Journal of Quality Measurement and Analysis
Jurnal Pengukuran Kualiti dan Analisis
JQMA 14(1) 2018, 67-79

ON THE NUMBER OF DISTINCT FUZZY SUBGROUPS OF DIHEDRAL GROUP OF ORDER 60

(Bilangan Subkumpulan Kabur yang Berbeza bagi Kumpulan Dwihedron Berdarjah 60)

OLAYIWOLA ABDULHAKEEM & ISYAKU BABANGIDA

ABSTRACT

In this paper, we compute the number of distinct fuzzy subgroups of dihedral group of order 60 with respect to a new equivalence relation existing in literature. Our computation shows that the number of distinct fuzzy subgroups of dihedral group of order 60 is 150.

Keywords: dihedral group; equivalence relation; Burnside's lemma

ABSTRAK

Dalam makalah ini, dikira bilangan subkumpulan kabur yang berbeza bagi kumpulan dwihedron berdarjah 60 dengan merujuk kepada suatu hubungan kesetaraan yang baharu yang terdapat dalam sorotan kajian. Pengiraan kami mendapati bahawa bilangan subkumpulan kabur yang berbeza bagi kumpulan dwihedron berdarjah 60 adalah sebanyak 150.

Kata kunci: kumpulan dwihedron; hubungan kesetaraan; lema Burnside

1. Introduction

After the introduction of fuzzy sets by Zadeh (1965) and the fuzzification group algebraic structure by Rosenfeld (1971), many more algebraic structures have been fuzzified, see Liu (1982).

Without any equivalence relation on fuzzy subgroups of a group, the number of fuzzy subgroups is infinite. Using poset diagrams and a suitable equivalence relation, Sulaiman and Ahmad (2010) computed the number of distinct fuzzy subgroups of symmetric groups S_2, S_3 and alternating group A_4. Furthermore, Sulaiman (2012) studied and constructed the number of distinct fuzzy subgroups of symmetric group S_4. Tarnauceanu (2012) used inclusion and exclusion principles in classifying fuzzy subgroups of some dihedral groups. Recent research had focused on classification of fuzzy subgroups of finite dihedral groups G, with respect to a new equivalence relation $\approx$ see Tarnuaceanu (2016). However, the number of distinct fuzzy subgroups of G of order 60 had not been computed. This study was therefore designed to compute the number of distinct fuzzy subgroups for G of order 60 with respect to the equivalence relation, $\approx$. In particular, D_{60} is the smallest dihedral group of the form $D_{2\times P1\times P2\times P3}$, where $P1, P2$ and $P3$ are distinct primes.

It is important to note that our ultimate aim for future work in this direction is to establish an explicit formula for counting distinct fuzzy subgroups of dihedral groups $G = D_{2\times P1\times P2\times \ldots\times Pn}$ where $P1, P2, \ldots, Pn$ are all distinct primes, with respect to the new equivalence relation $\approx$. However, D_{60} will serve as our beginning point. For example, the number of distinct fuzzy subgroups of D_8, was computed by Tarnuaceanu (2016).

The paper is divided into three sections. Section two gives an overview of the new equivalence relation as defined by Tarnuaceanu (2016). In section three, the main result is discussed in details. The notation used mainly in this work can be found in Tarnuaceanu (2016). For introduction of basic notions on fuzzy group see Moderson *et al.*(2005).

2. Preliminaries

As a way of getting acquainted with the new equivalence relation on fuzzy subgroups of a group we give an overview of the new equivalence that will be needed as we progress. Group action is a fundamental notion in algebra as it generalizes group multiplication. It serves as a tool for solving some problems in algebra. Let G be a group and X be an arbitrary set, a group action of an element $g\epsilon G$ and $x\epsilon X$ is a product gx has in X. It is well known that every group action of G on a set X induces an equivalence relation on X. The equivalence classes of X modulo the equivalence relation are called the orbits of X relative to the group action.

For any $g\epsilon G$, we denote by $Fix_X(g)$ the set of all elements of X which are fixed by g. If both G and X are finite, then the number of distinct orbits of X relative to group action is given by the equality;

$$N = \frac{1}{|G|}\sum_{g\epsilon G} |Fix_X (g)|.$$

The result is known as Burnside's lemma.

Let G be a group and $FL(G)$ be the lattice subgroups of G. An action of $Aut(G)$ on $FL(G)$ was defined using the concept of group action and that action is well defined. This action induces an equivalence relationon $FL(G)$. It is also known that every fuzzy subgroup of G determines a chain of subgroups of G which ends in G. The action can be seen in terms of chains of subgroups of G. Denote by $\bar{C}$ the set consisting of all chains of subgroups of G terminated in G. Then the previous action of $Aut(G)$ on $FL(G)$ can be seen as an action of $Aut(G)$ on $\bar{C}$and the previous equivalence relation is seen as equivalence relation induced by this action. An equivalence relation is then defined on $\bar{C}$ in the following manner: for two chains $C_1: H_1 \subset H_2 \subset \cdots \subset H_m = G$ and $C_2: K_1 \subset K_2 \subset \cdots \subset K_n = G$ of $\bar{C}$, we put $C_1 \approx C_2$ iff $m = n$ and the exists $f\epsilon\, Aut(G)$ such that $f(H_i) = K_{i,} i = \overline{i, n}$. The orbit of a chain in this situation is given by $C\epsilon\bar{C}\{f(C)|f\epsilon\, Aut(G)\}$, while the set of chains in $\bar{C}$ that are fixed by an automorphism of f of G is $Fix_{\bar{C}}(f) = \{C\, \epsilon\bar{C}|f(C) = C\}$. By applying Burnside's lemma, the number of distinct fuzzy subgroups N of G is given by

$$N = \frac{1}{|Aut(G)|}\sum_{f\epsilon\, Aut(G)} |Fix_{\bar{C}} (f)|.$$

3. Main Result

3.1. *Subgroups of D_{60}*

Dihedral group of order 60 with generators *a* and *b* can be presented as follows;

$$D_{60} = \langle a, b | a^{30} = b^2 = 1, bab^{-1} = a^{-1}\rangle.$$

It is well known in literature that dihedral groups D_{2n} of order $2n$ posses two subgroup structures, one of which is cyclic and isomorphic to $\mathbb{Z}_r$ and is of the form $H_0^r = \langle a^{\frac{n}{r}}\rangle$, where

r is a divisor of n and the other $\frac{n}{r}$ dihedral subgroups isomorphic to D_r, of the form $H_i^r = \langle a^{\frac{n}{r}}, a^{i-1}b\rangle$ where $i = 1,2,\dots,\frac{n}{r}$. Using existing literature and Tarnuaceanu (2016) the subgroups of D_{60} were generated and are given below:

- $H_0^1 = \langle e\rangle$
- $H_0^n = \langle a^{30/n}\rangle$, for $n = 2,3,5,6,10,15$ and 30.
- $H_n^1 = \langle a^{n-1}b\rangle$, for $n = 1,2,3,4,\dots,30$.
- $H_n^2 = \langle a^{15}, a^{n-1}b\rangle$, for $n = 1,2,3,4,\dots,15$.
- $H_n^3 = \langle a^{10}, a^{n-1}b\rangle$, for $n = 1,2,3,4,\dots,10$.
- $H_n^5 = \langle a^{6}, a^{n-1}b\rangle$, for $n = 1,2,3,4,5,6$.
- $H_n^6 = \langle a^{5}, a^{n-1}b\rangle$ for $n = 1,2,3,4,5$.
- $H_n^{10} = \langle a^{3}, a^{n-1}b\rangle$ for $n = 1,2,3$.
- $H_n^{15} = \langle a^{2}, a^{n-1}b\rangle$ for $n = 1,2$.
- $H_1^{30} = \langle a, b\rangle$.

It is important to compute these subgroups because they are essential in drawing subgroup lattice diagram and counting chains of subgroups of D_{60}.

The automorphism group of D_{60} is given by

$$Aut(D_{2(30)}) = \{f_{\alpha,\beta} \mid \alpha = \overline{0,29} \text{ with } (\alpha,30) = 1, \beta = \overline{0,29}\}$$

where $\overline{0,29}$ means integer number from 0 to 29, $f_{\alpha,\beta}: D_{60} \to D_{60}$ and $|Aut(D_{60})| = 240$.

3.2. *Subgroups Fixed by Each Automorphism*

In what follows, we give the subgroups of D_{60} that are invariant under the action of elements of $Aut(D_{60})$. That is,

$$Fix(f_{\alpha,\beta}) = \{H < D_{60} \mid f_{\alpha,\beta}(H) = H\}$$

A subgroup of D_{2n} of the form H_0^r belongs to $Fix(f_{\alpha,\beta})$ if and only if $(\alpha, r) = 1$, while a subgroup form H_i^r belongs to $Fix(f_{\alpha,\beta})$ if and only if $(\alpha, r) = 1$ and $\frac{n}{r}$ divides $(\alpha - 1)(i - 1) + \beta$, see Tarnuaceanu (2016). Hence we have the following table:

Table 1: Subgroups fixed by Aut($\boldsymbol{D_{60}}$)

α	β	$Fix(f_{\alpha,\beta})$
1	0	$\{H_1^{30}\}$
1	7, 11, 13, 17, 23, 29	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_0^1, H_1^{30}\}$
1	2, 4, 8, 14, 16, 22, 26, 28	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^5, H_2^5\}$
1	3, 9, 21, 27	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^{10}, H_2^{10}, H_3^{10}\}$
1	5, 25	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6\}$
1	6, 12, 18, 24	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^5, H_2^5, H_3^5, H_4^5, H_5^5, H_6^5, H_1^{10}, H_2^{10}, H_3^{10}, H_1^{15} H_2^{15}\}$
1	10, 20	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_6^3, H_7^3, H_8^3, H_9^3, H_{10}^3, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_1^{15}, H_2^{15}\}$
1	15	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^2, H_2^2, H_3^2, H_4^2, H_5^2, H_6^2, H_7^2, H_8^2, H_9^2, H_{10}^2, H_{11}^2, H_{12}^2, H_{13}^2, H_{14}^2, H_{15}^2, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
11	0, 10, 20	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^1, H_4^1, H_7^1, H_1^{10}, H_{13}^1, H_{16}^1, H_{19}^1, H_{22}^1, H_{25}^1, H_{28}^1, H_1^2, H_4^2, H_7^2, H_{10}^2, H_{13}^2, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_6^3, H_7^3, H_8^3, H_9^3, H_{10}^3, H_1^5, H_4^5, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_1^{10}, H_1^{15}, H_2^{15}\}$
11	1, 7, 13, 19	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_3^{10}\}$
11	2, 8, 14, 26	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_2^5, H_5^5, H_2^{10}, H_1^{15}, H_2^{15}\}$
11	3, 9, 21, 27	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^{10}\}$
11	4, 16, 22, 28	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_3^5, H_6^5, H_3^{10}, H_1^{15}, H_2^{15}\}$
11	5	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_2^2, H_5^2, H_8^2, H_1^{12}, H_1^{42}, H_2^{10}, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6\}$
11	6, 12, 18, 24	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^5, H_4^5, H_1^{10}, H_1^{15}, H_2^{15}\}$
11	10	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_3^1, H_6^1, H_9^1, H_{12}^1, H_{15}^1, H_{18}^1, H_{21}^1, H_{24}^1, H_{27}^1, H_{30}^1, H_3^2, H_6^2, H_9^2, H_{12}^2, H_{15}^2, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_6^3, H_7^3, H_8^3, H_9^3, H_{10}^3, H_3^5, H_6^5, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
11	11, 17, 23, 29	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_2^{10}\}$
11	15	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^2, H_4^2, H_7^2, H_{10}^2, H_{13}^2, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_1^{10}\}$

Cont'd.

Table 1*(Continued)*

11	20	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_2^1, H_5^1, H_8^1, H_{11}^1, H_{14}^1, H_{17}^1, H_{20}^1, H_{23}^1, H_{26}^1, H_{29}^1, H_2^2, H_5^2, H_8^2, H_{11}^2, H_{14}^2, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_6^3, H_7^3, H_8^3, H_9^3, H_{10}^3 H_2^5, H_5^5, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
11	25	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_3^2, H_6^2, H_9^2, H_{12}^2, H_{15}^2, H_1^6, H_2^6, H_3^6, H_4^6, H_5^6, H_3^{10}\}$
17	0	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_1^1, H_{16}^1, H_1^2, H_1^3, H_6^3, H_1^5, H_4^5, H_1^6, H_1^{10}, H_1^{15}, H_2^{15}\}$
23	0	
29	0	
17	1	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{15}^2, H_5^6, H_3^{10}\}$
23	7	
29	13	
17	2	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{14}^1, H_{29}^1, H_{14}^2, H_4^3, H_9^3, H_2^5, H_5^5, H_4^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
23	14	
29	26	
17	3	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{13}^2, H_3^6, , H_1^{10}\}$
23	21	
29	9	
17	4	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{12}^1, H_{27}^1, H_{12}^2, H_2^3, H_7^3, H_3^5, H_6^5, H_2^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
23	28	
29	22	
17	5	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{11}^2, H_1^6, H_2^{10}\}$
23	5	
29	5	
17	6	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_{10}^1, H_1^1, H_1^2, H_1^3, H_6^3, H_1^5, H_4^5, H_1^{10}, H_1^{15}, H_2^{15}\}$
23	12	
29	18	
17	7	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_9^2, H_4^6, H_3^{10}\}$
23	19	
29	1	

Cont'd.

Table 1*(Continued)*

17	8	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30}, H_8^1, H_{23}^1, H_8^2, H_3^3, H_8^3, H_2^5, H_3^5, H_3^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
23	26	
29	14	
17	9	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_7^2, H_2^6, H_1^{10}\}$
23	3	
29	27	
17	10	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_6^1, H_{21}^1, H_6^2, H_1^3, H_6^3, H_3^5, H_6^5, H_1^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
23	10	
29	10	
17	11	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^2, H_6^6, H_2^{10}\}$
23	17	
29	23	
17	12	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_4^1, H_{19}^1, H_4^2, H_4^3, H_9^3, H_1^5, H_4^5, H_4^6, H_1^{10}, H_1^{15}, H_2^{15}\}$
23	24	
29	6	
17	13	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^2, H_3^6, H_3^{10}\}$
23	1	
29	19	
17	14	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^1, H_{17}^1, H_2^2, H_2^3, H_7^3, H_2^5, H_5^5, H_2^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
23	8	
29	2	
17	15	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_1^2, H_1^6, H_1^{10}\}$
23	15	
29	15	
17	16	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{15}^1, H_{30}^1, H_{15}^2, H_5^3, H_{10}^3, H_3^5, H_6^5, H_3^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
23	22	
29	28	
17	17	

Cont'd.

Table 1*(Continued)*

23	29	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{14}^2, H_4^6, H_2^{10}\}$
29	11	
17	18	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{13}^1, H_{28}^1, H_{13}^2, H_3^3, H_8^3, H_1^5, H_4^5, H_3^6, H_1^{10}, H_1^{15}, H_2^{15}\}$
23	6	
29	24	
17	19	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{12}^2, H_2^6, H_3^{10}\}$
23	13	
29	7	
17	20	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{11}^1, H_{26}^1, H_{11}^2, H_1^3, H_6^3, H_2^5, H_5^5, H_1^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
23	20	
29	20	
17	21	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_{10}^2, H_5^6, H_1^{10}\}$
23	27	
29	3	
17	22	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_9^1, H_{24}^1, H_9^2, H_4^3, H_9^3, H_3^5, H_6^5, H_4^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
23	4	
29	16	
17	23	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_8^2, H_3^6, H_2^{10}\}$
23	11	
29	29	
17	24	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_7^1, H_{22}^1, H_7^2, H_2^3, H_7^3, H_1^5, H_4^5, H_2^6, H_1^{10}, H_1^{15}, H_2^{15}\}$
23	18	
29	12	
17	25	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_6^2, H_1^6, H_3^{10}\}$
23	25	
29	25	
17	26	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^1, H_{20}^1, H_5^2, H_5^3, H_{10}^3, H_2^5, H_5^5, H_5^6, H_2^{10}, H_1^{15}, H_2^{15}\}$
23	2	

Cont'd.

Table 1*(Continued)*

29	18	
17	27	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_4^2, H_4^6, H_1^{10}\}$
23	9	
29	21	
17	28	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^1, H_{18}^1, H_3^2, H_3^3, H_8^3, H_3^5, H_6^5, H_3^6, H_3^{10}, H_1^{15}, H_2^{15}\}$
23	16	
29	4	
17	29	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^2, H_2^6, H_3^{10}\}$
23	23	
29	17	
7	0	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_1^1, H_6^1, H_{11}^1, H_{16}^1, H_{21}^1, H_1^2, H_6^2, H_{11}^2, H_1^3, H_6^3, H_1^5, H_2^5, H_3^5, H_4^5, H_5^5, H_6^5, H_1^6, H_1^{10}, H_2^{10}, H_3^{10}, H_1^{15}, H_2^{15}\}$
13	0	
19	0	
7	13, 23	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^6\}$
13	1, 11	
19	19, 11	
7	9	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^2, H_7^2, H_{12}^2, H_2^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
13	3	
19	27	
7	2, 22	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_4^3, H_9^3, H_4^6, H_1^{15}, H_2^{15}\}$
13	4, 14	
19	16, 26	
7	5, 25	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_1^6\}$
13	5, 25	
19	5, 25	
7	18	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^1, H_8^1, H_{13}^1, H_{18}^1, H_{23}^1, H_{28}^1, H_3^2, H_8^2, H_{13}^2, H_3^3, H_8^3, H_1^5, H_2^5 H_3^5, H_4^5, H_5^5, H_6^5, H_3^6, H_1^{10}, H_2^{10}, H_3^{10}, H_1^{15}, H_2^{15}\}$
13	6	
19	4	
7	1, 11	

Cont'd.

Table 1*(Continued)*

13	7, 17	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^6\}$
19	13, 23	
7	4, 14	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^3, H_7^3, H_2^6, H_{16}^1, H_{21}^1, H_{26}^1, H_1^2, H_6^2, H_{11}^2, H_1^3, H_6^3, H_1^5, H_2^5, H_3^5, H_4^5, H_5^5, H_6^5, H_1^6, H_1^{10}, H_2^{10}, H_3^{10}, H_1^{15}, H_2^{15}\}$
13	8, 28	
19	2, 22	
7	27	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_4^2, H_9^2, H_{14}^2, H_4^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
13	9	
19	21	
7	10, 20	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_1^3, H_6^3, H_1^6, H_1^{15}, H_2^{15}\}$
13	10, 20	
19	10, 20	
7	6	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^1, H_{10}^1, H_{15}^1, H_{20}^1, H_{25}^1, H_{30}^1, H_5^2, H_{10}^2, H_{15}^2, H_5^3, H_{10}^{10}, H_1^5, H_2^5, H_3^5, H_4^5, H_5^5, H_6^5, H_5^6, H_1^{10}, H_2^{10} H_3^{10}, H_1^{15}, H_2^{15}\}$
13	12	
19	18	
7	19, 29	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^6\}$
13	13, 23	
19	7, 17	
7	15	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_1^2, H_6^2, H_{11}^2, H_1^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
13	15	
19	15	
7	8, 28	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^3, H_8^3, H_3^6, H_1^{15}, H_2^{15}\}$
13	16, 26	
19	14, 4	
7	24	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_2^1, H_7^1, H_{12}^1, H_{17}^1, H_{22}^1, H_{27}^1, H_2^2, H_7^2, H_{12}^2, H_2^3, H_7^3, H_1^5, H_2^5, H_3^5, H_4^5, H_5^5, H_6^5, H_2^6, H_1^{10}, H_2^{10}, H_3^{10}, H_1^{15}, H_2^{15}\}$
13	18	
19	12	
7	7, 17	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_4^6\}$
13	29, 19	

Cont'd.

Table 1*(Continued)*

19	1, 11	
7	3	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^2, H_8^2, H_{13}^2, H_3^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
13	21	
19	9	
7	16, 26	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^3, H_{10}^3, H_5^6, H_1^{15}, H_2^{15}\}$
13	2, 22	
19	8, 28	
7	12	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_3^2, H_8^2, H_{13}^2, H_3^6, H_1^{10}, H_2^{10}, H_3^{10}\}$
13	24	
19	6	
7	21	$\{H_0^1, H_0^2, H_0^3, H_0^5, H_0^6, H_0^{10}, H_0^{15}, H_0^{30}, H_1^{30} H_5^3, H_{10}^3, H_5^6, H_1^{15}, H_2^{15}\}$
13	27	
19	3	

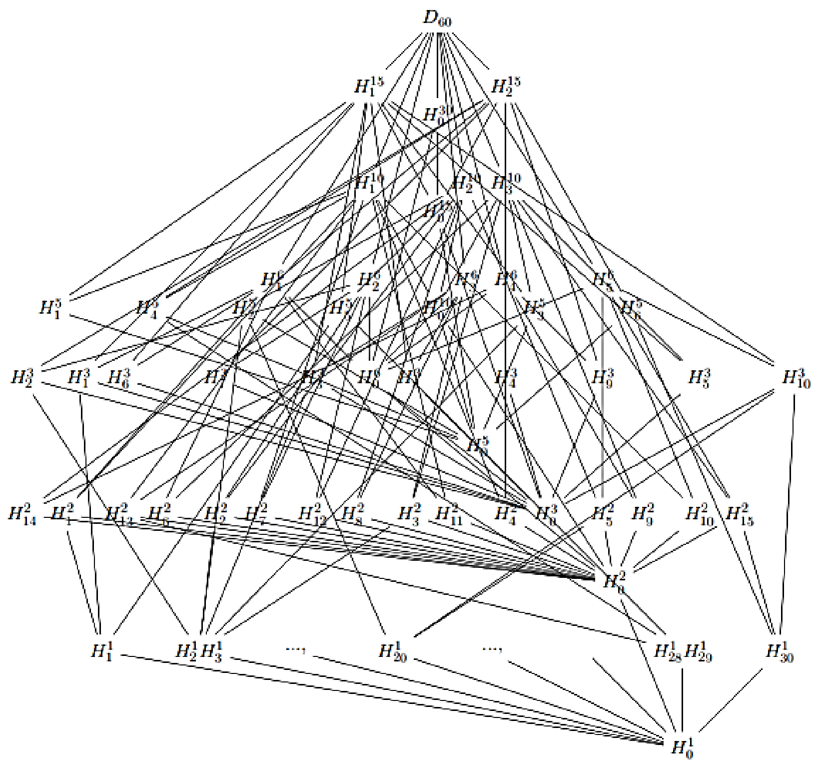

Figure 1: Subgroup lattice diagram of D_{60}

Figure 1 displays all the subgroups of D_{60}. Each subgroup is placed at a node according to its order and connected by lines. These connections helps in computing the chains, that is $|Fix(f_{\alpha,\beta})|$. Now, computing from subgroup lattice from Figure 1, we have the following values:

- $|Fix_{\bar{C}}(f_{1,0})| = 1324$

- $Fix_{\bar{C}}(f_{1,1})| = |Fix_{\bar{C}}(f_{1,7})| = |Fix_{\bar{C}}(f_{1,11})| = |Fix_{\bar{C}}(f_{1,13})| = |Fix_{\bar{C}}(f_{1,19})| =$
  $|Fix_{\bar{C}}(f_{1,23})| = |Fix_{\bar{C}}(f_{1,29})| = 52$

- $|Fix_{\bar{C}}(f_{1,2})| = |Fix_{\bar{C}}(f_{1,4})| = |Fix_{\bar{C}}(f_{1,8})| = |Fix_{\bar{C}}(f_{1,14})| = |Fix_{\bar{C}}(f_{1,16})| =$
  $|Fix_{\bar{C}}(f_{1,22})| = |Fix_{\bar{C}}(f_{1,26})| = |Fix_{\bar{C}}(f_{1,28})| = 76$

- $|Fix_{\bar{C}}(f_{1,3})| = |Fix_{\bar{C}}(f_{1,9})| = |Fix_{\bar{C}}(f_{1,21})| = |Fix_{\bar{C}}(f_{1,27})| = 88$

- $|Fix_{\bar{C}}(f_{1,5})| = |Fix_{\bar{C}}(f_{1,25})| = 112$

- $|Fix_{\bar{C}}(f_{1,6})| = |Fix_{\bar{C}}(f_{1,12})| = |Fix_{\bar{C}}(f_{1,18})| = |Fix_{\bar{C}}(f_{1,24})| = 184$

- $|Fix_{\bar{C}}(f_{1,10})| = |Fix_{\bar{C}}(f_{1,20})| = 256$

- $|Fix_{\bar{C}}(f_{1,15})| = 328$

- $|Fix_{\bar{C}}(f_{23,0})| = |Fix_{\bar{C}}(f_{17,0})| = |Fix_{\bar{C}}(f_{29,0})| = |Fix_{\bar{C}}(f_{23,14})| = |Fix_{\bar{C}}(f_{17,2})| =$
  $|Fix_{\bar{C}}(f_{29,26})| = |Fix_{\bar{C}}(f_{23,28})| = |Fix_{\bar{C}}(f_{17,4})| = |Fix_{\bar{C}}(f_{29,22})| = |Fix_{\bar{C}}(f_{23,12})| =$
  $|Fix_{\bar{C}}(f_{17,6})| = |Fix_{\bar{C}}(f_{29,18})| = |Fix_{\bar{C}}(f_{23,26})| = |Fix_{\bar{C}}(f_{17,8})| = |Fix_{\bar{C}}(f_{29,14})| =$
  $|Fix_{\bar{C}}(f_{23,10})| = |Fix_{\bar{C}}(f_{17,10})| = |Fix_{\bar{C}}(f_{29,10})| = |Fix_{\bar{C}}(f_{23,24})| = |Fix_{\bar{C}}(f_{17,12})|$
  $= |Fix_{\bar{C}}(f_{29,6})| = |Fix_{\bar{C}}(f_{23,8})| = |Fix_{\bar{C}}(f_{17,14})| = |Fix_{\bar{C}}(f_{29,2})| = |Fix_{\bar{C}}(f_{23,22})| =$
  $|Fix_{\bar{C}}(f_{17,16})| = |Fix_{\bar{C}}(f_{29,28})| = |Fix_{\bar{C}}(f_{23,6})| = |Fix_{\bar{C}}(f_{17,18})| = |Fix_{\bar{C}}(f_{29,24})| =$
  $|Fix_{\bar{C}}(f_{23,20})| = |Fix_{\bar{C}}(f_{17,20})| = |Fix_{\bar{C}}(f_{29,20})| = |Fix_{\bar{C}}(f_{23,4})| = |Fix_{\bar{C}}(f_{17,22})| =$
  $|Fix_{\bar{C}}(f_{29,16})| = |Fix_{\bar{C}}(f_{23,18})| = |Fix_{\bar{C}}(f_{17,24})| = |Fix_{\bar{C}}(f_{29,12})| = |Fix_{\bar{C}}(f_{23,2})| =$
  $|Fix_{\bar{C}}(f_{17,26})| = |Fix_{\bar{C}}(f_{29,18})| = |Fix_{\bar{C}}(f_{23,16})| = |Fix_{\bar{C}}(f_{17,28})| = |Fix_{\bar{C}}(f_{29,4})| = 212$

- $|Fix_{\bar{C}}(f_{23,7})| = |Fix_{\bar{C}}(f_{17,1})| = |Fix_{\bar{C}}(f_{29,13})| = |Fix_{\bar{C}}(f_{23,21})| = |Fix_{\bar{C}}(f_{17,3})| =$
  $|Fix_{\bar{C}}(f_{29,9})| = |Fix_{\bar{C}}(f_{23,5})| = |Fix_{\bar{C}}(f_{17,5})| = |Fix_{\bar{C}}(f_{29,5})| = |Fix_{\bar{C}}(f_{23,19})| =$
  $|Fix_{\bar{C}}(f_{17,7})| = |Fix_{\bar{C}}(f_{29,1})| = |Fix_{\bar{C}}(f_{23,3})| = |Fix_{\bar{C}}(f_{17,9})| = Fix_{\bar{C}}(f_{29,27})| =$
  $|Fix_{\bar{C}}(f_{23,17})| = |Fix_{\bar{C}}(f_{17,11})| = |Fix_{\bar{C}}(f_{29,23})| = |Fix_{\bar{C}}(f_{23,1})| = |Fix_{\bar{C}}(f_{17,13})| =$
  $|Fix_{\bar{C}}(f_{29,19})| = |Fix_{\bar{C}}(f_{23,15})| = |Fix_{\bar{C}}(f_{17,15})| = |Fix_{\bar{C}}(f_{29,15})| = |Fix_{\bar{C}}(f_{23,29})| =$
  $|Fix_{\bar{C}}(f_{17,17})| = |Fix_{\bar{C}}(f_{29,11})| = |Fix_{\bar{C}}(f_{23,13})| = |Fix_{\bar{C}}(f_{17,19})| = |Fix_{\bar{C}}(f_{29,7})| =$
  $|Fix_{\bar{C}}(f_{23,27})| = |Fix_{\bar{C}}(f_{17,21})| = |Fix_{\bar{C}}(f_{29,3})| = |Fix_{\bar{C}}(f_{23,11})| = |Fix_{\bar{C}}(f_{17,23})| =$
  $|Fix_{\bar{C}}(f_{29,29})| = |Fix_{\bar{C}}(f_{23,25})| = |Fix_{\bar{C}}(f_{17,25})| = |Fix_{\bar{C}}(f_{29,25})| = |Fix_{\bar{C}}(f_{23,9})| =$
  $|Fix_{\bar{C}}(f_{17,27})| = |Fix_{\bar{C}}(f_{29,21})| = |Fix_{\bar{C}}(f_{23,23})| = |Fix_{\bar{C}}(f_{17,29})| = |Fix_{\bar{C}}(f_{29,17})| = 88$

- $|Fix(f_{19,0})| = |Fix(f_{13,0})| = |Fix(f_{7,0})| = |Fix(f_{19,4})| = |Fix(f_{13,6})| = |Fix_{\bar{C}}(f_{7,18})| = |Fix_{\bar{C}}(f_{19,18})| = |Fix_{\bar{C}}(f_{13,12})| = |Fix_{\bar{C}}(f_{7,6})| = |Fix_{\bar{C}}(f_{19,12})| = |Fix_{\bar{C}}(f_{13,18})| = |Fix_{\bar{C}}(f_{7,24})| = |Fix_{\bar{C}}(f_{19,6})| = |Fix_{\bar{C}}(f_{13,24})| = |Fix_{\bar{C}}(f_{7,12})| = 412$

- $|Fix_{\bar{C}}(f_{19,19})| = |Fix_{\bar{C}}(f_{19,29})| = |Fix_{\bar{C}}(f_{13,1})| = |Fix_{\bar{C}}(f_{13,11})| = |Fix_{\bar{C}}(f_{7,13})| = |Fix_{\bar{C}}(f_{7,23})| = |Fix_{\bar{C}}(f_{19,5})| = |Fix_{\bar{C}}(f_{19,25})| = |Fix_{\bar{C}}(f_{13,5})| = |Fix_{\bar{C}}(f_{13,25})| = |Fix_{\bar{C}}(f_{7,5})| = |Fix_{\bar{C}}(f_{7,25})| = |Fix_{\bar{C}}(f_{19,7})| = |Fix_{\bar{C}}(f_{19,17})| = |Fix_{\bar{C}}(f_{13,13})| = |Fix_{\bar{C}}(f_{13,23})| = |Fix_{\bar{C}}(f_{7,19})| = |Fix_{\bar{C}}(f_{7,29})| = |Fix_{\bar{C}}(f_{19,13})| = |Fix_{\bar{C}}(f_{19,23})| = |Fix_{\bar{C}}(f_{13,17})| = |Fix_{\bar{C}}(f_{13,7})| = |Fix_{\bar{C}}(f_{7,1})| = |Fix_{\bar{C}}(f_{7,11})| = |Fix_{\bar{C}}(f_{19,1})| = |Fix_{\bar{C}}(f_{19,11})| = |Fix_{\bar{C}}(f_{13,19})| = |Fix_{\bar{C}}(f_{13,29})| = |Fix_{\bar{C}}(f_{7,7})| = |Fix_{\bar{C}}(f_{7,17})| = 64$

- $|Fix_{\bar{C}}(f_{19,27})| = |Fix_{\bar{C}}(f_{13,3})| = |Fix_{\bar{C}}(f_{7,9})| = |Fix_{\bar{C}}(f_{19,21})| = |Fix_{\bar{C}}(f_{13,9})| = |Fix_{\bar{C}}(f_{7,27})| = |Fix_{\bar{C}}(f_{19,5})| = |Fix_{\bar{C}}(f_{13,15})| = |Fix_{\bar{C}}(f_{7,15})| = |Fix_{\bar{C}}(f_{19,9})| = |Fix_{\bar{C}}(f_{13,21})| = |Fix_{\bar{C}}(f_{7,3})| = |Fix_{\bar{C}}(f_{19,3})| = |Fix_{\bar{C}}(f_{13,27})| = |Fix_{\bar{C}}(f_{7,21})| = 136$

- $|Fix(f_{19,10})| = |Fix(f_{19,20})| = |Fix(f_{13,10})| = |Fix(f_{13,20})| = |Fix(f_{7,10})| = |Fix(f_{7,20})| = |Fix(f_{19,28})| = |Fix(f_{19,8})| = |Fix(f_{13,22})| = |Fix(f_{13,2})| = |Fix(f_{7,26})| = |Fix(f_{7,16})| = |Fix(f_{19,2})| = |Fix(f_{19,22})| = |Fix(f_{13,28})| = |Fix(f_{13,8})| = |Fix(f_{7,4})| = |Fix(f_{7,14})| = Fix(f_{19,16})| = |Fix(f_{19,26})| = |Fix(f_{13,14})| = |Fix(f_{13,4})| = |Fix(f_{7,2})| = |Fix(f_{7,22})| = |Fix(f_{19,4})| = |Fix(f_{19,14})| = |Fix(f_{13,26})| = |Fix(f_{13,16})| = |Fix(f_{7,8})| = |Fix(f_{7,28})| = 112$

- $|Fix(f_{11,0})| = |Fix(f_{11,10})| = |Fix(f_{11,20})| = 612$

- $|Fix(f_{11,1})| = |Fix(f_{11,7})| = |Fix_{\bar{C}}(f_{11,13})| = |Fix_{\bar{C}}(f_{11,19})| = |Fix_{\bar{C}}(f_{11,11})| = |Fix_{\bar{C}}(f_{11,17})| = |Fix_{\bar{C}}(f_{11,23})| = |Fix_{\bar{C}}(f_{11,29})| = |Fix_{\bar{C}}(f_{11,3})| = |Fix_{\bar{C}}(f_{11,9})| = |Fix_{\bar{C}}(f_{11,21})| = |Fix_{\bar{C}}(f_{11,27})| = 64$

- $|Fix_{\bar{C}}(f_{11,2})| = |Fix_{\bar{C}}(f_{11,8})| = |Fix_{\bar{C}}(f_{11,14})| = |Fix_{\bar{C}}(f_{11,26})| = |Fix_{\bar{C}}(f_{11,4})| = |Fix_{\bar{C}}(f_{11,16})| = |Fix_{\bar{C}}(f_{11,22})| = |Fix_{\bar{C}}(f_{11,28})| = |Fix_{\bar{C}}(f_{11,6})| = |Fix_{\bar{C}}(f_{11,12})| = |Fix_{\bar{C}}(f_{11,18})| = |Fix_{\bar{C}}(f_{11,24})| = 112$

- $|Fix_{\bar{C}}(f_{11,5})| = |Fix_{\bar{C}}(f_{11,15})| = |Fix_{\bar{C}}(f_{11,25})| = 184.$

Theorem 3.3. *The number N of distinct fuzzy subgroups of finite dihedral group of order 60 with respect to the equivalence relation ≈ is 150.*

Proof. Now, from the formula in Tarnauceanu (2016) we have that the number of distinct fuzzy subgroups of any finite group is given by

$$N = \frac{1}{|Aut(G)|} \sum_{f \epsilon\, Aut(G)} |Fix_{\bar{C}}(f)|.$$

From computations above and by substitution we have the following

$$N = \frac{1}{240}[1324 + (52 \times 8) + (76 \times 8) + (88 \times 4) + (112 \times 2) + (256 \times 2) +$$

$$(184 \times 4) + (328 \times 1)(412 \times 15) + (64 \times 30) + (112 \times 30) + (136 \times 15) + (612 \times 3) + (64 \times 12) + (112 \times 12) + (184 \times 3)]$$
$$= \frac{36000}{240}$$
$$= 150. \square$$

4. Conclusion

Notice that $D_{60} = D_{2(2\times3\times5)}$, where 2, 3 and 5 are three distinct primes. One is tempted to ask; Will computation yield the same result when these distinct primes are different from 2, 3 and 5? We answer the question in affirmative. For example $D_{210} = D_{2(3\times5\times7)}, D_{770} = D_{2(5\times7\times11)}$ and $D_{330} = D_{2(3\times5\times11)}$ all have 150 distinct fuzzy subgroups. Finally, these computations show that, the equivalence relation ≈ used in this classification and counting is independent on the distinct primes but dependent on the number of distinct primes for this class of dihedral groups. Dihedral groups of this form have similar structures, hence they yield the same number of distinct fuzzy subgroups with respect to the equivalence relation ≈.

References

Liu W. J. 1982. Fuzzy invariant subgroups and fuzzy ideals. *Fuzzy Set and Systems* **8**: 133-139.
Mordeson J. N., Bhutani K. R. & Rosenfeld A. 2005. *Fuzzy Group Theory*. Berlin Heidelberg New York: Springer-Verlag.
Rosenfeld A. 1971. Fuzzy groups. *J. Math. Anal. Appl* **35**: 512-517.
Sulaiman R. & Abd Ghafur A. 2010. Counting fuzzy subgroups of symmetric groups S_2, S_3 and alternating group A_4. *Journal of Quality Measurement and Analysis* **6**(1): 57-63.
Sulaiman R. 2012. Constructing fuzzy subgroups of symmetric group S_4. *Int. Journal of Algebra* **6**(1):23-28.
Tarnauceanu M. 2012. Classifying fuzzy subgroups of finite non Abelian groups. *Iranian Journal of Fuzzy Systems* **9**(4): 33-43.
Tarnauceanu M. 2016. A new equivalence relation to classify fuzzy subgroups of finite groups. *Fuzzy Sets and Systems* **289**: 113-121.
Zadeh L. 1965. Fuzzy sets. *Information and Control* **8**: 338-353.

Department of Mathematics & Computer Science
SuleLamido University
Kafin Hausa Jigawa State +234, NIGERIA
*E-mail: olayiwola.abdhakm@gmail.com**, bangis4u@gmail.com*

*Corresponding author