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Simplicial cohomology of band semigroup algebras

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We establish the simplicial triviality of the convolution algebra $\ell^1(S)$, where S is a band semigroup. This generalizes some results of Choi (*Glasgow Math. J.* **48** (2006), 231–245; *Houston J. Math.* **36** (2010), 237–260). To do so, we show that the cyclic cohomology of this algebra vanishes in all odd degrees, and is isomorphic in even degrees to the space of continuous traces on $\ell^1(S)$. Crucial to our approach is the use of the structure semilattice of S, and the associated grading of S, together with an inductive normalization procedure in cyclic cohomology. The latter technique appears to be new, and its underlying strategy may be applicable to other convolution algebras of interest.

1. Introduction

Computing the Hochschild cohomology of Banach algebras has remained a difficult task, even when restricted to the class of ℓ^1-convolution algebras of semigroups (see [1, 6] for earlier work on various examples, albeit only in low dimensions). Choi [3, 4] has shown that the simplicial cohomology of the semigroup algebra $\ell^1(S)$ vanishes when S is a *normal band*. However, the techniques were unable to handle the case of general band semigroups. (We note that bands comprise a rich and interesting class of semigroups: particular kinds of band have been studied both in abstract semigroup theory, and also in operator-theoretic settings [11, 12].)

Here, we calculate all the cyclic and simplicial cohomology groups of $\ell^1(S)$, where S is an *arbitrary* band semigroup. More precisely, we shall show the following:

- the cyclic cohomology of $\ell^1(S)$ is isomorphic in even degrees to the space of continuous traces on $\ell^1(S)$, and vanishes in odd degrees (theorem 7.2);
- the simplicial cohomology of $\ell^1(S)$ vanishes in all strictly positive degrees (theorem 7.4).

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The techniques used in establishing these results resemble those in the earlier work of Gourdeau *et al.* [7], in that one performs explicit calculations with cyclic cochains, and then uses the Connes–Tzygan long exact sequence to calculate the simplicial cohomology. As in that paper, the decision to work with cyclic cohomology is forced upon us by the nature of our construction (see corollary 5.17), and is not merely incidental.

Some of our results appear to generalize to the setting of Banach algebras which are ℓ^1-graded over a semilattice. In particular, it seems that similar calculations would provide an alternative approach to some of Choi's existing results for Clifford semigroups in [4]. However, we shall focus throughout on the case of band semigroup algebras in order to keep the exposition reasonably self-contained.

One approach which one might be tempted to adopt, in order to prove that band semigroup algebras have trivial cyclic cohomology, is to exhaust the band by finitely generated bands and cobound the cocycle on increasingly large sets.

This is even more tempting when one recalls that finitely generated bands are finite [10, theorem 4.4.9] (or see [2] for a short, direct proof). However, one encounters problems with this approach. It is difficult to obtain uniform control of the norms of the coboundaries as we take increasingly large generating sets for these bands. This is true even in the commutative case, which corresponds to the setting of [3]. Another feature is that finite band algebras are, in general, neither semisimple nor amenable, which makes their trivial simplicial cohomology surprising.

It should nevertheless be noted that, by specializing the present arguments to the case of a *semilattice* L, one obtains a direct calculation of the cyclic cohomology of $\ell^1(L)$. Previously, this was only known by applying the Connes–Tzygan exact sequence and using the main result of [3]. Moreover, in order to apply the Connes–Tzygan exact sequence, one first has to show that certain obstruction groups vanish, and the only previous proof that these obstructions vanish relied indirectly on other results from [3]. Thus, the methods of the present paper give a much more accessible proof that $\ell^1(L)$ has the same cyclic cohomology as the ground field.

Remark 1.1. A feature which may be of wider interest is that, rather than constructing a splitting homotopy directly on the cyclic cochain complex, we construct maps which split 'modulo terms of lower order' in a particular filtration, and then employ an iterative procedure to move progressively further down the filtration. Some of these arguments could be cast in terms of a more general theory of cohomology of filtered complexes. However, this seems to bring little extra advantage or clarity for the present problem, so we shall carry out our iterative reduction in a hands-on fashion.

2. Notation and preliminaries

2.1. Cohomology

Since this paper is only concerned with simplicial and cyclic cohomology, rather than Hochschild cohomology with more general coefficients, we shall present a fairly minimal set of definitions that is sufficient for our purposes. Our terminology is that of [7], but with some small differences of notation.

Let $\mathcal{A}$ be a Banach algebra and regard $\mathcal{A}'$, the dual space of $\mathcal{A}$, as a Banach $\mathcal{A}$-bimodule in the usual way. As in [7, § 1], for $n \geqslant 0$, $\mathcal{C}^n(\mathcal{A}, \mathcal{A}')$ denotes the space of n-

cochains, $\mathcal{Z}^n(\mathcal{A},\mathcal{A}')$ denotes the subspace of n-*cocycles*, and $\mathcal{B}^n(\mathcal{A},\mathcal{A}') \subseteq \mathcal{Z}^n(\mathcal{A},\mathcal{A}')$ denotes the subspace of n-*coboundaries*. Note that, by convention, $\mathcal{C}^0(\mathcal{A},\mathcal{A}') = \mathcal{A}'$ and $\mathcal{C}^n(\mathcal{A},\mathcal{A}') = 0$ for negative n. Our notation for the corresponding cohomology groups differs from that of [7]. We shall write $\mathcal{HH}^n(\mathcal{A})$ for the quotient space $\mathcal{Z}^n(\mathcal{A},\mathcal{A}')/\mathcal{B}^n(\mathcal{A},\mathcal{A}')$. This is the nth *simplicial cohomology group* of $\mathcal{A}$.

We need to specify some notation for the *Hochschild coboundary operator*

$$\delta^n\colon \mathcal{C}^n(\mathcal{A},\mathcal{A}') \to \mathcal{C}^{n+1}(\mathcal{A},\mathcal{A}').$$

Recall (see [7]) that an n-cochain is a bounded n-linear map $T\colon \mathcal{A}^n \to \mathcal{A}'$, and that the $(n+1)$-cochain $\delta^n T$ is defined by

$$\begin{aligned}(\delta^n T)(a_1,\dots,a_{n+1})(a_{n+2}) = {}& T(a_2,a_3,\dots,a_{n+1})(a_{n+2}a_1)\\ &+ \sum_{j=1}^{n}(-1)^j T(a_1,a_2,\dots,a_ja_{j+1},\dots,a_{n+1})(a_{n+2})\\ &+ (-1)^{n+1}T(a_1,\dots,a_n)(a_{n+1}a_{n+2}),\end{aligned}$$

where $a_1,\dots,a_{n+2} \in \mathcal{A}$. We will usually omit the superscript and write δ for δ^n.

For each n, elements of $\mathcal{C}^n(\mathcal{A},\mathcal{A}')$ may be regarded as bounded linear functionals on the space $\mathcal{C}_n(\mathcal{A}) := A^{\widehat{\otimes} n+1}$, the $(n+1)$-fold completed projective tensor product of $\mathcal{A}$; if we do this, then the coboundary operator $\delta\colon \mathcal{C}^n(\mathcal{A},\mathcal{A}') \to \mathcal{C}^{n+1}(\mathcal{A},\mathcal{A}')$ is clearly the adjoint of the operator $d\colon \mathcal{C}_{n+1}(\mathcal{A}) \to \mathcal{C}_n(\mathcal{A})$ given by

$$d(a_1\otimes\cdots\otimes a_{n+2}) = a_2\otimes\cdots\otimes a_{n+1}\otimes a_{n+2}a_1 + \sum_{j=1}^{n+1}(-1)^j a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_{n+2}$$

for $a_1,\dots,a_{n+2} \in \mathcal{A}$. This point of view will be more convenient when $\mathcal{A} = \ell^1(S)$ for a semigroup S. This is the case because, since there is a well-known isometric isomorphism of Banach spaces

$$\ell^1(I) \mathbin{\widehat{\otimes}} \ell^1(J) \cong \ell^1(I\times J) \quad \text{for any index sets } I \text{ and } J,$$

in what follows we shall identify $\ell^1(S)^{\widehat{\otimes} n}$ with $\ell^1(S^n)$.

Simplicial cohomology is closely linked to *cyclic cohomology*, which we now introduce. Denote by $\boldsymbol{t}$ the *signed cyclic shift operator* on the simplicial chain complex:

$$\boldsymbol{t}(a_1\otimes\cdots\otimes a_{n+1}) = (-1)^n(a_{n+1}\otimes a_1\otimes\cdots\otimes a_n). \tag{2.1}$$

By an abuse of notation, we also write $\boldsymbol{t}$ for the adjoint operator on the simplicial *cochain* complex. The n-cochain T (in $\mathcal{C}^n(\mathcal{A},\mathcal{A}')$) is called *cyclic* if $\boldsymbol{t}T = T$ and the linear space of all cyclic n-cochains is denoted by $\mathcal{CC}^n(\mathcal{A})$.

It is well known that the cyclic cochains $\mathcal{CC}^n(\mathcal{A})$ form a subcomplex of $\mathcal{C}^n(\mathcal{A},\mathcal{A}')$, that is, $\delta(\mathcal{CC}^n(\mathcal{A})) \subseteq \mathcal{CC}^{n+1}(\mathcal{A})$, and this allows us to define cyclic versions of the spaces defined above, denoted here by $\mathcal{ZC}^n(\mathcal{A})$, $\mathcal{BC}^n(\mathcal{A})$ and $\mathcal{HC}^n(\mathcal{A})$. Under certain conditions on the algebra $\mathcal{A}$ [9], the cyclic and simplicial cohomology groups are connected via the Connes–Tzygan long exact sequence

$$\cdots \to \mathcal{HH}^n(\mathcal{A}) \xrightarrow{B} \mathcal{HC}^{n-1}(\mathcal{A}) \xrightarrow{S} \mathcal{HC}^{n+1}(\mathcal{A}) \xrightarrow{I} \mathcal{HH}^{n+1}(\mathcal{A}) \to \cdots, \tag{2.2}$$

where the maps B, S and I all behave naturally with respect to algebra homomorphisms. (Although we use S to denote both the shift map in cyclic cohomology and a band semigroup, this should not lead to any confusion.) The reader is referred to [9] for more details.

We now introduce some definitions and notation which will be useful in our work.

Definition 2.1 (cyclic cocycles arising from traces). Let A be a Banach algebra. Given $\psi \in A'$ and $n \geqslant 0$, let $\psi^{(n)} \in \mathcal{C}^n(A, A')$ be the cochain defined by

$$\psi^{(n)}(a_1, \dots, a_n)(a_{n+1}) := \psi(a_1 \cdots a_{n+1}).$$

Lemma 2.2. *If τ is a continuous trace on A, then $\tau^{(2n)}$ is a cyclic cocycle.*

This is easily verified by a direct calculation, and we omit the proof.

Definition 2.3. Two chains $x, y \in \mathcal{C}_n(\mathcal{A})$ are *cyclically equivalent* if

$$x - y \in (I - \boldsymbol{t})\mathcal{C}_n(\mathcal{A}).$$

Notation 2.4. Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1} \in \mathcal{C}_n(\mathcal{A})$ be an elementary tensor, and suppose we group terms in the tensor together as $\boldsymbol{x} = w_1 \otimes \cdots \otimes w_j$, where $j \geqslant 2$. We then denote by $d_c(w_l)$ the restriction of d to w_l when seen as a part of $\boldsymbol{x}$, meaning that

$$d_c(w_l) = \sum_{i=\beta_k}^{\beta_k+\alpha_k-2} (-1)^i x_{\beta_k} \otimes \cdots \otimes x_i \cdot x_{i+1} \otimes \cdots \otimes x_{\beta_k+\alpha_k-1},$$

where α_l is the length of the subtensor w_l, and β_l the rank of its first element. If w_l has length 1, i.e. $\alpha_l = 1$, then we define $d_c(w_l)$ to be 0.

Note that the introduction of d_c is a notational device and does not define a map on subtensors as the signs are tributary to the position of this subtensor in the tensor. With this notation, we can write

$$\begin{aligned} d(w_1 \otimes \cdots \otimes w_j) = x_2 \otimes \cdots \otimes x_{\alpha_1} \otimes w_j \cdot x_1 & \\ + \sum_{k=1}^{j-1} (-1)^{\beta_{k+1}-1} w_k \cdot w_{k+1} j & \\ + \sum_{k=1}^{j} w_1 \otimes \cdots \otimes d_c(w_k) \otimes \cdots \otimes w_j. & \qquad (2.3) \end{aligned}$$

Notation 2.5. In later sections, many of the calculations involve elementary tensors in $\mathcal{A}^{\hat{\otimes} n+1}$, of the form $x_1 \otimes \cdots \otimes x_{n+1}$, and their images under certain maps, which have the form

$$x_1 \otimes \cdots \otimes x_{i-1} \otimes f(x_i, x_{i+1}) \otimes g(x_i, x_{i+1}) \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} \qquad (2.4)$$

for certain functions $f, g\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. As a notational shorthand, we will often denote such an expression in the abbreviated form

$$\underset{i-1}{\bullet} \otimes f(x_i, x_{i+1}) \otimes g(x_i, x_{i+1}) \otimes \underset{n-i}{\bullet}.$$

2.2. Band semigroups

Definition 2.6. A semigroup S formed only of idempotents is a *band semigroup.*

Rectangular bands are of particular importance.

Definition 2.7. A *rectangular band* is a semigroup in which the identity $a = aba$ always holds.

Note that, in any rectangular band, the identity

$$abc = (aca)bc = a(cabc) = ac$$

holds for arbitrary elements a, b and c. This is particularly clear if one takes the following description of rectangular bands [10, theorem 1.1.3].

Theorem 2.8. *Let S be a semigroup. The the following conditions are equivalent:*

(i) *S is a rectangular band;*

(ii) *S is isomorphic to a semigroup of the form $A \times B$, where A and B are non-empty sets, and where the multiplication is given by*

$$(a_1, b_1)(a_2, b_2) = (a_1, b_2), \quad \textit{where } a_1, a_2 \in A \textit{ and } b_1, b_2 \in B.$$

A commutative band semigroup is called a *semilattice*, and carries a natural and useful partial order defined by $\alpha \preceq \beta \iff \alpha\beta = \alpha$. Semilattices are important in the study of general bands because of the following structure theorem.

Theorem 2.9 (see [10, theorem 4.4.5]). *Any band semigroup S can be represented as a disjoint union $\coprod_{\alpha \in L} R_\alpha$, where L is a semilattice, each R_α is a rectangular band given by $A_\alpha \times B_\alpha$, the left and right index sets, and the following properties are satisfied:*

(i) *$R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all $\alpha, \beta \in L$;*

(ii) *for $x = (a_1, b_1) \in R_\alpha$ and $y = (a_2, b_2) \in R_\beta$ with $\alpha \preceq \beta$, xy and y have the same right index (i.e. $xy = (\cdot, b_2)$), while yx and y have the same left index (i.e. $yx = (a_2, \cdot)$);*

(iii) *the product is associative.*

Note that condition (iii) is needed to ensure that such a construction gives a band semigroup.

Example 2.10 (normal bands). A band S is said to be a *normal band* if $xaby = xbay$ for all $a, b, x, y \in S$. In this case, the structure theorem can be sharpened significantly; not only do we get a decomposition $S = \coprod_{\alpha \in L} R_\alpha$ into rectangular bands, but this decomposition turns out to exhibit S as a 'strong semilattice of rectangular bands' (see [10, proposition 4.6.14] for a proof and relevant definitions). In [4] this stronger decomposition theorem was used to calculate the simplicial cohomology of $\ell^1(S)$; in the present, more general case, new techniques are needed.

Although bands do not in general have units, we will define *left coherent units* for each element as follows. Given S, for each rectangular band R_α, fix an element $y_\alpha \in R_\alpha$ and define $\langle x] = xy_\alpha$ for each $x \in R_\alpha$. Then the function $\langle\cdot]\colon S \to S$ has the following properties:

- for each $\alpha \in L$, $\langle R_\alpha] \subseteq R_\alpha$;
- for each $\alpha \in L$ and each $x \in R_\alpha$, $\langle x]x = x$;
- for each $\alpha, \beta \in L$ such that $\alpha \preceq \beta$, and each $x \in R_\alpha$ and $y \in R_\beta$, $\langle xy] = \langle x]$.

Notation 2.11. As is often done, if $x \in S$, we denote the point mass at x, as an element of $\ell^1(S)$, by x itself. This should not cause confusion and follows the notation used more generally. Throughout, we write $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ for an elementary tensor in which each x_i is a point mass at $x_i \in S$. *Henceforth, when we speak of an elementary tensor $\boldsymbol{x} \in \mathcal{C}_n(\mathcal{A})$, we shall always mean one of this restricted form.*

To show that one can cobound any $\phi \in \mathcal{ZC}^n(\mathcal{A})$, it suffices to show that one can do it on these elementary tensors (with a uniform bound). Our strategy will be to proceed by steps, expanding the set of elementary tensors on which one can cobound at each step.

To do so, at each step we define $s\colon \mathcal{CC}_{n-1}(\mathcal{A}) \to \mathcal{CC}_n(\mathcal{A})$ in such a way that, denoting by E_0 the set of elementary tensors on which one has cobounded ϕ at the previous step, and by E_1 the set on which one wishes to cobound, $(ds+sd)(E_0) \subset E_0$ and $\boldsymbol{x} - (ds+sd)(\boldsymbol{x}) \in E_0$ for $\boldsymbol{x} \in E_1$. Then, given $\phi \in \mathcal{ZC}^n(\mathcal{A})$ such that $\phi_0 = \phi - \delta(\psi)$ vanishes on E_0, defining ψ_1 on E_1 by $\psi_1(\boldsymbol{x}) = \phi_0(s(\boldsymbol{x}))$ (for $\boldsymbol{x} \in \mathcal{CC}_{n-1}(\mathcal{A})$) gives

$$\begin{aligned}(\phi_0 - \delta\psi_1)(\boldsymbol{x}) &= \phi_0(\boldsymbol{x}) - \psi_1(d\boldsymbol{x})\\ &= \phi_0(\boldsymbol{x}) - \phi_0(sd\boldsymbol{x})\\ &= \phi_0(\boldsymbol{x} - (ds+sd)\boldsymbol{x}),\end{aligned}$$

which still vanishes on E_0 and now vanishes on E_1. Note that if $E_0 \subset E_1$, then it is sufficient to verify that $\boldsymbol{x} - (ds+sd)(\boldsymbol{x}) \in E_0$ for $\boldsymbol{x} \in E_1$.

3. A first normalization step

3.1. Cobounding cyclically with norm control

Our first observation is that, if R is a rectangular band, then $\ell^1(R)$ is a 1-biprojective Banach algebra. That is, there exists[1] an $\ell^1(R)$-bimodule map

$$\sigma\colon \ell^1(R) \to \ell^1(R) \mathbin{\widehat{\otimes}} \ell^1(R),$$

which has norm 1 and which is right inverse to the product map $\ell^1(R) \mathbin{\widehat{\otimes}} \ell^1(R) \to \ell^1(R)$. (To see how this definition relates to the original homological one, see [8, ch. IV, § 5].)

[1] In fact, we can take $\sigma(x) = xe \otimes ex$, where e is a fixed element in R, although we do not need to know this for the arguments which follow.

As observed in [4, lemma 7.5], one can use σ to construct a splitting homotopy for the simplicial chain complex, and thus show directly that $\mathcal{HH}^n(\ell^1(R)) = 0$ for all $n \geqslant 1$. The corresponding result for cyclic cohomology is more complicated, but can nevertheless be deduced using the Connes–Tzygan long exact sequence for Banach algebras [9].

Now consider a general band S which decomposes into rectangular band components as $S = \coprod_\alpha R_\alpha$. We wish to use this decomposition to reduce our cohomology problem to the case of rectangular bands. Although we are ultimately interested in simplicial cohomology, it seems necessary at certain points in our reduction technique to be working with cyclic cochains. Thus, we shall need to consider the *cyclic* cohomology of $\ell^1(R_\alpha)$, for each $\alpha \in L$. Since we need to deal with all the R_α simultaneously, it no longer suffices to appeal to [9, theorem 25]. A more precise version of that result is needed, as follows.

Theorem 3.1. *Let A be a biflat Banach algebra, with biflatness constant $K \geqslant 1$. Let $m \geqslant 0$.*

(i) *For every $\psi \in \mathcal{ZC}^{2m+1}(A)$ there exists $\chi \in \mathcal{CC}^{2m}(A)$ such that $\psi = \delta\chi$; moreover, χ may be chosen to satisfy the bound*

$$\|\chi\| \leqslant 2(m+1)^3 K^{4m} \|\psi\|.$$

(ii) *For every $\psi \in \mathcal{ZC}^{2m+2}(A)$ there exists $\chi \in \mathcal{CC}^{2m+1}(A)$ and $\tau \in \mathcal{CC}^0(A)$ such that $\psi = \tau^{(2m+2)} + \delta\chi$; moreover, τ and χ may be chosen to satisfy the bounds*

$$\|\tau\| \leqslant K^{2(m+1)} \|\psi\|, \qquad \|\chi\| \leqslant 2(m+1)^3 K^{2(2m+1)} \|\psi\|.$$

Theorem 3.1 may well be implicitly known to specialists; a fairly direct and self-contained proof can be found in [5]. The important aspect, for our purposes, is that the constants which control the cobounding depend only on the degree of the cocycle and on the biflatness constant K.

3.2. Initializing a cyclic cocycle on $\ell^1(S)$

Given a cyclic cocycle ψ, we are trying to find a cyclic cochain χ such that $\psi - \delta\chi$ vanishes on a conveniently large set. This will be done in stages: the precise definition for our first step is as follows.

Definition 3.2. Let $\phi \in \mathcal{ZC}^n(\ell^1(S))$. We say that ϕ is *rectangular-band-normalized*, or *R-normalized* for short, if it vanishes on $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ whenever all x_i are in the same rectangular band component of S.

Lemma 3.3.

(i) *If R is a rectangular band, every continuous trace on $\ell^1(R)$ is a scalar multiple of the augmentation character*

$$\epsilon\colon \ell^1(R) \to \mathbb{C}, \quad \text{where } \epsilon\left(\sum_{s\in R} \lambda_s s\right) = \sum \lambda_s.$$

(ii) *If S is an arbitrary band, decomposed canonically as $\coprod_{\alpha\in L} R_\alpha$, where each R_α is rectangular, then $\mathcal{Z}^0(\ell^1(S),\ell^1(S)')$ is isomorphic to $\ell^\infty(L)$.*

A sketch of the proof. Part (i) is proved by fixing $e\in R$ and noting that any trace τ on $\ell^1(R)$ must satisfy $\tau(x)=\tau(xeex)=\tau(exxe)=\tau(e)$ for all $x\in R$. Part (ii) follows by considering the restriction of a trace $\tau\in\mathcal{Z}^0(\ell^1(S),\ell^1(S)')$ to each subalgebra $\ell^1(R_\alpha)$. □

Proposition 3.4 (normalization on each rectangular component). *Let $n\geqslant 1$.*

(i) *For every $\psi\in\mathcal{ZC}^{2n-1}(\ell^1(S))$, there exists $\chi\in\mathcal{CC}^{2n-2}(\ell^1(S))$ such that $\psi-\delta\chi$ is R-normalized.*

(ii) *For every $\psi\in\mathcal{ZC}^{2n}(\ell^1(S))$, there exist*

$$\tau\in\mathcal{Z}^0(\ell^1(S),\ell^1(S)') \quad\text{and}\quad \chi\in\mathcal{CC}^{2n-1}(\ell^1(S))$$

such that $\psi-\tau^{(2n)}-\delta\chi$ is R-normalized.

Proof. We recall that, for each α, the algebra $\ell^1(R_\alpha)$ is biprojective with constant 1.

The case of odd degree is straightforward. Given $\psi\in\mathcal{ZC}^{2n-1}(\ell^1(S))$, let ψ_α denote the restriction of ψ to the subalgebra $\ell^1(R_\alpha)$. Then, by theorem 3.1, for each α there exists $\chi_\alpha\in\mathcal{CC}^{2n-2}(\ell^1(R_\alpha))$ such that $\delta\chi_\alpha=\psi_\alpha$ and

$$\|\chi_\alpha\|\leqslant 2n^3\|\psi_\alpha\|\leqslant K_n'\|\psi\|,$$

where the constant K_n' does not depend on α. Given $(x_1,\dots,x_{2n-1})\in S^{2n-1}$, we define

$$\chi(x_1,\dots,x_{2n-2})(x_{2n-1}):=\begin{cases}\chi_\alpha(x_1,\dots,x_{2n-2})(x_{2n-1}) & \text{if } x_1,\dots,x_{2n-1}\in R_\alpha \\ & \quad\text{for some } \alpha\in L,\\ 0 & \text{otherwise.}\end{cases}$$

Then χ extends to a bounded $(2n-1)$-multilinear functional on $\ell^1(S)$, which is clearly a cyclic cochain since each χ_α is. By construction, if $x_1,\dots,x_{2n}\in R_\alpha$ for some α, then

$$(\psi-\delta\chi)(x_1,\dots,x_{2n-1})(x_0)=(\psi_\alpha-\delta\chi_\alpha)(x_1,\dots,x_{2n-1})(x_{2n})=0,$$

and thus $\psi-\delta\chi$ is R-normalized.

The case of even degree is similar, except that we have to deal with cocycles arising from traces. As before, let $\psi\in\mathcal{ZC}^{2n}(\ell^1(S))$ and, for each α, let $\psi_\alpha\in\mathcal{ZC}^{2n}(\ell^1(R_\alpha))$ be the restriction of ψ to $\ell^1(R_\alpha)$ in each variable. By theorem 3.1, for each α there exist $\chi_\alpha\in\mathcal{CC}^{2n-1}(\ell^1(R_\alpha))$ and $\tau_\alpha\in\mathcal{ZC}^0(\ell^1(R_\alpha))$ such that

$$\delta\chi_\alpha+\tau_\alpha^{(2n)}=\psi_\alpha \quad\text{with } \|\tau_\alpha\|\leqslant K_n''\|\psi\|,\ \|\chi_\alpha\|\leqslant K_n''\|\psi\|,$$

where the constant K_n'' does not depend on α.

By lemma 3.3, each τ_α is constant, with value c_α, say. Let $\tau\colon S\to\mathbb{C}$ be defined by $\tau\colon R_\alpha\to\{c_\alpha\}$. Then τ is a bounded trace on $\ell^1(S)$, and the restriction of $\tau^{(2n)}$ to $\ell^1(R_\alpha)$ is clearly just $\tau_\alpha^{(2n)}$.

Also, given $(x_1, \dots, x_{2n}) \in S^{2n}$, we define

$$\chi(x_1, \dots, x_{2n-1})(x_{2n}) := \begin{cases} \chi_\alpha(x_1, \dots, x_{2n-1})(x_{2n}) & \text{if } x_1, \dots, x_{2n} \in R_\alpha \\ & \quad \text{for some } \alpha \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Then χ extends to a well-defined cyclic $(2n-1)$-cochain on $\ell^1(S)$ and, by construction, we find that, for each α and every $x_1, \dots, x_{2n+1} \in R_\alpha$,

$$(\psi - \tau^{(2n)} - \delta\chi)(x_1, \dots, x_{2n})(x_{2n+1}) = (\psi_\alpha - \tau_\alpha^{(2n)} - \delta\chi_\alpha)(x_1, \dots, x_{2n})(x_{2n+1})$$

as required. □

4. A sufficient condition for using the Connes–Tzygan sequence

DEFINITION 4.1. First, given a Banach algebra B, the *reduced Hochshild complex* $\mathrm{CR}_*(B)$ is the following chain complex of Banach spaces:

$$0 \longleftarrow B \xleftarrow{d} B^{\widehat{\otimes}2} \xleftarrow{d} \cdots \xleftarrow{d} B^{\widehat{\otimes}n} \xleftarrow{d} \cdots,$$

where the boundary map d is defined by

$$d(b_1 \otimes \cdots \otimes b_{n+1}) = \sum_{j=1}^{n} (-1)^j \underset{j-1}{\bullet} \otimes b_j b_{j+1} \otimes \underset{n-j}{\bullet}, \quad b_1, \dots, b_{n+1} \in B.$$

Then, for each $n \geqslant 0$, we write $\mathrm{ZR}_n(B)$ for the kernel of $d\colon B^{\widehat{\otimes}n+1} \to B^{\widehat{\otimes}n}$ and $\mathrm{BR}_n(B)$ for the image of $d\colon B^{\otimes n+2} \to B^{\widehat{\otimes}n+1}$.

Consider the case where $B = \ell^1(S)$. In order to construct the Connes–Tzygan long exact sequence for $\ell^1(S)$, we need to know that the complex $\mathrm{CR}_*(\ell^1(S))$ is exact, i.e. that $\mathrm{BR}_n(\ell^1(S)) = \mathrm{ZR}_n(\ell^1(S))$ for all $n \geqslant 0$ (see [9, theorem 11]). This is the aim of the current section.

REMARK 4.2. Even in the case where $B = L$, i.e. when our band is a semilattice, the result is not immediately obvious; hitherto, the only known proof used a special case of the main results in [3].

The case where $n = 0$ is trivial, since $\mathrm{ZR}_0(\ell^1(S)) = \ell^1(S)$, and for

$$a = \sum_{s \in S} \lambda_s s \in \ell^1(S)$$

we have

$$d\biggl(\sum_{s \in S} \lambda_s \langle s] \otimes s\biggr) = a.$$

We therefore restrict attention in what follows to the case where $n \geqslant 1$.

Left-coherent units will play a key role in our proof. In particular, we need the following lemma in several places.

Lemma 4.3. *Let $y, z \in S$. Then $\langle\!\langle y]\!] = \langle y]$ and $\langle y]\langle yz] = \langle yz]$.*

Proof. Both identities follow from our explicit construction of the function $\langle\cdot]$. They can also be deduced from the coherence properties that were observed earlier.

The first identity follows since $\langle x] = \langle xy]$ for all $x \in R_\alpha$ and $y \in R_\beta$ with $\alpha \preceq \beta$, so that taking $x = \langle y]$ does the job. For the second identity, note that $x\langle x] = (\langle x]x)\langle x] = \langle x]$, the last equality following because x and $\langle x]$ lie in the same rectangular band. In particular, taking $x = yz$ yields

$$\langle y]\langle yz] = \langle y](yz\langle yz]) = yz\langle yz] = \langle yz]$$

as required. □

For $1 \leqslant k \leqslant n+1$, let $s_k \colon \ell^1(S^{n+1}) \to \ell^1(S^{n+2})$ be defined by

$$s_k(x_1 \otimes \cdots \otimes x_{n+1}) = (-1)^k x_1 \otimes \cdot \otimes x_{k-1} \otimes \langle x_k] \otimes x_k \otimes \cdot \otimes x_{n+1}. \tag{4.1}$$

Then set

$$Q_k := ds_k + s_k d - I \colon \ell^1(S^{n+1}) \to \ell^1(S^{n+1}).$$

If $z \in \mathrm{ZR}_n(\ell^1(S))$, then a straightforward calculation shows that $Q_k \cdots Q_1(z)$ is homologous to $(-1)^k z$ (that is, the two tensors differ by an element of BR_n). The work lies in obtaining a formula for $Q_n \cdots Q_1$, which will allow us to see that $Q_n \cdots Q_1(z)$ is homologous to zero.

Remark 4.4. We give some motivation for the introduction of the maps s_k and Q_k, and the attention paid to $Q_n \cdots Q_1$. When B is a Banach algebra with identity $\mathbf{1}$, it is well known that the chain complex $\mathrm{CR}_*(B)$ is exact, and that one can construct an explicit contracting homotopy σ. The maps $\sigma_{n+1} \colon B^{\widehat{\otimes} n+1} \to B^{\widehat{\otimes} n+2}$ are given by $\sigma(b_1 \otimes \cdots \otimes b_{n+1}) = -\mathbf{1} \otimes b_1 \otimes \cdots \otimes b_{n+1}$, and one can check directly that $d\sigma + \sigma d = I$.

In the present case, $\ell^1(S)$ might not even have a bounded approximate identity. Nevertheless, since we do have local left-coherent units $\langle x]$, it is natural to see how far $ds_1 + s_1 d$ is from being the identity map, i.e. how far Q_1 is from being zero.

Although $Q_1(\boldsymbol{x})$ is in general non-zero (see (4.2) below), it is a tensor with more 'structure' than $\boldsymbol{x}$ in some sense. When we successively apply the maps $Q_2, \ldots, Q_n$, at each stage we increase the amount of structure present. Thus, given $z \in \mathrm{ZR}_n(\ell^1(S))$, the tensor $Q_n \cdots Q_1(z)$, which, as previously noted, is homologous to $(-1)^n z$, will be so highly structured that it falls into $\mathrm{BR}_n(\ell^1(S))$.

Let us start our argument by calculating $Q_1(\boldsymbol{x})$, where $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$. Since

$$ds_1(\boldsymbol{x}) = x_1 \otimes \cdots \otimes x_{n+1} + \sum_{j=1}^{n} (-1)^j \langle x_1] \otimes x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \underset{n-j}{\bullet}$$

and

$$s_1 d(\boldsymbol{x}) = \langle x_1 x_2] \otimes x_1 x_2 \otimes \underset{n-1}{\bullet} + \sum_{j=2}^{n} (-1)^{j-1} \langle x_1] \otimes x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \underset{n-j}{\bullet},$$

we have

$$Q_1(\boldsymbol{x}) = -\langle x_1] \otimes x_1x_2 \otimes \underset{n-1}{\bullet} + \langle x_1x_2] \otimes x_1x_2 \otimes \underset{n-1}{\bullet}. \tag{4.2}$$

Note that $\operatorname{ran}(Q_1)$ is contained in the kernel of the map

$$\rho_1 \colon y_1 \otimes \cdots \otimes y_{n+1} \mapsto y_1\langle y_2] \otimes y_2 \otimes \underset{n-1}{\bullet}.$$

Now let $\boldsymbol{y} = y_1 \otimes \cdots \otimes y_{n+1}$, where $y_1, \ldots, y_{n+1} \in S$. A similar calculation to that for Q_1 shows that

$$\begin{aligned} Q_2(\boldsymbol{y}) = -\rho_1(\boldsymbol{y}) &- y_1 \otimes \langle y_2] \otimes y_2y_3 \otimes \underset{n-2}{\bullet} \\ &- y_1y_2 \otimes \langle y_3] \otimes y_3 \otimes \underset{n-2}{\bullet} + y_1 \otimes \langle y_2y_3] \otimes y_2y_3 \otimes \underset{n-2}{\bullet}. \end{aligned}$$

Then, since $\rho_1 Q_1(\boldsymbol{x}) = 0$, we find after some calculation that

$$\begin{aligned} Q_2Q_1(\boldsymbol{x}) = \langle x_1] &\otimes \langle x_1x_2] \otimes x_1x_2x_3 \otimes \underset{n-2}{\bullet} \\ &- \langle x_1] \otimes \langle x_1x_2x_3] \otimes x_1x_2x_3 \otimes \underset{n-2}{\bullet} \\ &- \langle x_1x_2] \otimes \langle x_1x_2] \otimes x_2x_2x_3 \otimes \underset{n-2}{\bullet} \\ &+ \langle x_1x_2] \otimes \langle x_1x_2x_3] \otimes x_1x_2x_3 \otimes \underset{n-2}{\bullet}. \end{aligned} \tag{4.3}$$

By lemma 4.3, $\langle x_1x_2]\langle x_1x_2x_3] = \langle x_1x_2x_3]\langle x_1x_2x_3]$. Hence, $Q_2Q_1(\boldsymbol{x})$ lies in the kernel of the map

$$\rho_2 \colon x_1 \otimes \cdots \otimes x_{n+1} \mapsto x_1 \otimes x_2\langle x_3] \otimes x_3 \otimes \underset{n-3}{\bullet}.$$

It also lies in $\ker(\rho_1)$, as can be shown using lemma 4.3 again.

By continuing in this way, one could calculate $Q_n \cdots Q_1(\boldsymbol{x})$ directly, but it would become harder to keep track of the terms involved and how they cancel. To do the necessary bookkeeping, we write the boundary operator as an alternating sum of *face maps*. That is, for $n \geqslant 1$ and $1 \leqslant i \leqslant n$, let $\partial_i \colon \ell^1(S^{n+1}) \to \ell^1(S^n)$ denote the map defined by

$$\partial_i \colon x_1 \otimes \cdots \otimes x_{n+1} \mapsto \underset{i-1}{\bullet} \otimes x_ix_{i+1} \otimes \underset{n-i}{\bullet},$$

so that

$$d = \sum_{i=1}^{n} (-1)^i \partial_i : \ell^1(S^{n+1}) \to \ell^1(S^n).$$

The following identities are easily verified by checking on elementary tensors:

$$\partial_i s_k = -s_{k-1}\partial_i \quad \text{if } i+2 \leqslant k \leqslant n+1, \tag{4.4 a}$$

$$\partial_i s_i = (-1)^i I, \tag{4.4 b}$$

$$\partial_i s_k = s_k \partial_{i-1} \quad \text{if } k+2 \leqslant i \leqslant n+1. \tag{4.4 c}$$

Using these identities, for $1 \leqslant k \leqslant n$ we may rewrite Q_k as

$$\left.\begin{aligned}
Q_k &= \sum_{i=1}^{n+1}(-1)^i \partial_i s_k + \sum_{i=1}^{n}(-1)^i s_k \partial_i - I \\
&= \sum_{i=1}^{k+1}(-1)^i \partial_i s_k + \sum_{i=1}^{k}(-1)^i s_k \partial_i - I && \text{(by (4.4 } c\text{))} \\
&= \sum_{i=1}^{k-1}(-1)^i \partial_i s_k + (-1)^{k+1}\partial_{k+1} s_k + \sum_{i=1}^{k}(-1)^i s_k \partial_i && \text{(by (4.4 } b\text{))} \\
&= \sum_{i=1}^{k-2}(-1)^{i+1} s_{k-1}\partial_i - \rho_{k-1} \\
&\qquad + (-1)^{k+1}\partial_{k+1} s_k + \sum_{i=1}^{k}(-1)^i s_k \partial_i && \text{(by (4.4 } a\text{)),}
\end{aligned}\right\} \tag{4.5}$$

where $\rho_i = (-1)^{i+1}\partial_i s_{i+1}$, i.e.

$$\rho_i(x_1 \otimes \cdots \otimes x_{n+1}) = \underset{i-1}{\bullet} \otimes x_i \langle x_{i+1}] \otimes x_{i+1} \otimes \underset{n-i}{\bullet}. \tag{4.6}$$

Note that $\partial_i \rho_i = \partial_i$, which implies that $\ker \rho_i \subseteq \ker \partial_i$.

Next, for $1 \leqslant k \leqslant n$, let $\tilde{Q}_k := (-1)^{k+1}\partial_{k+1}s_k + (-1)^k s_k \partial_k$, so that

$$\tilde{Q}_k(\boldsymbol{x}) = -\underset{k-1}{\bullet} \otimes \langle x_k] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} + \underset{k-1}{\bullet} \otimes \langle x_k x_{k+1}] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet}. \tag{4.7}$$

We then have the following result.

Proposition 4.5. *Let $1 \leqslant r \leqslant n$. Then*

(i) $Q_r \cdots Q_1 = \tilde{Q}_r \cdots \tilde{Q}_1$,

(ii) $\operatorname{ran}(Q_r \cdots Q_1) \subseteq (\ker \rho_1) \cap \cdots \cap (\ker \rho_r)$.

Proof. The proof is by induction. When $r = 1$, part (i) is trivial and part (ii) was proved above (see (4.2)). Suppose both parts hold true for $r = k-1$, where $2 \leqslant k \leqslant n$. Then, since $\ker \rho_i \subseteq \ker \partial_i$ for all i, and since (ii) holds for $r = k-1$,

$$\partial_i(Q_{k-1} \cdots Q_1) = 0 \text{ for } 1 \leqslant i \leqslant k-2 \quad \text{and} \quad \rho_{k-1}(Q_{k-1} \cdots Q_1) = 0.$$

Comparing this with (4.5), we see that $Q_k Q_{k-1} \cdots Q_1 = \tilde{Q}_k Q_{k-1} \cdots Q_1$. Since (i) holds for $r = k-1$, we have $Q_k \cdots Q_1 = \tilde{Q}_k \cdots \tilde{Q}_1$. Thus, part (i) holds for $r = k$.

To complete the inductive step, we must show that (ii) holds for $r = k$, i.e. that $\tilde{Q}_k(\ker \rho_i) \subseteq \ker \rho_i$ for all $1 \leqslant i \leqslant k$.

For $1 \leqslant i \leqslant k-2$ this is straightforward, since a direct check on elementary tensors shows that ρ_i commutes with $\tilde{Q}_k$. For $i = k-1$, by using lemma 4.3 we obtain

$$\begin{aligned}
\rho_{k-1}\tilde{Q}_k(\boldsymbol{x}) &= \rho_{k-1}\Big(- \underset{k-2}{\bullet} \otimes x_{k-1} \otimes \langle x_k] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \\
&\qquad\qquad + \underset{k-2}{\bullet} \otimes x_{k-1} \otimes \langle x_k x_{k+1}] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \Big) \\
&= - \underset{k-2}{\bullet} \otimes x_{k-1}\langle x_k] \otimes \langle x_k] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \\
&\quad + \underset{k-2}{\bullet} \otimes x_{k-1}\langle x_k x_{k+1}] \otimes \langle x_k x_{k+1}] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \\
&= \rho_{k-1}\Big(- \underset{k-2}{\bullet} \otimes x_{k-1}\langle x_k] \otimes \langle x_k] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \\
&\qquad\qquad + \underset{k-2}{\bullet} \otimes x_{k-1}\langle x_k] \otimes \langle x_k x_{k+1}] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet} \Big) \\
&= \rho_{k-1}\tilde{Q}_k\Big(\underset{k-2}{\bullet} \otimes x_{k-1}\langle x_k] \otimes x_k \otimes x_{k+1} \otimes \underset{n-k}{\bullet} \Big) \\
&= \rho_{k-1}\tilde{Q}_k\rho_{k-1}(\boldsymbol{x}).
\end{aligned}$$

Finally, another direct calculation on elementary tensors, using lemma 4.3, shows that $\rho_k\tilde{Q}_k = 0$. This completes the inductive step. □

Lemma 4.6. *We have* $(s_n d + d s_{n+1} - I)\tilde{Q}_n = 0$.

Proof. Using the identities (4.4 a) and (4.4 b), we have

$$\begin{aligned}
(s_n d + d s_{n+1} - I)(\boldsymbol{y}) &= \sum_{j=1}^{n}(-1)^j s_n \partial_j(\boldsymbol{y}) + \sum_{k=1}^{n+1}(-1)^k \partial_k s_{n+1}(\boldsymbol{y}) - \boldsymbol{y} \\
&= (-1)^n s_n \partial_n(\boldsymbol{y}) + (-1)^n \partial_n s_{n+1}(\boldsymbol{y}) \\
&= \underset{n-1}{\bullet} \otimes \langle y_n y_{n+1}] \otimes y_n y_{n+1} - \underset{n-1}{\bullet} \otimes y_n\langle y_{n+1}] \otimes y_{n+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(s_n d + d s_{n+1} - I)\tilde{Q}_n(\boldsymbol{x}) \\
&\quad = -(s_n d + d s_{n+1} - I)\Big(\underset{n-1}{\bullet} \otimes \langle x_n] \otimes x_n x_{n+1} \Big) \\
&\qquad\quad + (s_n d + d s_{n+1} - I)\Big(\underset{n-1}{\bullet} \otimes \langle x_n x_{n+1}] \otimes x_n x_{n+1} \Big) \\
&\quad = - \underset{n-1}{\bullet} \otimes \langle x_n x_{n+1}] \otimes x_n x_{n+1} + \underset{n-1}{\bullet} \otimes \langle x_n]\langle x_n x_{n+1}] \otimes x_n x_{n+1} \\
&\qquad\quad + \underset{n-1}{\bullet} \otimes \langle x_n x_{n+1}] \otimes x_n x_{n+1} - \underset{n-1}{\bullet} \otimes \langle x_n x_{n+1}]\langle x_n x_{n+1}] \otimes x_n x_{n+1}.
\end{aligned}$$

Since $\langle x_n]\langle x_n x_{n+1}] = \langle x_n x_{n+1}] = \langle x_n x_{n+1}]\langle x_n x_{n+1}]$, from lemma 4.3, these four terms cancel pairwise to give 0. □

Theorem 4.7. *The complex* $\mathrm{CR}_*(\ell^1(S))$ *is exact.*

Proof. Let $n \geqslant 1$. As already mentioned, it suffices to prove that $\mathrm{ZR}_n(\ell^1(S)) = \mathrm{BR}_n(\ell^1(S))$. Thus, let $z \in \mathrm{ZR}_n(\ell^1(S))$. A simple induction using the definition of the Q_i shows that, for $1 \leqslant k \leqslant n$,

$$Q_k \cdots Q_1(z) = (ds_k - I)\cdots(ds_1 - I)(z) \in \mathrm{ZR}_n(\ell^1(S)).$$

Hence,

$$(s_n d + ds_{n+1} - I)Q_n \cdots Q_1(z) = (ds_{n+1} - I)(ds_n - I)\cdots(ds_1 - I)(z).$$

Now, combining proposition 4.5 and lemma 4.6 yields

$$(s_n d + ds_{n+1} - I)Q_n \cdots Q_1 = (s_n d + ds_{n+1} - I)\tilde{Q}_n \cdots \tilde{Q}_1 = 0.$$

Thus, $(ds_{n+1} - I)(ds_n - I)\cdots(ds_1 - I)(z) = 0$. Expanding out, we deduce that $z \in \mathrm{BR}_n(\ell^1(S))$, as required. □

REMARK 4.8. Although the present work is focused on band semigroups, it should nevertheless be noted that the calculations of this section apply equally well to a Clifford semigroup.

For our present purposes (see [10, theorem IV.2.1]), we say that a semigroup $\mathbb{G}$ is a *Clifford semigroup* if it decomposes as a disjoint union $\mathbb{G} = \coprod_{\alpha \in L} G_\alpha$ of sub-semigroups, where the indexing set L is a semilattice, each G_α is a group with identity element e_α, and $G_\alpha \cdot G_\beta \subseteq G_{\alpha\beta}$ for all α and β. We can define an analogous local left unit function $\langle\cdot]\colon \mathbb{G} \to \mathbb{G}$, which sends $x \in G_\alpha$ to e_α.

If we were then to repeat the calculations of this section, we would find that everything goes through (with slight simplifications, in fact), and we would thus obtain a direct proof that the complex $\mathrm{CR}_*(\ell^1(\mathbb{G}))$ is exact. This implies, for instance, that we have a Connes–Tzygan long exact sequence for $\ell^1(\mathbb{G})$, so that the results of [4] for the simplicial cohomology of $\ell^1(\mathbb{G})$ could be applied to obtain results for its *cyclic* cohomology.

5. Inductively reducing down to the R-normalized case

5.1. Minimal elements

DEFINITION 5.1 (degree of elements and tensors). For $a \in S$, let $[a]$ denote its *degree* in S, that is, if $a \in R_\alpha$, then $[a] := \alpha$. Then, given an elementary (sub)tensor of point masses $w = x_k \otimes \cdots \otimes x_l$, define the *degree of* w, denoted by $[w]$, to be $[x_k x_{k+1} \cdots x_l]$.

In this section, we shall prove that one can cobound any R-normalized $\phi \in \mathcal{ZC}_n(\mathcal{A})$ on those elementary tensors $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ such that $[x_i] = [\boldsymbol{x}]$ for some i.

We work cyclically with indices when dealing with cyclic cohomology: for example, the interval $[n-1, 2]$ is the set $\{n-1, n, n+1, 1, 2\}$ and we call $x_{n-1} \otimes \cdots \otimes x_{n+1} \otimes x_1 \otimes x_2$ a subtensor. We will sometimes emphasize this by describing these as *cyclic intervals* or *cyclic subtensors*.

DEFINITION 5.2. An elementary tensor $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ has a *minimal element* x_i, for some i, if $[x_i] = [\boldsymbol{x}]$. (The degree of such an element is a minimum.) A *minimal block* is a (cyclic) subtensor $x_k \otimes \cdots \otimes x_l$ such that $[x_i] = [\boldsymbol{x}]$ for all i in the cyclic interval $[k, l]$, and $x_{k-1} \neq [\boldsymbol{x}] \neq x_{l+1}$.

Note that if $\boldsymbol{x}$ is a minimal block itself, then all x_i are in the same rectangular band; the assumption that ϕ is R-normalized therefore implies that it vanishes on such an $\boldsymbol{x}$.

For elementary tensors with at least one minimal element, let

$$J_{\boldsymbol{x}} = \{i \in \{1, 2, \ldots, n+1\} \colon x_i \text{ is the first component of a minimal block}\}.$$

$J_{\boldsymbol{x}}$ is the set of the indices of all *initial* points of minimal blocks. Given $i \in J_{\boldsymbol{x}}$, define $s_i \colon \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A})$ by

$$s_i(x_1 \otimes \cdots \otimes x_{n+1}) = (-1)^i (x_1 \otimes \cdots \otimes x_{i-1} \otimes \langle x_i] \otimes x_i \otimes \cdots \otimes x_{n+1}),$$

and then define $s \colon \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A})$ on $\boldsymbol{x}$ with j minimal blocks by

$$s(\boldsymbol{x}) = \frac{1}{j} \sum_{i \in J_{\boldsymbol{x}}} s_i(\boldsymbol{x}). \tag{5.1}$$

If there are no minimal blocks ($J_{\boldsymbol{x}}$ is empty), set $s(\boldsymbol{x}) = 0$.

Dualizing this operator then yields $\sigma \colon \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}') \to \mathcal{C}^n(\mathcal{A}, \mathcal{A}')$, defined on $\phi \in \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}')$ by

$$\sigma\phi(\boldsymbol{x}) = \phi(s(\boldsymbol{x})). \tag{5.2}$$

We now wish to show that σ takes cyclic cochains to cyclic cochains.

Lemma 5.3. *If ϕ is cyclic, then so is $\sigma\phi$.*

Proof. The key point is that the definition of minimal blocks is equivariant with respect to cyclic shifts, that is,

$$i \in I_{(x_1 \otimes \cdots \otimes x_{n+1})} \iff i + 1 \in I_{(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n)},$$

where this is understood cyclically in the case $i = n+1$.

If $i \in I_{(x_1 \otimes \cdots \otimes x_{n+1})}$ and $1 \leqslant i \leqslant n$, then

$$\begin{aligned} \boldsymbol{t}s_i(x_1 \otimes \cdots \otimes x_{n+1}) &= (-1)^i \boldsymbol{t}(x_1 \otimes \cdots \otimes x_{i-1} \otimes \langle x_i] \otimes x_i \otimes \cdots \otimes x_{n+1}) \\ &= (-1)^{n+1+i}(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes \langle x_i] \otimes x_i \otimes \cdots \otimes x_n) \\ &= (-1)^n s_{i+1}(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n). \end{aligned}$$

On the other hand, if $n + 1 \in I_{x_1 \otimes \cdots \otimes x_{n+1}}$, then

$$\begin{aligned} \boldsymbol{t}^2 s_{n+1}(x_1 \otimes \cdots \otimes x_{n+1}) &= (-1)^{n+1} \boldsymbol{t}^2 (x_1 \otimes \cdots \otimes \langle x_{n+1}] \otimes x_{n+1}) \\ &= (-1)^{n+1}(\langle x_{n+1}] \otimes x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n) \\ &= (-1)^n s_1(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n). \end{aligned}$$

Thus, if $\psi \in \mathcal{CC}^{n+1}(\mathcal{A})$, so that $\psi \circ \boldsymbol{t} = \psi$, we find that

$$\begin{aligned} \boldsymbol{t}\sigma\psi(x_1 \otimes \cdots \otimes x_{n+1}) &= (-1)^n \psi(s(x_{n+1} \otimes x_1 \otimes \cdots \otimes x_n)) \\ &= \psi(s(x_1 \otimes \cdots \otimes x_{n+1})) \\ &= \sigma\psi(x_1 \otimes \cdots \otimes x_{n+1}) \end{aligned}$$

as required. □

Proposition 5.4. *For any R-normalized $\phi \in \mathcal{ZC}^n(\mathcal{A})$, there exists $\psi \in \mathcal{CC}^{n-1}(\mathcal{A})$, such that $(\phi - \delta\psi)(\boldsymbol{x}) = 0$ for all elementary tensors $\boldsymbol{x}$ with some minimal element.*

Proof. Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ be a tensor with some minimal element and let $m_{\boldsymbol{x}}$ be the sum of the length of its minimal blocks. Since we are working in *cyclic* cohomology, by cycling our tensor we can assume without loss of generality that $1 \in J_{\boldsymbol{x}}$. It will be convenient to write $\boldsymbol{x} = u_1 \otimes v_1 \otimes \cdots \otimes u_l \otimes v_l$, where u_l, $l = 1, \ldots, j$ are the j minimal blocks.

If all elements are minimal, we say that $m_{\boldsymbol{x}} = n+1$. Then, since ϕ is R-normalized, it will be assumed to vanish on $\boldsymbol{x}$.

Suppose we can cobound on $\boldsymbol{x}$ such that $m_{\boldsymbol{x}} \geqslant K$. Let $\boldsymbol{x}$ be such that $m_{\boldsymbol{x}} = K-1$ and consider $(ds+sd)\boldsymbol{x}$. In the notation of (2.3), in $d(\boldsymbol{x})$ there are terms with $d_c(v_l)$, $u_l \cdot v_l$ and $v_l \cdot u_{l+1}$ that all have $K-1$ minimal elements, and those with $d_c(u_l)$ that have $K-2$ minimal elements. Applying s increases the number of minimal elements by one in all terms, and, therefore, by induction it suffices to consider only those terms in $sd(\boldsymbol{x})$ of the form

$$(-1)^i u_1 \otimes v_1 \otimes \cdots \otimes \langle x_i] \otimes d_c(u_l) \otimes v_l \otimes \cdots \otimes u_j \otimes v_j,$$

where x_i is the first element of u_l (and where we have used that if $[x] \preceq [y]$, then $\langle xy] = \langle x]$). Note that d_c is applied to u_l as a subtensor of $\boldsymbol{x}$.

Similarly, in $ds(\boldsymbol{x})$, we only need to consider terms of the form

$$(-1)^i u_1 \otimes v_1 \otimes \cdots \otimes d_c'(\langle x_i] \otimes u_l) \otimes v_l \otimes \cdots \otimes u_j \otimes v_j,$$

where d_c' is applied to $\langle x_i] \otimes u_l$ as a subtensor of $s_i(\boldsymbol{x})$. This effectively changes the signs when comparing to terms in $sd(\boldsymbol{x})$. When summing, all terms cancel except

$$u_1 \otimes v_1 \otimes \cdots \otimes \langle x_i] \cdot u_l \otimes v_l \otimes \cdots \otimes u_j \otimes v_j,$$

which is $\boldsymbol{x}$. □

5.2. Without minimal elements

The procedure for handling tensors without minimal elements is much more involved. Crucial to our construction is the following definition.

Definition 5.5. Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1} \in S^{n+1}$ be without minimal element. We say that a subtensor $x_k \otimes \cdots \otimes x_l$ has a *minimal left element* if $[x_k] \preceq [x_i]$ for all i in the cyclic interval $[k, l]$. A subtensor is a *left-block* if it has a minimal left element and is not strictly included in another subtensor which has a minimal left element.

Clearly, a tensor $\boldsymbol{x} \in S^{n+1}$ can have at most $n+1$ left-blocks. Note that a tensor $\boldsymbol{x}$ with a minimal element informally corresponds to having only one left-block: extending the definition to this case leads to confusion as the initial element of such a left-block may not be well defined. Nevertheless, if $\boldsymbol{x}$ does not have at least two left-blocks, then it has a minimal element.

We stress again that we consider tensors like $x_n \otimes x_{n+1} \otimes x_1 \otimes x_2$ as subtensors of $\boldsymbol{x}$ and, therefore, as potential left-blocks.

Notation 5.6. For $2 \leqslant j \leqslant n+1$, denote by $\mathcal{F}_n^j$ the set of all elementary tensors in S^{n+1} with at most j left-blocks. We write $\mathcal{F}_n^1$ for the subset of elementary tensors with a minimal element.

These subsets give us a filtration

$$\mathcal{F}_n^1 \subset \mathcal{F}_n^2 \subset \cdots \subset \mathcal{F}_n^{n+1} = S^{n+1},$$

where $\mathcal{F}_n^{n+1}$ has dense linear span in $\mathcal{C}_n(\mathcal{A})$. Crucially, each face map

$$\partial_i \colon \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n-1}(\mathcal{A})$$

cannot increase the number of left-blocks, and hence maps $\mathcal{F}_n^j$ to $\mathcal{F}_{n-1}^j$.

We have seen in the previous sections that if ψ is an R-normalized n-cocycle, it is equivalent in cyclic cohomology to one that vanishes on $\mathcal{F}_n^1$.

THEOREM 5.7. *Let $2 \leqslant j \leqslant n+1$ and let $\psi \in \mathcal{ZC}^n(\mathcal{A})$. Suppose that ψ vanishes on $\mathcal{F}_n^{j-1}$. It is then equivalent in cyclic cohomology to a cocycle that vanishes on $\mathcal{F}_n^j$.*

Theorem 5.7 will allow us, by an inductive argument, to conclude that if ψ is an R-normalized cocycle, there exists a cyclic cochain ϕ such that $\psi = \delta\phi$. The proof of this theorem will take up the rest of this section and the following one.

NOTATION 5.8. For elementary tensors without a minimal element, it is easy to see that any tensor $\boldsymbol{x}$ has a unique decomposition into left-blocks. Therefore, we can define

$$I_{\boldsymbol{x}} = \{i \in \{1, 2, \ldots, n+1\} \colon x_i \text{ is the first component of a left-block}\}.$$

$I_{\boldsymbol{x}}$ is the set of the indices of all *initial* points of left-blocks.

As in § 5.1, we shall now define an insertion operator in terms of this block structure. (This operator will also be denoted by s, but this abuse of notation should not cause any confusion with the insertion operator that was considered in § 5.1.) Given an elementary tensor $\boldsymbol{x} \in \mathcal{C}_n(\mathcal{A})$ and $i \in I_{\boldsymbol{x}}$, define

$$s_i(\boldsymbol{x}) = (-1)^i \Big(\underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i \otimes \underset{n-i}{\bullet} \Big) \in \mathcal{C}_{n+1}(\mathcal{A}),$$

and then define $s \colon \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A})$ by

$$s(\boldsymbol{x}) = \sum_{i \in I_{\boldsymbol{x}}} s_i(\boldsymbol{x}). \tag{5.3}$$

If there are no left-blocks, ($I_{\boldsymbol{x}}$ is empty), set $s(\boldsymbol{x}) = 0$.

Dualizing this operator, we define $\sigma \colon \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}') \to \mathcal{C}^n(\mathcal{A}, \mathcal{A}')$ by

$$\sigma\phi(\boldsymbol{x}) = \phi(s(\boldsymbol{x})) \quad \text{for } \phi \in \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}'). \tag{5.4}$$

The proof of lemma 5.3 shows that we also have the following.

LEMMA 5.9. *If ϕ is cyclic, then so is $\sigma\phi$.*

Two parameters will be important for our approach in this section and the next: the *degree* of a left-block, and the *height* of a elementary tensor. The degree of a (sub)tensor was defined earlier (definition 5.1). Note that the degree of a left-block will be the same as the degree of its initial element.

Definition 5.10. If T is a finite semilattice, and $\alpha \in T$, the *height of* α *in* T is the length of the longest descending chain in T which starts at α. That is,

$$\mathrm{ht}_T(\alpha) = \sup\{m\colon \text{there exist } t_0, \dots, t_m \in T \text{ with } \alpha = t_m \succ t_{m-1} \succ \cdots \succ t_0\}.$$

If $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ is an elementary tensor in $\mathcal{C}_n(\mathcal{A})$, let $L(\boldsymbol{x})$ be the sub-semilattice of L that is generated by the set $\{[x_1], \dots, [x_{n+1}]\}$, and define the *height of* $\boldsymbol{x}$ to be

$$\mathrm{ht}(\boldsymbol{x}) := \sum_{i=1}^{n+1} \mathrm{ht}_{L(\boldsymbol{x})}([x_i]).$$

Denote by $\mathcal{F}_n^{j,h}$ the set of elementary tensors with at most j left-blocks and with height at most h. Note for later reference that, if $\boldsymbol{x}$ has no minimum element, then there are crude bounds

$$n + 1 \leqslant \mathrm{ht}(\boldsymbol{x}) \leqslant n(n+1).$$

We now define linear spaces which will be key to our induction:

$$G_{n,j,h} = \mathrm{lin}\,\mathcal{F}_n^{j,h} + \mathrm{lin}\,\mathcal{F}_n^{j-1} \quad \text{and} \quad H_{n,j,h} = (I - \boldsymbol{t})\mathcal{C}_n(\mathcal{A}) + G_{n,j,h}. \tag{5.5}$$

Lemma 5.11. *Let T be a finite semilattice and let $F \subseteq T$ be a sub-semilattice.*

(i) *If $\alpha \in F$ then $\mathrm{ht}_F(\alpha) \leqslant \mathrm{ht}_T(\alpha)$.*

(ii) *If $\alpha, \beta \in T$ and $\alpha \prec \beta$ then $\mathrm{ht}_T(\alpha) < \mathrm{ht}_T(\beta)$.*

The proofs of both parts are clear.

Proposition 5.12. *Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ be an elementary tensor in $\mathcal{F}_n^{j,h}$. Then*

$$\begin{aligned}(sd + ds)(\boldsymbol{x}) \equiv \sum_{i \in I_{\boldsymbol{x}}} \Big[\boldsymbol{x} + \underset{i-1}{\bullet} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \\ - \underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \Big] \quad \mathrm{mod}\ G_{n,j,h-1}.\end{aligned} \tag{5.6}$$

Remark 5.13. When $n+1 \in I_{\boldsymbol{x}}$, the corresponding term in square brackets should be interpreted as

$$\begin{aligned}\boldsymbol{x} + (-1)^n x_2 \otimes \cdots \otimes x_n \otimes \langle x_{n+1} x_1] \otimes x_{n+1} x_1 \\ + (-1)^{n+1} x_2 \otimes \cdots \otimes x_n \otimes \langle x_{n+1}] \otimes x_{n+1} x_1.\end{aligned} \tag{5.7}$$

The proof of proposition 5.12 is long, and will therefore be deferred to § 6. Assuming for the moment that the proposition holds, let us continue with the proof of theorem 5.7.

For $k = 1, \dots, j$, let

$$P_k = I - k^{-1}(sd + ds).$$

By construction, if $\psi \in \mathcal{ZC}^n(\mathcal{A})$, then $\psi - P_k^* \psi = k^{-1} \delta\sigma\psi \in \mathcal{BC}^n(\mathcal{A})$, and so applying P_k^* to a cyclic cocycle does not change its cyclic cohomology class. Proposition 5.12 suggests that, by repeatedly applying P_k to an elementary tensor in $\mathcal{F}_n^j$, for varying k, one would eventually obtain a linear combination of tensors in $\mathcal{F}_n^{j-1}$.

To prove that this hope can be realized (at least, if we work up to cyclic equivalence, see definition 2.3) we must analyse the surviving terms in (5.6) in more detail. Left-blocks of length 1 will play a special role and we adopt the following definitions.

DEFINITION 5.14. A left-block of length 1 is called a 1-*block*. A 1-block x_k in an elementary tensor $\boldsymbol{x}$ is called a *block-unit* if $x_k = \langle x_k]$ and $x_k x_{k+1} = x_{k+1}$.

REMARK 5.15. Since block-units are left-blocks of length 1, there are certainly no more than $j-1$ of them when $j < n+1$. In fact, if $j = n+1$, then this is still true, for if there were only block-units, the degree of each would lie above that of the following block-unit, and hence the tensor would have a minimal element (and therefore no left-blocks).

Given an elementary tensor $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$, let

$$\mathsf{R}_{\boldsymbol{x}} = \{i\colon x_i \text{ is a 1-block but not a block-unit, and } [x_i] \succ [x_{i+1}]\},$$

where we allow $n+1 \in \mathsf{R}_{\boldsymbol{x}}$. Clearly, $\mathsf{R}_{\boldsymbol{x}}$ is a proper subset of the set $I_{\boldsymbol{x}}$ of all initial points of left-blocks; it may even be empty.

LEMMA 5.16. *Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ be an elementary tensor in $\mathcal{F}_n^j$, and let $i \in I_{\boldsymbol{x}}$. Then precisely one of the following four cases can occur:*

(i) *x_i is not a 1-block in $\boldsymbol{x}$, in which case,*

$$\underset{i-1}{\bullet} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} = \underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet}; \tag{5.8}$$

(ii) *x_i is a 1-block and $[x_i] \not\succeq [x_{i+1}]$, in which case, the tensor*

$$\underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \tag{5.9}$$

either has fewer left-blocks, or lower height (and the same number of left-blocks), than $\boldsymbol{x}$;

(iii) *x_i is a block-unit, in which case $\bullet_{i-1} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \bullet_{n-i} = \boldsymbol{x}$;*

(iv) *$i \in \mathsf{R}_{\boldsymbol{x}}$.*

In cases (ii)–(iv), the tensor $\bullet_{i-1} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$ has fewer left-blocks than $\boldsymbol{x}$.

Proof. If x_i is not a 1-block, then since i is initial we must have $[x_i] \preceq [x_{i+1}]$. Equation (5.8) then follows from the left-coherent property of the function $\langle\cdot]$, as described in § 2.

If x_i is a 1-block, we split into two cases. The first is when $[x_i]$ does not lie above $[x_{i+1}]$, i.e. case (ii) of the lemma. In this case, $x_i x_{i+1}$ has strictly smaller degree than x_{i+1}, and so (5.9) has height at most

$$\mathrm{ht}(\boldsymbol{x}) - \mathrm{ht}_{L(\boldsymbol{x})}([x_{i+1}]) + \mathrm{ht}_{L(\boldsymbol{x})}([x_i x_{i+1}]) < \mathrm{ht}(\boldsymbol{x})$$

as claimed. The second case is when $[x_i] \succeq [x_{i+1}]$ (note that, since x_i is assumed here to be a 1-block, it then has to lie *strictly* above x_{i+1}). There are now two subcases:

either x_i is a block-unit, in which case the claim in (iii) follows immediately from the definition of a block-unit (and the fact that $\langle\!\langle\cdot]\!] = \langle\cdot]$), or it is not, in which case i is by definition a member of $R_{\boldsymbol{x}}$, so that we are in case (iv).

Finally, if we are not in case (i), i.e. if x_i is a 1-block, then $\bullet_{i-1} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$ clearly has fewer left-blocks than $\boldsymbol{x}$ (more precisely, the left-blocks which started in position i and position $i+1$ have merged). □

The previous lemma motivates the following notation. Define a map Err on elementary tensors by

$$\operatorname{Err}(\boldsymbol{x}) = \sum_{i \in R_{\boldsymbol{x}}} \underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet}, \tag{5.10}$$

where if $R_{\boldsymbol{x}} = \emptyset$, we define $\operatorname{Err}(\boldsymbol{x}) := 0$, and extend Err by linearity and continuity to a bounded linear map on $\mathcal{C}_n(\mathcal{A})$. It is easily checked from the definitions in (5.5) that Err maps $G_{n,j,h}$ into itself, and hence maps $H_{n,j,h}$ into itself.

Corollary 5.17. *If $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ is an elementary tensor with height h, and with j left-blocks, exactly r of which are block-units, then*

$$P_k(\boldsymbol{x}) \equiv \left(1 - \frac{j-r}{k}\right)\boldsymbol{x} + \frac{1}{k}\operatorname{Err}(\boldsymbol{x}) \mod H_{n,j,h-1}. \tag{5.11}$$

Proof. Fix $i \in I_{\boldsymbol{x}}$ and consider the corresponding terms enclosed by square brackets on the right-hand side of (5.6) (or, if $i = n+1$, the terms in (5.7)).

If x_i is a block-unit and $1 \leqslant i \leqslant n$, then, by lemma 5.16(iii), the first and third of these terms cancel out, while the middle term is equal to $\bullet_{i-1} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$ and so lies in $\mathcal{F}_n^{j-1}$. If $i = n+1$ and x_{n+1} is a block-unit, we have to consider (5.7). There, the first and third terms will now cancel out *modulo cyclic equivalence*, while the middle term is cyclically equivalent to $x_{n+1}x_1 \otimes \bullet_{n-1} \otimes \langle x_{n+1}x_1]$ and so lies in $\mathcal{F}_n^{j-1}$ as before.

If x_i is not a block-unit, then we get $\boldsymbol{x}$, together with two other terms. These two will cancel if x_i is not a 1-block (lemma 5.16(i)), while if x_i is a 1-block that does not lie above its successor, these terms will have either fewer left-blocks or lower height than $\boldsymbol{x}$ (lemma 5.16(ii)). That only leaves the case where $i \in R_{\boldsymbol{x}}$, when one of the terms will have fewer left-blocks and the other will form part of $\operatorname{Err}(\boldsymbol{x})$.

Summing over all $i \in I_{\boldsymbol{x}}$, we obtain from (5.6) that

$$\begin{aligned}(ds + sd)(\boldsymbol{x}) \equiv &\left(\sum_{i \in I_{\boldsymbol{x}},\, x_i \text{ not a block-unit}} \boldsymbol{x}\right) \\ &- \left(\sum_{i \in R_{\boldsymbol{x}}} \underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet}\right) \mod H_{n,j,h-1}\end{aligned}$$

The first sum in brackets is equal to $(j-r)\boldsymbol{x}$; the second is equal to $\operatorname{Err}(\boldsymbol{x})$. Rearranging gives us the desired identity. □

At this point, note that the degree of a left-block and the height of a tensor, which both play a pivotal role in our analysis, depend only on the degrees of terms in an elementary tensor, i.e. those elements of the structure semilattice L which index these terms. This motivates the following definition.

DEFINITION 5.18. Given an elementary tensor $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$, the *shape* of $\boldsymbol{x}$ is the tensor $[x_1] \otimes \cdots \otimes [x_{n+1}] \in \ell^1(L^{n+1})$.

Clearly, the number of left-blocks, the location of the initial points of left-blocks and the height are each dependent only on the shape of a tensor. It is also clear that if $i \in R_{\boldsymbol{x}}$, then $\bullet_{i-1} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$ has the same shape as $\boldsymbol{x}$. Consequently, each term in $P_k(\boldsymbol{x})$ has the same shape as $\boldsymbol{x}$, has lower height or has fewer left-blocks.

Given an elementary tensor $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ without minimal element, define a *descending block* in $\boldsymbol{x}$ to be a cyclic subtensor $x_k \otimes \cdots \otimes x_l$ with the property that $[x_k] \succ [x_{k+1}] \succ \cdots \succ [x_l]$, while $[x_{k-1}] \not\succ [x_k]$ and $[x_l] \not\succ [x_{l+1}]$. Since the entries of $\boldsymbol{x}$ can strictly decrease at most n times, a descending block in $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ can have length at most n, and so has a well-defined first element and last element. In particular, for each i, we can define the *descent of x_i in $\boldsymbol{x}$* to be $l - i$, where x_l is the last element in the unique descending block that contains x_i. (This is interpreted cyclically, so that if $x_n \otimes x_{n+1} \otimes x_1$ is a descending block, then the descent of x_{n+1} is 1.) We denote the descent of x_i in $\boldsymbol{x}$ by $\operatorname{desc}_i(\boldsymbol{x})$, and now define the *descent of $\boldsymbol{x}$* to be

$$\operatorname{desc}(\boldsymbol{x}) := \sum_{i \in R_{\boldsymbol{x}}} \operatorname{desc}_i(\boldsymbol{x}).$$

Since $\operatorname{desc}_i(\boldsymbol{x}) \leqslant n-1$ for all i and $|R_{\boldsymbol{x}}| \leqslant j-1$, there is a crude upper bound $\operatorname{desc}(\boldsymbol{x}) \leqslant (j-1)(n-1)$. Moreover, since $\operatorname{desc} x_i \geqslant 1$ for each $i \in R_{\boldsymbol{x}}$ (recall that if $i \in R_{\boldsymbol{x}}$ then x_i lies *strictly* above its successor), there is a lower bound $\operatorname{desc}(\boldsymbol{x}) \geqslant |R_{\boldsymbol{x}}|$.

The idea behind the next lemma is that, given $\boldsymbol{x}$ with $R_{\boldsymbol{x}}$ non-empty, each term in $\operatorname{Err}(\boldsymbol{x})$ either has one more block-unit, or else has one of the block-units shifted one place to the left. Since each such term has the same shape as $\boldsymbol{x}$, this process must terminate after a finite number of steps.

LEMMA 5.19. *Let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_{n+1}$ be a tensor without minimal element, such that $R_{\boldsymbol{x}}$ is non-empty, and let $i \in R_{\boldsymbol{x}}$. Then*

$$\operatorname{desc}\Big(\underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \Big) < \operatorname{desc}(\boldsymbol{x}).$$

Proof. To allay potential concerns with degenerate cases, we start by observing that, since $\boldsymbol{x}$ has no minimal element but $R_{\boldsymbol{x}}$ is non-empty, we must have $n \geqslant 2$. Next, since the definition of descent is cyclically invariant, we may as well cycle our tensor so that $2 \leqslant i \leqslant n$ (this just simplifies some of the notational bookkeeping).

Set $\boldsymbol{y} = \bullet_{i-1} \otimes \langle x_i] \otimes x_i x_{i+1} \otimes \bullet_{n-i}$. Note that $\boldsymbol{y}$ coincides with $\boldsymbol{x}$ in the first $i-1$ and last $n-i$ entries, and $[y_k] = [x_k]$ for all $k \in \{1, \dots, n+1\}$. Given $r \in \{1, \dots, i-1\} \cup \{i+2, \dots, n+1\}$, it follows that y_r is a 1-block and non-block-unit lying strictly above its successor in $\boldsymbol{y}$ if and only if x_r is a 1-block and non-block-unit lying strictly above its successor in $\boldsymbol{x}$; moreover, if this is the case, then the descent of y_r is equal to that of x_r. It follows from the definition of descent that

$$\operatorname{desc}(\boldsymbol{x}) - \sum_{r \in R_{\boldsymbol{x}} \cap \{i, i+1\}} \operatorname{desc}_r(\boldsymbol{x}) = \operatorname{desc}(\boldsymbol{y}) - \sum_{r \in R_{\boldsymbol{y}} \cap \{i, i+1\}} \operatorname{desc}_r(\boldsymbol{y}).$$

Now, by hypothesis $i \in R_{\boldsymbol{x}}$, and since y_i is a block-unit in $\boldsymbol{y}$, we have $i \notin R_{\boldsymbol{y}}$. Moreover, since $[x_i] \succeq [x_{i+1}]$, it is clear that the descent of x_i in $\boldsymbol{x}$ is strictly greater than the descent of $y_{i+1} = x_i x_{i+1}$ in $\boldsymbol{y}$. Therefore,

$$\sum_{r \in R_{\boldsymbol{x}} \cap \{i,i+1\}} \mathrm{desc}_r(\boldsymbol{x}) \geqslant \mathrm{desc}_i(\boldsymbol{x}) > \mathrm{desc}_{i+1}(\boldsymbol{y}) \geqslant \sum_{r \in R_{\boldsymbol{y}} \cap \{i,i+1\}} \mathrm{desc}_r(\boldsymbol{y}),$$

and hence $\mathrm{desc}(\boldsymbol{x}) > \mathrm{desc}(\boldsymbol{y})$ as claimed. □

COROLLARY 5.20. *Let $\boldsymbol{x} \in \mathcal{F}_n^{j,h}$. Then $(P_j \cdots P_1)(\boldsymbol{x})$ is congruent* $\bmod H_{n,j,h-1}$ *to a linear combination of elementary tensors which have smaller descent than $\boldsymbol{x}$. In particular, if $N \geqslant j(n-1)n^2$, then $(P_j \cdots P_1)^N(\boldsymbol{x})$ is cyclically equivalent to a tensor in* $\mathrm{lin}\,\mathcal{F}_n^{j-1}$.

Proof. Let $h := \mathrm{ht}(\boldsymbol{x})$. First, note that the operators $P_1, \dots, P_j$ are pairwise commuting (as they are just linear combinations of I and $sd + ds$). Note also that, by corollary 5.17 and the remarks preceding it, each P_i maps $H_{n,j,h-1}$ to itself.

Now, if r is the number of block-units in $\boldsymbol{x}$, let

$$Q_{j-r} = P_j \cdots P_{j-r+1} P_{j-r-1} \cdots P_1.$$

We note that Q_{j-r} maps $H_{n,j,h-1}$ to itself. Hence, recalling that $0 \leqslant r \leqslant j-1$, it follows from corollary 5.17 that

$$P_j \cdots P_1(\boldsymbol{x}) = Q_{j-r} P_{j-r}(\boldsymbol{x}) \equiv Q_{j-r}(\mathrm{Err}(\boldsymbol{x})) \mod H_{n,j,h-1}. \tag{5.12}$$

Let $\boldsymbol{y}$ be an elementary tensor in $G_{n,j,h-1}$. For arbitrary k, the identity (5.11) also implies that the tensor $P_k(\boldsymbol{y})$ is cyclically equivalent to a linear combination of a term in $G_{n,j,h-1}$, some scalar multiple of $\boldsymbol{y}$, and some scalar multiple of $\mathrm{Err}(\boldsymbol{y})$; in particular, $\bmod H_{n,j,h-1}$, $P_k(\boldsymbol{y})$ is a linear combination of terms whose descent does not exceed $\mathrm{desc}(\boldsymbol{y})$. (This uses lemma 5.19 applied to $\boldsymbol{y}$.)

Since Q_{j-r} is a product of various P_k, the same is true of $Q_{j-r}(\boldsymbol{y})$, and so the descent of *each term* in $Q_{j-r}(\mathrm{Err}(\boldsymbol{x}))$ is bounded above by $\mathrm{desc}(\mathrm{Err}(\boldsymbol{x}))$, which is in turn strictly less than $\mathrm{desc}(\boldsymbol{x})$. Combining this with (5.12), we see that, $\bmod H_{n,j,h-1}$, the tensor $P_j \cdots P_1(\boldsymbol{x})$ is a linear combination of terms with descent *strictly* less than $\mathrm{desc}(\boldsymbol{x})$.

Now let $P = (P_j \cdots P_1)^{j(n-1)}$. Since $\mathrm{desc}(\boldsymbol{x}) \leqslant (j-1)(n-1) \leqslant j(n-1) - 1$, the previous paragraph implies that

$$P(\boldsymbol{x}) \equiv 0 \mod H_{n,j,h-1}.$$

That is, $P(\boldsymbol{x})$ is cyclically equivalent to a linear combination of terms that have at most $j-1$ left-blocks, together with terms that have height strictly less than $\mathrm{ht}(\boldsymbol{x})$. Finally, we can iterate again, using the fact that $n+1 \leqslant \mathrm{ht}(\boldsymbol{x}) \leqslant n(n+1)$, to deduce that if we apply P to $\boldsymbol{x}$ at least $n(n+1) - (n+1) + 1 = n^2$ times, then the resulting tensor will be cyclically equivalent to one in $\mathrm{lin}\,\mathcal{F}_n^{j-1}$. This concludes the proof. □

To prove theorem 5.7, we take $N = j(n-1)n^2$. If $\psi \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}')$ vanishes on all tensors in $\mathcal{F}_n^{j-1}$, the cochain $\psi_1 := [(P_j \cdots P_1)^N]^*(\psi)$ vanishes on all tensors in $\mathcal{F}_n^j$, by the previous corollary. By our earlier remarks, ψ_1 is in the same cyclic cohomology class as ψ, and we have proved theorem 5.7, provided we take for granted the proof of proposition 5.12.

6. Proof of proposition 5.12

Throughout, $\boldsymbol{x}$ denotes a fixed elementary tensor $x_1 \otimes \cdots \otimes x_{n+1}$ which has exactly j left-blocks.

As in earlier sections, it will be useful to regard the boundary operator d as an alternating sum of *face maps*.

Definition 6.1 (face maps on $\mathcal{C}_*(\mathcal{A})$). For $i = 0, \dots, n$, define the *face maps* from $\mathcal{C}_n(\mathcal{A})$ to $\mathcal{C}_{n-1}(\mathcal{A})$ by

$$\begin{aligned} \partial_0(x_1 \otimes \cdots \otimes x_{n+1}) &= x_2 \otimes \cdots \otimes x_n \otimes x_{n+1}x_1, \\ \partial_i(x_1 \otimes \cdots \otimes x_{n+1}) &= \underset{i-1}{\bullet} \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \quad \text{for } 1 \leqslant i \leqslant n. \end{aligned}$$

An easy but key observation. If $a, b, c \in S$ and $[a] \preceq [bc]$, then $[a] \preceq [b]$ and $[a] \preceq [c]$. Thus, if $\boldsymbol{x} = w_1 \otimes \cdots \otimes w_j$, where each w_l is a left-block, and if x_r and x_{r+1} are contained in the same left-block w_l, then $\partial_r(\boldsymbol{x})$ also has exactly j left-blocks, and has the form

$$\partial_r(\boldsymbol{x}) = w_1 \otimes \cdots \otimes w_{l-1} \otimes w_l' \otimes w_{l+1} \otimes \cdots \otimes w_j,$$

where the only new left-block, w_l', is just w_l with x_r and x_{r+1} multiplied together. The important point is that w_l' does not become part of a larger left-block.

If, on the other hand, the face map takes the product of the end of one left-block with the start of the next left-block, then the resulting tensor might have j left-blocks, but might have fewer. The following example illustrates some possibilities.

Example 6.2 (an illustration of complications). Let S be the free semilattice on four generators, labelled as g_1, g_2, g_3 and g_4. Consider

$$\boldsymbol{x} = \overbrace{g_1g_2 \otimes g_2} \otimes \overbrace{g_1g_2g_3 \otimes g_1g_2} \otimes \overbrace{g_3g_4} \otimes \overbrace{g_1} \otimes \overbrace{g_1g_3 \otimes g_3} \in \mathcal{C}_7(\mathcal{A}),$$

which consists of five left-blocks as indicated (so that $I_{\boldsymbol{x}} = \{1, 3, 5, 6, 7\}$). Then

$$\partial_4(\boldsymbol{x}) = g_1g_2 \otimes g_2 \otimes g_1g_2g_3 \otimes g_1g_2g_3g_4 \otimes g_4 \otimes g_1g_2 \otimes g_1g_3 \otimes g_3 \in \mathcal{C}_6(\mathcal{A})$$

contains a *minimal element*, so that all entries lie in the same left-block. Note that the set I_4 of initial points in $\partial_4(\boldsymbol{x})$ is just $\{4\}$.

For the purposes of comparison, note that

$$\begin{aligned} \partial_5(\boldsymbol{x}) &= \overbrace{g_1g_2 \otimes g_2} \otimes \overbrace{g_1g_2g_3 \otimes g_1g_2} \otimes \overbrace{g_1g_3g_4 \otimes g_1g_3 \otimes g_3}, & I_5 &= \{1, 3, 5\}, \\ \partial_6(\boldsymbol{x}) &= \overbrace{g_1g_2 \otimes g_2} \otimes \overbrace{g_1g_2g_3 \otimes g_1g_2} \otimes \overbrace{g_3g_4} \otimes \overbrace{g_1g_3 \otimes g_3}, & I_6 &= \{1, 3, 5, 6\}, \\ \partial_2(\boldsymbol{x}) &= \overbrace{g_1g_2} \otimes \overbrace{g_1g_2g_3 \otimes g_1g_2} \otimes \overbrace{g_3g_4} \otimes \overbrace{g_1} \otimes \overbrace{g_1g_3 \otimes g_3}, & I_2 &= \{1, 2, 4, 5, 6\}. \end{aligned}$$

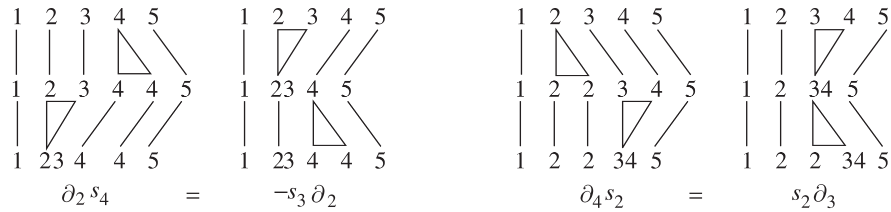

Figure 1. Individual pairs of cancelling terms.

With this warning example in mind, we start on the proof. Define indexing sets $\mathrm{SD} \subseteq \{1,\dots,n\}\times\{0,\dots,n\}$ and $\mathrm{DS} \subseteq \{0,\dots,n+1\}\times\{1,\dots,n+1\}$ as

$$\mathrm{SD} = \{(i,p)\colon 0 \leqslant p \leqslant n \text{ and } i \in I_{\partial_p(\boldsymbol{x})}\} = \coprod_{0\leqslant p\leqslant n} I_{\partial_p(\boldsymbol{x})} \times \{p\},$$

$$\mathrm{DS} = \{(r,k)\colon 0 \leqslant r \leqslant n+1 \text{ and } k \in I_{\boldsymbol{x}}\} = \{0,\dots,n+1\}\times I_{\boldsymbol{x}}.$$

Then

$$(sd+ds)(\boldsymbol{x}) = \sum_{(i,p)\in\mathrm{SD}} (-1)^p s_i\partial_p(\boldsymbol{x}) + \sum_{(r,k)\in\mathrm{DS}} (-1)^r \partial_r s_k(\boldsymbol{x}),$$

and the first task in evaluating this tensor is to show that most terms on the right-hand side either cancel pairwise, or have fewer than j left-blocks, or have lower height than $\boldsymbol{x}$. Much of this takes place in greater generality, without using the properties of the left-coherent units that are inserted.

Lemma 6.3.

(A) *Let $1 \leqslant i < p \leqslant n$. Then*

$$s_i\partial_p(\boldsymbol{x}) = (-1)^i \underset{i-1}{\bullet} \otimes \langle x_i] \otimes x_i \otimes \cdots \otimes x_{p-1} \otimes x_p x_{p+1} \otimes \underset{n-p-1}{\bullet} = \partial_{p+1}s_i(\boldsymbol{x}). \tag{6.1 a}$$

(B) *Let $1 \leqslant p < i \leqslant n$. Then*

$$s_i\partial_p(\boldsymbol{x}) = (-1)^i \underset{p-1}{\bullet} \otimes x_p x_{p+1} \otimes \cdots \otimes x_i \otimes \langle x_{i+1}] \otimes x_{i+1} \otimes \underset{n-i}{\bullet} = -\partial_p s_{i+1}(\boldsymbol{x}). \tag{6.1 b}$$

(C) *Let $1 \leqslant i \leqslant n-1$. Then*

$$s_i\partial_0(\boldsymbol{x}) = (-1)^i x_2 \otimes \cdots \otimes x_i \otimes \langle x_{i+1}] \otimes x_{i+1} \otimes \underset{n-i}{\bullet} \otimes x_{n+1}x_1 = -\partial_0 s_{i+1}(\boldsymbol{x}). \tag{6.1 c}$$

Proof. This is a direct computation. We omit the details (see figure 1 for a diagram that illustrates how this works in cases A and B). □

We have to keep track of which terms in a corresponding pair, as in lemma 6.3, actually occur when we expand out $(sd+ds)(\boldsymbol{x})$. More notation will be useful. Let

$$\begin{aligned} A &= \{(i,p)\colon 1 \leqslant i < p \leqslant n\}, & A' &= \{(r,k)\colon 1 \leqslant k < r-1 \leqslant n\},\\ B &= \{(i,p)\colon 1 \leqslant p < i \leqslant n\}, & B' &= \{(r,k)\colon 1 \leqslant r < k-1 \leqslant n\},\\ C &= \{(i,0)\colon 1 \leqslant i \leqslant n-1\}, & C' &= \{(0,k)\colon 2 \leqslant k \leqslant n\}, \end{aligned}$$

so that A, B and C are pairwise disjoint subsets of $\{1,\dots,n\}\times\{0,\dots,n\}$ and A', B' and C' are pairwise disjoint subsets of $\{0,\dots,n+1\}\times\{1,\dots,n+1\}$. Now define

$$\mathrm{DS}_j := \{(r,k)\in\mathrm{DS}\colon \partial_r s_k(\boldsymbol{x}) \text{ has exactly } j \text{ left-blocks}\},$$
$$\mathrm{SD}_j := \{(i,p)\in\mathrm{SD}\colon \partial_p(\boldsymbol{x}) \text{ has exactly } j \text{ left-blocks}\},$$

and let $\mathrm{DS}_j^* := \mathrm{DS}_j \cap (A'\sqcup B'\sqcup C')$ and $\mathrm{SD}_j^* := \mathrm{SD}_j\cap(A\sqcup B\sqcup C)$.

It turns out that the obvious maps $A\leftrightarrow A'$, $B\leftrightarrow B'$ and $C\leftrightarrow C'$ restrict to give a bijection between DS_j^* and SD_j^* (which shows that most terms in $\sum_{\mathrm{DS}_j}+\sum_{\mathrm{SD}_j}$ cancel pairwise). To do this precisely, we have a lemma.

LEMMA 6.4. *Let $m\geqslant 1$, and let $\boldsymbol{y}=y_1\otimes\cdots\otimes y_{m+1}$ be an elementary tensor with j left-blocks. Let $1\leqslant p\leqslant m$. Then $\partial_p(\boldsymbol{y})\in\mathcal{F}_n^{j-1}$ if and only if one (or both) of the following holds: either*

(i) *y_p is a 1-block in $\boldsymbol{y}$, or*

(ii) *$[y_py_{p+1}]\preceq[w]$, where w is the left-block immediately following the one which contains y_{p+1}.*

An analogous result holds for $\partial_0(\boldsymbol{y})$, provided that condition (i) is interpreted as 'y_{m+1} is a 1-block in $\boldsymbol{y}$', and condition (ii) as '$[y_{m+1}y_1]\preceq[w]\dots$'.

Proof. We first note that the case $p=0$ is not really distinct from the cases $1\leqslant p\leqslant m$, once we interpret 'position 0' in a tensor of length $m+1$ as being position $m+1$. Next, we may assume without loss of generality that $1\in I_{\boldsymbol{y}}$ (if not, then by applying a suitable power of $\boldsymbol{t}$ we obtain a tensor $\boldsymbol{y}'$ in which 1 is an initial point, and work with $\boldsymbol{y}'$ instead). Let $\boldsymbol{y}=w_1\otimes\cdots\otimes w_j$ be the decomposition of $\boldsymbol{y}$ into its constituent left-blocks. Let w_k be the left-block which contains y_{p+1}.

If (i) holds, then w_{k-1} just consists of the single element y_p, and

$$\partial_p(\boldsymbol{y}) = w_1\otimes\cdots\otimes w_{k-2}\otimes y_p\cdot w_k\otimes\cdots\otimes w_j.$$

Thus, two left-blocks have been merged together, and there are now at most $j-1$ of them. If (ii) holds, then every element of w and every element of w_k lies above $[y_py_{p+1}]$, so that in $\partial_p(\boldsymbol{y})$ these two left-blocks are merged into a single one; thus, once again, the number of left-blocks has decreased.

Conversely, suppose that $\partial_p(\boldsymbol{y})$ has fewer than j left-blocks, and suppose (ii) does not hold. Then the left-blocks $w_1,\dots,w_{k-2}$ and $w_{k+1},\dots,w_j$ remain left-blocks in $\partial_p(\boldsymbol{y})$. Therefore, $w_{k-1}\cdot w_k$ must form a single left-block. If y_r denotes the initial element of w_{k-1}, and $r<p$, then this implies that $[y_r]\preceq[y_py_{p+1}]\preceq[y_{p+1}]$ and this contradicts the fact that w_{k-1} and w_k are disjoint left-blocks. The only remaining possibility is that w_{k-1} is a 1-block, with y_p as its sole element, and so (i) holds. $\square$

In view of condition (ii) in this lemma, we say that the tensor $\boldsymbol{y}$ has a *dead spot at* $p+1$, for $0\leqslant p\leqslant m$, if $[x_px_{p+1}]\preceq[w]$, where w is the left-block immediately following the one which contains x_{p+1}. Once again, this definition should be interpreted cyclically, so that having a dead spot at 1 means that $[x_{m+1}x_1] \preceq [w]$, etc.

Proposition 6.5. *Define* $\phi\colon \mathrm{DS}_j^* \to \{1,\dots,n\}\times\{1,\dots,n\}$ *as*

$$\phi(r,k) = \begin{cases} (k-1,r) & \text{if } (r,k)\in B'\cup C', \\ (k,r-1) & \text{if } (r,k)\in A'. \end{cases} \tag{6.2}$$

Then ϕ *maps* SD_j^* *bijectively onto* DS_j^*. *Consequently,*

$$\sum_{(r,k)\in\mathrm{DS}_j^*} (-1)^r \partial_r s_k(\boldsymbol{x}) + \sum_{(i,p)\in\mathrm{SD}_j^*} (-1)^p s_i \partial_r(\boldsymbol{x}) = 0. \tag{6.3}$$

Proof. Start by noting that ϕ is the restriction of obvious bijections from A', B' and C' to A, B and C, respectively.

Moreover, the identities in lemma 6.3 show that if $(r,k)\in\mathrm{DS}_j^*$, then $\phi(r,j)\in\mathrm{SD}_j^*$. (The point is that if, say, $1\leqslant k\leqslant r-2$ and $\partial_r s_k(\boldsymbol{x})$ has j left-blocks, then the identity (6.1 b) shows that $s_{k-1}\partial_r(\boldsymbol{x})$ has j left-blocks, so $\partial_r(\boldsymbol{x})$ must have j left-blocks.) Thus, $\operatorname{ran}\phi\subseteq\mathrm{SD}_j^*$.

To show the converse inclusion, let $(i,p)\in\mathrm{SD}_j^*$. Then, by lemma 6.4 (with $m=n$), x_p is not a 1-block in $\boldsymbol{x}$ and $p+1$ is not a dead spot in $\boldsymbol{x}$. (If $p=0$ this means x_{n+1} is not a 1-block, etc.) Therefore, by the other direction of lemma 6.4 (with $m=n+1$), we have the following.

- If $1\leqslant i\leqslant p-1$, and we consider $s_i(\boldsymbol{x})$, then x_p (occurring in position $p+1$) is not a 1-block in $s_i(\boldsymbol{x})$ and $p+2$ is not a dead spot in $s_i(\boldsymbol{x})$, so that $(i,p+1)\in\mathrm{DS}_j^*$.
- If $2\leqslant p+1\leqslant i\leqslant n$, and we consider $s_{i+1}(\boldsymbol{x})$, then x_p (occurring in position p) is not a 1-block in $s_{i+1}(\boldsymbol{x})$ and $p+1$ is not a dead spot in $s_{i+1}(\boldsymbol{x})$, so that $(i+1,p)\in\mathrm{DS}_j^*$.
- If $p=0$ and $1\leqslant i\leqslant n$, and we consider $s_{i+1}(\boldsymbol{x})$, then x_{n+1} (occurring in position $n+2$) is not a 1-block in $s_{i+1}(\boldsymbol{x})$ and 1 is not a dead spot in $s_{i+1}(\boldsymbol{x})$, so that $(i+1,0)\in\mathrm{DS}_j^*$.

In each case, $(i,p)\in\operatorname{ran}\phi$ as required. □

We now continue with the proof of proposition 5.12. It follows from (6.3) that

$$(sd+ds)(\boldsymbol{x}) \equiv \sum_{(r,k)\in\mathrm{DS}_j\setminus\mathrm{DS}_j^*} (-1)^r\partial_r s_k(\boldsymbol{x}) + \sum_{(i,p)\in\mathrm{SD}_j\setminus\mathrm{SD}_j^*} (-1)^p s_i\partial_p(\boldsymbol{x}) \mod (\operatorname{lin}\mathcal{F}_n^{j-1}). \tag{6.4}$$

Expanding out the terms on the right-hand side gives

$$\sum_{1\leqslant k\leqslant n+1:\ (k-1,k)\in \mathrm{DS}_j} (-1)^{k-1}\partial_{k-1}s_k(\boldsymbol{x}) \tag{6.5 a}$$

$$+ \sum_{1\leqslant k\leqslant n+1:\ (k,k)\in \mathrm{DS}_j} (-1)^{k}\partial_{k}s_k(\boldsymbol{x}) \tag{6.5 b}$$

$$+ \sum_{1\leqslant k\leqslant n:\ (k+1,k)\in \mathrm{DS}_j} (-1)^{k}\partial_{k+1}s_k(\boldsymbol{x}) \tag{6.5 c}$$

$$+ \sum_{1\leqslant i\leqslant n:\ (i,i)\in \mathrm{SD}_j} (-1)^{i}s_i\partial_i(\boldsymbol{x}) \tag{6.5 d}$$

$$+ R_{\mathrm{DS}}(\boldsymbol{x}) + R_{\mathrm{SD}}(\boldsymbol{x}), \tag{6.5 e}$$

where the terms $R_{\mathrm{DS}}(\boldsymbol{x})$ and $R_{\mathrm{SD}}(\boldsymbol{x})$ are defined as

$$R_{\mathrm{DS}}(\boldsymbol{x}) = \begin{cases} \partial_0 s_{n+1}(\boldsymbol{x}) & \text{if } (0,n+1)\in \mathrm{DS}_j, \\ 0 & \text{otherwise,}\end{cases}$$

$$R_{\mathrm{SD}}(\boldsymbol{x}) = \begin{cases} s_n\partial_0(\boldsymbol{x}) & \text{if } (n,0)\in \mathrm{SD}_j, \\ 0 & \text{otherwise.}\end{cases}$$

Lemma 6.6.

(i) *If $0\leqslant k\leqslant n$ and $k+1\in I_{\boldsymbol{x}}$, then $\partial_k s_{k+1}(\boldsymbol{x})$ has strictly lower height than $\boldsymbol{x}$.*

(ii) *If $1\leqslant k\leqslant n$ and $k+1\in I_{\boldsymbol{x}}$, then $s_k\partial_k(\boldsymbol{x})$ has strictly lower height than $\boldsymbol{x}$. If $1\in I_{\boldsymbol{x}}$, then $s_n\partial_0(\boldsymbol{x})$ has strictly lower height than $\boldsymbol{x}$.*

Proof. First suppose that $1\leqslant k\leqslant n$. Then the corresponding terms in (i) and (ii) expand out to be

$$\boldsymbol{a} = (-1)^{k+1} \underset{k-1}{\bullet} \otimes x_k\langle x_{k+1}] \otimes x_{k+1} \otimes \underset{n-k}{\bullet},$$
$$\boldsymbol{b} = (-1)^{k} \underset{k-1}{\bullet} \otimes \langle x_k x_{k+1}] \otimes x_k x_{k+1} \otimes \underset{n-k}{\bullet},$$

respectively. Since $k+1$ is initial, $[x_k]\not\preceq[x_{k+1}]$ and so $\mathrm{ht}_{L(\boldsymbol{b})}([b_k]) < \mathrm{ht}_{L(\boldsymbol{x})}([x_k])$. It follows that $\mathrm{ht}(\boldsymbol{b}) < \mathrm{ht}(\boldsymbol{x})$, since $\boldsymbol{b}$ agrees with $\boldsymbol{x}$ in all other entries. A similar argument shows that $\mathrm{ht}(\boldsymbol{a}) < \mathrm{ht}(\boldsymbol{x})$.

In the case where $k=0$ (and $1\in I_{\boldsymbol{x}}$), set $\boldsymbol{b} = s_n\partial_0(\boldsymbol{x})$, $\boldsymbol{b}' = \boldsymbol{t}^{-1}(\boldsymbol{b})$ (see (2.1)), and $\boldsymbol{a} = \partial_0 s_1(\boldsymbol{x})$. Then

$$\boldsymbol{a} = \partial_0 s_1(\boldsymbol{x}) = -x_1 \otimes \underset{n-1}{\bullet} \otimes \langle x_{n+1}]x_1 \quad\text{and}\quad \boldsymbol{b}' = x_{n+1}x_1 \otimes \underset{n-1}{\bullet} \otimes \langle x_{n+1}x_1].$$

The same arguments as in the first part of the proof show that $\mathrm{ht}(\boldsymbol{b}')$ and $\mathrm{ht}(\boldsymbol{a})$ are both strictly less than $\mathrm{ht}(\boldsymbol{x})$; it remains only to note that since the height of an elementary tensor is unchanged by cyclic shifts, $\mathrm{ht}(\boldsymbol{b}) = \mathrm{ht}(\boldsymbol{b}')$. $\square$

Lemma 6.7. *Let $1\leqslant i\leqslant n$ and suppose $(i,i)\in \mathrm{SD}_j$. Then either i or $i+1$ lies in $I_{\boldsymbol{x}}$. If $(n,0)\in \mathrm{SD}_j$, then either $n+1$ or 1 lies in $I_{\boldsymbol{x}}$.*

Consequently, if $\boldsymbol{x}$ has height h, then

$$R_{\mathrm{SD}}(\boldsymbol{x}) + \sum_{1\leqslant i\leqslant n:\ (i,i)\in \mathrm{SD}_j} s_i\partial_i(\boldsymbol{x}) \equiv \sum_{i\in I_{\boldsymbol{x}}} \underset{i-1}{\bullet} \otimes \langle x_i x_{i+1}] \otimes x_i x_{i+1} \otimes \underset{n-i}{\bullet} \mod G_{n,j,h-1}, \tag{6.6}$$

where, if $n+1 \in I_{\boldsymbol{x}}$, the corresponding term on the right-hand side of (6.6) is interpreted as $(-1)^n x_2 \otimes \cdots \otimes x_n \otimes \langle x_{n+1}x_1] \otimes x_{n+1}x_1$.

Proof. Let $1 \leqslant i \leqslant n$. Write $s_i\partial_i(\boldsymbol{x}) = \boldsymbol{a}$, as defined in the proof of lemma 6.6. By assumption, $\boldsymbol{a}$ has j left-blocks and one of them starts in position i. If neither i nor $i+1$ were initial in $\boldsymbol{x}$, then x_i and x_{i+1} would both lie in the same left-block of $\boldsymbol{x}$, whose initial point is some $k < i$, and so a_k would also mark the start of a left-block in $\boldsymbol{a}$ which contains $a_i = \langle x_i x_{i+1}]$. Since we originally assumed that $i \in I_{\boldsymbol{a}}$, this yields a contradiction.

A similar argument, with slight adjustments to the notation, shows that if $n \in I_{\partial_0(\boldsymbol{x})}$, then either $n+1$ or 1 must have been initial in $\boldsymbol{x}$. This completes the proof of the first part of the lemma.

For the second part of the lemma, suppose that $i \in I_{\boldsymbol{x}}$ with $1 \leqslant i \leqslant n$, and note that there are two possibilities. Either $\partial_i(\boldsymbol{x})$ has fewer than j left-blocks, in which case $s_i\partial_i(\boldsymbol{x}) \in \mathcal{F}_n^{j-1}$, or $\partial_i(\boldsymbol{x})$ has exactly j left-blocks, in which case one of them must start in position i, and so $(i,i) \in \mathrm{SD}_j$. By the first part of the lemma, the only other $(k,k) \in \mathrm{SD}_j$ with $1 \leqslant k \leqslant n$ must arise from having $k+1 \in I_{\boldsymbol{x}}$, but then, by lemma 6.6(ii), such terms have height at most $h-1$.

It remains to deal with the case where $n+1 \in I_{\boldsymbol{x}}$. If $\partial_0(\boldsymbol{x})$ has fewer than j left-blocks, then $R_{\mathrm{SD}}(\boldsymbol{x}) = 0$; if it has exactly j left-blocks, then $(n,0) \in \mathrm{SD}_j$ and so

$$R_{\mathrm{SD}}(\boldsymbol{x}) = (-1)^n x_2 \otimes \cdots \otimes x_{n-1} \otimes \langle x_{n+1}x_1] \otimes x_{n+1}x_1,$$

as required. Equation (6.6) now follows. □

In summary, all terms in (6.5 a) have strictly lower height than $\boldsymbol{x}$; the terms in (6.5 b) each give $\boldsymbol{x}$, since $\langle x_i]x_i = x_i$ for all i, and lemma 6.7 tells us the sum of terms in (6.5 d) with $R_{\mathrm{SD}}(\boldsymbol{x})$, provided we work modulo terms of fewer left-blocks or lower height. Therefore, the right-hand side of (6.4) is equal to

$$\sum_{k\in I_{\boldsymbol{x}}} \boldsymbol{x} + \sum_{k\in I_{\boldsymbol{x}}} (-1)^{k+1}\partial_{k+1}s_k(\boldsymbol{x}) + \sum_{k\in I_{\boldsymbol{x}}} (-1)^k s_k\partial_k(\boldsymbol{x}) \mod G_{n,j,h-1},$$

provided that we interpret the case $n+1 \in I_{\boldsymbol{x}}$ appropriately. Expanding this out gives exactly what is claimed in proposition 5.12, and so completes the proof.

7. Tying things together

The inductive calculations carried out in the previous sections give us the following result.

THEOREM 7.1. *Let $n \geqslant 1$, and let $\psi \in \mathcal{ZC}^n(\ell^1(S))$ be an R-normalized cyclic n-cocycle. Then ψ is a cyclic coboundary.*

Combining this with proposition 3.4, we finally obtain our main result.

THEOREM 7.2. *The cyclic cohomology of $\ell^1(S)$ is zero in all odd degrees, whereas, in even degrees, it is the space* $\{[\tau^{(2n)}] : \tau \in \mathcal{Z}^0(\ell^1(S), \ell^1(S)')\}$.

As promised earlier, we can use theorem 7.2 to determine the simplicial cohomology of $\ell^1(S)$ via the Connes–Tzygan long exact sequence. This requires one last fact about how the cohomology classes $[\tau^{(2n)}]$ transform under the shift map S.

LEMMA 7.3. *Let $n \geqslant 1$. There exists a non-zero constant λ_n such that, for any Banach algebra A and $\tau \in \mathcal{Z}^0(A, A')$, the shift map $S\colon \mathcal{HC}^{2n-2}(A) \to \mathcal{HC}^{2n}(A)$ satisfies $S(\tau^{(2n-2)}) = \lambda_n \tau^{(2n)}$.*

This lemma seems to be folklore, to some extent (for a direct proof that does not rely on [9], see [5]). The value of λ_n depends on a choice of scalar normalization of S when one constructs the Connes–Tzygan exact sequence. In [5] the formulae are chosen so that $\lambda_n = 1$ for all n, but if one uses the formulae of [9], then different scaling factors will appear.

THEOREM 7.4. *The simplicial cohomology of $\ell^1(S)$ is zero in degrees* 1 *and above.*

Proof. By theorem 4.7 and [9, theorem 11], the Connes–Tzygan sequence for $\ell^1(S)$ exists. Then, since the cyclic cohomology of $\ell^1(S)$ vanishes in all odd degrees, the long exact sequence breaks up to give exact sequences

$$0 \to \mathcal{HH}^{2n-1}(\ell^1(S)) \xrightarrow{B} \mathcal{HC}^{2n-2}(\ell^1(S)) \xrightarrow{S} \mathcal{HC}^{2n}(\ell^1(S)) \to \mathcal{HH}^{2n}(\ell^1(S)) \to 0$$

for all $n \geqslant 1$. Moreover, the shift map is surjective: by theorem 7.2, every cyclic $2n$-cocycle is cohomologous to one of the form $\tau^{(2n)}$ for some trace τ, and by lemma 7.3 we have $\tau^{(2n)} = \lambda_n^{-1} S\tau^{(2n-2)}$. Thus, $\mathcal{HH}^{2n}(\ell^1(S)) = 0$ for all $n \geqslant 1$.

To conclude, it suffices to show that the shift map is injective (which will imply that B is the zero map, and hence that $\mathcal{HH}^{2n-1}(\ell^1(S)) = 0$). As already observed in this proof, $\mathcal{HC}^{2n-2}$ is generated by cohomology classes of the form $[\tau^{(2n-2)}]$, where $\tau \in \mathcal{Z}^0(\mathcal{A}, \mathcal{A}')$. Consider $S([\tau^{(2n-2)}]) = \lambda_n[\tau^{(2n)}]$ and suppose that $\tau^{(2n)} = \delta\varphi$ for some $\varphi \in \mathcal{CC}^{2n-1}(A)$. For each idempotent $e \in \mathcal{A}$, direct calculation gives $\tau(e) = \delta\varphi(e, \dots, e)(e) = \varphi(e, \dots, e)(e)$. But, since φ is *cyclic*,

$$\varphi(e, \dots, e)(e) = -\varphi(e, \dots, e)(e) = 0.$$

Thus, τ vanishes on each idempotent in $\mathcal{A}$, and since $\mathcal{A} = \ell^1(S)$ where S is a band, continuity forces τ to vanish identically. □

8. Conclusion

We have had to work quite hard to establish that the cyclic and simplicial cohomology of a band ℓ^1-semigroup algebra behave as one would hope. Our methods would simplify in the case where the band is a semilattice, and in that case they would give an alternative approach to the main result of [3]. Note that, in [3], Choi was unable to obtain explicit formulae for cobounding a given cocycle in high degrees, since the contracting homotopy in that setting was only given recursively. Here, we have an explicit algorithm for cobounding a given *cyclic* cocycle, but once again we do not have a reasonable formula for cobounding arbitrary cocycles in high degrees, even for the case of a semilattice, owing to the reliance on the Connes–Tzygan exact sequence.

We feel that the tactics used in establishing the main result may be of wider interest when interpreted in a broader sense. A general picture seems to be emerging: in order to obtain vanishing results for simplicial cohomology of Banach algebras, unless the geometry of the underlying Banach spaces intervenes helpfully, one has to replace the exhaustion arguments that are commonly found in 'purely algebraic' cohomology of algebras with more careful approaches, and these new arguments seem to depend on the *local relations* between entries of a given elementary tensor in the Hochschild chain complex, rather than on how such entries factorize in terms of *global generators* for the algebra.

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