $R[x]$, then $0 \neq x^n I \subseteq I \cap T$.

The following is an immediate consequence of Theorem 2.

Corollary 11. Assume that S_0 is an essential subring of R. If P is an S_0-disjoint prime of T with $R[x]x^n \nsubseteq P$, then there exists an R-disjoint prime ideal L of $R[x]$ with $R[x]x \nsubseteq L$ such that $L \cap T = P$.

For a prime ring R, let Q be the right (resp. left, symmetric) Martindale ring of quotients of R and C its extended centroid. The following proposition characterizes S_0-disjoint prime ideals of T when S_0 is an essential subring of R.

Corollary 12. Let R be a prime ring and S_0 an essential subring of R. If P is an S_0-disjoint prime ideal of T with $R[x]x^n \nsubseteq P$, then $P = Q[x]f_0 \cap T$, for some monic irreducible polynomial $f_0 \in C[x]$.

Proof. It follows easily from Corollary 11 and Corollary 2.6 of [2]. $\square$

It is well-known that a non-zero R-disjoint prime ideal P of $R[x]$ is maximal with respect to $P \cap R = 0$. The next example shows that, in general, S_0-disjoint prime ideals of T are not necessarily maximal in the set of all S_0-disjoint ideals of T.

Example 13. Let K be a field, $R = K[x_1, x_2, ..., x_n]$, $T = K + R[x]x$ and $P_i = (K[x_1, ..., x_i]x_i + ... + K[x_1, ..., x_n]x_n)[x] \cap T$, for $1 \le i \le n$. Thus we have a chain of K-disjoint prime ideals $P_1 \subset P_2 \subset ... \subset P_n$ of T.

Next we will show that the result mentioned above holds if S_0 is an essential subring of R. First we need the following:

Lemma 14. Assume that R is a prime ring and S_0 is an essential subring of R. If P is an S_0-disjoint prime ideal of T with $R[x]x^n \nsubseteq P$, then $P \nsubseteq S_1x + ... + S_{n-1}n^{n-1} + R[x]x^n$.

Proof. Put $S = S_1x+...+S_{n-1}n^{n-1}+R[x]x^n$ and suppose, by contradiction, that $P \subseteq S$. By Corollary 11 there exists an R-disjoint prime ideal L of $R[x]$ with $L \cap T = P$ and $R[x]x \nsubseteq L$. If $s \in (L + R[x]x^n) \cap S_0$, then $s = f + g$, where $f \in L$ and $g \in R[x]x^n$. Thus $s - g = f \in L \cap T = P \subseteq S$ and so $s \in S$. Hence $s = 0$ and consequently $(L + R[x]x^n) \cap S_0 = 0$. Since S_0 is an essential subring of R it follows that $(L + R[x]x^n) \cap R = 0$ and therefore $L + R[x]x^n = L$, a contradiction because $R[x]x^n \nsubseteq L$ □

Theorem 15. *Assume that S_0 is an essential prime subring of R. Then a non-zero ideal S_0-disjoint P of T is a prime ideal of T if and only if P is maximal in the set of all S_0-disjoint ideals of T*

Proof. It is easy see that R is prime. Also, if P is maximal in the set of S_0-disjoint ideals of T, then P is prime.

Conversely, assume that P is a prime ideal of T which is S_0-disjoint. If $R[x]x^n \subseteq P$, then by Theorem 6, P is maximal in the set of all S_0-disjoint ideals of T. Now suppose that $R[x]x^n \nsubseteq P$. By Corollary 11 there exists an R-disjoint prime ideal L of $R[x]$ such that $R[x]x \nsubseteq L$ and $L \cap T = P$. Let P' be a maximal S_0-disjoint ideal de T with $P \subseteq P'$. By the first part it follows that P' is a prime ideal of T. Moreover, Lemma 14 implies that $R[x]x^n \nsubseteq P'$ because $P \nsubseteq S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$. Finally, applying again by Corollary 11 it follows that there exists an R-disjoint prime ideal $L' \supseteq L$ of $R[x]$ such that $L' \cap T = P'$. Therefore $L = L'$ and consequently $P = P'$. □

Recall that the Brown-McCoy radical $\mathcal{U}(R)$ of a ring R is defined as the intersection of all ideals I of R such that R/I is a simple ring with an identity. In particular, a ring is a Brown-McCoy radical ring if it cannot be homomorphically mapped onto a simple ring with an identity.

In [5], Krempa proved that for every ring R, $\mathcal{U}(R[x]) = (\mathcal{U}(R[x]) \cap R)[x]$. In the following we denote by U the ideal $\mathcal{U}(R[x]) \cap R$ of R.

Proposition 16. For the ring T we have

$$\mathcal{U}(T) = U \cap \mathcal{U}(S_0) + (U \cap S_1)x + ... + (U \cap S_{n-1})x^{n-1} + U[x]x^n.$$

Proof. Let M be an ideal of T such that T/M is a simple ring with an identity. If $R[x]x^n \nsubseteq M$, then by Theorem 2, $U(R[x]) \cap T \subseteq M$. In the other case, if $R[x]x^n \subseteq M$, we have that $M = (M \cap S_0) + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n$ and $T/M \simeq S_0/M \cap S_0$. Hence

$$U \cap \mathcal{U}(S_0) + (U \cap S_1)x + ... + (U \cap S_{n-1})x^{n-1} + U[x]x^n \subseteq \mathcal{U}(T).$$

To get the converse inclusion, let $f = a_0 + a_1x + ... + a_kx^k \in \mathcal{U}(T)$ and L an ideal of $R[x]$ such that $R[x]/L$ is a simple ring with an identity. If $R[x]x \nsubseteq L$, then Theorem 2 implies that $f \in L \cap T$. If $R[x]x \subseteq L$, then $L = L \cap R + R[x]x$ and so $fx \in L$. It follows that $fx \in U[x]$. Hence $a_i \in U \cap S_i$, for every $i \in \{0, ..., k\}$, and so by the above inclusion we have that $a_0 \in \mathcal{U}(T)$. Also it is clear that $a_0 \in N$ for any ideal N of S_0 such that S_0/N is simple with identity. Consequently $a_0 \in \mathcal{U}(S_0)$ and the proof is complete. □

The following proposition extends ([7], Corollary 3).

Proposition 17. (i) If R is a nil ring, then T is a Brown-McCoy radical ring.
(ii) Let R be a simple ring without identity. If S_0 is either a nil ring or a simple subring of R without identity, then T is a Brown-McCoy radical ring.

Proof. (*i*) By the way of contradiction, suppose that there exists an ideal M of T such that T/M is a simple ring with an identity. Then Theorem 2 and Corollary 3 of [7] implies that $R[x]x^n \subseteq M$. Thus by Theorem 6 $S_0/(M \cap S_0) \backsimeq T/M$. This gives a contradiction since S_0 is nil ring.

(*ii*) The proof is similar. □

The following examples show that T is not, in general, a Brown-McCoy radical ring provided that either R is a simple ring with identity or S_0 is a simple ring with identity.

Example 18. Assume that R is a simple ring with identity element and let S be a subring of R simple without identity. Then $S + R[x]x$ is not a Brown-McCoy radical ring.

In fact, note that in this case $R[x]$ is not a Brown-McCoy radical ring. To see that $S + R[x]x$ is not a Brown-McCoy radical ring take a maximal ideal of $R[x]$ which does not contain x and apply Theorem 2.

Example 19. Let R be a simple ring without identity element and S a simple subring of R with identity element. Then $S + R[x]x$ is not a Brown-McCoy radical ring.

The pseudo-radical $ps(R)$ of a ring R is defined as the intersection of all non-zero prime ideals of R (see [3], Section 2).

It is well-known that if there exists an R-disjoint maximal ideal of $R[x]$, then the pseudo radical of R is non-zero. This is not true, in general, for T:

Example 20. Let K a field and $A = S + R[x]x$, where $S = K \times \{0\}$ and $R = K \times K$. Then $M = R[x]x$ is an S-disjoint maximal ideal of A and $ps(R) = 0$.

Now we show that the above result holds for T, provided that S_0 is an essential subring of R.

Corollary 21. Assume that S_0 is an essential subring of R. If M is an ideal of T which is S_0-disjoint and T/M is a simple ring with identity element, then $ps(R) \neq 0$.

Proof. By assumption M is a prime ideal of T. If $R[x]x^n \subseteq M$, by Theorem 6 we have that S_0 is a simple ring and so $I \cap S_0 = S_0$, for any non-zero prime ideal I of R. Now suppose that $R[x]x^n \nsubseteq M$. Then by Proposition 4 and Corollary 11 there exists an ideal L of $R[x]$ which is R-disjoint and $R[x]/L \simeq T/M$. Consequently $ps(R) \neq 0$ by Corollary 2.2 of [3]. □

We denote by ρ the class of all non-zero prime rings R such that for every non-zero ideal I of R, $I \cap Z(R) \neq 0$, where $Z(R)$ is the center of R.

The next corollary extends ([4] , Theorem 4.8).

Corollary 22. Assume that S_0 is a essential subring of a ring R. Then the following conditions are equivalent:
(i) T has an S_0-disjoint ideal M such that $R[x]x^n \nsubseteq M$ and T/M is a simple ring with identity.
(ii) $R \in \rho$ and $ps(R) \neq 0$.
(iii) R is a prime and $ps(R) \cap Z(R) \neq 0$.

Proof. Applying Theorem 2 and Corollary 21 it easily follows from Theorem 4.8 of [4]. □

Recall that the Jacobson radical $J(R)$ of R is equal to the intersection of all (right) primitive ideals of R. It is well-known that the Jacobson radical of the polynomial ring $R[x]$ is equal to $(J(R[x]) \cap R)[x]$. In the next result we denote by J the ideal $J(R[x]) \cap R$ of R. Finally, recall also that a ring R is a Jacobson radical ring if $J(R) = R$.

Proposition 23. $J(T) = J(R[x]) \cap T = \sum_{0 \leq i \leq n-1} (J \cap S_i)[x] + J[x]x^n$.

Proof. Let P be a primitive ideal of T such that $R[x]x^n \nsubseteq P$. Then Theorem 2 implies that $J(R[x]) \cap T \subseteq P$. If $R[x]x^n \subseteq P$, Theorem 6 implies that $J(R[x]) \cap S_0 \subseteq P$. Consequently $J(R[x]) \cap T \subseteq J(T)$.

To get the other inclusion, let $f = a_0 + a_1x + ... + a_kx^k \in J(T)$ and let L be a primitive ideal of $R[x]$. If $R[x]x \nsubseteq L$, then by Theorem 2 $J(T) \subseteq L \cap T$. If $R[x]x \subseteq L$, then $L = L \cap R + R[x]x$. Thus $J(T)x \subseteq J(R[x])$ and it follows that $fx \in J(R[x]) = (J(R[x]) \cap R)[x]$. Hence $f \in J(R[x]) \cap T$ and consequently $J(T) \subseteq J(R[x]) \cap T$. This shows the first equality. The second equality is clear. $\square$

The next proposition extends ([8], Corollary 1).

Proposition 24. If R is nil ring, then T cannot be homomorphically mapped onto a simple primitive ring.

Proof. By contradiction, assume that there exists an ideal P of T such that T/P is a simple primitive ring. If $R[x]x^n \subseteq P$ we have that $S_0/P \cap S_0$ is a primitive nil ring, a contradiction by [8], Corollary 1. Hence $R[x]x^n \nsubseteq P$ and by Proposition 4 there exists an ideal L of $R[x]$ such that $L \cap T = P$ and $R[x]/L \simeq T/P$. Consequently $R[x]/L$ is a simple primitive ring, again a contradiction. $\square$

Proposition 25. Let R be a nil ring. Then T is a Jacobson radical ring if and only if $R[x]$ is Jacobson radical ring.

Proof. First suppose that T is a Jacobson radical ring and there exists a primitive ideal L of $R[x]$. Then $R[x]x \nsubseteq L$, since R is a nil ring. Thus Theorem 2 implies that $L \cap T$ is a primitive ideal of T, which is a contradiction. The converse is similar. $\square$

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