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Stitching B-Spline Curves Symbolically

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ABSTRACT

We present an algorithm for stitching B-spline curves, which is different from the generally used least square method. Our aim is to find a symbolic solution for unifying the control polygons of arcs separately described as 4th degree B-spline curves. We show the effect of interpolation conditions and fairing functions as well.

Key words: B-spline curve, B-spline surface, merging, interpolation, fairing

MSC2010: 65D17, 65D05, 65D07, 68U05, 68U07

Simboličko spajanje B-splajn krivulja

SAŽETAK

Predstavljamo algoritam za spajanje B-splajn krivulja, koji se razlikuje od općenito upotrebljavane metode najmanjih kvadrata. Naš cilj je naći simboličko rješenje za ujedinjavanje kontrolnih poligona lukova koji se svaki zasebno opisuju kao B-splajn krivulje 4. stupnja. Također pokazujemo utjecaj uvjeta interpolacije i postizanja glatkih funkcija.

Ključne riječi: B-splajn krivulja, B-splajn ploha, integriranje, interpolacija, postizanje glatkoće

1 Introduction

Stitching or merging B-spline curves is a frequently used technique in geometric modeling, and is usually implemented in CAD-systems. These algorithms are basically numerical interpolations using the least squares method. The problem, how to replace two or more curves which are generated separately and defined as B-spline curves, has well functioning numerical solutions, therefore, relatively few papers have been published about this topic. In [6] and [3] methods for approximate merging of B-spline curves and surfaces are given. In [4] one of the symbolical algorithms is described, which extends B-spline curves by adding more interpolation points one by one at the end of the curve. In [5] the construction of a covering surface is shown for unifying more B-spline surfaces.

We approach the stitching problem from a geometrical point of view, and represent a symbolical solution to compute the control points of the new curve from the control points of the two given curve segments and appropriate interpolation conditions. This symbolical solution is stable, it can be used generally for any two given curves. The error of the interpolation depends on the curvatures of the input curves. Larger difference in their curvatures raises the error. In order to reduce the error, two of the new control points are adjusted by fairing conditions using the concrete numerical data. This computation requires minimization of quadratic functions leading to solve linear equations. In this way we avoid non-linear optimization problems. Applying fairing functions for modifying the shape and the properties of curves and surfaces is a standard technique. In [7], [8] and [9] constructions of B-spline surfaces with boundary conditions are presented using fairing functions. Finally, merging of B-spline surface patches are shown applying the developed curve stitching method for their parameter curves.

2 Symbolical solution for stitching two B-spline curve segments

In our symbolical solution for stitching two given curves we assume that they are represented by B-spline segments of degree 4 with uniform periodic knot vectors. The one-parameter vector function of such a curve is

$$\mathbf{r}(t) = \begin{pmatrix} t^4 & t^3 & t^2 & t & 1 \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix}, 0 \le t \le 1,$$

where

$$\mathbf{M} = \frac{1}{24} \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}.$$

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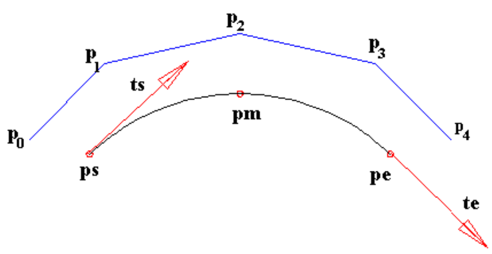

Figure 1: *Input data and control points of one curve segment*

We recall the symbolical solution of the interpolation problem ([10]), where the input data are the interpolation points **ps**, **pm**, **pe**, and the derivatives at the endpoints **ts** and **te** (Fig. 1). The output is the 5 control points $\mathbf{p}_i$, $i = 0, \dots, 4$ computed from the conditions

$$\mathbf{r}(0) = \mathbf{ps},\ \mathbf{r}(0.5) = \mathbf{pm},\ \mathbf{r}(1) = \mathbf{pe},\ \dot{\mathbf{r}}(0) = \mathbf{ts},\ \dot{\mathbf{r}}(1) = \mathbf{te}.$$

The control points are expressed by the input data as the solution of this system of linear equations.

$$\begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \mathbf{p}_4 \end{pmatrix} = \begin{pmatrix} -30.1667\mathbf{pe} - 46.1667\mathbf{ps} + 77.3333\mathbf{pm} + 6.33333\mathbf{te} - 16.3333\mathbf{ts} \\ 7.83333\mathbf{pe} + 11.8333\mathbf{ps} - 18.6667\mathbf{pm} - 1.66667\mathbf{te} + 2.66667\mathbf{ts} \\ -6.16667\mathbf{pe} - 6.16667\mathbf{ps} + 13.3333\mathbf{pm} + 1.33333\mathbf{te} - 1.33333\mathbf{ts} \\ 11.8333\mathbf{pe} + 7.83333\mathbf{ps} - 18.6667\mathbf{pm} - 2.66667\mathbf{te} + 1.66667\mathbf{ts} \\ -46.1667\mathbf{pe} - 30.1667\mathbf{ps} + 77.3333\mathbf{pm} + 16.3333\mathbf{te} - 6.33333\mathbf{ts} \end{pmatrix}$$

In order to demonstrate the behaviour of this symbolic interpolation method we approximated a circular arc $\mathbf{c}(t)$ with central angle $\leq \pi/3$ interpolated by the curve $\mathbf{r}(t)$. The numerical error measured by $\int_0^1 (\mathbf{c}(t) - \mathbf{r}(t))^2 dt$ is less than 10^{-28}, i.e approximately zero.

We use this experience for stitching two joining B-spline curve segments. In that algorithm we will use also similar interpolation data and B-spline functions of degree 4.

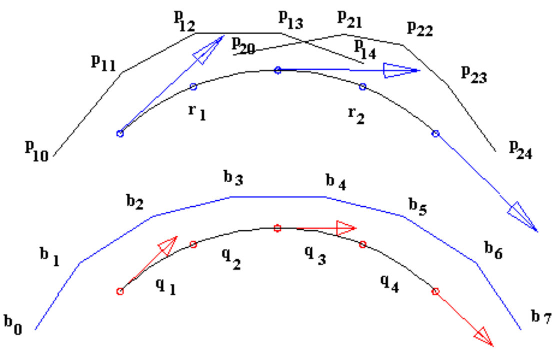

Figure 2: *Merging two curves into 4 B-spline segments*

We assume that the two input segments are given by B-spline functions, one by $\mathbf{r}_1(t)$ with control points $\mathbf{p}_{1j}$ and the other by $\mathbf{r}_2(t)$ with control points $\mathbf{p}_{2j}$, $(j = 0, \dots, 4)$.

We generate the resulting B-spline curve with 4 segments $\mathbf{q}_i(t)$, $0 \leq t \leq 1$, $(i = 1, \dots, 4)$ determined by 8 control points $\mathbf{b}_j$, $(j = 0, \dots, 7)$.

The interpolation conditions are 5 points + 3 tangent vectors (Fig. 2).

$$\mathbf{q}_1(0) = \mathbf{r}_1(0),\ \mathbf{q}_2(0) = \mathbf{r}_1(0.5),\ \mathbf{q}_2(1) = \mathbf{r}_1(1),$$
$$\mathbf{q}_3(1) = \mathbf{r}_2(0.5),\ \mathbf{q}_4(1) = \mathbf{r}_2(1)$$
$$\dot{\mathbf{q}}_1(0) = \dot{\mathbf{r}}_1(0),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_1(1),\ \dot{\mathbf{q}}_4(1) = \dot{\mathbf{r}}_2(1)$$

These 8 equations are linear in the unknown control points of the new B-spline curve. The solution of the system results in the required control points $\mathbf{b}_j$, $(j = 0, \dots, 7)$ expressed as linear combinations of the given control points $\mathbf{p}_{1i}$, $\mathbf{p}_{2i}$, $(i = 0, \dots, 4)$.

Especially,

$$\begin{aligned}
\mathbf{b}_0 = {} & 1.07083\mathbf{p}_{10} + 2.0166\mathbf{p}_{11} - 4.2305\mathbf{p}_{12} + 3.7694\mathbf{p}_{13} + 1.4069\mathbf{p}_{14} \\
& -0.0090\mathbf{p}_{20} - 0.6500\mathbf{p}_{21} - 1.8236\mathbf{p}_{22} - 0.5416\mathbf{p}_{23} - 0.0090\mathbf{p}_{24} \\
\mathbf{b}_1 = {} & -0.01527\mathbf{p}_{10} + 0.5944\mathbf{p}_{11} + 1.0083\mathbf{p}_{12} - 0.9638\mathbf{p}_{13} - 0.3236\mathbf{p}_{14} \\
& +0.0020\mathbf{p}_{20} + 0.1500\mathbf{p}_{21} + 0.4208\mathbf{p}_{22} + 0.1250\mathbf{p}_{23} + 0.0020\mathbf{p}_{24} \\
\mathbf{b}_2 = {} & 0.0090\mathbf{p}_{10} + 0.2055\mathbf{p}_{11} + 0.2930\mathbf{p}_{12} + 0.7444\mathbf{p}_{13} + 0.2145\mathbf{p}_{14} \\
& -0.0013\mathbf{p}_{20} - 0.1000\mathbf{p}_{21} - 0.2805\mathbf{p}_{22} - 0.0833\mathbf{p}_{23} - 0.0013\mathbf{p}_{24} \\
\mathbf{b}_3 = {} & -0.0020\mathbf{p}_{10} + 0.1833\mathbf{p}_{11} + 0.9152\mathbf{p}_{12} - 0.3555\mathbf{p}_{13} - 0.2076\mathbf{p}_{14} \\
& +0.0013\mathbf{p}_{20} + 0.1000\mathbf{p}_{21} + 0.2805\mathbf{p}_{22} + 0.0833\mathbf{p}_{23} + 0.0013\mathbf{p}_{24} \\
\mathbf{b}_4 = {} & 0.0013\mathbf{p}_{10} - 0.1222\mathbf{p}_{11} + 0.0750\mathbf{p}_{12} + 1.4361\mathbf{p}_{13} + 0.3097\mathbf{p}_{14} \\
& -0.0020\mathbf{p}_{20} - 0.1500\mathbf{p}_{21} - 0.4208\mathbf{p}_{22} - 0.1250\mathbf{p}_{23} - 0.0020\mathbf{p}_{24} \\
\mathbf{b}_5 = {} & -0.0013\mathbf{p}_{10} + 0.1222\mathbf{p}_{11} - 0.1861\mathbf{p}_{12} - 1.6305\mathbf{p}_{13} - 0.3375\mathbf{p}_{14} \\
& +0.0090\mathbf{p}_{20} + 0.6500\mathbf{p}_{21} + 1.8236\mathbf{p}_{22} + 0.5416\mathbf{p}_{23} + 0.0090\mathbf{p}_{24} \\
\mathbf{b}_6 = {} & 0.0020\mathbf{p}_{10} - 0.1833\mathbf{p}_{11} + 0.3069\mathbf{p}_{12} + 2.4944\mathbf{p}_{13} + 0.5131\mathbf{p}_{14} \\
& -0.01527\mathbf{p}_{20} - 0.8500\mathbf{p}_{21} - 1.3361\mathbf{p}_{22} + 0.0833\mathbf{p}_{23} - 0.01527\mathbf{p}_{24} \\
\mathbf{b}_7 = {} & -0.0090\mathbf{p}_{10} + 0.7944\mathbf{p}_{11} - 1.4041\mathbf{p}_{12} - 10.9388\mathbf{p}_{13} - 2.2423\mathbf{p}_{14} \\
& +0.0708\mathbf{p}_{20} + 3.3500\mathbf{p}_{21} + 6.0583\mathbf{p}_{22} + 4.2500\mathbf{p}_{23} + 1.0708\mathbf{p}_{24}
\end{aligned}$$

The range of magnitudes of the coefficients show that the solution is stable. The corresponding vector equation of the unified B-spline curve is

$$\mathbf{q}_i(t) = \begin{pmatrix} t^4 & t^3 & t^2 & t & 1 \end{pmatrix} \cdot \mathbf{M} \cdot \begin{pmatrix} \mathbf{b}_{i+j-1} \\ \mathbf{b}_{i+j} \\ \mathbf{b}_{i+j+1} \\ \mathbf{b}_{i+j+2} \\ \mathbf{b}_{i+j+3} \end{pmatrix}, 0 \leq t \leq 1,$$

where $i = 1, \dots, 4$ (4 segments), $j = 0, \dots, 4$ (each with 5 control points).

The examples show that the interpolation error depends on the variation of the curvatures of the given input curves. This error is measured by the integrated sum of the quadratic difference between the corresponding points of the given and the computed new curves, while each segment is parametrized on the $[0, 1]$ interval. That is, the error

$$\text{error} = \sum_{\textit{all segments}} \int_0^1 (\mathbf{r}_{ik}(t) - \mathbf{q}_j(t))^2\, dt,$$

$$(i = 1, 2,\ k = 1, 2,\ j = 1 \dots 4).$$

Fig. 2 shows the segmentation of the curves, where each of the given curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ is divided into two parts.
In Fig. 3 the result of stitching two circular arcs of equal curvatures is shown with the computed control points. The interpolation points are marked by circles. The new B-spline curve interpolates the given arcs practically with zero (10^{-26}) error. If the curvatures of the two arcs are different, the error is larger (Fig. 4 and Fig. 5, the given curves are drawn lighter, the interpolation points are marked with circles, the interpolation derivatives are not shown). Moreover, the resulting curve shows a wavy shape due to the low number of interpolation conditions.
The interpolation error can be reduced by prescribing more interpolation conditions. The shape of the curve can be improved by fairing (smoothing) conditions.

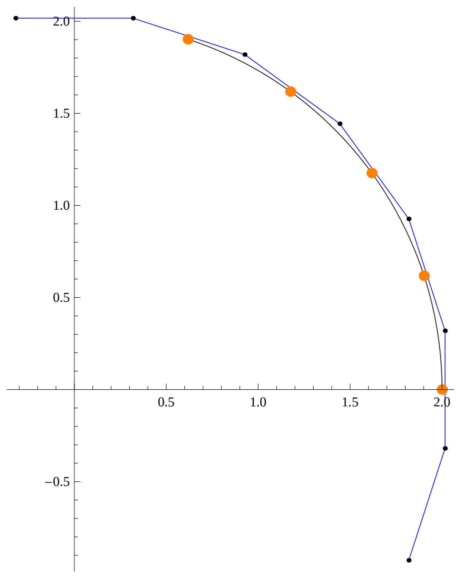

Figure 3: *Merged circular arcs, error* ≈ 0

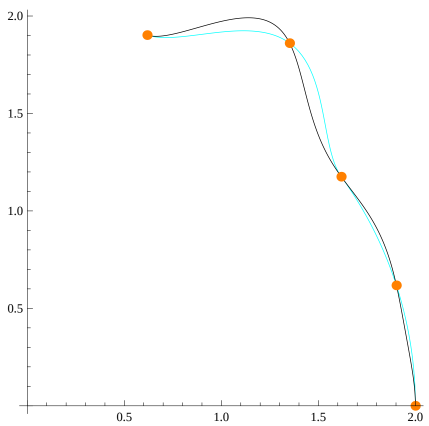

Figure 4: *Merged curves with different curvatures, error*$= 0,0066$

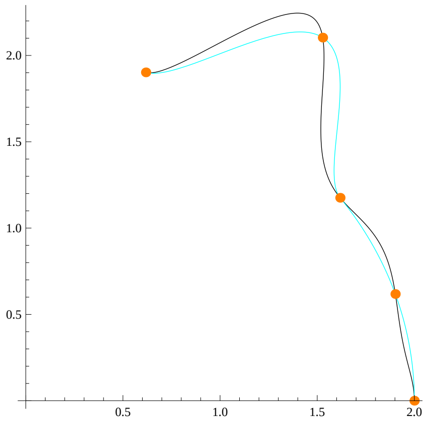

Figure 5: *Merged curves with more different curvatures, error*$= 0,026$

3 The effect of fairing conditions

The solution, where 8 control points are determined from 2×5 given control points, result in a uniquely determined B-spline curve with 4 segments. In order to apply fairing conditions free control points are necessary. Therefore, the prescribed 8 interpolation conditions have to be relaxed. In our investigation we have deleted two interpolation points (the midpoints of the input curves), and have chosen two variable control points $\mathbf{b}_3$ and $\mathbf{b}_4$ for modifying the shape of the resulting curve. In this case 3 points and 3 derivatives are prescribed,

$$\mathbf{q}_1(0) = \mathbf{r}_1(0),\ \mathbf{q}_2(1) = \mathbf{r}_1(1),\ \mathbf{q}_4(1) = \mathbf{r}_2(1)$$
$$\dot{\mathbf{q}}_1(0) = \dot{\mathbf{r}}_1(0),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_1(1),\ \dot{\mathbf{q}}_2(1) = \dot{\mathbf{r}}_2(1)$$

We consider the same integrated sum of the quadratic differences between the given and required B-spline curve segments, which measures the interpolation error, but now it contains two free control points, and is considered as target function to be minimized.

$$F(\mathbf{b}_3, \mathbf{b}_4) = \sum_{\textit{all segments}} \int_0^1 (\mathbf{r}_{ik}(t) - \mathbf{q}_j(t))^2\, dt,$$

$$(i = 1,2,\ k = 1,2,\ j = 1\ldots 4).$$

This function is quadratic in the variables. Therefore, the minimization leads to a system of linear equations. The minimal value measures the interpolation error.

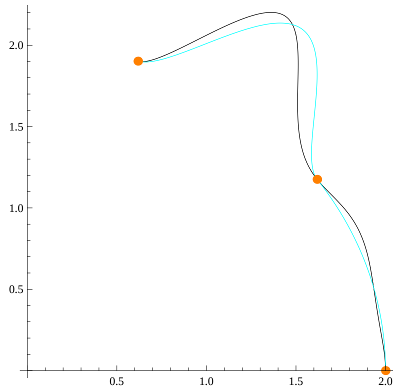

Figure 6: *The error*$= 0,024$

Though the error has been reduced, but the shape is still wavy. In order to get smoother curve, we add to the target function two additional terms. One is for minimizing the difference of the derivatives between the given and required curves, the other for minimizing the variation of the second derivative of the middle curve segments $\mathbf{q}_2(t)$ and $\mathbf{q}_3(t)$, where the curvatures of the given curves show larger difference.
The extended target function is

$$\begin{gathered}\Sigma_{all\,segments}[\int_0^1(\mathbf{r}_{ik}(t)-\mathbf{q}_j(t))^2\,dt\\ +0,2\cdot\int_0^1(\dot{\mathbf{r}}_{ik}(t)-\dot{\mathbf{q}}_j(t))^2]\,dt\\ +0,1\cdot\Sigma_{j=2}^{3}\int_0^1\ddot{\mathbf{q}}_j(t)^2\,dt\end{gathered}$$

The minimization of this target function results in a smoother curve and larger error. The coefficients 0,2 and 0,1 are chosen by experiments. If the weight of the third term is larger, the upper bump in the new merged curve disappears and the error is growing (Fig. 7). It is obvious that smoothing requires more interpolation conditions.

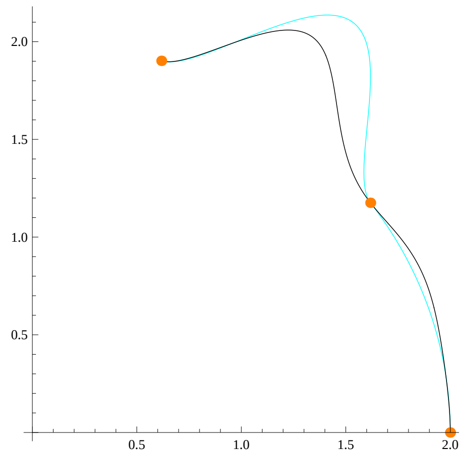

Figure 7: *After fairing the error*$= 0,048$

4 Improving the solution

More interpolation conditions lead to more control points, therefore, the resulting curve will consist of more curve segments. After several experiments our solution will have 8 curve segments with 12 new control points (Fig. 8). Accordingly, the input curves have to be segmented each into 4 parts.

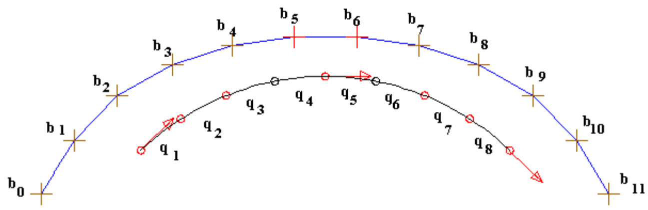

Figure 8: *Segmentation of the merged curve*

First, all the 12 new control points are computed from 12 interpolation conditions, which are 7 interpolation points and 5 tangent vectors. The interpolation points are points of the input curves corresponding to the starting point of the curve segment $\mathbf{q}_1(t)$ and to the end points of the 1., 2., 4., 6., 7., 8. curve segments (Fig. 8). The first derivatives are prescribed at the two end points, at the midpoints and at the joining point of the given curves. These conditions expressed with the B-spline vector functions are linear in the unknown control points $\mathbf{b}_i$, $i = 0,\ldots,11$. The solution is expressed by linear combinations of the given control points $\mathbf{p}_{1j}$ and $\mathbf{p}_{2j}$, $j = 0,\ldots,4$. This symbolical solution is shown in Fig. 9. The error has been succesfully reduced from 0,026 to 0,0035.

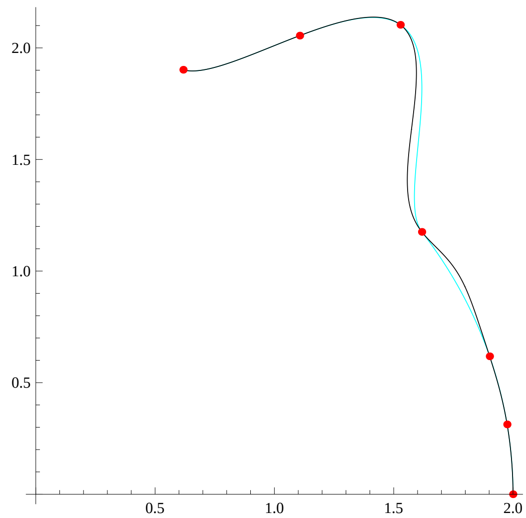

Figure 9: *Merged curve computed by the symbolical solution. The error*$= 0,0035$.

If the given curves do not join, but there is a gap between them, the interpolation point is the midpoint between the two end points of the given curves and similarly, the interpolation tangent vector at this point is the middle value of the two end tangents. In this way a B-spline curve can be determined which replaces parts of the two given curves and connect them smoothly (Fig. 10).

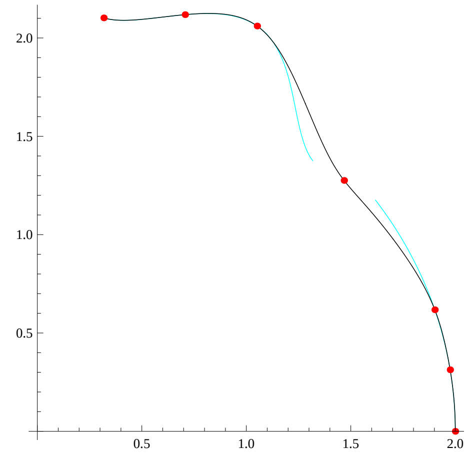

Figure 10: *Stitching two curves with a gap*

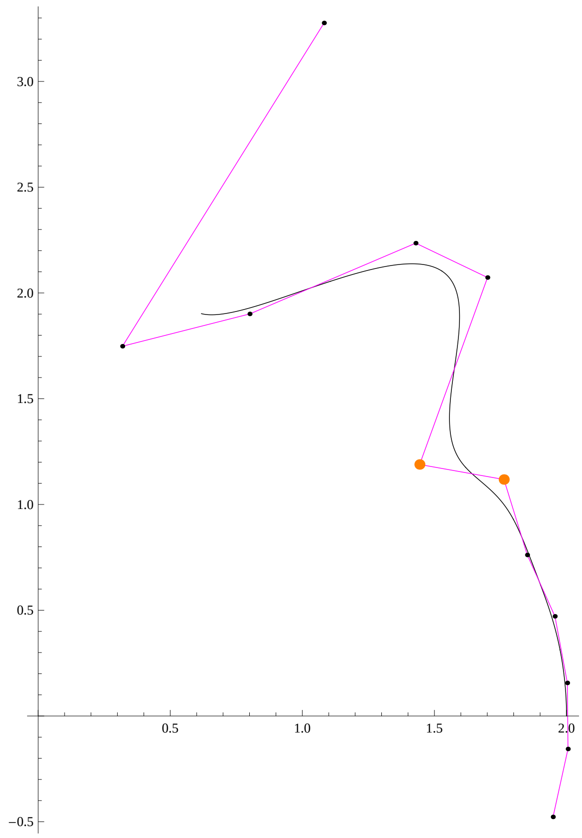

Figure 11: *The control polygon with two variable control points*

The shape of the solution can be improved by applying fairing conditions. Our investigations have shown that two variable control points provide satisfactory solutions. Fig. 11 shows the control polygon of the merged B-spline curve with 10 precomputed and 2 free control points. The interpolation conditions are now 7 points and 3 tangent vectors, and the fairing condition is given by the same target function as in Section 2, but with 8 curve segments. The solution of minimization results in a slightly smoother curve. The interpolation error slightly increased in this case from 0,0035 to 0,004. The picture of the curve looks like in Fig. 9, the difference is not visible.

The symbolical solution (without fairing) leads to smoother curve and smaller error, if the variation of the curvatures of the given curve is smaller. On the base of this experience we have applied it for stitching B-spline patches.

5 Stitching two B-spline patches

We assume that the surface patches are represented by two-parameter vector functions of 4×4 degree with periodical uniform knot vectors. The matrix form is

$$\mathbf{r}(u,v) = (u^4\ u^3\ u^2\ u\ 1)\cdot\mathbf{M}\cdot\mathbf{B}\cdot\mathbf{M}^T\cdot(v^4\ v^3\ v^2\ v\ 1)^T,$$

$$(u,v)\in[0,1]\times[0,1]$$

and

$$\mathbf{M} = \frac{1}{24}\begin{pmatrix} 1 & -4 & 6 & -4 & 1\\ -4 & 12 & -12 & 4 & 0\\ 6 & -6 & -6 & 6 & 0\\ -4 & -12 & 12 & 4 & 0\\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}.$$

The geometric data are the points of the control net denoted by

$$\mathbf{B} = [\mathbf{b}[i,j]]\ , \quad i=0,\ldots 4, \quad j=0,\ldots 4.$$

In the computation of merging two given B-spline patches we apply the symbolical solution shown for merging two B-spline curve segments. Each given control net consists of 5×5 control points. The new control net of 5×12 control points are computed row by row by the same scheme applied for curves. The resulting surface has 1×8 patches joining with third order continuity, if there are no multiple control points and knot values.

In Fig. 12 two B-spline surface patches are shown defined separately. In Fig. 13 the merged surface is shown. The interpolation error has been computed by numerical integration of the squared differences between the points of the given and the resulting surfaces at the same parameter values. This estimated error is 0,0032.

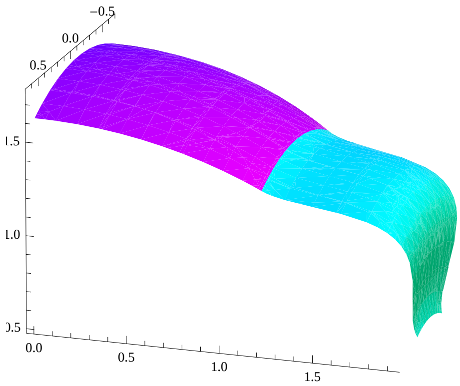

Figure 12: *Two given surface patches*

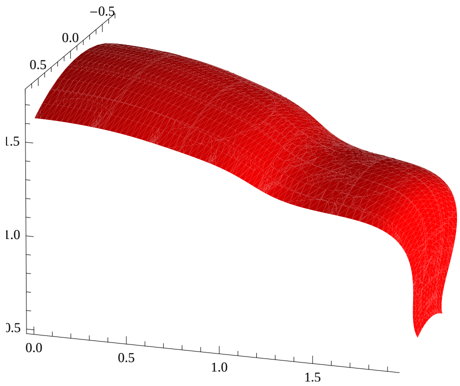

Figure 13: *The merged surface*

Stitching of separately defined B-spline patches ensures higher order continuity along the joining curves than known constructions. In several applications surfaces are determined by local geometric data ([10], [11]). This is the case for example in surface manufacturing in a neighborhood of a processing tool, however, a smooth resulting surface is required. A series in a stripe of separately generated surface patches are shown in [12].

6 Conclusions

We have presented stitching algorithms for two given B-spline curve segments. Our final symbolical solution generates a B-spline curve with 8 segments independently from the numerical input data. This continuous curve approximates the two separately defined (even not joining) curve segments. We have proposed an additional fairing method to improve the shape of the resulting curve. We have also analyzed the interpolation error in many different cases. The proposed algorithm gave the most satisfactory result. This has been applied for merging two B-spline surface patches.

Our aim is to extend this method for stitching more B-spline curves.

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