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Inflationary buildup of a vector field condensate and its cosmological consequences

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Abstract. Light vector fields during inflation obtain a superhorizon perturbation spectrum when their conformal invariance is appropriately broken. Such perturbations, by means of some suitable mechanism (e.g. the vector curvaton mechanism), can contribute to the curvature perturbation in the Universe and produce characteristic signals, such as statistical anisotropy, on the microwave sky, most recently surveyed by the Planck satellite mission. The magnitude of such characteristic features crucially depends on the magnitude of the vector condensate generated during inflation. However, in the vast majority of the literature the expectation value of this condensate has so-far been taken as a free parameter, lacking a definite prediction or a physically motivated estimate. In this paper, we study the stochastic evolution of the vector condensate and obtain an estimate for its magnitude. Our study is mainly focused in the supergravity inspired case when the kinetic function and mass of the vector boson is time-varying during inflation, but other cases are also explored such as a parity violating axial theory or a non-minimal coupling between the vector field and gravity. As an example, we apply our findings in the context of the vector curvaton mechanism and contrast our results with current observations.

Keywords: inflation, physics of the early universe, cosmology of theories beyond the SM

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1 Introduction

The recent cosmological observations of the Planck satellite mission have largely confirmed the generic predictions of cosmic inflation, even though they have put substantial tension to and even excluded specific classes of inflationary models [1]. Apart from a red spectral index (which was already known by the WMAP observations) and the constraints on non-Gaussianity [2] the other significant deviation from the so-called vanilla predictions of inflation (such as adiabaticity, Gaussianity and scale invariance) which was constrained by Planck was statistical anisotropy [3]. This amounts to the possibility that there may be a preferred direction in space, which is difficult to account for in the traditional inflationary paradigm, because the latter utilises only scalar fields, which cannot break isotropy (see however, ref. [4]).

This is why, in recent years, there is growing interest of the possible role that vector fields may play in the physics of inflation. Vector fields naturally break isotropy and are also a necessary ingredient of fundamental physics and all the theories beyond the standard model [5]. However, until recently, their role in inflation has been ignored. The pioneering work in ref. [6] was the first to consider the possible contribution of vector fields to the curvature perturbation in the Universe. It was soon realised that such a contribution could be inherently anisotropic and can give rise to statistical anisotropy [7, 8] as demonstrated via the δN-formalism in ref. [9]. The degree of statistical anisotropy due to the direct contribution of the anisotropic perturbations of a vector field, is crucially determined by the magnitude of the vector boson condensate, which corresponds to the homogeneous background zero mode of the vector field. This may generate indirectly additional statistical anisotropy, by mildly anisotropising the inflationary expansion leading to anisotropic inflation [10–17], which renders the perturbations of the scalar inflaton field themselves statistically anisotropic. In this case too, the degree of the anisotropy is determined by the magnitude of the vector condensate. The latter remains significant and it is not diluted by inflationary expansion only because it is replenished by continuous vector field particle production during inflation. The magnitude of the condensate, however, has been taken as a free parameter in the vast majority of considerations so far (see however, ref. [18]). This, not only is incomplete and unrealistic but also it removes constraining power from vector field models, which otherwise can shed some light on the total duration of inflation, necessary in order to have the required condensate created. This adds onto the fact that, as mentioned, the presence of a vector field condensate renders the inflationary expansion mildly anisotropic, which in turn evades the no-hair theorem and opens potentially a window to the initial conditions of inflation [10–17].

In this paper we develop in detail the techniques necessary to calculate the stochastic buildup of an Abelian vector field condensate during inflation and provide specific predictions of the magnitude of such a condensate. We focus mostly in the case of a time-varying kinetic function and mass, because this corresponds to a system which is drastically different from the well-known buildup of a scalar field condensate in ref. [19][1] and can be also motivated by supergravity considerations (see for example [20–23]). We apply our findings in the vector curvaton mechanism of ref. [6] (for a review see refs. [24, 25]) as an example of the predictive power of our results. However, we also look into other models such as a massless Maxwell vector field with varying kinetic function (confirming the result in ref. [18]) (as used in refs. [10–17] for example), an Abelian vector field non-minimally coupled to gravity through a coupling of the form RA^2 [26] and an axial theory, which also considers the effect of the $\propto F\tilde{F}$ term, in the buildup of the vector field condensate [27]. At first approximation we consider quasi-de Sitter inflation, with a subdominant Abelian spectator field.

Throughout our paper, we use natural units with $c=\hbar=k_B=1$ and $8\pi G=m_P^{-2}$, where G is Newton's gravitational constant and $m_P = 2.4 \times 10^{18}$ GeV is the reduced Planck mass.

2 The importance of a vector field condensate

In this paper we study in detail the stochastic buildup of a vector field condensate during inflation. The existence of such a condensate may affect the inflationary expansion and render it mildly anisotropic, thereby evading the no-hair theorem and generating statistical

[1] In contrast to ref. [18], where a massless vector field is considered and the resulting condensate is, in many ways, identical to the scalar field case.

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anisotropy in the inflaton's perturbations [10–17]. Moreover, since the buildup of the condensate is based on the particle production of the vector field perturbations, the condensate is essential in order to quantify the effect on the curvature perturbation that the vector perturbations can have directly. To demonstrate this consider that the statistical anisotropy in the spectrum of the curvature perturbations can be parametrised as [28]

$$\mathcal{P}_\zeta(\boldsymbol{k}) = \mathcal{P}_\zeta^{\rm iso}(k)[1 + g(k)(\boldsymbol{d}\cdot\hat{\boldsymbol{k}})^2 + \cdots], \tag{2.1}$$

where "iso" denotes the isotropic part, $\boldsymbol{d}$ is the unit vector depicting the preferred direction, $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$ is the unit vector along the wavevector $\boldsymbol{k}$ (with k being the modulus of the latter), the ellipsis denotes higher orders and $g(k)$ is the so-called anisotropy parameter, which quantifies the statistical anisotropy in $\mathcal{P}_\zeta$. The latest observations from the Planck satellite suggest that g can be at most 2% [29].

For example, if the curvature perturbation is affected by a single scalar and a single vector field then [9]

$$g = \xi\frac{\mathcal{P}_\| - \mathcal{P}_+}{\mathcal{P}_\phi + \xi\mathcal{P}_+}\,, \tag{2.2}$$

where $\mathcal{P}_\phi$ and $\mathcal{P}_\|$ denote the power spectra of the scalar field ϕ (e.g. the inflaton) and the longitudinal component of the vector field W_μ respectively, while $\mathcal{P}_+ \equiv \frac{1}{2}(\mathcal{P}_L + \mathcal{P}_R)$ with $\mathcal{P}_L$ and $\mathcal{P}_R$ being the spectra of the left and right polarisations of the transverse components of the vector field respectively. The parameter ξ is defined as $\xi \equiv N_W^2/N_\phi^2$, where N_ϕ denotes the amount of modulation of the number of elapsing e-foldings because of the scalar field $N_\phi \equiv \partial N/\partial\phi$, while similarly N_W denotes the amount of modulation of the number of elapsing e-foldings because of the vector field: $N_W = |\boldsymbol{N_W}|$, where $N_W^i \equiv \partial N/\partial W_i$. According to the δN-formalism [9], the curvature perturbation is given by $\zeta = N_\phi\delta\phi + N_W^i\delta W_i + \cdots$, where Einstein summation over the spatial indices $i = 1, 2, 3$ is assumed. Therefore, the value of N_W is necessary to quantify g (through ξ). This value, in turn, is partly determined by the value of the vector field condensate, which we investigate in this paper.

For example, in the vector curvaton scenario [6] we have [9, 24, 25]

$$N_W^i = \frac{2}{3}\hat{\Omega}_{\rm dec}\frac{W_i}{W^2}\,, \tag{2.3}$$

where $\hat{\Omega}_{\rm dec} = \frac{3\Omega_{\rm dec}}{4-\Omega_{\rm dec}} \sim \Omega_{\rm dec}$, with $\Omega_{\rm dec}$ denoting the vector field density parameter at the time of the vector field decay. In the above $W = |\boldsymbol{W}|$ is the magnitude of the vector field condensate and W_i its components.

Similarly, in the end of inflation mechanism, the waterfall at the end of Hybrid Inflation can be modulated by a vector field [7, 8], whose condensate determines N_W. Indeed, in this case [9]

$$N_W^i = N_c\frac{\lambda_W}{\lambda_\phi}\frac{W_i}{\phi_c}\,, \tag{2.4}$$

where λ_ϕ $\{\lambda_W\}$ is the coupling of the interaction term between the waterfall field and the inflaton $\{$vector$\}$ field and $N_c = \partial N/\partial\phi_c$ with ϕ_c being the critical value of the inflaton when the waterfall occurs. Thus, we see again that $N_W^i \propto W_i$, i.e. N_W is determined by the magnitude of the condensate components.

In both the above examples to determine g it is necessary to know W_μ. The value of the latter until now has been taken as a free parameter (see however, ref. [18]). In this paper we calculate it explicitly by considering the stochastic formation of the condensate through

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particle production. Finally, it is important to point out that, apart from g, the components of the condensate also determine the preferred direction itself, because $d = \hat{\boldsymbol{N}}_{\boldsymbol{W}} \equiv \boldsymbol{N}_{\boldsymbol{W}}/N_W$ in eq. (2.1) [9].

3 Our model

In this section we introduce the vector field model which we want to study. To illustrate the growth of the vector condensate we consider the model [21, 22] (see also refs. [18, 30, 31])

$$\mathcal{L} = -\frac{1}{4} f F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \,, \tag{3.1}$$

where f is the kinetic function, m is the mass of A_μ and the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. During inflation, $f = f(t)$ and $m = m(t)$ can be functions of cosmic time t. Following the convention in refs. [21, 22] we consider

$$f \propto a^\alpha, \qquad m \propto a^\beta. \tag{3.2}$$

The motivation for the above model is ample. In supergravity the fundamental functions of the theory are the scalar and Kähler potentials and the gauge kinetic function f of the gauge fields, which is, in principle, a holomorphic function of the scalar fields of the theory. Now, due to Kähler corrections, the scalar fields obtain masses of order the Hubble scale [32, 33] so they are expected to fast-roll down the slopes of the scalar potential leading to a sizable modulation of f. The same is true in the context of superstrings. Thus, time dependence of the vector field kinetic function is natural to consider during inflation.[2] Similar considerations also apply for the masses of vector fields, which can be modulated by varying scalar fields as well. A D-brane inflation example of this model can be seen in refs. [36, 37].

In the context of this paper, though, we will refrain to be grounded on a specific theoretical background, albeit generic. Instead, we will consider that $f = f(t)$ and $m = m(t)$ only, and explore particle production and the formation of a vector field condensate in its own right.[3] The reason, as we will show, is that the model demonstrates an untypical behaviour with the condensate never equilibrating and being dominated by the longitudinal modes, whose stochastic variation is diminishing with time, in contrast to the scalar field case, where the variation is $H/2\pi$ per Hubble time and which equilibrates to the value $\sim H^2/m$ over long enough time [19]. The value of the accumulated condensate is essential in determining observables, such as statistical anisotropy, in all cases (either when the vector field contributes to the curvature perturbation directly or indirectly through rendering the Universe expansion mildly anisotropic).

To study the field dynamics we consider an isotropic inflationary background of quasi-de Sitter kind, i.e. $H \simeq$ cte. Assuming $\dot{H} \simeq 0$, the equations for the temporal (A_t) and spatial ($\boldsymbol{A}$) components of the vector field A_μ are

$$\boldsymbol{\nabla} \cdot \dot{\boldsymbol{A}} - \nabla^2 A_t + \frac{(am)^2}{f} A_t = 0 \tag{3.3}$$

[2]The above reasons led a plethora of authors to consider such a model in cosmology, either to generate a primordial magnetic field [34, 35], or to give rise to anisotropic inflation [10–17] or to directly affect the curvature perturbation [7, 8, 20–22].

[3]Hence, we will not consider effects due to the coupling of vector and scalar field perturbations, which may enhance statistical anisotropy [18, 38].

and

$$\ddot{\boldsymbol{A}} + \left(H + \frac{\dot{f}}{f}\right)\dot{\boldsymbol{A}} + \frac{m^2}{f}\boldsymbol{A} - a^{-2}\nabla^2\boldsymbol{A} = \left(\frac{\dot{f}}{f} - 2\frac{\dot{m}}{m} - 2H\right)\nabla A_t\,. \tag{3.4}$$

Since inflation homogenises the vector field A_μ, we impose the condition $\partial_i A_\mu = 0$, which then translates into $A_t = 0$ [6]. Nevertheless, particle production during inflation gives rise to perturbations of the vector field $\delta A_\mu \equiv (\delta A_t, \delta\boldsymbol{A})$

$$A_\mu(t,\boldsymbol{x}) = A_\mu(t) + \delta A_\mu(t,\boldsymbol{x})\,, \tag{3.5}$$

which we expand in Fourier modes $\delta\mathcal{A}_\mu \equiv (\delta\mathcal{A}_t, \delta\boldsymbol{\mathcal{A}})$ as

$$\delta A_\mu(t,\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}}\,\delta\mathcal{A}_\mu(t,\boldsymbol{k})\,\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,. \tag{3.6}$$

Because $A_t = 0$ for the background vector field, the temporal component is itself a perturbation, i.e. $A_t(t,\boldsymbol{x}) = \delta A_t(t,\boldsymbol{x})$, determined by the spatial field perturbations

$$\delta\mathcal{A}_t + \frac{i\partial_t\left(\boldsymbol{k}\cdot\delta\boldsymbol{\mathcal{A}}\right)}{k^2 + (am)^2/f} = 0\,. \tag{3.7}$$

At this point we introduce the physical vector field

$$\boldsymbol{W} \equiv \sqrt{f}\boldsymbol{A}/a\,. \tag{3.8}$$

Writing eq. (3.4) in terms of the physical vector field perturbation $\delta\boldsymbol{W}$ we have

$$\delta\ddot{\boldsymbol{W}} + 3H\delta\dot{\boldsymbol{W}} + \left[-\frac{1}{4}(\alpha+4)(\alpha-2)H^2 + M^2 - a^{-2}\nabla^2\right]\delta\boldsymbol{W} = p(t)\boldsymbol{\nabla}A_t\,, \tag{3.9}$$

where

$$p(t) = \left(\frac{\dot{f}}{f} - \frac{2\dot{m}}{m} - 2H\right)\sqrt{f}\,a^{-1} \tag{3.10}$$

and

$$M \equiv m/\sqrt{f} \propto a^{\beta-\alpha/2} \tag{3.11}$$

is the time-dependent effective mass of the physical vector field.

To study the quantum production of the vector field we introduce creation/annihilation operators for each polarisation as follows

$$\delta\boldsymbol{W}(t,\boldsymbol{x}) = \sum_{\lambda=L,R,\parallel}\delta\boldsymbol{W}_\lambda(t,\boldsymbol{x})\,, \tag{3.12}$$

where

$$\delta\boldsymbol{W}_\lambda(t,\boldsymbol{x}) \equiv \int\frac{d^3k}{(2\pi)^3}\left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})w_\lambda(t,k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})w^*_\lambda(t,k)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{3.13}$$

and where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$, $k \equiv |\boldsymbol{k}|$ and $\lambda = L, R, \parallel$ labels the *Left* and *Right* transverse and longitudinal polarisations respectively. The polarisation vectors are

$$\boldsymbol{e}_L \equiv \frac{1}{\sqrt{2}}(1,i,0), \qquad \boldsymbol{e}_R = \frac{1}{\sqrt{2}}(1,-i,0), \qquad \boldsymbol{e}_\parallel = (0,0,1)\,. \tag{3.14}$$

The perturbation $\delta\boldsymbol{W}(t,\boldsymbol{x})$ is quantised imposing equal-time commutation relations

$$[\hat{a}_\alpha(\boldsymbol{k}), \hat{a}^\dagger_\beta(\boldsymbol{k}')] = (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\alpha\beta}\,, \tag{3.15}$$

whereby quantum particle production is uncorrelated among different polarisation modes, i.e. $\langle\delta\boldsymbol{W}_\alpha(t,\boldsymbol{x})\,\delta\boldsymbol{W}_\beta(t',\boldsymbol{y})\rangle \propto \delta_{\alpha\beta}$.

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3.1 Case $f \propto a^{-4}$ and $m \propto a$

The reason to focus our attention in this case is twofold. Firstly, the vector field obtains a nearly scale-invariant spectrum of superhorizon perturbations. This means that its effects, e.g. by generating statistical anisotropy in the curvature perturbation, are apparent (and the same) in all scales. Also, there is no "special time" during inflation (i.e. no fine-tuning), when particle production is more pronounced. Thus, the only relevant variable is the total inflationary e-foldings. Apart from simplicity, however, it has been shown that the above behaviour can be an attractor solution if f and m are modulated by the rolling inflaton field, because vector backreaction can adjust the variation of the inflaton accordingly [23].

The second reason to consider such a choice is that it constitutes substantial deviation with respect to the case of a minimally coupled, light scalar field [44]. As we show later on, and in contrast to the case of a light scalar field, the vector field features a non-trivial superhorizon evolution. Moreover, the longitudinal and transverse modes of the vector field obtain different superhorizon perturbation spectra, which then must be treated separately.

Introducing the expansion (3.13) into eq. (3.9) and taking into account that $\alpha = -4$ we obtain the evolution equations for the transverse and longitudinal mode functions [21, 22]

$$\ddot{w}_{L,R} + 3H\dot{w}_{L,R} + \left(\frac{k^2}{a^2} + M^2\right) w_{L,R} = 0\,, \tag{3.16}$$

$$\ddot{w}_{\parallel} + \left(3 + \frac{8}{1+r^2}\right) H\dot{w}_{\parallel} + \left[\frac{24}{1+r^2}H^2 + \left(\frac{k}{a}\right)^2 (1+r^2)\right] w_{\parallel} = 0\,. \tag{3.17}$$

where $r \equiv \frac{aM}{k}$. In the limit $r \gg r_c \gg 1$, where r_c is defined for a given k by the condition that the terms in the square brackets in eq. (3.17) become comparable [21, 22], the equations for $w_{L,R}$ and $w_{\parallel}$ coincide. The solutions to the above equations in the aforementioned limit are found to be

$$w_{L,R}(t,k) = a^{-3/2}\left[\widehat{c}_1 J_{1/2}\left(\frac{M}{3H}\right) + \widehat{c}_2 J_{-1/2}\left(\frac{M}{3H}\right)\right] \tag{3.18}$$

$$w_{\parallel}(t,k) = a^{-3/2}\left[\widehat{c}_3 J_{1/2}\left(\frac{M}{3H}\right) + \widehat{c}_4 J_{-1/2}\left(\frac{M}{3H}\right)\right], \tag{3.19}$$

where the constants $\widehat{c}_i$ are determined by

$$\widehat{c}_1 = \frac{i}{2}\sqrt{\frac{\pi}{H}}\left(\frac{aH}{k}\right)^{3/2}\left(\frac{3H}{M}\right)^{1/2}, \qquad \widehat{c}_2 = \frac{1}{6}\sqrt{\frac{\pi}{H}}\left(\frac{k}{aH}\right)^{3/2}\left(\frac{M}{3H}\right)^{1/2}, \tag{3.20}$$

$$\widehat{c}_3 = i\,\widehat{c}_2, \qquad \widehat{c}_4 = i\,\widehat{c}_1\,. \tag{3.21}$$

In view of eq. (3.16) the transverse modes $w_{L,R}$ behave like a minimally coupled, massive scalar field. Therefore, provided $M \ll H$, the transverse modes $w_{L,R}$ cease to oscillate on superhorizon scales ($k/aH \ll 1$) and obtain an expectation value

$$w_{L,R} \simeq \frac{i}{\sqrt{2k}}\left(\frac{H}{k}\right). \tag{3.22}$$

Also, the first and second derivatives give

$$\dot{w}_{L,R} \simeq -\frac{M^2}{9H}\,w_{L,R}, \qquad \ddot{w}_{L,R} \simeq -\frac{2M^2}{3}\,w_{L,R}\,. \tag{3.23}$$

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Regarding eq. (3.17), although this coincides with eq. (3.16) in the limit $r \gg r_c \gg 1$ (as previously noticed), the longitudinal mode function $w_\parallel$ does not feature the same superhorizon evolution as $w_{L,R}$ due to the different boundary conditions imposed in the subhorizon limit $k/aH \to \infty$ [21, 22]. Owing to this, and in contrast to $w_{L,R}$ (determined by the growing mode $\propto J_{1/2}(M/3H)$), the superhorizon evolution of $w_\parallel$ is dominated by the decaying mode $\propto J_{-1/2}(M/3H)$. In the limit $r \gg r_c \gg 1$ we find

$$w_\parallel \simeq i\, w_{L,R} \left(\frac{3H}{M}\right). \tag{3.24}$$

Since particle production demands $M \ll H$, the above implies $|w_\parallel| \gg |w_{L,R}|$, and the vector field is approximately curl-free on superhorizon scales. Moreover, since $M(t) \propto a^3$ the longitudinal modes feature a fast-roll evolution on superhorizon scales. Owing to this non-trivial evolution we find

$$\dot{w}_\parallel \simeq -3Hw_\parallel, \qquad \ddot{w}_\parallel \simeq 9H^2 w_\parallel\,. \tag{3.25}$$

A similar result arises in the case of a heavy scalar field. If we consider a scalar field ϕ with mass $m_\phi \gg H$, the amplitude of the mode ϕ_k varies as $a^{-3/2}$ on superhorizon scales. In fact, such a scaling begins when the mode is still subhorizon. Consequently, one finds $2\dot{\phi}_k \simeq -3H\phi_k$ and $4\ddot{\phi}_k \simeq 9H^2\phi_k$, similarly to eq. (3.25). However, a heavy scalar field does not become classical on superhorizon scales [39–43]. In our case, though, the vector field (which remains light) indeed becomes classical because the occupation number of the k-modes grows larger than unity. Moreover, owing to the factor $3H/M \gg 1$ in eq. (3.24), the occupation number for longitudinal modes is much larger than the corresponding to transverse modes.

4 Stochastic formalism

The stochastic approach to inflation [44–46] describes the evolution of the inflaton field on patches of superhorizon size during inflation from a probabilistic point of view. The probabilistic nature of the field's evolution on superhorizon scales owes to the quantum particle production undergone by the inflaton field during inflation. Quantum particle production, however, is not exclusive of the inflaton field, but it can be undergone by any light field during inflation as long as it is not conformally coupled to gravity [47]. Consequently, the stochastic approach to the inflaton can be extended to fields other than the inflaton, even if such fields are subdominant during inflation. This is the case we consider in this paper: the vector field A_μ remains subdominant during inflation.[4]

The essence of the stochastic approach to inflation consists in establishing a divide to separate the long and short distance behavior of the field. Such a long/short wavelength decomposition is carried out by introducing a time dependent cut-off scale $k_s \equiv \epsilon a(t)H$, where ϵ is an auxiliary parameter that determines the scale at which the separation is performed. Such scale is usually referred to as the *smoothing* or *coarse-graining* scale. In the simplest approach, which we follow here, the long ($k \ll k_s$) and short ($k \gg k_s$) wavelength parts of the field are split up through a top-hat window function, which implies a sharp transition

[4]We do not consider here the case of vector inflation, which assumes that inflation is driven by hundreds of randomly oriented vector fields [48, 49]. Neither do we consider gauge-flation [50–54]. However, our results are readily extendable in these cases, assuming that there is a quasi-de Sitter background.

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between the two regimes.[5] Following this approach we decompose the physical vector field $\boldsymbol{W}(t,\boldsymbol{x})$ as follows

$$\boldsymbol{W}(t,\boldsymbol{x}) = \boldsymbol{W}_c(t,\boldsymbol{x}) + \boldsymbol{W}_q(t,\boldsymbol{x}) \,, \tag{4.1}$$

$$\boldsymbol{W}_q(t,\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\,\theta(k-k_s)\sum_\lambda \left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})w_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})w^*_\lambda e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{4.2}$$

where $\boldsymbol{W}_c$ $\{\boldsymbol{W}_q\}$ represents the long $\{$short$\}$ wavelength part of the field. Although $\boldsymbol{W}_c$ is only approximately homogeneous, for it contains modes with $0 \le k \le k_s(t)$, according to the separate Universe approach [56] and for the sake of simplicity we disregard its spatial dependence and consider it homogeneous in patches of superhorizon size. Introducing the decomposition (4.1) into eq. (3.4) we arrive at the effective equation of motion for $\boldsymbol{W}_c$

$$\ddot{\boldsymbol{W}}_c + 3H\dot{\boldsymbol{W}}_c + \left(M^2 - a^{-2}\nabla^2\right)\boldsymbol{W}_c - p(t)\boldsymbol{\nabla}(A_t)_c = \boldsymbol{\xi}(t,\boldsymbol{x}) \,, \tag{4.3}$$

where the source term $\boldsymbol{\xi}(t,\boldsymbol{x})$ encodes the behavior of short-wavelength modes and is determined by

$$\boldsymbol{\xi}(t,\boldsymbol{x}) = -\left\{\ddot{\boldsymbol{W}}_q + 3H\dot{\boldsymbol{W}}_q + \left[M(t) - a^{-2}\nabla^2\right]\boldsymbol{W}_q - p(t)\boldsymbol{\nabla}(A_t)_q\right\}. \tag{4.4}$$

In turn, this can be expressed as the superposition of a number of sources (one per polarisation mode) such that $\boldsymbol{\xi}(t,\boldsymbol{x}) = \sum_\lambda \boldsymbol{\xi}_\lambda$, where

$$\begin{aligned}\boldsymbol{\xi}_\lambda(t,\boldsymbol{x}) \equiv -\int \frac{d^3k}{(2\pi)^3}\Bigg\{&\left[\ddot{\theta}(k-k_s) + 3H\dot{\theta}(k-k_s)\right]\left[\boldsymbol{e}_\lambda\hat{a}_\lambda(\boldsymbol{k})w_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda\hat{a}^\dagger_\lambda(\boldsymbol{k})w^*_\lambda e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\\ &+ 2\dot{\theta}(k-k_s)\left[\boldsymbol{e}_\lambda\,\hat{a}_\lambda(\boldsymbol{k})\dot{w}_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda\hat{a}^\dagger_\lambda(\boldsymbol{k})\dot{w}^*_\lambda e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]\Bigg\}.\end{aligned} \tag{4.5}$$

As already mentioned, the probabilistic nature of the field's evolution stems from the quantum production of field perturbations, which in turn originate from the field's vacuum fluctuation.[6] Since the probability distribution of the vacuum fluctuation is gaussian with zero mean, the field's probabilistic evolution can be accounted for by considering a stochastic source of white noise with zero mean, i.e. $\boldsymbol{\xi}(t,\boldsymbol{x})$ is such that

$$\langle\boldsymbol{\xi}(t,\boldsymbol{x})\rangle = 0, \qquad \langle\boldsymbol{\xi}(t,\boldsymbol{x})\,\boldsymbol{\xi}(t',\boldsymbol{y})\rangle \propto \delta(\boldsymbol{x}-\boldsymbol{y})\,\delta(t-t') \,. \tag{4.6}$$

Since we are following the separate Universe approach, we are entitled to neglect the gradient term $a^{-2}\nabla^2\boldsymbol{W}_c$ in eq. (4.3), which is the usual strategy when dealing with scalar fields. Nevertheless, in our case another gradient term appears in the evolution equation: $\boldsymbol{\nabla}(A_t)_c$. Although it is reasonable to expect that the term in $\boldsymbol{\nabla}(A_t)_c$ can be neglected as well in eq. (4.3), it is instructive to compute such a term explicitly and compare it with the rest of the terms in (4.3). We perform this in appendix A.

[5] Other window functions, when applied to separate the long/short distance behavior of the inflaton field, have been shown to modify the CMB angular power spectrum at low multipoles [55].

[6] We do not consider here that the field perturbations correspond to amplifications of excited states, as e.g. in ref. [57].

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After neglecting the gradient terms, the approximate equation of motion for the coarse-grained vector field $\boldsymbol{W}_c$ is

$$\ddot{\boldsymbol{W}}_c + 3H\dot{\boldsymbol{W}}_c + M^2\,\boldsymbol{W}_c = \boldsymbol{\xi}(t,\boldsymbol{x})\,. \tag{4.7}$$

Although this equation is formally the same as that of a coarse-grained massive scalar field, the evolution for the vector field requires careful attention given the existence of polarisation modes and the different perturbation spectra and superhorizon evolutions. In the next section we explain how to circumvent such a difficulty and study the stochastic field evolution for the different polarisation modes in a unified manner.

4.1 Effective evolution equations

Since different polarisation modes obey different equations, it is convenient to separate their contribution to the coarse-grained field. We then introduce the λ-polarised coarse-grained vector field $\boldsymbol{W}_\lambda$ as follows

$$\boldsymbol{W}_\lambda \equiv \int \frac{d^3k}{(2\pi)^3}\,\theta(k_s-k)\left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})w_\lambda(t,k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})w^*_\lambda(t,k)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{4.8}$$

such that $\boldsymbol{W}_c = \sum_\lambda \boldsymbol{W}_\lambda$. Owing to the linearity of eq. (4.7) we obtain a decoupled system of equations, one for each polarisation

$$\ddot{\boldsymbol{W}}_\lambda + 3H\dot{\boldsymbol{W}}_\lambda + M^2\boldsymbol{W}_\lambda = \boldsymbol{\xi}_\lambda\,. \tag{4.9}$$

At this point it is important to recall that, owing to the different boundary conditions obeyed by the various polarisation modes w_λ, the growing {decaying} part of the longitudinal modes ($w_\parallel$) behaves as the decaying {growing} part of the transverse modes ($w_\perp$) on superhorizon scales [cf. eqs. (3.18) and (3.19)]. Therefore, on superhorizon scales the growing {decaying} mode dominates the superhorizon evolution of the transverse {longitudinal} part of the field. This reversal of roles between the longitudinal and transverse modes on superhorizon scales renders eq. (4.9) inappropriate to describe the stochastic evolution of the longitudinal vector $\boldsymbol{W}_\parallel$. This is an important difficulty since the evolution of the classical vector field $\boldsymbol{W}_c$ is dominated by the longitudinal part $\boldsymbol{W}_\parallel$. The reason behind this failure is that the growing part of the solution to eq. (4.9) (for the longitudinal component) is sourced by $\boldsymbol{\xi}_\lambda$, which, in turn, is determined by the decaying mode. Being constant on superhorizon scales, the growing mode soon dominates the evolution of $\boldsymbol{W}_\lambda$, thus leading to an incorrect evolution. In summary, encoding the short-distance behaviour of a massive vector field by means of a stochastic noise source characterised by its two-point function only results in a loss of information, concerning the boundary conditions imposed on the various polarisation modes in the subhorizon regime, that is crucial to properly describe the evolution of the classical field $\boldsymbol{W}_c$. Of course, one can always find the particular solution to eq. (4.9) and remove the growing mode by hand, which solves the problem in a rather ad hoc manner.

Apart from the above, and as anticipated at the end of section 3.1, there exists yet another complication related to the left-hand side of eq. (4.9). The stochastic growth of fields during inflation proceeds due to quantum particle production, which in turn demands that $M \ll H$. In the scalar field case, particle production thus implies a slow-roll motion that allows us to neglect second time derivatives on superhorizon scales. Nevertheless, in the case of a massive vector field, when the longitudinal component is physical, one cannot afford such a carelessness. The reason is that $\ddot{\boldsymbol{W}}_\lambda$ results in a term of order $M^2\boldsymbol{W}_\lambda$ for the transverse components and of order $H^2\boldsymbol{W}_\lambda$ for the longitudinal one. Using eqs. (3.23) and (3.25) as guidance, a rough estimate suggests that $\ddot{\boldsymbol{W}}_{L,R} \sim -\frac{2M^2}{3}\boldsymbol{W}_{L,R}$ and $\ddot{\boldsymbol{W}}_\parallel \sim 9H^2\boldsymbol{W}_\parallel$. Consequently, $\ddot{\boldsymbol{W}}_\parallel$ cannot be absentmindedly thrown away even if the vector field is light enough to be produced during inflation. Despite this shortcoming, one might still insist in using eq. (4.9) as a starting point for the stochastic analysis. The basis to stick to this point of view relies on the fact that, on sufficiently superhorizon scales, the evolution equation of transverse and longitudinal modes coincides. Therefore, consistency demands that the second order equation for the various polarisation modes $\boldsymbol{W}_\lambda$ be the same.

Our purpose now is to develop the stochastic formalism for vector fields able to address the aforementioned difficulties while using the same second order equation for $\boldsymbol{W}_\lambda$ as a starting point. The approach followed below consists in introducing the coarse-grained conjugate momentum $\boldsymbol{\Pi}_\lambda$, thus reducing the second order equation to a system of first order equations, and then eliminating $\boldsymbol{\Pi}_\lambda$ utilising the superhorizon behaviour of the perturbation modes w_λ. Following this procedure we manage to arrive at a *single* first order equation providing a correct description of the stochastic evolution of $\boldsymbol{W}_\lambda$. We want to emphasise that our method goes beyond the Hamiltonian description of stochastic inflation [58–61]. Following the latter, one arrives at first order system leading to a Fokker-Planck equation in the variables $\boldsymbol{W}_\lambda$ and $\boldsymbol{\Pi}_\lambda$. In our case, however, we manage to obtain a *single* first order equation for $\boldsymbol{W}_\lambda$ leading to a Fokker-Planck equation in the variable $\boldsymbol{W}_\lambda$ only. Apart from this, our procedure can be successfully applied to scalar fields with a non-negligible scale-dependence (i.e. the case of a heavy field) and also away from the slow-roll regime when second-time derivatives cannot be neglected [62]. In the following we provide the details of our method.

As advertised, our approach towards a *single* first order equation for $\boldsymbol{W}_\lambda$ consists in introducing the coarse-grained conjugate momentum

$$\boldsymbol{\Pi}_\lambda \equiv \int \frac{d^3k}{(2\pi)^3}\,\theta(k_s - k)\left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})\dot{w}_\lambda(t,k)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})\dot{w}^*_\lambda(t,k)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{4.10}$$

After neglecting gradient terms, the equivalent first order stochastic equations are[7]

$$\dot{\boldsymbol{\Pi}}_\lambda + 3H\boldsymbol{\Pi}_\lambda + M^2\boldsymbol{W}_\lambda = \boldsymbol{\xi}_{\pi_\lambda}\,, \tag{4.11}$$

$$\dot{\boldsymbol{W}}_\lambda = \boldsymbol{\Pi}_\lambda + \boldsymbol{\xi}_{W_\lambda} \tag{4.12}$$

where

$$\boldsymbol{\xi}_{\pi_\lambda} = -\int \frac{d^3k}{(2\pi)^3}\,\dot{\theta}(k - k_s)\left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})\dot{w}_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})\dot{w}^*_\lambda e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right], \tag{4.13}$$

$$\boldsymbol{\xi}_{W_\lambda} = -\int \frac{d^3k}{(2\pi)^3}\,\dot{\theta}(k - k_s)\left[\boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\hat{a}_\lambda(\boldsymbol{k})w_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\hat{a}^\dagger_\lambda(\boldsymbol{k})w^*_\lambda e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right] \tag{4.14}$$

are stochastic noise sources for $\boldsymbol{\Pi}_\lambda$ and $\boldsymbol{W}_\lambda$, respectively. Comparing with eq. (4.5) it is straightforward to show that $\boldsymbol{\xi}_\lambda = a^{-3}(a^3\boldsymbol{\xi}_{W_\lambda})^{\cdot} + \boldsymbol{\xi}_{\pi_\lambda}$.

Instead of deriving a Fokker-Planck equation for the probability density (as a function of $\boldsymbol{W}_\lambda$ and $\boldsymbol{\Pi}_\lambda$) from the first order system (4.11)–(4.12), our approach consists in using the solution to the mode functions w_λ to reduce the first order system. Indeed, solving the equation of motion for w_λ amounts to solving eq. (4.11), whereas eq. (4.12) becomes an

[7] In the context of scalar fields, separate stochastic equations for the field and its conjugate momentum were introduced in refs. [58–61].

identity. Nevertheless, we proceed to manipulate eq. (4.12) to eliminate $\mathbf{\Pi}_\lambda$, thus arriving at a first order equation in $\boldsymbol{W}_\lambda$. The essence of our method boils down to utilise the superhorizon behavior of w_λ to find the function

$$\mathcal{F}_\lambda \equiv - \lim_{k/aH \to 0} \frac{\dot{w}_\lambda}{w_\lambda} \,, \tag{4.15}$$

in general time-dependent, that allows us to write $\mathbf{\Pi}_\lambda$ in terms of $\boldsymbol{W}_\lambda$. Consequently, eq. (4.12) can be rewritten as

$$\dot{\boldsymbol{W}}_\lambda + \mathcal{F}_\lambda(t)\, \boldsymbol{W}_\lambda \simeq \boldsymbol{\xi}_{W_\lambda} \,, \tag{4.16}$$

thus dispatching non-negligible second derivatives in eq. (4.9) and paving the road towards a Fokker-Planck equation for the probability density as a function of $\boldsymbol{W}_\lambda$ only. Of course, we could proceed similarly eliminating $\boldsymbol{W}_\lambda$ from eq. (4.11) to obtain

$$\dot{\mathbf{\Pi}}_\lambda + \left(3H - M^2 \mathcal{F}_\lambda^{-1}\right) \mathbf{\Pi}_\lambda \simeq \boldsymbol{\xi}_{\pi_\lambda} \,, \tag{4.17}$$

and derive a Fokker-Planck equation in the variable $\mathbf{\Pi}_\lambda$. However, in the following we are simply concerned with $\boldsymbol{W}_\lambda$, and hence we will work with eq. (4.16) only.

For the transverse components, since eq. (3.23) implies $\mathbf{\Pi}^{(s)}_{L,R} \simeq -\frac{M^2}{9H} \boldsymbol{W}_{L,R}$ we arrive at $\mathcal{F}_{L,R}(t) \equiv \frac{M^2}{9H}$, whereas for the longitudinal component eq. (3.25) implies $\mathbf{\Pi}^{(s)}_{\parallel} \simeq -3H\boldsymbol{W}_{\parallel}$ and we obtain $\mathcal{F}_{\parallel} \equiv 3H$.

Regarding the stochastic source, using eq. (4.14) and the commutation relations in eq. (3.15) the self-correlation function can be readily computed to be

$$\langle \boldsymbol{\xi}_{W_\alpha}(t)\, \boldsymbol{\xi}_{W_\beta}(t') \rangle = \mathcal{D}_\alpha\, \delta_{\alpha\beta}\, \delta(t-t') \,, \tag{4.18}$$

where

$$\mathcal{D}_\alpha \equiv H \left(\lim_{k \to k_s} \frac{1}{2\pi^2} k^3 |w_\alpha|^2 \right) \tag{4.19}$$

is the diffusion coefficient of the λ-polarised vector. The parenthesis represents the perturbation spectrum of $\boldsymbol{W}_\lambda$ at the coarse-graining scale, and coincides with the spectrum at horizon crossing when perturbation spectrum is flat. Using the superhorizon behavior of the mode functions $w_{L,R}$ and $w_\parallel$ in eqs. (3.22) and (3.24) we obtain

$$\langle \boldsymbol{\xi}_{W_{L,R}}(t)\, \boldsymbol{\xi}_{W_{L,R}}(t') \rangle \simeq \frac{H^3}{4\pi^2}\, \delta(t-t') \tag{4.20}$$

for the transverse noise source and

$$\langle \boldsymbol{\xi}_{W_\parallel}(t)\, \boldsymbol{\xi}_{W_\parallel}(t') \rangle \simeq \frac{H^3}{4\pi^2} \left(\frac{3H}{M} \right)^2 \delta(t-t') \tag{4.21}$$

for the longitudinal one.

4.2 Fokker-Planck equation

In this section we obtain and solve the Fokker-Planck equations that follow from eq. (4.16). Given such a stochastic differential equation, it is a standard procedure to derive the corresponding Fokker-Planck equation [63]. To do so, we first introduce an arbitrary basis of orthonormal vectors $\boldsymbol{u}_i$ $(i = 1, 2, 3)$ in position-space. In such basis, the components $W_i^\lambda \equiv \boldsymbol{W}_\lambda \cdot \boldsymbol{u}_i$ of the λ-polarised vector are

$$W_i^\lambda(t) = \int \frac{d^3k}{(2\pi)^3}\, \theta(k_s - k) \left[\boldsymbol{u}_i \cdot \boldsymbol{e}_\lambda(\hat{\boldsymbol{k}})\, \hat{a}_\lambda(\boldsymbol{k}) w_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{u}_i \cdot \boldsymbol{e}^*_\lambda(\hat{\boldsymbol{k}})\, \hat{a}^\dagger_\lambda(\boldsymbol{k}) w_\lambda e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right]. \tag{4.22}$$

It can be shown that the operators $W^\lambda_{1,2,3}$ are formally the same, and therefore we can write the vector $\boldsymbol{W}_\lambda$ in terms of a scalar-like operator W_λ defined as follows

$$\boldsymbol{W}_\lambda \equiv W_\lambda\,(1,1,1)\,. \tag{4.23}$$

The expectation value of W_λ thus determines the modulus of the λ-polarised vector $\boldsymbol{W}_\lambda$. Of course, the same applies to the stochastic source $\boldsymbol{\xi}_{W_\lambda}$, which can be expressed as

$$\boldsymbol{\xi}_{W_\lambda} \equiv \xi_{W_\lambda}\,(1,1,1) \tag{4.24}$$

after introducing the scalar-like noise ξ_{W_λ}. Using the above, the vector eq. (4.16) can be rewritten as

$$\dot{W}_\lambda + \mathcal{F}_\lambda(t) W_\lambda \simeq \xi_{W_\lambda}\,. \tag{4.25}$$

In appendix B we compute the mean-square field using the solution to eq. (4.25) and compare the result with the obtained by solving the Fokker-Planck equation in this section.

Similarly to eq. (4.18), to determine the magnitude of the self-correlation for the scalar-like sources ξ_{W_λ} we introduce the diffusion coefficients $D_\lambda(t)$ as follows

$$\langle \xi_{W_\alpha}(t)\, \xi_{W_\beta}(t')\rangle = D_\alpha(t)\, \delta_{\alpha\beta}\delta(t-t')\,. \tag{4.26}$$

Taking into account (4.24) and comparing eqs. (4.18) and (4.26) we find

$$D_\lambda(t) = \frac{1}{3}\,\mathcal{D}_\lambda(t)\,, \tag{4.27}$$

which results in the following transverse and longitudinal coefficients

$$D_{L,R} = \frac{1}{3}\frac{H^3}{4\pi^2}, \qquad D_\parallel(t) = \frac{1}{3}\frac{H^3}{4\pi^2}\left(\frac{3H}{M}\right)^2. \tag{4.28}$$

The Fokker-Planck equation that follows from eqs. (4.25) and (4.26) is [63]

$$\frac{\partial \rho_\lambda}{\partial t} = \frac{\partial}{\partial W_\lambda}\left(\mathcal{F}_\lambda W_\lambda \rho_\lambda\right) + \frac{1}{2}\frac{\partial^2}{\partial W_\lambda^2}\Big(D_\lambda\, \rho_\lambda\Big)\,, \tag{4.29}$$

which determines the probability density $\rho_\lambda(W_\lambda, t)$ for the expectation value of the scalar-like operator W_λ, and hence the modulus of $\boldsymbol{W}_\lambda$. The solution to eq. (4.29) can be readily obtained by Fourier transforming ρ_λ. Using

$$\rho_\lambda(W_\lambda, t) = \int_{-\infty}^{\infty} e^{isW_\lambda}\widetilde{\rho_\lambda}(t,s)\, ds\,, \tag{4.30}$$

the equation for $\widetilde{\rho_\lambda}$ is

$$\dot{\widetilde{\rho_\lambda}} = -\mathcal{F}_\lambda(t)\, s\, \partial_s \widetilde{\rho_\lambda} - \frac{s^2}{2}\, D_\lambda(t) \widetilde{\rho_\lambda}\,. \tag{4.31}$$

To solve this equation we consider that the expectation value of W_λ begins sharply peaked around the value $W_\lambda = W_\lambda(0)$, which translates into $\rho_\lambda(W_\lambda, 0) = \delta(W_\lambda - W_\lambda(0))$. Imposing such condition, the solution to eq. (4.31) is

$$\widetilde{\rho_\lambda}(t) = \frac{1}{2\pi} \exp\left[-i\mu_\lambda(t)s - \frac{\sigma_\lambda^2(t)}{2}s^2\right] \tag{4.32}$$

where

$$\mu_\lambda(t) \equiv W_\lambda(0) \exp\left[-\int_0^t \mathcal{F}_\lambda(\tau)d\tau\right] \tag{4.33}$$

and

$$\sigma_\lambda^2(t) \equiv \int_0^t \exp\left[-2\int_{\bar\tau}^t \mathcal{F}_\lambda(\tau)d\tau\right] D_\lambda(\bar\tau)d\bar\tau\,. \tag{4.34}$$

Integrating now eq. (4.30) we find

$$\rho_\lambda(W_\lambda, t) = \frac{1}{\sqrt{2\pi}\,\sigma_\lambda(t)} \exp\left[-\frac{(W_\lambda - \mu_\lambda(t))^2}{2\sigma_\lambda^2(t)}\right], \tag{4.35}$$

i.e. a Gaussian distribution with mean $\mu_\lambda(t)$ and variance $\sigma_\lambda^2(t)$. Therefore,

$$\langle W_\lambda\rangle = \mu_\lambda(t) \quad \text{and} \quad \langle W_\lambda^2\rangle = \mu_\lambda^2(t) + \sigma_\lambda^2(t)\,. \tag{4.36}$$

4.2.1 Transverse vector $W_\perp$

Using $\mathcal{F}_{L,R} = \frac{M^2}{9H}$ and $D_{L,R} = \frac{1}{3}\frac{H^3}{4\pi^2}$ for the transverse polarisations we find

$$\mu_{L,R}(t) = \exp\left[-\frac{\left(1-a^{-6}\right)M^2}{54H^2}\right] W_{L,R}(0) \simeq \exp\left(-\frac{M^2}{54H^2}\right) W_{L,R}(0) \tag{4.37}$$

and

$$\sigma_{L,R}^2(t) = \frac{H^2 \exp\left(-\frac{M^2}{27H^2}\right)\left[Ei\left(\frac{M^2}{27H^2}\right) - Ei\left(\frac{M_0^2}{27H^2}\right)\right]}{72\pi^2}\,, \tag{4.38}$$

where $Ei(x) = -\int_{-x}^{\infty} t^{-1}e^{-t}dt$ is the exponential integral [64]. When $M^2 \ll H^2$, using the expansion $Ei(x>0) = \gamma + \ln x + \sum_{n=1}^{\infty}\frac{x^n}{n\,n!}$ we find

$$\mu_{L,R} \simeq W_{L,R}(0), \qquad \sigma_{L,R}^2(t) \simeq \frac{H^3 t}{12\pi^2}\,, \tag{4.39}$$

whereas for $M^2 \gg H^2$, using the asymptotic expansion $Ei(x) \simeq \frac{e^x}{x}\sum_{n=0}^{\infty}\frac{n!}{x^n}$ for $x \gg 1$ [64] we obtain

$$\mu_{L,R} \simeq 0, \qquad \sigma_{L,R}^2(t) \simeq \frac{H^2}{72\pi^2}\left(\frac{27H^2}{M^2}\right) \ll H^2\,, \tag{4.40}$$

Using eq. (4.23) and summing over polarisations we can translate the above results in terms of the transverse vector $\boldsymbol{W}_\perp \equiv \boldsymbol{W}_L + \boldsymbol{W}_R$. Considering the case $M^2 \ll H^2$ only we have

$$\langle \boldsymbol{W}_\perp\rangle \simeq \boldsymbol{W}_\perp(0), \qquad \langle \boldsymbol{W}_\perp^2\rangle \simeq \boldsymbol{W}_\perp^2(0) + 2\left(\frac{H^3 t}{4\pi^2}\right). \tag{4.41}$$

Although the variance computed in eq. (4.39) grows linearly with time for $M^2 \ll H^2$, our result does not feature an asymptotic value corresponding to an equilibrium state, known to exist in the case of light scalar fields [19]. Since the transverse modes of the massive vector field behave like a light scalar field, an analogous equilibrium value might be expected. The reason for its non-appearance in eqs. (4.39) is clear. Firstly, the effective mass of the physical vector field grows as $M \propto a^3$. And secondly, inflation can proceed even if $M \gg H$ since the vector field does not play the role of inflaton (and in any case, its energy density is kept constant). If inflation continues after $M^2 \sim H^2$, the rapid scaling of M makes $M^2 \gg H^2$ in less than one e-folding. Consequently, at sufficiently long times the field fluctuations do not approach any asymptotic value, but undergo exponential suppression as shown in eq. (4.40).

Although no equilibrium fluctuation appears in eq. (4.39), it is instructive to compare the mean-square field $\langle W_{L,R}^2\rangle$ with the *instantaneous* equilibrium value. By such instantaneous equilibrium we refer to the asymptotic value which the mean-square would feature for a certain value of M, namely $\langle W_{L,R}^2\rangle_{\rm eq} = \frac{H^4}{8\pi^2 M^2} \propto a^{-6}$ (see eq. (5.7)). At a given time t before the end of inflation we have

$$\frac{\langle W_{L,R}^2\rangle_{\rm eq}}{\langle W_{L,R}^2\rangle} \sim \left(\frac{H}{M_e}\right)^2 \frac{e^{6N_e}}{N}\,, \tag{4.42}$$

where $M_{\rm e}$ is the effective mass at the end of inflation, N is the number of e-foldings elapsed since the beginning of inflation and N_e the number of e-foldings remaining until the end of inflation. If $M_e \sim H$, at the end of inflation N corresponds to the number of the total inflationary e-foldings $N = N_{\rm tot}$ and the mean-square becomes $\langle W_{L,R}^2(t_e)\rangle \sim N_{\rm tot}\langle W_{L,R}^2(t_e)\rangle_{\rm eq}$, and hence much larger than the equilibrium value corresponding to the field's effective mass at the end of inflation, M_e. Although this result may seem surprising, it clearly follows because $\langle W_{L,R}^2\rangle \propto N$ [cf. eqs. (4.36) and (4.39)] while the scaling of M makes the instantaneous equilibrium value decrease as a^{-6}. Provided M_e is sufficiently close to H, only a moderate amount of inflation is needed for $\langle W_{L,R}^2\rangle$ to be above its instantaneous equilibrium value by the end of inflation. On the contrary, if $M_e \ll H$, the mean-square $\langle W_{L,R}^2\rangle$ remains well below its final equilibrium amplitude unless an exponentially large number of e-foldings is considered. This may be the case if eternal inflation is considered [65]. Finally, if $M^2 \gg H^2$ during inflation the condensate becomes exponentially suppressed very quickly, as indicated in eq. (4.40).

4.2.2 Longitudinal vector $W_\parallel$

Using now $\mathcal{F}_\parallel = 3H$ and $D_{L,R} = \frac{1}{3}\frac{H^3}{4\pi^2}\left(\frac{3H}{M}\right)^2$ for the longitudinal polarisation we find

$$\mu_\parallel(t) = \frac{M_0}{M}\,W_\parallel(0) \tag{4.43}$$

and

$$\sigma_\parallel^2(t) = \frac{1}{3}\left(\frac{3H}{M}\right)^2 \frac{H^3 t}{4\pi^2}\,. \tag{4.44}$$

Using eqs. (4.23) and (4.36) to translate the above results in terms of the longitudinal vector $\boldsymbol{W}_\parallel$ we find

$$\langle \boldsymbol{W}_\parallel\rangle \simeq \frac{M_0}{M}\,\boldsymbol{W}_\parallel(0) \tag{4.45}$$

and

$$\langle \boldsymbol{W}_\parallel^2\rangle \simeq \left(\frac{M_0}{M}\right)^2 \boldsymbol{W}_\parallel^2(0) + \left(\frac{3H}{M}\right)^2 \frac{H^3 t}{4\pi^2}\,. \tag{4.46}$$

Similarly to the transverse field, owing to the scaling of M the variance in eq. (4.44) does not exhibit an asymptotic equilibrium value. Comparing the mean-square $\langle W_\parallel^2 \rangle$ (obtained from eq. (4.36)) with its instantaneous equilibrium amplitude $\langle W_\parallel^2 \rangle_{\rm eq} = \left(\frac{3H}{M}\right)^2 \frac{H^4}{8\pi^2 M^2}$ (see eq. (5.8)) we obtain

$$\frac{\langle W_\parallel^2 \rangle_{\rm eq}}{\langle W_\parallel^2 \rangle} \sim \left(\frac{H}{M_e}\right)^2 \frac{e^{6N_e}}{N}\,, \tag{4.47}$$

after neglecting the pre-inflationary fluctuation in the longitudinal vector. Since this equation coincides with (4.42) the conclusions that apply for the transverse vector (see below eq. (4.42)) are also valid for the longitudinal vector.

5 Other cases of interest

5.1 Scale invariant spectrum with $f \propto a^2$ and $m \propto a$

A nearly scale-invariant spectrum of superhorizon perturbations can also be achieved provided the kinetic function f and the mass m vary as [21, 22]

$$f \propto a^2 \quad \text{and} \quad m \propto a\,, \tag{5.1}$$

which corresponds to $\alpha = 2$ and $\beta = 1$. In this case, the effective mass M remains constant.[8]

The equation that follows from eq. (3.9) for the transverse mode is

$$\ddot{w}_{L,R} + 3H\dot{w}_{L,R} + \left(\frac{k^2}{a^2} + M^2\right) w_{L,R} = 0\,. \tag{5.2}$$

Imposing that $w_{L,R}$ matches the Bunch-Davies (BD) vacuum solution in the subhorizon regime $k/aH \to \infty$ we obtain

$$w_{L,R} = a^{-3/2}\sqrt{\frac{\pi}{4H}}\, e^{i\pi(\nu+1/2)/2} H_\nu^{(1)}(k/aH)\,, \tag{5.3}$$

where $\nu^2 = 9/4 + M^2/H^2$. On the other hand, the longitudinal mode function satisfies

$$\ddot{w}_\parallel + \left(3 + \frac{2}{1+r^2}\right) H\dot{w}_\parallel + \left(\frac{k^2}{a^2} + M^2\right) w_\parallel = 0\,, \tag{5.4}$$

which coincides with eq. (5.2) when $r \gg 1$, attained on superhorizon scales. However, on superhorizon scales we have $w_\parallel \simeq -\frac{H}{\sqrt{2}k^{3/2}}\frac{3H}{M}$ for $M \ll 3H$, thus remaining approximately constant. Nevertheless, the amplitude of $w_\parallel$ is larger than the transverse function $w_{L,R}$ by a factor of $\frac{3H}{M} \gg 1$, hence $\boldsymbol{W}_c$ is approximately longitudinal. Also on superhorizon scales we find

$$\dot{w}_\lambda \simeq -\frac{M^2}{3H} w_\lambda, \qquad \ddot{w}_\lambda \simeq \left(\frac{M}{3H}\right)^2 M^2 w_\lambda\,, \tag{5.5}$$

where λ labels now any of the three polarisations.

[8]This case cannot correspond to a gauge field, since, were it the case, the kinetic function would be inversely proportional to the gauge coupling $f \propto e^{-2}$, which would render the theory strongly coupled during inflation, as $f = 1$ at the end. However, note that a massive Abelian vector boson does not need necessarily to be a gauge field as it is renormalisable [66].

Using the above and proceeding similarly to the previous case we find $\mathcal{F}_\lambda = \frac{M^2}{3H}$ in eq. (4.25), whereas the diffusion coefficients are the same as in eq. (4.28). Using now eqs. (4.33) and (4.34) we obtain the mean field

$$\mu_\lambda(t) = W_\lambda(0)\, e^{-\frac{M^2 t}{3H}} \tag{5.6}$$

for each polarisation, and the variances[9]

$$\sigma^2_{L,R} = \frac{H^4}{8\pi^2 M^2}\left(1 - e^{-\frac{2M^2 t}{3H}}\right) \tag{5.7}$$

and

$$\sigma^2_{\parallel} = \frac{H^4}{8\pi^2 M^2}\left(\frac{3H}{M}\right)^2\left(1 - e^{-\frac{2M^2 t}{3H}}\right). \tag{5.8}$$

At sufficiently early times $M^2 t \ll H$, the computed variances grow linearly with time, approaching their equilibrium amplitude at late times $M^2 t \gg H$. As discussed in section 4.2, the equilibrium amplitude becomes apparent now thanks to the constancy of M.

5.2 Massless vector field

We consider a massless vector field with a time-dependent Maxwell term

$$\mathcal{L} = -\frac{1}{4} f F_{\mu\nu}F^{\mu\nu}\,, \tag{5.9}$$

with $f \propto a^\alpha$. Systems similar to this have been studied in [67] and have been extensively considered for the formation of a primordial magnetic field [34, 35], or the creation of a vector field condensate in order to render inflation mildly anisotropic [10–17]. The buildup of a vector field condensate, in this case, has been considered in ref. [18] for $\alpha = -4$. We generalise and confirm this result.

The equation for the potential vector field is

$$\ddot{\boldsymbol{A}} + \left(H + \frac{\dot f}{f}\right)\dot{\boldsymbol{A}} - a^{-2}\nabla^2 \boldsymbol{A} = 0\,, \tag{5.10}$$

which follows from eq. (3.4) after taking $m = 0$ and $A_t = 0$. Since a massless vector field has two physical degrees of freedom only, the sum in eq. (3.12) runs over transverse polarisations, i.e. $\lambda = L, R$. Also, the commutation relations satisfied by the transverse modes are as in eq. (3.15).

Using $M = 0$ and $A_t = 0$ in eq. (3.9) we obtain the equation of motion for the transverse modes of the physical vector field

$$\ddot w_{L,R} + 3H\dot w_{L,R} + \left(m^2_{\rm eff} + \frac{k^2}{a^2}\right) w_{L,R} = 0\,, \tag{5.11}$$

[9]Note that since the variances in eqs. (5.7) and (5.8) refer to one of the three possible polarisations of the massive vector field, a factor of three is missing with respect to the scalar field case, for which $\langle\phi^2\rangle = \frac{3H^4}{8\pi^2 m_\phi^2}$ in the equilibrium state [19].

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where $m_{\rm eff}^2 \equiv -\frac{1}{4}(\alpha+4)(\alpha-2)H^2$. Demanding that $w_{L,R}$ matches the BD vacuum solution in the subhorizon limit, the solutions to the above equation are [6]

$$w_{L,R} = a^{-3/2}\sqrt{\frac{\pi}{4H}}\, e^{i\pi(\nu+1/2)/2} H_\nu^{(1)}(k/aH)\,, \tag{5.12}$$

where now $\nu^2 = 9/4 - m_{\rm eff}^2/H^2 = (1+\alpha)^2/4 > 0$. For the particular values $\alpha = -4, 2$, both corresponding to $\nu = 3/2$, a flat perturbation spectrum follows. Allowing α to take on any other value we find

$$\dot{w}_{L,R} \simeq \left(\nu - \frac{3}{2}\right) H\, w_{L,R}, \qquad \ddot{w}_{L,R} \simeq \left(\nu - \frac{3}{2}\right)^2 H^2\, w_{L,R} \tag{5.13}$$

in the superhorizon regime $k/aH \to 0$.

Performing the coarse-graining of the physical vector field, the equation of motion for $\boldsymbol{W}_{L,R}$ is given by eq. (4.9) with M^2 replaced by $m_{\rm eff}^2$. Using the expression for $\dot{w}_{L,R}$ in eq. (5.13) to compute $\boldsymbol{\Pi}_{L,R}$ (see eq. (4.10)) and comparing with eq. (4.16) we obtain $\mathcal{F}_{L,R} = \left(\frac{3}{2} - \nu\right) H$. Using eqs. (4.19) and (5.12) we find the diffusion coefficient

$$\mathcal{D}_{L,R} = \left(\frac{H^3}{4\pi^2}\right) \frac{4^{\nu-1/2}\Gamma^2(\nu)\, \epsilon^{3-2\nu}}{\pi}\,. \tag{5.14}$$

We note that the above grows unbounded as ν increases. This is because for large $\nu^2 \gg 1$ the physical field becomes tachyonic in the subhorizon regime. Therefore, by the time of horizon exit the amplitude w_λ has grown exponentially. Using eqs. (4.27), (4.33) and (4.34) we obtain the mean field

$$\mu_\lambda(t) = e^{(\nu-3/2)Ht} W_\lambda(0) \tag{5.15}$$

and the variance

$$\sigma_\lambda^2(t,k) = \frac{2^{-3+2\nu}\left[1 - e^{-Ht(3-2\nu)}\right] H^2\Gamma(\nu)^2\epsilon^{3-2\nu}}{\pi^3(3-2\nu)}\,. \tag{5.16}$$

The case of an exactly massless field ($m_{\rm eff}^2 = 0$, i.e. $\alpha = -1 \pm 3$) for which $\sigma_\lambda^2 = \frac{H^3 t}{4\pi^2}$ is trivially recovered in the limit $\nu \to 3/2$, confirming thereby the findings of ref. [18].

5.3 Non-minimally coupled vector field

We consider now a vector field non-minimally coupled to gravity. This theory has been studied, for example, in refs. [9, 26]. In refs. [68–70] the theory has been criticised for giving rise to ghosts, corresponding to the longitudinal perturbations, when subhorizon. However, the existence of ghosts and their danger to the stability of the theory is still under debate, see for example ref. [71].

Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}\,\gamma R A_\mu A^\mu\,, \tag{5.17}$$

where R is the Ricci scalar and γ is a constant. During de Sitter inflation $R \simeq -12H^2$, and the effective mass for the vector field is

$$m_{\rm eff}^2 = m^2 + \gamma R \simeq m^2 - 12\gamma H^2. \tag{5.18}$$

After taking $f = 1$ and substituting $m^2 \to m^2_{\rm eff}$, the former Lagrangian density can be considered a special case of the more general Lagrangian density in section 3, which then simplifies the study of the condensate formation. The equation for the vector field $\boldsymbol{A}$ is

$$\ddot{\boldsymbol{A}} + H\dot{\boldsymbol{A}} + m^2_{\rm eff}\boldsymbol{A} - a^{-2}\nabla^2\boldsymbol{A} = -2H\nabla A_t \tag{5.19}$$

and the evolution for the perturbation $\delta\boldsymbol{W}$ is obtained by replacing $M^2 \to m^2_{\rm eff}$ and taking $\alpha = 0$ in eq. (3.9).

5.3.1 Transverse modes

For the transverse mode functions $w_{L,R}$ we have

$$\ddot{w}_{L,R} + 3H\dot{w}_{L,R} + \left(2H^2 + m^2_{\rm eff} + \frac{k^2}{a^2}\right) w_{L,R} = 0\,, \tag{5.20}$$

whose solution, while matching the vacuum in the subhorizon limit, is given by eq. (5.12) with $\nu^2 \equiv 1/4 - m^2_{\rm eff}H^2$.

With $\gamma \approx 0$, the perturbation spectrum is $\mathcal{P}_{L,R} \propto k^2$ when $m \ll H$, thus reproducing the vacuum value. On the other hand, if $m \gg H$, the vector is a heavy field and the buildup of fluctuations becomes suppressed. Only when $m^2 \approx -2H^2$ can the vector field be substantially produced during inflation [6]. In such case, the evolution of $w_{L,R}$ is determined by eq. (5.12) with $\nu \approx 3/2$ (which corresponds to either $\alpha \approx -4$ or $\alpha \approx 2$). When $\gamma \neq 0$, the vector field obtains a flat perturbation spectrum ($\nu \approx 3/2$) provided γ is tuned according to

$$\gamma \approx \frac{1}{6}\left(1 + \frac{m^2}{2H^2}\right). \tag{5.21}$$

In both cases, the mode functions $w_{L,R}$ satisfy eq. (5.13) in the superhorizon regime. Consequently, the mean and variance of the transverse vector condensate are given by eqs. (5.15) and (5.16) using $\nu^2 = 1/4 - m^2_{\rm eff}H^2$.

5.3.2 Longitudinal modes

The evolution equation for the longitudinal modes is

$$\ddot{w}_\parallel + \left(3 + \frac{2k^2}{k^2 + a^2 m^2_{\rm eff}}\right) H\dot{w}_\parallel + \left(2H^2 + m^2_{\rm eff} + \frac{2H^2k^2}{k^2 + a^2m^2_{\rm eff}} + \frac{k^2}{a^2}\right) w_\parallel = 0\,. \tag{5.22}$$

If $k^2 \ll a^2 m^2_{\rm eff}$ during inflation, this equation becomes identical to eq. (5.20). Consequently, the results that apply for the transverse field are valid for the longitudinal component too. In the opposite regime ($k^2 \gg a^2m^2_{\rm eff}$) the vector field indeed obtains a flat perturbation spectrum. When $0 < m^2_{\rm eff} \ll H^2$ satisfying the condition $k^2 \gg a^2m^2_{\rm eff}$ approximates $m^2_{\rm eff} \approx 0$, when the longitudinal component decouples from the theory and is unphysical. However, if $m^2_{\rm eff} \approx -2H^2$ the longitudinal component can be produced while attaining a flat perturbation spectrum. Writing $m^2_{\rm eff} = -2H^2$, the solution to eq. (5.22) is [9, 24, 25]

$$w_\parallel = \frac{e^{ik/aH}H}{2k^{3/2}}\left(-2 + 2i\frac{k}{aH} + \frac{k^2}{a^2H^2}\right). \tag{5.23}$$

Deriving and taking the limit $k/aH \to 0$ we find[10]

$$\dot{w}_\parallel \simeq \frac{i}{2}\left(\frac{k}{aH}\right)^3 H w_\parallel, \qquad \ddot{w}_\parallel \simeq -3H\dot{w}_\parallel\,. \tag{5.25}$$

Substituting now $\dot{w}_\parallel$ in eq. (4.10) gives rise to gradient terms in eq. (4.12). After neglecting these, eq. (4.16) becomes

$$\dot{\boldsymbol{W}}_\parallel \simeq \boldsymbol{\xi}_{W_\parallel}(t)\,, \tag{5.26}$$

hence $\mathcal{F}_\parallel = 0$. Using that $w_\parallel \simeq -\frac{H}{k^{3/2}}$ on superhorizon scales (implying $\mathcal{P}_\parallel = 2\mathcal{P}_{L,R}$ [9, 24, 25]) we find the diffusion coefficient $D_\parallel = 2D_{L,R} = \frac{1}{3}\frac{H^3}{2\pi^2}$. Finally, using eqs. (4.27), (4.33) and (4.34) we obtain the mean field and variance

$$\mu_\parallel(t) = W_\lambda(0), \qquad \sigma_\parallel^2(t) = \frac{1}{3}\frac{H^3 t}{2\pi^2}\,. \tag{5.27}$$

5.4 Parity violating vector field

Recently, a parity violating, massive vector field has been considered in the context of the vector curvaton mechanism [27] in the effort to generate parity violating signatures on the microwave sky (see also refs. [50–54, 72, 73]).

The Lagrangian density considered is

$$\mathcal{L} = -\frac{1}{4}fF_{\mu\nu}F^{\mu\nu} - \frac{1}{4}hF_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu. \tag{5.28}$$

Since the axial term does not affect the longitudinal component of the perturbation [27], taking $f \propto a^{-1\pm3}$ and $m \propto a$ we obtain a scale invariant perturbation spectrum for the longitudinal component with amplitude

$$\mathcal{P}_\parallel = \left(\frac{3H}{M}\right)^2\left(\frac{H}{2\pi}\right)^2. \tag{5.29}$$

The equation of motion for the transverse polarisations is

$$\ddot{w}_\pm + 3H\dot{w}_\pm + \left(\frac{k^2}{a^2} + M^2 \pm Q^2\right)w_\pm = 0 \tag{5.30}$$

where $Q^2 \equiv \frac{k}{a}\frac{|\dot{h}|}{f}$. If $\dot{h}$ is positive during inflation the subscripts $+ \equiv R$ and $- \equiv L$, whereas $+ \equiv L$ and $- \equiv R$ if $\dot{h}$ is negative. We focus on the case when Q^2 is the dominant term in the above brackets on superhorizon scales. The case when M^2 is the dominant term is studied in [21, 22]. We further assume that M^2 never dominates in the subhorizon regime. Upon parametrising the time-dependence of Q^2 as $Q \propto a^c$, the equation

$$\ddot{w}_\pm + 3H\dot{w}_\pm \pm Q^2 w_\pm = 0 \tag{5.31}$$

[10] Using eq. (5.23) it can be checked that $\ddot{w}_\parallel$ exactly satisfies $\ddot{w}_\parallel + \left(3 + \frac{ik}{aH}\right)H\dot{w}_\parallel = 0$, which can be used to rewrite eq. (5.22) as a first order equation

$$\left(\frac{2k^2}{k^2 - 2a^2H^2} - \frac{ik}{aH}\right)H\dot{w}_\parallel + \left(\frac{2H^2k^2}{k^2-2a^2H^2} + \frac{k^2}{a^2}\right)w_\parallel = 0\,. \tag{5.24}$$

can be solved exactly. The arbitrary constants in the general solution are chosen so that $w_\pm$ and $\dot{w}_\pm$ match the BD vacuum solution in the subhorizon limit $k/aH \to \infty$.

In order to describe the evolution of an individual mode, four cases are identified according to the magnitude of Q/H during inflation. In what follows, we illustrate the buildup of the condensate when $Q_e \ll H$. Of course, when $Q_e \gg H$ the amplitude of the vector fluctuations are suppressed. Regarding the evolution of $w_\pm$ subject to the condition $Q_e \ll H$, two cases can be considered:

- **Case I:** $Q \ll H$ at all times. Defining the scale factor a_X by the condition $k/a_X = Q$, and a_* by $k/a_* = H$ (horizon crossing), the condition $Q \ll H$ can be rewritten as $a_* \ll a_X$. This implies that the mode functions $w_\pm$ behave as a light field at all times during inflation. To zero order in Q/H, the growth of fluctuations proceeds as if the parity violating term were absent. Parity violating effects appear as higher order corrections in Q/H, which can be neglected to estimate the magnitude of the condensate at the end of inflation. To zero order in Q/H, the power spectrum for these modes is

$$\mathcal{P}^{(\mathrm{I})}_{w_\pm}(k) = \left(\frac{H}{2\pi}\right)^2 . \tag{5.32}$$

- **Case II:** $Q \gg H$ during an earliest stage of inflation, but $Q_e \ll H$. In this case, the mode function $w_\pm$ behaves as follows: for $a < a_X$, the mode functions $w_\pm$ approach the BD vacuum solution, thus behaving as modes of an effectively massless field. For $a_X < a < a_*$, $w_\pm$ behave as modes of a heavy field. Consequently, the amplitude of the vacuum fluctuations at horizon crossing is suppressed. For $a_* < a < a_H$, where a_H is defined by $Q = H$, the modes continue behaving as those of a heavy field, thus oscillating and reducing the amplitude of their vacuum fluctuation. During the final stage of inflation: $a_H < a < a_e$, the mode ceases to oscillate and obtains an expectation value. If the previous phase of oscillations is long-lasting, and depending on the value of c, the amplitude of the mode can become very suppressed by the end of inflation. To order zero in Q/H, the power spectrum for these modes is scale independent when $c = -1/2$, which can be naturally realised when string axions are considered [27]:

$$\mathcal{P}^{(\mathrm{II})}_{w_+} = \frac{4}{\pi}\left(\frac{Hf}{\dot{h}}\right)^3\left(\frac{H}{2\pi}\right)^2 , \qquad \mathcal{P}^{(\mathrm{II})}_{w_-} = \frac{1}{2}\,\mathcal{P}^{(\mathrm{II})}_{w_+}\,\exp\left(\frac{4\dot{h}}{Hf}\right) . \tag{5.33}$$

Although cases I and II describe the evolution of a single mode, the condensate formed during inflation contains a collection of modes which can span many orders of magnitude in momentum space. Consequently, in the most general case the condensate encompasses modes which have undergone different evolution, and therefore their amplitudes can be much different. For example, if $Q \ll H$ at all times during inflation, the evolution of the modes in the condensate is dictated by case I only. Nevertheless, if $Q \gg H$ initially, the condensate at the end of inflation is made up of modes with evolution dictated by case II (modes exiting the horizon before $Q = H$) and by case I (modes exiting the horizon after $Q = H$). This is in contrast to the cases previously studied, for which all the modes in the condensate undergo the same evolution.

To compute the mean square field we simply add up the square amplitude of the modes that are superhorizon at the end of inflation and disregard the contribution from modes that are superhorizon at the beginning. Bearing in mind the foregoing discussion and using eqs. (5.32) and (5.33) we find

$$\langle \boldsymbol{W}_+^2 \rangle \simeq \frac{1}{3}\left(\int_H^{He^{N_{\rm II}}} \mathcal{P}^{\rm (II)}_{w_+}(k)\,\frac{dk}{k} + \int_{He^{N_{\rm II}}}^{He^{N_{\rm I}}} \mathcal{P}^{\rm (I)}_{w_+}(k)\,\frac{dk}{k}\right) = \frac{H^2}{12\pi^2}\left[N_{\rm I} + \frac{4N_{\rm II}}{\pi}\left(\frac{Hf_e}{\dot{h}_e}\right)^3\right], \tag{5.34}$$

where $N_{\rm II}$ is the number of e-foldings from the beginning of inflation until $Q = H$ and $N_{\rm I}$ is the remaining number of e-foldings until the end of inflation. To estimate the length of inflation while $Q > H$ we take into account that the first mode that crosses outside the horizon during inflation is $k_0/a_0 \sim H$. Therefore, at the beginning of inflation we can estimate $Q_0 \simeq (H|\dot{h}_0|/f_0)^{1/2}$. Using now that $Q \propto a^c$, the number of e-foldings until $Q = H$ is $N_{\rm II} \simeq \frac{1}{c}\ln\frac{H}{Q_0} \simeq \frac{1}{2c}\ln\frac{Hf_0}{|\dot{h}_0|}$. Writing the total number of e-foldings as $N_{\rm tot} = N_{\rm I} + N_{\rm II}$ we have

$$\langle \boldsymbol{W}_+^2 \rangle \simeq \frac{H^2}{12\pi^2}\left\{N_{\rm tot} + \left[\frac{4}{\pi}\left(\frac{Hf_e}{\dot{h}_e}\right)^3 - 1\right]\frac{1}{2c}\ln\left(\frac{Hf_0}{|\dot{h}_0|}\right)\right\}. \tag{5.35}$$

Proceeding similarly we find the mean square for the mode w_-

$$\langle \boldsymbol{W}_-^2 \rangle \simeq \frac{H^2}{12\pi^2}\left\{N_{\rm tot} + \left[\frac{2}{\pi}\left(\frac{Hf_e}{\dot{h}_e}\right)^3 \exp\left(\frac{4\dot{h}}{Hf}\right) - 1\right]\frac{1}{2c}\ln\left(\frac{Hf_0}{|\dot{h}_0|}\right)\right\}. \tag{5.36}$$

The exponential amplification with respect to $\boldsymbol{W}_+^2$ is due to the fact that, when the effective potential for w_- becomes tachyonic at $a = a_X$, the mode undergoes a fast-roll motion [74] until its evolution becomes overdamped at $a = a_H$. Consequently, the amplitude of the mode w_- undergoes exponential amplification for $a_X < a < a_H$. The immediate consequence of this fact is the subsequent exponential amplification of the transverse vector condensate, which may well overwhelm the longitudinal component and dominate the entire condensate.

As discussed in ref. [27] (see also ref. [76]), parity violating signals cannot source parity violating statistical anisotropy in the power spectrum of the curvature perturbation, because the latter (i.e. g) depends only on the even combination $\mathcal{P}_+$ of the transverse spectra (cf. eq. (2.2)). Parity violating signals appear only in higher order correlators of the curvature perturbation such as the bispectrum, trispectrum etc. However, the observations of the Planck satellite have not detected any significant non-Gaussianity as yet [75].

From the above we see that, even though parity violation is hard to observe at the moment in the cosmological perturbations, the parity violating axial model can result in exponential amplification of the vector field condensate. This, in turn, can have drastic implications on observables stemming from the existence of such a vector condensate, such as statistical anisotropy, as we discussed in section 2.

6 Classical versus quantum evolution

We now return to the varying kinetic function and mass theory, discussed in section 3. From the results in section 4 we can obtain the mean-square of coarse-grained vector $\boldsymbol{W}_c$. Using eqs. (4.41) and (4.45) we have

$$\langle \boldsymbol{W}_c^2 \rangle \simeq \boldsymbol{W}_\perp(0)^2 + \left(\frac{M_0}{M}\right)^2 \boldsymbol{W}_\parallel(0)^2 + 2\,\frac{H^3 t}{4\pi^2} + \left(\frac{3H}{M}\right)^2 \frac{H^3 t}{4\pi^2}\,. \tag{6.1}$$

Given that $M \ll 3H$, the coarse-grained vector is dominated by the longitudinal modes (see e.g eqs. (3.22) and (3.24)), which allow us to disregard the contribution from the transverse modes for the most part of inflation. Consequently, and introducing $W \equiv \boldsymbol{W}_c$ and $W_0 \equiv \boldsymbol{W}_\parallel(0)$ for notational simplicity, for the vector field we can write

$$\langle W^2 \rangle = \left(\frac{M_0}{M}\right)^2 W_0^2 + \left(\frac{3H}{M}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Delta N \,, \tag{6.2}$$

where $\Delta N = H\Delta t$ denotes the elapsing e-foldings and $M \propto a^3$. From the above equation we see that, while the homogeneous "zero"-mode of the vector field (square-root of first term) scales as $\propto a^{-3}$ during inflation (before the possible onset of its oscillations), the region of the "diffusion zone" in field space, which corresponds to the accumulated fluctuations (square-root of second term), scales as $\propto a^{-3}\sqrt{\ln a}$, since $\Delta N \propto \ln a$. This means that the diffusion zone diminishes slightly slower than the amplitude of the "quantum kick" $\delta W \sim H^2/M \propto a^{-3}$. As a result, given enough e-folds, the vector field condensate will assume a large value which will dominate over subsequent "quantum kicks". In a sense, once the condensate is $W \gg \delta W$, the "quantum kicks" become irrelevant to its evolution, which follows the classical equations of motion. This is analogous to the scalar field case. Indeed, when the scalar potential is flat, the scalar field condensate due to the accumulated fluctuations, grows as $\langle \phi^2 \rangle \sim H^3 \Delta t \propto \Delta N$ [46], so it can, in time, become much larger than the value of the "quantum kick" $\delta\phi = H/2\pi$.

Another consequence of the fact that the diffusion zone diminishes slower than the zero-mode is that the initial value of the vector field condensate is, in time, overwhelmed by the quantum diffusion contribution, and can, eventually, be ignored. Thus, when the cosmological scales exit the horizon we can consider only the last term in the above equation, giving

$$\langle W_*^2 \rangle = \left(\frac{3H}{M_*}\right)^2 \left(\frac{H}{2\pi}\right)^2 N_p \,, \tag{6.3}$$

where with N_p we denote the number of e-foldings which have passed since the beginning of inflation until the time when the cosmological scales exit the horizon, i.e. $N_p = N_{\rm tot} - N_*$, with N_* being the number of the remaining e-foldings of inflation when the cosmological scales leave the horizon.

We can now use the above value as our initial homogeneous "zero"-mode and follow the development of the condensate after the exit of the cosmological scales. Employing eq. (6.2), we find

$$\langle W^2 \rangle = \left(\frac{M_*}{M}\right)^2 W_*^2 + \left(\frac{3H}{M}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Delta N_* = \left(\frac{3H}{M}\right)^2 \left(\frac{H}{2\pi}\right)^2 (N_p + \Delta N_*) \,, \tag{6.4}$$

where ΔN_* denotes the elapsing e-folds *after* the cosmological scales exit the horizon. Since $\Delta N_* \leq N_*$ we can safely assume that the amount contributed by the quantum diffusion to $\langle W^2 \rangle$ from t_* can be ignored if $N_* < N_p$ or equivalently if $N_{\rm tot} > 2N_*$. This is a reasonable assumption to make, given that inflation can be long-lasting. If this is the case then, after the cosmological scales exit the horizon and for all intends and purposes, the value of the vector field condensate scales as $W \propto a^{-3}$, while we can take

$$W_* \simeq \frac{H^2}{\varepsilon M_*} \,. \tag{6.5}$$

where we have defined

$$\varepsilon \equiv \frac{2\pi}{3\sqrt{N_p}} \ll 1\,. \tag{6.6}$$

The value of the vector field condensate W_* when the cosmological scales exit the horizon was considered a free parameter in all studies until now, as explained in section 2, and results were expressed in terms of it. However, in this paper we have managed to produce an estimate of this quantity based on physical reasoning, which is given in eq. (6.5).[11] Using this equation, we now investigate whether the desired observational outcomes (e.g. observable statistical anisotropy) can be obtained with realistic values of the remaining free parameter $\varepsilon \sim N_p^{-1/2}$.

The first bound we can obtain for ε comes from the requirement that the density of the vector boson should not dominate the density of inflation.[12] As shown in refs. [21, 22] the density of the vector field is $\rho_W = M^2W^2 = \text{cte}$. Evaluating this at the horizon exit of the cosmological scales we have

$$(M_*W_*)^2 = \rho_W < \rho_{\rm inf} = 3H^2m_P^2 \Rightarrow \varepsilon > \frac{1}{\sqrt{3}}\frac{H}{m_P}\,. \tag{6.7}$$

This implies that the inflationary period cannot be arbitrary large. Indeed, the range of N_p values is

$$N_* < N_p < \frac{4\pi^2}{3}\left(\frac{m_P}{H}\right)^2. \tag{6.8}$$

7 Vector curvaton physics

In this section we apply the above into the vector curvaton scenario following the findings of refs. [21, 22]. We consider a massive vector field with varying kinetic function $f \propto a^{-4}$ and mass $M \propto a^3$. The vector field is subdominant during inflation and light when the cosmological scales exit the horizon. Afterwards, it becomes heavy (this can occur even before the end of inflation) and undergoes coherent oscillations, during which it behaves as pressureless and *isotropic* matter [6]. Hence, after inflation, its density parameter grows in time and has a chance of contributing significantly to the curvature perturbation in the Universe, generating for example observable statistical anisotropy. For a review of the mechanism see refs. [24, 25].

7.1 Light vector field

As before, by "light" we mean a vector field whose mass M remains $M < H$ until the end of inflation. At the end of inflation we assume that the vector field becomes canonically normalised (i.e. $f = 1$) and M assumes a constant value $M_{\rm end} \equiv m$. As discussed in refs. [21, 22], in this case the vector field undergoes strongly anisotropic particle production so that its role can only be to generate statistical anisotropy in the curvature perturbation ζ, while leaving the dominant contribution to the spectrum of ζ to be accounted for by some other isotropic source, e.g. the inflaton scalar field.

In this case, the anisotropy parameter g, which quantifies the statistical anisotropy in the spectrum, is related to ζ as [21, 22]

$$\zeta \sim \frac{1}{\sqrt{g}}\Omega_{\rm dec}\zeta_W\,, \tag{7.1}$$

[11] This is analogous to the well-known Bunch-Davis result, where the initial condition of a light scalar field in a quadratic potential was found to be $\langle\phi^2\rangle \sim H^4/m_\phi^2$, with m_ϕ being the mass of the scalar field [19].

[12] We do not consider vector inflation here [48, 49].

where $\Omega_{\rm dec} \equiv (\rho_W/\rho)_{\rm dec}$ is the density parameter of the vector field at the time of its decay and

$$\zeta_W \sim \left.\frac{\delta W}{W}\right|_{\rm end} \tag{7.2}$$

is the curvature perturbation attributed to the vector field. Using that $\delta W = (\frac{3H}{M})(\frac{H}{2\pi})$ and that $M \propto W^{-1} \propto a^3$ we find

$$\zeta_W \sim \varepsilon\,. \tag{7.3}$$

In refs. [21, 22] it is shown that this scenario generates predominantly anisotropic non-Gaussianity, which peaks in the equilateral configuration. In this configuration, we have [21, 22]

$$\frac{6}{5}|f_{\rm NL}^{\rm equil}| = \frac{1}{4}\frac{g^2}{\Omega_{\rm dec}}\,. \tag{7.4}$$

According to the latest Planck data $|f_{\rm NL}^{\rm equil}| \lesssim 120$ (at 95% CL) [75]. Using this bound and eq. (7.1), it is easy to find that $g < 24\sqrt{\Omega_{\rm dec}}$ and also $\zeta^4 \gtrsim 10^{-3}\varepsilon^4\Omega_{\rm dec}^3$. Combining this with eq. (7.1) we obtain

$$g \lesssim (10^3\zeta/\varepsilon)^{2/3} \simeq 0.05 N_p^{1/3} \tag{7.5}$$

Thus, we see that we can obtain observable statistical anisotropy in the spectrum even with $\varepsilon \sim 1$ (i.e. N_p of a few), where we saturated the non-Gaussianity bound and used that $\zeta = 4.8 \times 10^{-5}$. From eqs. (6.7) and (7.5) we obtain

$$\frac{H}{m_P} < \varepsilon \sim \frac{1}{\sqrt{N_p}} \lesssim 10^3\zeta g^{-3/2}, \tag{7.6}$$

where we also considered eq. (6.6). If we take statistical anisotropy to be observable (g of a few per cent), then the above becomes

$$1 \lesssim N_p < \left(\frac{m_P}{H}\right)^2, \tag{7.7}$$

which incorporates the entire allowed range for N_p shown in eq. (6.8). This means that observable statistical anisotropy in the spectrum of ζ is quite possible. For example, from eq. (7.5), saturating the non-Gaussianity bound, we have $g \lesssim 0.05\, N_p^{1/3}$, which can easily allow observable statistical anisotropy ($g \simeq 2\%$ [29]), since $N_p > N_* \approx 60$.[13]

7.2 Heavy vector field

We now consider the possibility that the final value of the mass of our vector boson is $m \gtrsim H$. In this case, as shown in refs. [21, 22], particle production is rendered isotropic by the end of inflation.[14] This means that the vector field alone can generate the observed curvature perturbation without the need for the direct contribution of any other source such as a scalar field. The generated curvature perturbation is [21, 22]

$$\zeta \sim \Omega_{\rm dec}\zeta_W\,. \tag{7.8}$$

[13]The case when $f \propto a^2$ and $H \gg M =$cte also leads to scale invariant anisotropic particle production [21, 22] as also discussed in section 5.1. In this case, we can still use eq. (6.5) as the initial condition with $M_* = M =$cte. The results are identical to the ones in section 7.1.

[14]This may not be true if the variation of the kinetic function and mass are due to a rolling scalar field, which also undergoes particle production. Then, the cross-coupling of the vector and scalar perturbations introduces an additional source term that may enhance statistical anisotropy [38]. We do not consider this possibility here.

The vector field condensate can begin oscillating a few e-folds ($\lesssim 4$) before the end of inflation [21, 22]. In this case, we have

$$\zeta_W \sim \left.\frac{\delta W}{W}\right|_{\rm osc} \sim \varepsilon\,, \tag{7.9}$$

where the subscript 'osc' denotes the onset of the oscillations and we considered eq. (6.5) and that $M_{\rm osc} \simeq H$.

The generated non-Gaussianity in this case is [21, 22]

$$f_{\rm NL} = \frac{5}{4\Omega_{\rm dec}}\,, \tag{7.10}$$

as in the scalar curvaton case. Since observations suggest $|f_{\rm NL}^{\rm local}| \lesssim 8$ [75], we find $\Omega_{\rm dec} \gtrsim 0.1$ Thus, because of the observed value of ζ, we see that $\varepsilon \lesssim 10^{-4}$.

In refs. [21, 22] it is shown that a heavy vector curvaton with prompt reheating satisfies

$$\frac{H}{m_P} \gtrsim \sqrt{\Omega_{\rm dec}}\,\zeta_W \left(\frac{\Gamma_W}{H}\right)^{1/4}, \tag{7.11}$$

where Γ_W is the vector curvaton's decay rate. Assuming that the vector curvaton decays at least through gravitational couplings we have $\Gamma_W \gtrsim m^3/m_P^2$, which simplifies the above to

$$\frac{H}{m_P} \gtrsim \frac{\zeta^2}{\Omega_{\rm dec}} \sim \varepsilon^2\Omega_{\rm dec}\,, \tag{7.12}$$

where we also used eqs. (7.8) and (7.9) and considered that $m \geq H$ for a heavy field. Using the fact that $\Omega_{\rm dec} \gtrsim 0.1$ to avoid excessive non-Gaussianity, we obtain[15]

$$\varepsilon \lesssim \sqrt{\frac{H}{m_P}}\,. \tag{7.13}$$

Thus, in view of eq. (6.7), we find

$$\frac{H}{m_P} < \varepsilon \sim \frac{1}{\sqrt{N_p}} \lesssim \min\left\{10^{-4}; \sqrt{\frac{H}{m_P}}\right\}. \tag{7.14}$$

Therefore, inflation has to be much more long-lasting ($N_p \gtrsim 10^8$) for this possibility to be realised, compared to the case of a light vector field.

8 Summary and conclusions

In this paper we have studied in detail the inflationary buildup of an Abelian vector boson condensate. Such a condensate, as we outlined in section 2, may be responsible for the quantitative predictions of a cosmological model, which involves vector fields, such as statistical anisotropy, either by mildly anisotropising the inflationary expansion [10–17] or by involving directly the anisotropic vector field perturbations in the curvature perturbation [6–9].

In our treatment, we have mainly focused in the case of a vector field with a time-varying kinetic function $f(t)$ and mass $m(t)$. This was partly motivated by supergravity but

[15]This bound is further strengthened if reheating is not prompt and $\Gamma_W \gg m^3/m_P^2$.

it was also motivated by the peculiar type of particle production of vector boson perturbations, which could be drastically different from the case of a scalar field. We put emphasis on the possibility that $f \propto a^{-4}$ and $m \propto a$, which results in a flat superhorizon spectrum of perturbations for both longitudinal and transverse components, and may be an attractor if time-variation is due to the rolling inflaton [23]. The flat superhorizon spectrum of perturbations is dominated by the longitudinal modes and, in contrast to the scalar field case, its amplitude is decreasing with time even though it remains flat. As a result, the condensate builds up onto a decreasing core as shown in eq. (6.2). Also, the condensate never equilibrates, albeit the vector field being massive, in contrast to the well known Bunch-Davies result [19] $\langle \phi^2 \rangle \sim H^4/m^2$, for a massive but light ($0 < m < H$) scalar field. We have applied our findings to the vector curvaton mechanism as an example, and showed that, if the condensate buildup is considered, we obtain constraints on the total duration of inflation, as encoded in eq. (7.7), if we want to generate observable statistical anisotropy. This demonstrates the predictive power of this approach, compared to the majority of the previous literature, which takes the value of the condensate as a free parameter (see however, ref. [18]).

We also studied the buildup of an Abelian vector boson condensate in other models of vector field particle production and found some interesting results. For example, we have looked into the time-varying f and m model when $f \propto a^2$, which also produces scale invariant spectra for the vector field components. In this case, we found that the condensate does equilibrate in a similar manner to the light massive scalar field case, because the mass of the physical vector boson is now constant. Another case we have looked into is the case of an Abelian vector field non-minimally coupled to gravity through an RA^2 term, where we found that the scale invariant case (coupling $\gamma \approx 1/6$) leads to a condensate buildup $\langle W^2 \rangle \sim H^3 t$, similar to the massless scalar field case [46] and also to the case of a massless Maxwell vector field with $f \propto a^{-1\pm3}$ [18]. Finally, we looked also into the case of an axial coupling and found that the vector condensate can be exponentially amplified in the string axion inspired case when the spectrum of the transverse vector field perturbations is flat and uneven.

Apart from the specific, model dependent results above, our work is a generic, comprehensive study of the inflationary buildup of a vector boson condensate and can be used as a blueprint by any future similar study (see also ref. [77]). We carry out our study by extending the methods of stochastic inflation (usually applied to scalar fields) to include vector fields. Owing to the different boundary conditions imposed on the various polarisation modes w_λ, we identify differences (with respect to the scalar field case) making necessary to modify the stochastic formalism to properly account for the evolution of the classical vector field $\boldsymbol{W}_c$. The bottom line of our method, developed in section 4.1, consists in introducing the conjugate momentum $\boldsymbol{\Pi}_\lambda$, to subsequently eliminate it in the equation for $\boldsymbol{W}_\lambda$ (eq. (4.12)) using the superhorizon behavior of the perturbation modes w_λ. Our method goes beyond the Hamiltonian description of stochastic inflation since we manage to obtain a *single* first order equation for $\boldsymbol{W}_\lambda$ (eq. (4.16)) which, in turn, leads to a Fokker-Planck equation in the variable $\boldsymbol{W}_\lambda$ only. Finally, we remark that our procedure can be successfully applied to scalar fields with a non-negligible scale-dependence (i.e. the case of a heavy field) and also to phases of inflation away from the slow-roll regime [62].

All in all, we have investigated in detail the buildup of a vector boson field condensate during inflation. We considered a multitude of Abelian vector field models, where the conformal invariance of the field is appropriately broken, but focused mostly onto the case of a time-varying kinetic function and mass. As an example, we have applied our findings onto the vector curvaton mechanism and obtained specific predictions about the duration and scale of inflation, which were previously ignored when the magnitude of the condensate was taken as a free parameter.

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A The case of $\boldsymbol{\nabla}(A_t)_c$

Expanding δA_μ using creation/annihilation operators as in eq. (3.13), the temporal component A_t is determined by [cf. eq. (3.7)]

$$A_t = \int \frac{d^3k}{(2\pi)^3} \frac{k}{k^2 + a^2M^2} \partial_t \left\{ f^{-1/2} a \left[\hat{a}_\parallel(k) w_\parallel \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger_\parallel(k) w^*_\parallel \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right] \right\}. \tag{A.1}$$

To compute $(\boldsymbol{\nabla} A_t)_c$ we multiply the above integrand by $\theta(k_s - k)$ to extract the long wavelength part and utilize the superhorizon limit of $w_\parallel$ in eq. (3.24). Taking the gradient and writing $\boldsymbol{k} = \boldsymbol{e}_\parallel \, k$ we arrive at

$$(\boldsymbol{\nabla} A_t)_c = \int \frac{d^3k}{(2\pi)^3} \theta(k_s - k) \frac{k^2 \partial_t \left[f^{-1/2} a \left(\boldsymbol{e}_\parallel \, \hat{a}_\parallel(k) w_\parallel \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{e}^*_\parallel \, \hat{a}^\dagger_\parallel(k) w^*_\parallel \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right) \right]}{k^2 + a^2M^2}. \tag{A.2}$$

Since the longitudinal modes dominate over the transverse ones [cf. eqs. (3.22) and (3.24)], eq. (3.25) indicates that the terms $\ddot{\boldsymbol{W}}_c$ and $3H\dot{\boldsymbol{W}}_c$ are of order $H^2\boldsymbol{W}_c \gg M^2\boldsymbol{W}_c$. Consequently, the term $p(t)\boldsymbol{\nabla}(A_t)_c$ in eq. (4.3) can be neglected provided that

$$|p(t)(\boldsymbol{\nabla} A_t)_c| \ll \left| M^2(t) \boldsymbol{W}_c \right| . \tag{A.3}$$

Since we can approximate $\boldsymbol{W}_c$ as the superposition of longitudinal modes only, i.e.

$$\boldsymbol{W}_c(t, \boldsymbol{x}) \simeq \int \frac{d^3k}{(2\pi)^3} \theta(k_s - k) \left[\boldsymbol{e}_\parallel \, \hat{a}_\parallel(\boldsymbol{k}) w_\parallel e^{i\boldsymbol{k}\boldsymbol{x}} + \boldsymbol{e}^*_\parallel \, \hat{a}^\dagger_\parallel(\boldsymbol{k}) w^*_\parallel e^{-i\boldsymbol{k}\boldsymbol{x}} \right], \tag{A.4}$$

to find out whether (A.3) is satisfied it suffices to compare the square brackets in eqs. (A.2) and (A.4). If we denote by $w_\parallel^{(d)}$ the decaying, albeit dominant on superhorizon scales, part of $w_\parallel$ (second term in eq. (3.19)), then we have $f^{-1/2} a w_\parallel^{(d)} \propto a^3 w_\parallel^{(d)} \simeq$ cte for modes above the coarse-graining scale. Consequently, only the growing part of $w_\parallel$ (first term in eq. (3.19)), which we denote by $w_\parallel^{(g)}$, contributes to the gradient operator in eq. (A.2). Operating the integrand in eq. (A.2) we obtain

$$\left| \frac{\partial_t \left[f^{-1/2} a \left(\hat{a}_\parallel(k) w_\parallel^{(g)} + \hat{a}^\dagger_\parallel(k) w_\parallel^{(g)*} \right) \right]}{1 + (aM/k)^2} p(t) \right| = \frac{24H^2 \left(\hat{a}_\parallel(k) w_\parallel^{(g)} + \hat{a}^\dagger_\parallel(k) w_\parallel^{(g)*} \right)}{1 + (aM/k)^2}, \tag{A.5}$$

where we used that $w_{\parallel}^{(g)} \simeq$ cte on superhorizon scales. Using now the expression of $w_{\parallel}^{(g)}$ that follows from eq. (3.19), the condition (A.3) translates into

$$(k/aH)^3 \ll 1 + (aM/k)^2, \tag{A.6}$$

which holds in the superhorizon regime. Therefore, we may neglect the gradient of the temporal component A_t to describe the evolution of $\boldsymbol{W}_c$. Note that this is an expected result since for sufficiently superhorizon scales ($r \gg r_c \gg 1$) the equations of motion for $w_{L,R}$ and $w_{\parallel}$ coincide [cf. eqs. (3.16) and (3.17)].

B Direct computation

The general solution to the non-homogeneous equation

$$\dot{W}_\lambda + \mathcal{F}_\lambda(t) W_\lambda \simeq \xi_{W_\lambda}, \tag{B.1}$$

can be easily obtained

$$W_\lambda(t) = \exp\left[-\int_0^t \mathcal{F}_\lambda(\tau)\, d\tau\right] W_\lambda(0) + \int_0^t \exp\left[-\int_{\bar\tau}^t \mathcal{F}_\lambda(\tau)\, d\tau\right] \xi_{W_\lambda}(\bar\tau)\, d\bar\tau\,. \tag{B.2}$$

If $\boldsymbol{\xi}_{W_\lambda}$ is a white noise source, i.e. $\langle \boldsymbol{\xi}_{W_\lambda}\rangle = 0$, the ensemble average (over independent representations of the stochastic source) is [cf. eq. (4.33)]

$$\langle W_\lambda(t)\rangle = \exp\left[-\int_0^t \mathcal{F}_\lambda(\tau)\, d\tau\right] W_\lambda(0)\,. \tag{B.3}$$

Using eq. (B.2), the two-point function is [cf. eqs. (4.34) and (4.36)]

$$\begin{aligned} \langle W_\lambda^2\rangle &= \langle W_\lambda\rangle^2 + \int_0^t \exp\left[\int_0^{\bar\tau} \mathcal{F}_\lambda(\tau)\, d\tau\right] \int_0^t \exp\left[\int_0^{\hat\tau} \mathcal{F}_\lambda(\tau)\, d\tau\right] \langle \xi_{W_\lambda}(\hat\tau)\, \xi_{W_\lambda}(\bar\tau)\rangle\, d\hat\tau\, d\bar\tau \\ &= \langle W_\lambda\rangle^2 + \int_0^t \exp\left[-2\int_{\bar\tau}^t \mathcal{F}_\lambda(\tau)\, d\tau\right] D_\lambda(\bar\tau)\, d\bar\tau\,. \end{aligned} \tag{B.4}$$

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