A Generalized Dual of the Tonnetz for Seventh Chords: Mathematical, Computational and Compositional Aspects

Sonia Cannas, Moreno Andreatta

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Sonia Cannas[1] and Moreno Andreatta[2]

[1]University of Pavia and University of Strasbourg; sonia.cannas01@universitadipavia.it
[2]CNRS/IRCAM/Sorbonne University and IRMA/USIAS/University of Strasbourg
moreno.andreatta@ircam.fr

Abstract

In Mathematical Music Theory, geometric models such as graphs and simplicial complexes are music-analytical tools which are commonly used to visualize and represent musical operations. The most famous example is given by the *Tonnetz*, a graph whose basic idea was introduced by Euler in 1739, and developed by several musicologists of the XIX$^{\text{th}}$ century, such as Hugo Riemann. The aim of this paper is to introduce a generalized *Chicken-wire Torus* (dual of the *Tonnetz*) for seventh chords and to show some possible compositional applications. It is a new musical graph representing musical operations between seventh chords, described from an algebraic point of view. As in the traditional *Tonnetz*, geometric properties correspond to musical properties and offer to the computational musicologists new and promising analytical tools.

Introduction

In the last 20 years, an increasing number of geometric structures such as graphs and simplicial complexes have been developed to display and study musical structures. In *A Geometry of Music*, the music theorist and composer Dmitri Tymoczko wrote:

> "Geometry can help to sensitize us to relationships that might not be immediately apparent in the musical score. Ultimately, this is because conventional music notation evolved to satisfy the needs of the performer rather than the musical thinker: it is designed to facilitate the translation of musical symbols into physical action, rather than to foment conceptual clarity"[1]

In fact, computer-aided music analysis and composition through software such as HexaChord [4] and Open-Music [3] are based on spatial representations of musical objects inspired by the *Tonnetz* and other geometric models. They are nowadays commonly taught in many Conservatories and music schools, especially in France. While algebra is useful to describe musical structures, geometric models have a double utility: they not only allow an easier visualization of musical structures which are useful in musical analysis, but they can also provide music-analytical tools for compositional applications.

In the first section we will review some musical background and notation used in Mathematical Music Theory. In second section, we will describe the *Tonnetz* and the three neo-Riemannian operations P, L and R. Starting from these, we will present an algebraic generalization of the neo-Riemannian operations for seventh chords [5], and we will use this to create a new generalized *Chicken-wire Torus*, corresponding to a generalized version of the dual of the *Tonnetz*. In the final section we study some Hamiltonian cycles and paths on it and suggest the potential interest as compositional tools.

[1]See [10], page 79.

Musical Background and Notation

One of the most important aspects that makes a musical piece interesting is the possibility to control the relations between the different musical lines, which is traditionally known as voice leading. It is in fact the interplay of two or more musical lines that realize chord progressions, according to the principles of the common-practice and counterpoint. This corresponds to the two main possible ways of analyzing a musical score, respectively horizontally (or melodically) and vertically (or harmonically).
In this paper, we will only focus on Western classical music and we recall that most of the Western music from the XVII$^{\text{th}}$ to XIX$^{\text{th}}$ century is based on four-part harmony, which means that every chord in the progression contain four tones. Therefore there are four melodic lines, conventionally known with the names of the four voices of the chant: soprano, alto, tenor, bass.
Voice leading is organized according to musical rules, and music theory also deals with the study of these compositional strategies. For composers music theory is fundamental as grammar for poets. In the last decades, Computational Music Analysis has become increasingly widespread, also thanks to the use of some different mathematical areas[2].

In Mathematical Music Theory the starting point is to consider two equivalence relations on the set of all pitches of the twelve-tone equal temperament.

Enharmonic equivalence : two tones are enharmonic equivalent if they have the same pitch, but named differently. Example: $C\sharp \sim D\flat$.

Octave equivalence : two pitches $x, y \in \mathbb{R}^+$ are equivalent if their interval distance is an octave. If the frequency of a tone is f, the frequency of the tone one octave above is $2f$, while the frequency of the tone one octave below is $\frac{1}{2}f$. Therefore the octave equivalence is mathematically described as follows:

$$x \sim y \qquad \Leftrightarrow \qquad y = 2^n x \qquad n \in \mathbb{Z}$$

From these equivalence relations we obtain twelve equivalence classes called *pitch-classes*, one for each note of the chromatic scale. We usually map these twelve pitch-classes to $\mathbb{Z}_{12}$, starting by mapping the pitch-class C to number 0.
A chord is a set of two or more pitches played simultaneously. Exploiting the tradition of musical set theory, we will consider chords as mathematical sets of pitch-classes. For our aim, we will continue to use the following cyclic notation defined in [5].

Definition 1. *We define a* cyclic marked chord $[\underline{x_1}, x_2, \ldots, x_n]$ *as a chord constituted by the n pitch-classes* $x_1, x_2, \ldots, x_n$*, so that acoustically* $[\underline{x_1}, x_2, \ldots, x_n] = [x_2, \ldots, x_n, \underline{x_1}] = \cdots = [x_n, \underline{x_1}, \ldots, x_2]$*, where* $x_i \in \mathbb{Z}_{12}$ *and the pitch-class corresponding to the root of the chord is underlined.*

The most used chords in voice leading theory, at least from the perspective of Western musical tradition, are triads (major and minor) and some seventh chords. A triad is a chord of three tones (trichord) where the intervals between adjacent tones comprise a minor third interval (formed by three semitones) or a major third (formed by four semitones). A major triad is obtained from the overlap between a major third (bottom) and a minor third (above). Conversely, a minor triad has a minor third at the bottom and a major third at the top. Using the definition 1, mathematically we can define them respectively as follows:

$$[\underline{x}, x+4, x+7] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(major triad)}$$
$$[\underline{x}, x+3, x+7] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(minor triad)}$$

[2]See [9] for a comprehensive perspective on Computational Music Analysis.

A seventh is a chord of four tones obtained by overlapping three intervals of third. We will consider the five classical types:

$$
\begin{aligned}
&[\underline{x}, x+4, x+7, x+10] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(dominant seventh)}\\
&[\underline{x}, x+3, x+7, x+10] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(minor seventh)}\\
&[\underline{x}, x+3, x+6, x+10] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(half-diminished seventh)}\\
&[\underline{x}, x+4, x+7, x+11] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(major seventh)}\\
&[\underline{x}, x+3, x+6, x+9] \pmod{12}, \qquad x \in \mathbb{Z}_{12}, \quad \text{(diminished seventh)}
\end{aligned}
$$

For example, by taking the note C as the root $\underline{x}$ of the chord we have the following chords expressed with the usual music notation:

$$
\begin{aligned}
C^7 &= [\underline{0}, 4, 7, 10] \quad (C \text{ dominant seventh})\\
C_m &= [\underline{0}, 3, 7, 10] \quad (C \text{ minor seventh})\\
C^{\emptyset} &= [\underline{0}, 3, 6, 10] \quad (C \text{ half-diminished seventh})\\
C^{\Delta} &= [\underline{0}, 4, 7, 11] \quad (C \text{ major seventh})\\
C^{o} &= [\underline{0}, 3, 6, 9] \quad (C \text{ diminished seventh})
\end{aligned}
$$

Tonnetz and Neo-Riemannian Operations

One of the most peculiar musical rules in Western music tradition is to organize voice leading making the least movement. This property of stepwise motion in a single voice is called *parsimonious voice leading*. Music theorists typically use graphs to describe parsimonious voice leading. Two different kinds of graphs are used: note-based and chord-based graphs. In the first ones each vertex represents a note, in the second ones, by contrast, each vertex represents a chord, and parsimonious voice leading correspond to short-distance motion along edges. Chord-based graphs are associated with note-based graphs by duality.

In music, the most famous example of a note-based graph is probably the *Tonnetz*, firstly introduced by Euler in 1739 and developed by several musicologists of the XIX$^{\text{th}}$ century, such as Hugo Riemann[3]. It is a note-based graph in which each triple of vertices adjacent two by two determines a triangle representing major or minor triads (see Fig. 1). Its geometrical dual, known as *Chicken-wire torus* [6], is a chord-based graph in which vertices represent major and minor triads (see Fig. 2).

Geometric models are related to musical operations of transformational theory, a branch of music theory introduced by David Lewin [8] based on the use of mathematical functions to define musical transformations, which are elements of algebraic groups. For example the edge-flips in the *Tonnetz* and the edges in the *Chicken-wire torus* represent the neo-Riemannian musical operations P (parallel), R (relative) and L (leading tone), commonly used in parsimonious voice leading. To define them let $S = \{[\underline{x_1}, x_2, x_3] | x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_1 + 7\}$ be the set of all 24 major and minor triads. The transformations $P, R, L\colon S \longrightarrow S$ are defined as follows:

$$P\colon [\underline{x}, x+4, x+7] \leftrightarrow [\underline{x}, x+3, x+7] \pmod{12} \tag{1}$$

$$R\colon [\underline{x}, x+4, x+7] \leftrightarrow [x, x+4, \underline{x+9}] \pmod{12} \tag{2}$$

$$L\colon [\underline{x}, x+4, x+7] \leftrightarrow [x-1, \underline{x+4}, x+7] \pmod{12} \tag{3}$$

Since paths in the *Chicken-wire torus* represent sequences through major and minor triads using P, L and R transformations, it is interesting to study particular cycles and paths. For example Albini and Antonini

[3]Not to be confused with the mathematician Bernhard Riemann.

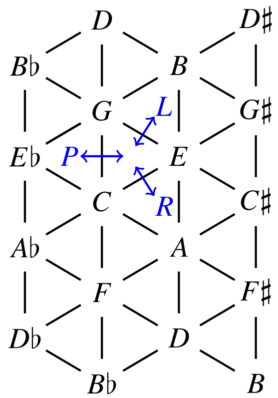

Figure 1: *The* neo-Riemannian Tonnetz *and the P, L and R operations*

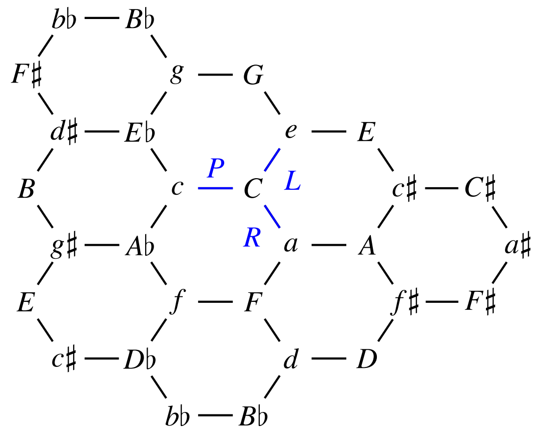

Figure 2: Chicken-wire torus

enumerated and studied all Hamiltonian cycles [1]. A Hamiltonian cycle in a graph G is a closed path that visits each vertex exactly once. These classes of cycles are a useful compositional device for contemporary art[4] and popular music. Some examples can be found in composition by Giovanni Albini and Moreno Andreatta[5].

Generalized Dual of the *Tonnetz* for Sevenths

Since seventh chords are often used in voice leading, we have classified all most parsimonious musical operations between seventh chords, similar to the P, L and R operations [5]. The most parsimonious transformations exchange two types of sevenths moving just a note by a semitone or a whole-tone. Let H be the set of all 60 dominant, minor, half-diminished, major and diminished sevenths. Since we want functions from $H \to H$ algebraically well-defined, we define them as transformations that exchange two types of sevenths and fix the other types. These transformations are 17, defined also as pairs of elements $(\sigma, v) \in S_5 \ltimes \mathbb{Z}_{12}^5$, where $\ltimes$ denotes the semi-direct product of two groups:

$$P_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+7, x+10] \quad (\sigma, v) = ((12), (0,0,0,0,0))$$
$$P_{14}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+4, x+7, x+11] \quad (\sigma, v) = ((14), (0,0,0,0,0))$$
$$P_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+10] \quad (\sigma, v) = ((23), (0,0,0,0,0))$$
$$P_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+9] \quad (\sigma, v) = ((35), (0,0,0,0,0))$$
$$R_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \quad (\sigma, v) = ((12), (-3,3,0,0,0))$$
$$R_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [x, x+3, x+7, \underline{x+9}] \quad (\sigma, v) = ((23), (0,-3,3,0,0))$$
$$R_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \quad (\sigma, v) = ((42), (0,3,0,-3,0))$$
$$R_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, x+6, \underline{x+9}] \quad (\sigma, v) = ((35), (0,0,-3,0,3))$$
$$R_{53}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, x+7, \underline{x+9}] \quad (\sigma, v) = ((53), (0,0,3,0,-3))$$
$$L_{13}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+2, \underline{x+4}, x+7, x+10] \quad (\sigma, v) = ((13), (4,0,-4,0,0))$$
$$L_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+1, \underline{x+4}, x+7, x+10] \quad (\sigma, v) = ((15), (4,0,0,0,-4))$$

[4] See the second movement of Beethoven's Ninth Symphony for an unconscious use of a portion of hamiltonian cycle.

[5] Several Hamiltonian compositions are available in their respective web pages. See: http://www.giovannialbini.it/opus/ and http://repmus.ircam.fr/moreno/music)

$$L_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x+2, \underline{x+4}, x+7, x+11] \quad (\sigma, v) = ((42), (0, -4, 0, 4, 0))$$
$$Q_{43}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [\underline{x+1}, x+4, x+7, x+11] \quad (\sigma, v) = ((43), (0, 0, -1, 1, 0))$$
$$Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x+1}, x+4, x+7, x+10] \quad (\sigma, v) = ((15), (1, 0, 0, 0, -1))$$
$$RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, \underline{x+6}, x+9] \quad (\sigma, v) = ((35), (0, 0, -6, 0, 6))$$
$$QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, \underline{x+2}, x+6, x+9] \quad (\sigma, v) = ((51), (-2, 0, 0, 0, 2))$$
$$N_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, \underline{x+5}, x+9] \quad (\sigma, v) = ((51), (-5, 0, 0, 0, 5))$$

Starting from these algebraic transformations[6], we can construct a generalized *Chicken-wire torus* for seventh chords, a chord-based graph in which each vertex represents a seventh chord and each edge identifies a parsimonious musical operation (see Fig. 3).

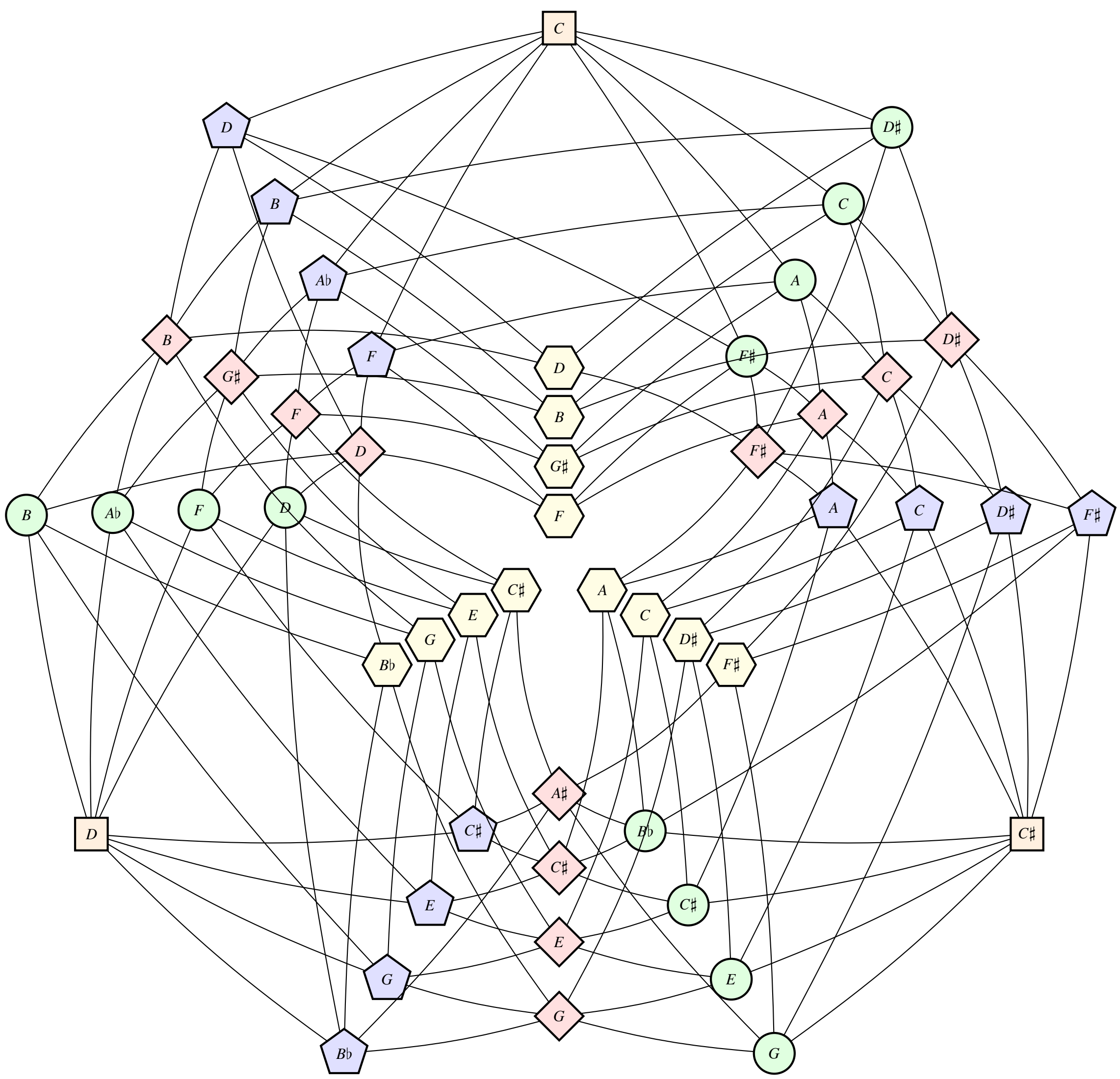

Figure 3: *The new generalized* Chicken-wire torus *we propose for seventh chords. Pentagonal, rhombic, circular, hexagonal and quadrangular vertices represent dominant, minor, half-diminished, major and diminished sevenths respectively.*

[6]For their connections with classical neo-Riemannian operations and other algebraic details see [5].

For algebraic reasons we have considered 60 sevenths, 12 for each type, but some diminished sevenths are enharmonic equivalent. In fact due to their particular symmetry of the interval structure, from an acoustical point of view we have only 3 diminished sevenths: $C^o = E\flat^o = G\flat^o = A^o = [0,3,6,9], C\sharp^o = E^o = G^o = B\flat^o = [1,4,7,10], D^o = F^o = A\flat^o = B^o = [2,5,8,11]$. Similarly, some of the 17 transformations are different from a theoretical and mathematical point of view, but enharmonic equivalent from an acoustical point of view: $P_{35} \sim R_{35} \sim R_{53} \sim RR_{35}$ and $L_{15} \sim Q_{15} \sim QQ_{51} \sim N_{51}$. Since our aim is to study paths in our graph and their musical interpretation up to enharmonic equivalences, we identify vertices and edges enharmonic equivalent. Therefore our graph has $12 \cdot 4 + 3 = 51$ vertices and $12 \cdot 11 = 132$ edges.

The famous *Power towers* (see Fig.4), a graph representing dominant, minor, half-diminished and diminished sevenths realized by Douthett and Steinbach [6], is a subgraph of our graph: vertices and edges in *Power towers* are also in our graph; the latter differs from the first one by an additional twelve vertices representing major sevenths and the related edges.

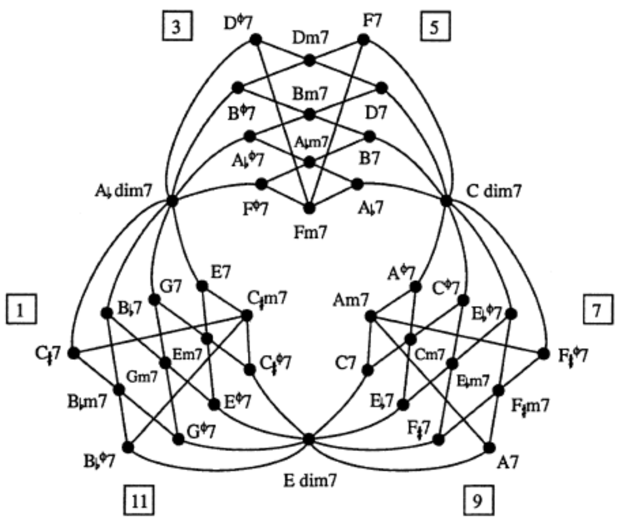

Figure 4: Power towers *by Douthett and Steinbach*

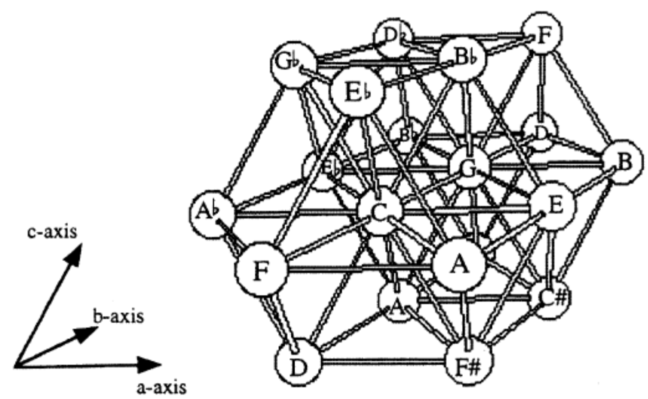

Figure 5: *Three-dimensional* Tonnetz *by Gollin*

The most famous graph representing seventh chords is the *generalized Tonnetz* by Gollin [7]. It is a note-based graph whose edges form tetrahedra representing dominant or diminished sevenths and pyramids (see Fig. 5). Given two tetrahedra sharing an edge, the "edge-flip" represent a musical transformation between the sevenths representing by the tetrahedra that share two tones. There are six type of transformations between tetrahedra sharing a common edge, in one of them the two tetrahedra share three tones. It corresponds to our transformation L_{13}, that in our graph is represented by the edges between green and blue vertices.

If we observe the drawing of our graph, it is evident that a rotation of the plane by $\frac{2\pi}{3}$ induces automorphisms on our graph. Also reflections of the plane through the axis passing through a diminished seventh and in the center of the graph induce automorphisms of which musical meaning is not clear at the moment.

Hamiltonian Cycles and Paths in the Generalized *Chicken-wire Torus*

As Albini and Bernardi observe [2], Hamiltonian cycles of chord-based graphs represent complete sequences through all the admitted chords which only consider certain types of transformations. Unfortunately we do not know necessary and sufficient conditions for Hamiltonicity, and proving whether a graph is Hamiltonian is an NP-complete problem. Our graph is Hamiltonian; in fact we have found the following Hamiltonian cycle:

$$
\begin{aligned}
&C\sharp^{\Delta} \xrightarrow{L_{42}} F_m \xrightarrow{R_{42}} G\sharp^{\Delta} \xrightarrow{Q_{43}} A^{\emptyset} \xrightarrow{R_{23}} C_m \xrightarrow{P_{23}} C^{\emptyset} \xrightarrow{Q_{43}} B^{\Delta} \xrightarrow{P_{14}} B^7 \xrightarrow{L_{13}} D\sharp^{\emptyset} \xrightarrow{P_{23}} D\sharp_m \xrightarrow{R_{42}} F\sharp^{\Delta} \rightarrow \\
&\xrightarrow{P_{14}} F\sharp^7 \xrightarrow{L_{13}} B\flat^{\emptyset} \xrightarrow{R_{23}} C\sharp_m \xrightarrow{P_{23}} C\sharp^{\emptyset} \xrightarrow{Q_{43}} C^{\Delta} \xrightarrow{P_{14}} C^7 \xrightarrow{L_{13}} E^{\emptyset} \xrightarrow{Q_{43}} D\sharp^{\Delta} \xrightarrow{P_{14}} D\sharp^7 \xrightarrow{L_{15}} C\sharp^o \rightarrow \\
&\xrightarrow{P_{35}} G^{\emptyset} \xrightarrow{P_{23}} G_m \xrightarrow{R_{12}} B\flat^7 \xrightarrow{P_{12}} A\sharp_m \xrightarrow{R_{12}} C\sharp^7 \xrightarrow{L_{13}} F^{\emptyset} \xrightarrow{P_{35}} D^o \xrightarrow{L_{15}} G^7 \xrightarrow{R_{12}} E_m \xrightarrow{R_{42}} G^{\Delta} \rightarrow \\
&\xrightarrow{Q_{43}} A\flat^{\emptyset} \xrightarrow{L_{13}} E^7 \xrightarrow{P_{14}} E^{\Delta} \xrightarrow{L_{42}} G\sharp_m \xrightarrow{P_{12}} A\flat^7 \xrightarrow{L_{15}} C^o \xrightarrow{L_{15}} F^7 \xrightarrow{P_{14}} F^{\Delta} \xrightarrow{L_{42}} A_m \xrightarrow{P_{12}} A^7 \rightarrow \\
&\xrightarrow{P_{14}} A^{\Delta} \xrightarrow{R_{42}} F\sharp_m \xrightarrow{P_{23}} F\sharp^{\emptyset} \xrightarrow{L_{13}} D^7 \xrightarrow{P_{14}} D^{\Delta} \xrightarrow{R_{42}} B_m \xrightarrow{P_{23}} B^{\emptyset} \xrightarrow{Q_{43}} B\flat\Delta \xrightarrow{L_{42}} D_m \xrightarrow{P_{23}} D^{\emptyset} \xrightarrow{Q_{43}} C\sharp^{\Delta}
\end{aligned}
$$

In this sequence of seventh chords there is not regularity. But with such a symmetrical structure in the graph we expect to find sequences[7] of sevenths or other musical structures with some regularity. It is the case of the following interesting Hamiltonian path:

$$
\begin{aligned}
&[B\flat^{\Delta} \xrightarrow{Q_{43}} B^{\emptyset} \xrightarrow{L_{13}} G^7 \xrightarrow{R_{12}} E_m] \xrightarrow{R_{42}} [G^{\Delta} \xrightarrow{Q_{43}} A\flat^{\emptyset} \xrightarrow{L_{13}} E^7 \xrightarrow{R_{12}} C\sharp_m] \rightarrow \\
&\xrightarrow{R_{42}} [E^{\Delta} \xrightarrow{Q_{43}} F^{\emptyset} \xrightarrow{L_{13}} C\sharp^7 \xrightarrow{R_{12}} A\sharp_m] \xrightarrow{R_{42}} (C\sharp^{\Delta} \xrightarrow{Q_{43}} D^{\emptyset} \xrightarrow{P_{35}} D^o \xrightarrow{L_{15}} B\flat^7 \xrightarrow{R_{12}} G_m) \rightarrow
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&\xrightarrow{R_{42}} [D\sharp^{\Delta} \xrightarrow{Q_{43}} E^{\emptyset} \xrightarrow{L_{13}} C^7 \xrightarrow{R_{12}} A_m] \xrightarrow{R_{42}} [C^{\Delta} \xrightarrow{Q_{43}} C\sharp^{\emptyset} \xrightarrow{L_{13}} A^7 \xrightarrow{R_{12}} F\sharp_m] \rightarrow \\
&\xrightarrow{R_{42}} [A^{\Delta} \xrightarrow{Q_{43}} B\flat^{\emptyset} \xrightarrow{L_{13}} F\sharp^7 \xrightarrow{R_{12}} D\sharp_m] \xrightarrow{R_{42}} (F\sharp^{\Delta} \xrightarrow{Q_{43}} G^{\emptyset} \xrightarrow{P_{35}} C\sharp^o \xrightarrow{L_{15}} D\sharp^7 \xrightarrow{R_{12}} C_m) \rightarrow
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&\xrightarrow{R_{42}} [G\sharp^{\Delta} \xrightarrow{Q_{43}} A^{\emptyset} \xrightarrow{L_{13}} F^7 \xrightarrow{R_{12}} D_m] \xrightarrow{R_{42}} [F^{\Delta} \xrightarrow{Q_{43}} F\sharp^{\emptyset} \xrightarrow{L_{13}} D^7 \xrightarrow{R_{12}} B_m] \rightarrow \\
&\xrightarrow{R_{42}} [D^{\Delta} \xrightarrow{Q_{43}} D\sharp^{\emptyset} \xrightarrow{L_{13}} B^7 \xrightarrow{R_{12}} G\sharp_m] \xrightarrow{R_{42}} (B^{\Delta} \xrightarrow{Q_{43}} C^{\emptyset} \xrightarrow{P_{35}} C^o \xrightarrow{L_{15}} A\flat^7 \xrightarrow{R_{12}} F_m)
\end{aligned}
\tag{6}
$$

The structure of this Hamiltonian path presents a sequence where the main motif (from $B\flat^{\Delta}$ to G_m) is repeated twice 5 semitones up: the first repetition is from $D\sharp^{\Delta}$ to C_m, the second one from $G\sharp^{\Delta}$ to F_m. This Hamiltoninan path is interesting not only because it corresponds to this sequence of sevenths, but also because within each of the three motifs we find another sequence. In fact, we consider the first of the three motifs: the musical phrase from $B\flat^{\Delta}$ to E_m is repeated three times 3 semitones down. And in the third repetition the repetition is modified: instead of connecting a half-diminished seventh with a dominant seventh through L_{13}, a diminished seventh is inserted and the connection L_{13} is substituted by $L_{15} \circ P_{35}$. This new phrase is indicated in parentheses. Musically it sounds like a passage that allows you to get to the next sequence creating a sense of variety. The harmonic sequence obtained in OpenMusic and the trace of the chord progression represented in HexaChord are shown in Fig.6.

Conclusions and Future Works

Voice leading is an essential component of music composition, at least whithin the Western tradition. Paths and cycles in chord-based graphs corresponds to sequences of chords useful for parsimonious voice leading. In particular Hamiltonian cycles and paths in our generalized dual of the *Tonnetz* can be a compositional device to compose parsimonious voice leading passing through all 51 seventh chords once. Our next aim is to classify all Hamiltonian cycles and paths and to study their possible applications in music analysis and composition.

[7]In music theory a *sequence* is the repetition of a motif or melodic element at a higher or lower pitch.

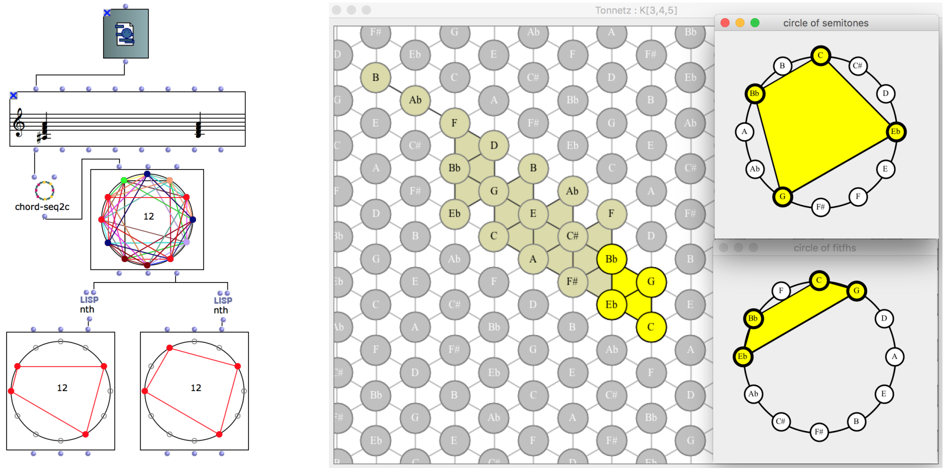

Figure 6: *On the left, a patch in OpenMusic showing the circular representation of the entire chord progression 4 of the previous list together with the circular representation of the first two chords. On the right, the trace of the chord progression represented in HexaChord together with two circular representations (chromatic and circle of fifths) of two seventh chord of the same Hamiltonian path.*

Acknowledgements

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